On set-valued intertwining duality for diffusion processes

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Based on joint works with
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Plan of the mini-lectures

1. **Strong stationary times for one-dimensional diffusions**
In particular we will see that a positive recurrent elliptic diffusion on $\mathbb{R}$ admits a strong stationary time, whatever its initial distribution, if and only if $-\infty$ and $+\infty$ are entrance boundaries.

2. **Duality and hypoellipticity for one-dimensional diffusions**
It will be shown that the convergence to equilibrium of hypo-elliptic diffusions on the circle can also be understood via intertwining relations.

3. **Stochastic evolutions of domains on manifolds**
We introduce stochastic modifications of mean curvature flows on manifolds and prove their existence at least for small times.

4. **Algebraic intertwining relations on manifolds**
We see how the previous evolutions serve as set-valued duals for diffusions on manifolds and present some of their properties.

Plan of the first lecture

Strong stationary times
for one-dimensional diffusions

Plan

Strong stationary times

Consider a Markov process $X := (X_t)_{t \geqslant 0}$. A finite stopping time τ (relative to the filtration generated by X and possibly some independent randomness) is said to be **strong** if τ and X_τ are independent. When furthermore X admits μ as invariant probability, if X_τ is distributed according to μ, then τ is called a **strong stationary time**. The notion was introduced by Aldous and Diaconis [1986] in the context of finite Markov chains, but an early example can be found in Dubins [1968], already for one-dimensional diffusions.

Dubins' example

Consider for X the Brownian motion on $[0,1]$, reflected at 0 and 1 and starting from $1/2$. It is positive recurrent with the restriction of the Lebesgue measure as invariant probability. A strong stationary time can be constructed as follows: let τ_1 be the first time X hits $1/4$ or $3/4$. Next let τ_2 be the first time after τ_1 that $X_{\tau_1} \pm 1/8$ is reached. Iteratively, τ_{n+1} is the first time after τ_n that $X_{\tau_n} \pm 1/2^{n+2}$ is hit. The limit $\tau := \lim_{n\to\infty} \tau_n$ exists a.s. and is a strong stationary time for X.

This construction can be extended to any initial distribution, by first waiting that $1/2$ is reached (not always smart, for instance if X_0 was already at equilibrium).

Intertwining

In the finite framework, Diaconis and Fill [1990] developed the tool of intertwining relations with absorbed Markov chains to construct strong stationary times. Intertwining of diffusions was also investigated by Rogers and Pitman [1981] and Carmona, Petit and Yor [1998], especially to deduce identities in law for particular processes. Pal and Shkolnikov [2013] studied some conditions insuring that there exists an intertwining between two Markov semi-groups and their article also provides a survey of applications of intertwining relations. Our goal is to come back to the investigation of strong stationary times through intertwining, but in the context of diffusions. This subject was also investigated by Fill and Lyzinski [2016].

One-dimensional diffusions

Consider on $\mathbb{R}$ the Markovian generator

$$L \quad := \quad a\partial^2 + b\partial$$

with regular coefficients $a > 0$ and b, and introduce

$$\forall\ x \in \mathbb{R}, \qquad c(x) \quad := \quad \int_0^x \frac{b(y)}{a(y)}\,dy$$
$$\mu(x) \quad := \quad \frac{\exp(c(x))}{a(x)}$$

(μ is the **speed density**, the **scale density** is $\exp(-c)$). We assume that μ has a finite mass (then it is renormalized into a probability measure) and furthermore that the process X associated to L is **positive recurrent** (a.k.a. **ergodic**):

$$\int_{-\infty}^0 \exp(-c(y))\,dy \;=\; +\infty \quad \text{and} \quad \int_0^\infty \exp(-c(y))\,dy \;=\; +\infty$$

Main result

Define

$$
\begin{aligned}
I_- &:= \int_{-\infty}^{0} \left(\int_x^0 \exp(-c(y))\, dy \right) \mu(dx) \\
I_+ &:= \int_{0}^{+\infty} \left(\int_0^x \exp(-c(y))\, dy \right) \mu(dx) \\
I &:= \max(I_-, I_+)
\end{aligned}
$$

Theorem 1

Assume that X is positive recurrent. There exists a strong stationary time for X, whatever its initial distribution, if and only if $I < +\infty$.

