On set-valued intertwining duality for diffusion processes

Laurent Miclo

Toulouse School of Economics
Institut de Mathématiques de Toulouse

Based on joint works with
Marc Arnaudon and Koléhè Coulibaly-Pasquier

Plan of the mini-lectures

1. **Strong stationary times for one-dimensional diffusions**

In particular we have seen that a positive recurrent elliptic diffusion on $\mathbb{R}$ admits a strong stationary time, whatever its initial distribution, if and only if $-\infty$ and $+\infty$ are entrance boundaries.

2. **Duality and hypoellipticity for one-dimensional diffusions**

It will be shown that the convergence to equilibrium of hypo-elliptic diffusions on the circle can also be understood via intertwining relations.

3. **Stochastic evolutions of domains on manifolds**

We introduce stochastic modifications of mean curvature flows on manifolds and prove their existence at least for small times.

4. **Algebraic intertwining relations on manifolds**

We see how the previous evolutions serve as set-valued duals for diffusions on manifolds and present some of their properties.

Duality and hypoellipticity for one-dimensional diffusions

1. Introduction: a toy model on $\mathbb{R}$
2. The toy model on $\mathbb{R}_+$
3. The toy model on $\mathbb{R}_-$
4. A toy example on the circle
5. The turning diffusion

Plan

Let $(P_t)_{t\geqslant 0}$ be an ergodic Markov semi-group on a manifold M with invariant probability π. Assume it can be intertwined with a Markov semi-group $(\bar{P}_t)_{t\geqslant 0}$ on $\mathcal{D}$, a set of "nice" subsets of M. In particular $\mathcal{D}$ should contain the singletons and for any $D \in \mathcal{D}$ which is not a singleton, we should have $\pi(D) > 0$. We furthermore make the assumption that the link Λ from $\mathcal{D}$ to M is given by: for any $D \in \mathcal{D}$ and any "Borelian $dy \subset M$",

$$\Lambda(D, dy) \quad := \quad \begin{cases} \delta_x(dy) & \text{, if } D = \{x\} \\ \frac{\pi(D\cap dy)}{\mu(D)} & \text{, otherwise} \end{cases}$$

Namely we have

$$\forall\ t \geqslant 0, \qquad \bar{P}_t \Lambda \quad = \quad \Lambda P_t$$

Applying these intertwining relations at the singleton $\{x\}$, we get for any $t > 0$ such that $\bar{P}_t(\{x\}, \{\{y\} : y \in M\}) = 0$,

$$P_t(x, \cdot) \quad = \quad \int_{\mathcal{D}\setminus\{\{y\}\,:\,y\in M\}} \bar{P}_t(\{x\}, dD)\Lambda(D, \cdot)$$

This decomposition shows that $P_t(x, \cdot)$ admits a density with respect to π, a first step toward more regularity results.
So we are led to wonder if there exist set-valued intertwining duals for hypoelliptic diffusions.

We will just investigate toy models of hypoellipticity in dimension 1. For a generator of the form

$$L \quad \coloneqq \quad a\partial^2 + b\partial$$

we allow the diffusion coefficient a to vanish at isolated points (contrary to the situation of the first lecture), but there the drift coefficient b should not vanish.

A toy model on $\mathbb{R}$

Consider the hypoelliptic s.d.e. on $X := (X(t))_{t\in[0,\tau)}$, with $\tau \in (0,+\infty]$ the potential explosion time, evolving as

$$\forall\ t \in [0,\tau), \qquad dX(t) \;=\; \sqrt{2}X^n(t)\,dB(t) + dt$$

where $n \in \mathbb{N} := \{1,2,3,...\}$ and $(B(t))_{t\geqslant 0}$ is a standard Brownian motion. We will often assume that $X(0)$ is a deterministic point.