Trace class

The same result holds for diffusions on the half-line $\mathbb{R}_+$, just replace I by I_+. In this context (but it should be also true on $\mathbb{R}$), Cheng and Mao [2013] have shown that the condition $I_+ < +\infty$ is equivalent to the strong ergodicity of X:

$$\exists\ C, \epsilon > 0\ :\ \forall\ \mathcal{L}(X_0),\ \forall\ t \geqslant 0,\ \|\mathcal{L}(X_t) - \mu\|_{\mathrm{tv}} \quad \leqslant \quad C\exp(-\epsilon t)$$

and to the centered Green operator G having a finite trace, where

$$\forall\ f \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}_+),\ \forall\ x \in \mathbb{R}_+,\ \ G[f](x) \quad := \quad \int_0^{+\infty} \mathbb{E}_x[f(X_t) - \mu[f]]\, dt$$

(namely L has no continuous spectrum and the sum of the inverses its non-zero eigenvalues converges). Nevertheless this equivalence is not true for general Markov processes.

Transposing to the setting of an ergodic diffusion generator L the program described by Diaconis and Fill [1990], we are looking for a state space E^*, a Markov kernel Λ from E^* to $\mathbb{R}$ and a Markovian generator L^* on E^* satisfying the intertwining relation $L^*\Lambda = \Lambda L$ on a sufficiently large domain of functions. Assume furthermore that the generator L^* leads to an absorbed process Z^*. By ergodicity of L, it follows that for any absorbing point $\infty \in E^*$, we have $\Lambda(\infty, \cdot) = \mu$. In principle and when $\mathcal{L}(X_0) = \mathcal{L}(Z_0^*)\Lambda$, one should be able to couple X and Z^* through a **probabilistic intertwining**, such that the absorption time of Z^* is a strong stationary time for X. At least this is always true for finite state spaces.

(i) For any $t \geqslant 0$, the piece of trajectory $Z^*_{[0,t]}$ is constructed from $X_{[0,t]}$ and independent randomness. So that any stopping time τ with respect to the filtration generated by the process Z^* is also a stopping time for X.

(ii) For any finite stopping time τ with respect to the filtration generated by the process Z^*:

$$\mathcal{L}(X_\tau | Z^*_{[0,\tau]}) = \Lambda(Z^*_\tau, \cdot)$$

In particular, under the previous conditions, if we consider the absorbing time τ^* of Z^*, then

$$\mathcal{L}(X_{\tau*} | Z^*_{[0,\tau*]}) = \mu$$

so that τ^* is a strong stationary time for X.

Here is a solution: E^* is the set of extended segments

$$\begin{aligned} E^* &:= \{(x,y) \,:\, x,y \in [-\infty,+\infty],\, x \leqslant y\} \setminus \{(-\infty,-\infty),(+\infty,+\infty)\} \\ \mathring{E}^* &:= \{(x,y) \in \mathbb{R}^2 \,:\, x < y\} \\ D^* &:= \{(x,x) \in E^* \,:\, x \in \mathbb{R}\} \end{aligned}$$

Λ is the conditioning of μ on these segments:

$$\Lambda((x,y),A) \quad := \quad \begin{cases} \delta_x(A) & \text{, if } y = x \\ \frac{\mu([x,y]\cap A)}{\mu([x,y])} & \text{, otherwise} \end{cases}$$

for any $(x,y) \in E^*$ and for any Borelian set $A \subset \mathbb{R}$.
The description of the diffusion generator L^* is more frightening:

A dual process

- on $\mathring{E}^*$,

$$L^* := (\sqrt{a(y)}\partial_y - \sqrt{a(x)}\partial_x)^2 + (a'(x)/2 - b(x))\partial_x + (a'(y)/2 - b(y))\partial_y$$
$$+2\frac{\sqrt{a(x)}\mu(x) + \sqrt{a(y)}\mu(y)}{\mu([x,y])}(\sqrt{a(y)}\partial_y - \sqrt{a(x)}\partial_x)$$

- on $\mathbb{R} \times \{+\infty\}$,

$$L^* := (\sqrt{a(x)}\partial_x)^2 + (a'(x)/2 - b(x))\partial_x - 2\frac{\sqrt{a(x)}\mu(x)}{\mu([x,+\infty))}\sqrt{a(x)}\partial_x$$

- on $\{-\infty\} \times \mathbb{R}$,

$$L^* := (\sqrt{a(y)}\partial_y)^2 + (a'(y)/2 - b(y))\partial_y + 2\frac{\sqrt{a(y)}\mu(y)}{\mu((-\infty,y])}\sqrt{a(y)}\partial_y$$

and a Dirichlet condition is put at $\infty := (-\infty, +\infty)$.

It is not necessary to make precise the boundary condition on the diagonal D^*, because it is an entrance boundary:

Proposition 2

For any initial distribution on E^*, there is a continuous Markov process $Z^* := (Z^*_t)_{t \geqslant 0}$:
- starting with this condition,
- whose generator is L^* (in the sense of martingale problems),
- satisfying for all $t > 0$, $Z^*_t \in E^* \backslash D^*$,
- which is absorbed at ∞ (if it reaches it).
Furthermore the law of such a Z^* is uniquely determined.

But the generator L^* is not the uniquely one which can be intertwined with L through Λ:

Other dual processes

This relation is also true if L^* is replaced by

$$\check{L}^* := (\sqrt{a(y)}\partial_y + \sqrt{a(x)}\partial_x)^2 + (a'(x)/2 - b(x))\partial_x + (a'(y)/2 - b(y))\partial_y$$
$$+2\frac{\sqrt{a(y)}\mu(y) - \sqrt{a(x)}\mu(x)}{\mu([x,y])}(\sqrt{a(y)}\partial_y + \sqrt{a(x)}\partial_x)$$

(on $\mathring{E}^*$ and its natural extensions on $\mathbb{R} \times \{+\infty\}$ and $\{-\infty\} \times \mathbb{R}$).
But D^* is no longer an entrance boundary, because the drift coefficient does not degenerate near D^*: an associated process starting in D^* stays in D^*...
For any $\alpha \in (0,1)$, the generator $L^*_\alpha := (1-\alpha)L^* + \alpha\check{L}^*$ also satisfies the intertwining relation and is elliptic. But this is not an advantage, as it can be shown that the associated strong stationary time (if it is finite) stochastically strictly dominates the one corresponding to $L^* = L^*_0$.

Explosion

Consider the (complete) explosion time for Z^*:

$$\tau^* \quad := \quad \inf\{t \geqslant 0 \,:\, Z_t^* = (-\infty, +\infty)\}$$

Up to the construction of the intertwining between X and Z^*, the previous arguments, the fact for any initial probability m_0 on $\mathbb{R}$, we can find a distribution m_0^* on E^* such that $m_0 = m_0^* \Lambda$ (take for instance $m_0^* := \int \delta_{(x,x)}\, m_0(dx)$) and the next result, provide the direct implication in Theorem 1:

Proposition 3

The random time τ^* is a.s. finite, whatever $\mathcal{L}(Z_0^*)$, if and only if $I < +\infty$. In this case, τ^* is a strong stationary time for the positive recurrent diffusion X.

Separation discrepancy

The **separation discrepancy** $\mathfrak{s}(\nu, \mu)$ between two probability measures ν and μ on E is defined by

$$\mathfrak{s}(\nu, \mu) \quad := \quad \sup_{x \in E} 1 - \frac{d\nu}{d\mu}(x)$$

The computations of Aldous and Diaconis [1986] show that for any strong stationary time τ for X, we have

$$\forall \ t \geqslant 0, \qquad \mathfrak{s}(\mathcal{L}(X_t), \mu) \quad \leqslant \quad \mathbb{P}[\tau > t]$$

These inequalities may be equalities for all times $t \geqslant 0$ and such times τ are then stochastically minimal among all strong stationary times. They are called **sharp stationary times**. The converse implication in Theorem 1 relies on the fact that for initial distributions of X of the form $\Lambda((-\infty, x), \cdot)$ and $\Lambda((x, +\infty), \cdot)$, with $x \in \mathbb{R}$, the random time τ^* defined as above is indeed a sharp stationary time.

When is this technique working? Consider Langevin diffusions: $a \equiv 1$ and $b = -U'$, where $U : \mathbb{R} \to \mathbb{R}$ is a smooth potential. The (density of the) invariant measure μ is then proportional to $\exp(-U)$. The condition $I < +\infty$ writes down

$$\max\left(\int_{-\infty}^{0} \mu((-\infty, x))\frac{1}{\mu(x)}\,dx, \int_{0}^{+\infty} \mu((x, +\infty))\frac{1}{\mu(x)}\,dx\right) \quad < \quad +\infty$$

If for $|x|$ large enough, $U(x) = |x|^\alpha$, with $\alpha > 0$, the above condition is satisfied if and only if $\alpha > 2$, in particular, the benchmark Ornstein-Uhlenbeck process is not covered. This could also have been guessed from $\sum_{n\in\mathbb{N}} 1/n = +\infty$.
We will see how to get around this difficulty by considering other strong times τ.