It can be transformed into a Stratanovitch s.d.e.: for $t \in [0,\tau)$,

$$dX(t) \;=\; \sqrt{2}X^n(t)\circ dB(t) + (1 - nX^{2n-1}(t))\,dt$$

The corresponding generator L acts on $\mathcal{C}^\infty(\mathbb{R})$ via

$$\forall\ f \in \mathcal{C}^\infty(\mathbb{R}), \forall\ x \in \mathbb{R}, \qquad L[f](x) \;:=\; x^{2n}\partial^2 f(x) + \partial f(x)$$

Hypoellipticity

It can be rewritten under the Hörmander [1965] form $L = V_1^2 + V_0$, where V_0 and V_1 are the vector fields on $\mathbb{R}$ given by

$$\forall\ x \in \mathbb{R}, \qquad \begin{cases} V_0(x) & := & (1 - nx^{2n-1})\partial \\ V_1(x) & = & x^n \partial \end{cases}$$

Define for all $l \in \mathbb{Z}_+$, the set of vector fields $\mathcal{V}_l$ through the iteration

$$\begin{array}{rcl} \mathcal{V}_0 & := & \{V_1\} \\ \forall\ l \in \mathbb{Z}_+, \quad \mathcal{V}_{l+1} & := & \mathcal{V}_l \cup \{[U, V] \,:\, U \in \mathcal{V}_l \text{ and } V \in \{V_0, V_1\}\} \end{array}$$

where $[\cdot, \cdot]$ stands for the usual Lie bracket. For any $x \in \mathbb{R}$, let $\mathcal{V}_l(x) := \{V(x) \,:\, V \in \mathcal{V}_l\}$. For any $x \in \mathbb{R}\backslash\{0\}$, we have $\mathcal{V}_0(x) \neq \{0\}$, so that L is elliptic on $\mathbb{R}\backslash\{0\}$. At 0, the first $l \in \mathbb{Z}_+$ such that $\mathcal{V}_l(0) \neq \{0\}$ is $l = n$, so that L is hypoelliptic of order n.

The link

Let $\mathcal{I}$ stand for the set of nonempty closed intervals from $[-\infty, +\infty]$, which are either included into $[-\infty, 0)$ or into $[0, +\infty]$ and which are different from $\{-\infty\}$ and $\{+\infty\}$, and $\mathcal{S} := \{\{x\} : x \in \mathbb{R}\}$. Consider μ_+ an μ_- the speed measures associated to X on $\mathbb{R}_+$ and $(-\infty, 0)$. We define a Markov kernel Λ from $\mathcal{I}$ to $\mathbb{R}$ by

$$\forall\ \iota \in \mathcal{I},\ \forall\ A \in \mathcal{B}(\mathbb{R}),$$

$$\Lambda(\iota, A) \quad := \quad \begin{cases} \delta_x(A) & \text{, when } \iota = \{x\}, \\ \frac{\mu_-(\iota \cap A)}{\mu_-(\iota)} & \text{, when } \iota \in \mathcal{I} \backslash \mathcal{S},\ \iota \subset [-\infty, 0), \\ \frac{\mu_+(\iota \cap A)}{\mu_+(\iota)} & \text{, when } \iota \in \mathcal{I} \backslash \mathcal{S},\ \iota \subset [0, +\infty], \end{cases}$$

where $\mathcal{B}(\mathbb{R})$ stands for the set of Borel subsets from $\mathbb{R}$.

Hypoelliptic intertwining

Theorem 1

There exists a process $I := (I(t))_{t\geqslant 0}$ taking values in $\mathcal{I}$ such that

$$\begin{array}{rrcl} & I(0) & = & \{X(0)\} \\ \forall\ t>0, & \mathbb{P}[I(t)\in\mathcal{S}] & = & 0 \\ \forall\ t\geqslant 0, & \mathcal{L}(X(t)|I[0,t]) & = & \Lambda(I(t),\cdot) \end{array}$$

In particular, we have the decomposition

$$\mathcal{L}(X(t)) \quad = \quad \int \Lambda(\iota,\cdot)\,\mathcal{L}(I(t))(d\iota)$$

and the r.h.s. is absolutely continuous with respect to the Lebesgue measure for $t>0$.

Behavior of I

I immediately grows into a segment with non-empty interior. But contrary to the elliptic case, where the dual process never return to $\mathcal{S}$, I collapses into the singleton $\{0\}$ at τ_0, the time when X hits 0 (this happens in finite positive time when $X(0) < 0$). The process I is continuous (for the Hausdorff topology on the compact subsets of $[-\infty, +\infty]$), except at τ_0, when I may be non left-continuous. The second point in the previous theorem will be deduced from the fact that the law of τ_0 has no atom outside 0.