Plan

Evolving sets

Consider the generator given on $\mathring{E}^*$ by

$$\widetilde{L} := (\sqrt{a(y)}\partial_y - \sqrt{a(x)}\partial_x)^2 + (a'(x)/2 - b(x))\partial_x + (a'(y)/2 - b(y))\partial_y$$

(and its natural extensions on $\mathbb{R} \times \{+\infty\}$ and $\{-\infty\} \times \mathbb{R}$). The diagonal is not an entrance boundary, impose Dirichlet boundary condition there, as well as on $(-\infty, +\infty)$, $(-\infty, -\infty)$ and $(+\infty, +\infty)$, to define an associated process. It is the continuous equivalent of the evolving sets introduced by Morris and Peres [2005] for denumerable Markov chains. Consider the mapping h defined on E^* by

$$\forall\ z = (x, y) \in E^*, \qquad h(z) \quad := \quad \mu([x, y])$$

Doob transform

It is not difficult to check that $\widetilde{L}[h] = 0$.
The generator L^* is the h-**transform** of $\widetilde{L}$:

$$\begin{aligned} L^*[\cdot] &= \frac{1}{h}\widetilde{L}[h\,\cdot\,] \\ &= \widetilde{L}[\cdot] + \widetilde{\Gamma}[\ln(h), \cdot] \end{aligned}$$

where $\widetilde{\Gamma}$ is the **carré du champ** associated to $\widetilde{L}$: for any smooth functions f, g defined on E^*,

$$\widetilde{\Gamma}[f, g] \quad := \quad \widetilde{L}[fg] - f\widetilde{L}[g] - g\widetilde{L}[f]$$

In particular we get $L^*[1/h] = 0$.

Thus if Z^* is started with a condition $z_0 \in \mathring{E}^*$, then $(1/h(Z_t^*))_{t\geqslant 0}$ is a positive (local) martingale. By the usual convergence theorem for such a martingale, Z^* cannot approach D^* and it can only exit $\mathring{E}^*$ through $(\mathbb{R}\times\{+\infty\}) \sqcup (\{-\infty\}\times\mathbb{R}) \sqcup \{(-\infty,+\infty)\}$. So there is no difficulty about the construction of Z^*. Writing $Z^* = (X^*, Y^*)$, it is given as the solution of the s.d.e.

$$\begin{aligned} dX_t^* &= -2\left(\frac{\sqrt{a(X_t^*)}\mu(X_t^*) + \sqrt{a(Y_t^*)}\mu(Y_t^*)}{\mu([X_t^*, Y_t^*])}\sqrt{a(X_t^*)}\right) dt \\ &\quad +(a'(X_t^*) - b(X_t^*))dt - \sqrt{2a(X_t^*)}\, dB_t \\ dY_t^* &= 2\left(\frac{\sqrt{a(X_t^*)}\mu(X_t^*) + \sqrt{a(Y_t^*)}\mu(Y_t^*)}{\mu([X_t^*, Y_t^*])}\sqrt{a(Y_t^*)}\right) dt \\ &\quad +(a'(Y_t^*) - b(Y_t^*))dt + \sqrt{2a(Y_t^*)}\, dB_t \end{aligned}$$

where $B = (B_t)_{t\geqslant 0}$ is a standard Brownian motion.

A Pitman-type property enables so solve almost all technical difficulties. For $z_0 \in \mathring{E}^*$, designate by $\mathbb{P}_{z_0}$ the law on the set of trajectories $\mathcal{C}(\mathbb{R}_+, E^*)$ of Z^* starting from z_0. Consider

$$\varsigma \;:=\; 2\int_0^{\tau^*} (\sqrt{a(X_s^*)}\mu(X_s^*) + \sqrt{a(Y_s^*)}\mu(Y_s^*))^2\, ds \;\in (0, +\infty]$$

and the time change $(\theta_t)_{t\in[0,\varsigma]}$ given by

$$2\int_0^{\theta_t} (\sqrt{a(X_s^*)}\mu(X_s^*) + \sqrt{a(Y_s^*)}\mu(Y_s^*))^2\, ds \;=\; t$$

We are interested in the process $R := (R_t)_{t \geqslant 0}$ given by

$$\forall\ t \geqslant 0, \qquad R_t \;\; := \;\; h(Z^*_{\theta_{t \wedge \varsigma}})$$

Proposition 4

Under $\mathbb{P}_{z_0}$ with $z_0 \in \mathring{E}^*$, R has the law of a Bessel process of dimension 3 starting from $h(z_0) \in (0,1)$ and stopped at 1. In particular ς is distributed as the first reaching time of 1 for this process.

Recall that the 3-dimensional Bessel process R solves the s.d.e.

$$dR_t \;\; = \;\; \frac{1}{R_t}\,dt + dW_t$$

The proof is based on usual stochastic calculus and on

$$L^*[h] = \frac{1}{h}(2h\widetilde{L}[h] + \widetilde{\Gamma}[h,h]) = \frac{1}{h}\widetilde{\Gamma}[h,h]$$

By taking into account that

$$\lim_{t\to\tau^*-} h(Z_t^*) \;=\; \lim_{t\to\varsigma-} R_t \;=\; 1$$

we get as a first consequence that (almost surely),

$$\begin{aligned}\lim_{t\to\tau^*-} X_t^* &= -\infty \\ \lim_{t\to\tau^*-} Y_t^* &= +\infty\end{aligned}$$

The question is now to determine if $\tau^* < +\infty$.
The Pitman property also enables to deduce the existence and uniqueness of the law of Z^* starting from a point of D^*, essentially due to the fact that 0 is an entrance boundary for R.

Plan

Our next goal is to check that $I < +\infty$ implies that $\tau^* < +\infty$ (a.s.).
By symmetry, it is sufficient to work with Y^* and to show that $\tau^+ := \inf\{t \geqslant 0 : Y^*_t = +\infty\} < +\infty$ if $I_+ < +\infty$. This leads to consider on $\mathbb{R}_+$ the reflected diffusion

$$dU_t := \big(a'(U_t) - b(U_t) + 2a(U_t)k'(U_t)\big)\, dt + \sqrt{2a(U_t)}\, dB_t + dl_t(U)$$

up to the explosion time $\tau(U) = \inf\{t \geqslant 0 : U_t = +\infty\}$, where $(l_t(U))_{t\geqslant 0}$ is the local time of U at 0 and where k is the mapping $\mathbb{R} \ni x \mapsto \ln(\mu((-\infty, x]))$. Indeed, a traditional comparison result says that if Y^* and U start from the same initial condition and are driven with the same Brownian motion, then U stays below Y^* up to the time when U reaches 0.

Taking into account a renewal property, it is enough to obtain:

Lemma 5

The explosion time $\tau(U)$ is finite almost surely if and only if $I_+ < +\infty$.

After symmetrization of U, the proof is based on the well-known criterion: if V is a diffusion solution of

$$dV_t = \widehat{b}(V_t)dt + \sqrt{2\widehat{a}(V_t^*)}\, dB_t$$

with odd/even coefficients, then $\inf\{t \geqslant 0 : \lim_{s \to t-} |V_s| = +\infty\}$ is a.s. finite if and only if

$$\int_0^{+\infty} \exp\left(-\int_0^x \frac{\widehat{b}(y)}{\widehat{a}(y)}\, dy\right) \int_0^x \exp\left(\int_0^z \frac{\widehat{b}(u)}{\widehat{a}(u)}\, du\right) \frac{dz}{\widehat{a}(z)}\, dx < +\infty$$

The reverse part is important when the initial law of Z^* is $(-\infty, y^*)$, with some $y^* > 0$: in this case $X^* \equiv -\infty$ and Y^* coincides with U, up to its reaching time of 0. If follows easily that $\tau^* < +\infty$ if and only if $I_+ < +\infty$.
In this particular situation, τ^* is a sharp stationary time, because $\mathcal{L}(X_t) = \mathbb{E}[\Lambda((-\infty, Y_t^*), \cdot)]$

$$\begin{aligned}
\mathfrak{s}(\mathcal{L}(X_t), \mu) &= \sup_{x\in\mathbb{R}} \mathbb{E}\left[1 - \frac{d\Lambda((-\infty, Y_t^*), \cdot)}{d\mu}(x)\right] \\
&= 1 - \lim_{x\to+\infty} \mathbb{E}\left[\frac{d\Lambda((-\infty, Y_t^*), \cdot)}{d\mu}(x)\right] \\
&= 1 - \mathbb{P}[Y_t^* = +\infty] \\
&= \mathbb{P}[\tau^* \leqslant t]
\end{aligned}$$

Thus if there exists a strong stationary time for X, τ^* must be finite a.s.