After τ_0 (or 0 if $X(0) \geqslant 0$), the behavior of I depends on n:

- For $n \in \mathbb{N}\backslash\{1\}$, in finite time the process I hits $[0, +\infty]$ and stays there afterward.
- For $n = 1$, the process I converges to $[0, +\infty]$ in large time, but never reaches it (starting from a singleton).

Plan

Scale and speed (1)

The situation here is similar to that treated in the first lecture, with $(0,+\infty)$ replacing $\mathbb{R}$. Indeed introduce the **scale** and **speed densities** associated to the restriction of L to $\mathbb{R}_+$: for any $x>0$,

$$\begin{aligned}\sigma_+(x) &:= \exp\left(-\int_1^x \frac{1}{u^{2n}}\,du\right) = \exp((x^{1-2n}-1)/(2n-1))\\ \mu_+(x) &:= \frac{1}{x^{2n}\sigma_+(x)} = \frac{1}{x^{2n}}\exp((1-x^{1-2n})/(2n-1))\end{aligned}$$

The interest of these functions is that on $(0,+\infty)$, we can write

$$L = \frac{1}{\mu_+}\partial\left(\frac{1}{\sigma_+}\partial\right)$$

The corresponding **scale** and **speed measures**, also written σ_+ and μ_+, are those admitting σ_+ and μ_+ for densities with respect to the Lebesgue measure.

We get that

$$\begin{aligned} \int_0^1 \sigma_+([0,x])\,\mu_+(x)dx &= +\infty \\ \int_0^1 \mu_+([0,x])\,\sigma_+(x)dx &= \int_0^1 v_+(x)\sigma_+(x)\,dx \;=\; 1 \;<\; +\infty \end{aligned}$$

From the book of Karlin and Taylor [1981], 0 is an entrance boundary: when $X(0)$ is distributed on $\mathbb{R}_+$, the positions of the process X are in $(0,+\infty)$ for any positive time.
The status of $+\infty$ can be investigated similarly, in particular

$$\int_1^{+\infty} \mu_+([x,+\infty))\,\sigma_+(x)dx \quad \begin{cases} = \;+\infty & \text{, if } n=1, \\ < \;+\infty & \text{, if } n \in \mathbb{N}\backslash\{1\}. \end{cases}$$

We deduce that $+\infty$ is a natural boundary if $n=1$ and an entrance boundary if $n \in \mathbb{N}\backslash\{1\}$. In both cases, $+\infty$ cannot be reached, so we have for the explosion time $\tau = +\infty$ a.s.

Consider

$$\begin{aligned} \mathcal{I}_+ &:= \{(y,z) \,:\, y,z \in [0,+\infty],\, y \leqslant z\} \backslash \{(+\infty,+\infty)\} \\ \mathring{\mathcal{I}}_+ &:= \{(y,z) \in (0,+\infty)^2 \,:\, y < z\} \\ \mathcal{S}_+ &:= \{(y,y) \,:\, y \in \mathbb{R}_+\} \subset \mathcal{I}_+ \end{aligned}$$

Recall that the element $(y,z) \in \mathcal{I}_+$ should be interpreted as the compact interval $[y,z]$ in $\mathbb{R}_+ \sqcup \{+\infty\}$ and the elements of $\mathcal{S}_+$, as singletons. Let Λ_+ be the Markov kernel Λ restricted from $\mathcal{I}_+$ to $\mathbb{R}_+$.

Let $\mathfrak{L}_+$ be the diffusion generator on $\mathring{\mathcal{I}}_+$ given by

$$\begin{aligned} \mathfrak{L}_+ &:= (z^n\partial_z - y^n\partial_y)^2 + (ny^{2n-1}-1)\partial_y + (nz^{2n-1}-1)\partial_z \\ &\quad +2\frac{y^n\mu_+(y) + z^n\mu_+(z)}{\mu_+([y,z])}(z^n\partial_z - y^n\partial_y) \end{aligned}$$