Plan

All the previous considerations are relevant if there exists an intertwining of X with Z^*. We begin with

Lemma 6

For any $f \in \mathcal{C}^2(\mathbb{R})$ such that f and $L[f]$ belong to $\mathbb{L}^1(\mu)$, we have

$$\forall\ z \in E^* \backslash (D^* \sqcup \{(-\infty, +\infty)\}), \qquad \Lambda[L[f]](z) \;=\; L^*[\Lambda[f]](z)$$

Indeed, in one hand, by definition,

$$L^*[\Lambda[f]](z) \;=\; \frac{1}{h(z)} \widetilde{L}[F](z)$$

where

$$\forall\ (x', y') \in E^*, \qquad F(x', y') \;:=\; \int_{x'}^{y'} f(u)\, \mu(du)$$

Since $\partial_x F(x,y) = -\mu(x)f(x)$ and $\partial_y F(x,y) = \mu(y)f(y)$, it follows from the expression for μ that for $(x,y) \in \mathring{E}^*$,

$$L^*[\Lambda[f]](x,y) = \frac{1}{h(x,y)} \left(a(y)\mu(y)\partial_y f(y) - a(x)\mu(x)\partial_y f(x)\right)$$

On the other hand, factorizing L under the form $\frac{1}{\mu}\partial(a\mu\partial\ \cdot)$, we get

$$\begin{aligned}\int_x^y L[f](u)\,\mu(du) &= \int_x^y \partial(a\mu\partial f)(u)\,du \\ &= a(y)\mu(y)\partial f(y) - a(x)\mu(x)\partial f(x)\end{aligned}$$

The commutation relation follows on $\mathring{E}^*$. Similar computations are valid on $\{-\infty\} \times \mathbb{R}$ and $\mathbb{R} \times \{+\infty\}$.

Commutation relations for the semi-groups

Writing $P_t = \exp(tL)$ and $P_t^* = \exp(tL^*)$, the next result could seem obvious:

Proposition 7

Assume that X is positive recurrent. Then for all $T \geqslant 0$ and all bounded and continuous function f on $\mathbb{R}$, we have

$$\forall\, z \in E^*, \qquad \Lambda[P_T[f]](z) \;=\; P_T^*[\Lambda[f]](z)$$

But technically it was not so simple, since we did not find an appropriate Banach setting for $(P_t^*)_{t \geqslant 0}$. Instead, we resorted to the classical trick of investigating the evolution of

$$[0, T] \ni t \;\mapsto\; P_t^*[\Lambda[P_{T-t}[f]]]$$

and to the martingale problem satisfied by Z^*.

Skeletons

Applied with $T = 2^{-N}$, the previous result enables to adapt the construction of Diaconis and Fill [1990], to obtain an intertwining Markov chain $(\bar{X}^{(N)}_{n2^{-N}}, \bar{Z}^{(N,*)}_{n2^{-N}})_{n\in\mathbb{Z}_+}$, assuming that $\mathcal{L}(X_0) = \mathcal{L}(Z^*_0)\Lambda$:

- $(\bar{X}^{(N)}_{n2^{-N}})_{n\in\mathbb{Z}_+}$ and $(X_{n2^{-N}})_{n\in\mathbb{Z}_+}$ have the same law
- $(\bar{Z}^{(N,*)}_{n2^{-N}})_{n\in\mathbb{Z}_+}$ and $(Z^*_{n2^{-N}})_{n\in\mathbb{Z}_+}$ have the same law
- $\forall\ m \in \mathbb{Z}_+$, the conditional law of $\bar{X}^{(N)}_{m2^{-N}}$ knowing
$$\bar{Z}^{(N,*)}_0, \bar{Z}^{(N,*)}_{2^{-N}}, \ldots, \bar{Z}^{(N,*)}_{m2^{-N}} \text{ is } \Lambda(\bar{Z}^{(N,*)}_{m2^{-N}}, \cdot\,)$$

- $\forall\ m \in \mathbb{Z}_+$, the conditional law of $(\bar{Z}^{(N,*)}_0, \bar{Z}^{(N,*)}_{2^{-N}}, \ldots, \bar{Z}^{(N,*)}_{m2^{-N}})$ knowing $(\bar{X}^{(N)}_{n2^{-N}})_{n\in\mathbb{Z}_+}$ only depends on $\bar{X}^{(N)}_0, \bar{X}^{(N)}_{2^{-N}}, \ldots, \bar{X}^{(N)}_{m2^{-N}}$

Considering the natural extension to continuous time:

$$\forall\ t \geqslant 0, \qquad (\bar{X}_t^{(N)}, \bar{Z}_t^{(N,*)}) := (\bar{X}_{\lfloor t2^N \rfloor 2^{-N}}^{(N)}, \bar{Z}_{\lfloor t2^N \rfloor 2^{-N}}^{(N,*)})$$

we get that the sequence of the laws of $(\bar{X}^{(N)}, \bar{Z}^{(N,*)})$, for $N \in \mathbb{N}$, on the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R} \times E^*)$, is relatively compact. We can thus extract a subsequence converging to a probability measure $\mathbb{P}$ which is necessarily supported by the set of continuous trajectories. Under this law, the canonical coordinate process $(\bar{X}_t, \bar{Z}_t^*)_{t \in \mathbb{R}_+}$ is a coupling of X with Z^* satisfying for all $t \in \mathbb{R}_+$,

- the conditional law of $\bar{X}_t$ knowing $\bar{Z}_{[0,t]}^*$ is $\Lambda(\bar{Z}_t^*, \cdot\,)$
- the conditional law of $\bar{Z}_{[0,t]}^*$ knowing $\bar{X}$ depends only on $\bar{X}_{[0,t]}$

This is the wanted intertwining relation.

Plan

Ornstein-Uhlenbeck process

An Ornstein-Uhlenbeck process X is a solution of

$$\forall\ t \geqslant 0, \qquad dX_t \;=\; -X_t\,dt + \sqrt{2}dB_t$$

and the variation of parameters method gives:

$$X_t \;=\; \exp(-t)X_0 + \sqrt{2}\int_0^t \exp(s-t)\,dB_s$$

Let us deal with the case $X_0 = 0$. Explicit computations furnish the exponential rate for the convergence in total variation:

Lemma 8

We have

$$\lim_{t\to+\infty} \frac{1}{t}\ln(\|\mathcal{L}(X_t) - \gamma\|_{\mathrm{tv}}) \;=\; -2$$

Dual diffusion

Despite that there is no strong stationary time, could this result be recovered with strong times? The previous constructions are still valid and we get by symmetry that $Z^* = (-Y^*, Y^*)$, with

$$\forall\ t > 0, \qquad dY_t^* \ = \ (Y_t^* + g(Y_t^*))\, dt + \sqrt{2}\, dB_t$$

where g is the mapping defined by

$$\forall\ y > 0, \qquad g(y) \ := \ 2\frac{\gamma(y)}{\gamma([0, y])}$$

X and Y^* can be intertwined as before: let $L^\dagger$ be the generator of Y^* and $\Lambda^\dagger$ be the kernel given by $\Lambda^\dagger(y^*, \cdot) := \Lambda((-y^*, y^*), \cdot)$ for $y^* \geqslant 0$. We have

$$L^\dagger \Lambda^\dagger \ = \ \Lambda^\dagger L$$

Strong times

From the intertwining, we deduce that for any $M > 0$,

$$\tau_M^* \quad := \quad \inf\{t \geqslant 0 \, : \, Y_t^* = M\}$$

is a strong time for X. Let $\gamma_{[-M,M]}$ be the conditioning of γ on the interval $[-M, M]$. We have

Lemma 9

For all $t \geqslant 0$ and $M > 0$, we have

$$\|m_t - \gamma\|_{\mathrm{tv}} \quad \leqslant \quad \mathbb{P}[\tau_M^* > t] + \left\|\gamma_{[-M,M]} - \gamma\right\|_{\mathrm{tv}}$$

The independence of τ_M^* and $X_{\tau_M^*}$ is crucial in the proof: it is a kind of stochastic renewal property, which enables to use after time τ^* the non-increasingness of the mapping

$$\mathbb{R}_+ \ni s \quad \mapsto \quad \|\mathcal{L}(X_s) - \gamma\|_{\mathrm{tv}}$$

A sub-optimal idea

The second term is easy to bound: for all $M > 0$,

$$\left\|\gamma_{[-M,M]} - \gamma\right\|_{\mathrm{tv}} \leqslant \frac{\sqrt{2}}{\sqrt{\pi} M} \exp(-M^2/2)$$

For the first term, we could use a comparison of Y^* with $|Y|$, where

$$\forall\ t \geqslant 0, \qquad dY_t = Y_t\, dt + \sqrt{2} dB_t$$

But this process is more lazy near 0 and this leads to the sub-optimal bound

$$\begin{aligned} \mathbb{P}[\tau_M^* > t] &\leqslant \mathbb{P}[\tau_M(|Y|) > t]] \\ &\leqslant \sqrt{\frac{2}{(1 - e^{-2t})\pi}} M e^{-t} \end{aligned}$$

$\mathbb{L}^2$ framework

To recover the exponent 2, we resort to a $\mathbb{L}^2$ point of view. Note that $L^\dagger = \exp(-V)\partial\exp(V)\partial$, which makes it apparent that $L^\dagger$ is symmetric in $\mathbb{L}^2(\nu)$, where ν is the σ-finite measure on $\mathbb{R}_+$ whose density is $\exp(V)$, with

$$\forall\ y \in \mathbb{R}_+, \qquad V(y) \ := \ \frac{y^2}{2} + 2\ln(\gamma([0,y]))$$

Thus $L^\dagger$ can be extended into its Freidrich extension in $\mathbb{L}^2(\nu)$. We will denote $(P_t^\dagger)_{t\geqslant 0}$ the associated semi-group.
Consider $(H_n)_{n\in\mathbb{Z}_+}$ the Hermite polynomials defined by

$$\forall\ n \in \mathbb{Z}_+,\ \forall\ x \in \mathbb{R}, \qquad H_n(x) \ := \ (-1)^n\exp(x^2/2)\partial^n\exp(-x^2/2)$$

They form a orthogonal basis of $\mathbb{L}^2(\gamma)$ and diagonalize L:

$$\forall\ n \in \mathbb{Z}_+, \qquad L[H_n] = -nH_n$$

Note that H_n is even (respectively odd) if n is even (resp. odd). It follows that $\Lambda^\dagger[H_n] = 0$ if n is odd. Since $H_0 \equiv 1$, we get that $\Lambda^\dagger[H_0] \equiv 1$ and this function does not belong to $\mathbb{L}^2(\nu)$. For the remaining Hermite polynomials, denote $H_{2n}^\dagger := \Lambda^\dagger[H_{2n}]$, for $n \in \mathbb{N}$. These functions can be computed explicitly: they belong to $\mathbb{L}^2(\nu)\setminus\{0\}$, and satisfy $L^\dagger H_{2n}^\dagger = -2nH_{2n}^\dagger$. Furthermore $(H_{2n}^\dagger)_{n\in\mathbb{N}}$ is an orthogonal Hilbertian basis of $\mathbb{L}^2(\nu)$. Thus the spectrum of $L^\dagger$ is $-2\mathbb{N}$. By self-adjointness, we deduce that

$$\forall\ t \geqslant 0, \forall\ f \in \mathbb{L}^2(\nu), \qquad \left\| P_t^\dagger[f] \right\|_{\mathbb{L}^2(\nu)} \leqslant \exp(-2t)\, \|f\|_{\mathbb{L}^2(\nu)}$$

This is the main ingredient in a series of classical computations leading to the existence of a constant $C > 0$ such that

$$\forall\ t \geqslant \sigma, \forall\ M > 1, \qquad \mathbb{P}_0[\tau_M^* > t] \leqslant CM^2 \exp(-2t)$$

Quasi-stationary measure

It remains to choose $M = 2\sqrt{t}$ to recover the rate 2 of exponential convergence in total variation.
Another related approach consists in remarking that the σ-finite measure η which admits the density $H_2^\dagger > 0$ with respect to ν is a quasi-stationary measure for $L^\dagger$ (η admits the density $\mathbb{R}_+ \ni y \mapsto y\gamma([0,y])$ with respect to the Lebesgue measure):
for any $t \geqslant 0$ and any measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$, we have (in $\mathbb{R}_+ \sqcup \{+\infty\}$),

$$\eta[P_t^\dagger[f]] \quad = \quad \exp(-2t)\eta[f]$$

Again up to a traditional series of computations, this can be transformed in the same bound as before on the queues of τ_M^*.

Plan

Historical references on intertwinings

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