Complete this definition on $\{0\} \times (0, +\infty)$ by

$$\mathfrak{L}_+ \quad := \quad (z^n \partial_z)^2 + (nz^{2n-1} - 1)\partial_z + 2\frac{z^{2n}\mu_+(z)}{\mu_+([0,z])}\partial_z,$$

on $(0, +\infty) \times \{+\infty\}$ by

$$\mathfrak{L}_+ \quad := \quad (y^n \partial_y)^2 + (ny^{2n-1} - 1)\partial_y - 2\frac{y^{2n}\mu_+(y)}{\mu_+([y,+\infty))}\partial_y$$

and on $(0, +\infty) \in \mathcal{I}_+$ by

$$\mathfrak{L}_+ \quad := \quad 0$$

namely $(0, +\infty)$ (alias $[0, +\infty]$) is absorbing for $\mathfrak{L}_+$.
More precisely, $\mathfrak{L}_+$ is defined on $\mathfrak{D}_+$, the set of continuous and bounded functions on $\mathcal{I}_+$ which are smooth on each of the subsets $\mathring{\mathcal{I}}_+$, $\{0\} \times (0, +\infty)$ and $(0, +\infty) \times \{+\infty\}$.

Theorem 2

For any probability distribution $\mathfrak{m}_0$ on $\mathcal{I}_+$, there is a unique (in law) continuous Markov process $I := (Y(t), Z(t))_{t \geqslant 0}$ whose initial distribution is $\mathfrak{m}_0$ and whose generator is $\mathfrak{L}_+$ in the sense of martingale problems: for any $F \in \mathfrak{D}_+$, the process $M^F := (M^F(t))_{t \geqslant 0}$ defined by

$$\forall\ t \geqslant 0, \qquad M^F(t) \ := \ F(Y(t), Z(t)) - F(Y(0), Z(0)) - \int_0^t \mathfrak{L}_+[F](Y(s), Z(s))\, ds$$

is a local martingale. Furthermore the diagonal $\mathcal{S}_+$ is an ***entrance boundary*** *for I: for any $t > 0$, we have $(Y(t), Z(t)) \notin \mathcal{S}_+$.*

On $\mathcal{I}_+ \backslash \mathcal{S}_+$, the process $(Y(t), Z(t))_{t \geqslant 0}$ is constructed as a solution to the s.d.e.'s associated to the generator $\mathfrak{L}_+$. For instance on $\mathring{\mathcal{I}}_+$, we have, up to the corresponding explosion time,

$$\begin{aligned}
dY(t) &= -\sqrt{2}Y^n(t)dB(t) \\
&\quad + \left(nY^{2n-1}(t) - 1 - 2\frac{\underline{\mu}_+(\{Y(t), Z(t)\})}{\mu_+([Y(t), Z(t)])}\right) Y^n(t)\, dt \\
dZ(t) &= \sqrt{2}Z^n(t)dB(t) \\
&\quad + \left(nZ^{2n-1}(t) - 1 + 2\frac{\underline{\mu}_+(\{Y(t), Z(t)\})}{\mu_+([Y(t), Z(t)])}\right) Z^n(t)\, dt
\end{aligned}$$

where $(B(t))_{t \geqslant 0}$ is a standard Brownian motion and where

$$\underline{\mu}_+ \quad := \quad \sum_{x \in (0, +\infty)} x^n \mu_+(x) \delta_x$$

i.e. $\underline{\mu}_+(\{Y(t), Z(t)\}) = Y^n(t)\mu_+(Y(t)) + Z^n(t)\mu_+(Z(t))$.

Pitman's property (1)

Introduce

$$\varsigma_+ \quad := \quad 2\int_0^{+\infty} \underline{\mu}_+(\partial I(s))^2\, ds$$

(where $\partial I(s)) = \{Y(s), Z(s)\}$).
Let the **time change** $(\theta_+(t))_{t\in[0,\varsigma_+]}$ be defined by

$$\forall\ t \in [0, \varsigma_+), \qquad 2\int_0^{\theta_+(t)} \underline{\mu}_+(\partial I(s))^2\, ds \quad = \quad t$$

and $\theta_+(\varsigma_+) := \lim_{t\to(\varsigma_+)_-} \theta_+(t)$.
We are interested in the process $R_+ := (R_+(t))_{t\geqslant 0}$ given by

$$\forall\ t \geqslant 0, \qquad R_+(t) \quad := \quad \mu_+(I(\theta_+(t \wedge \varsigma_+)))$$

Proposition 3

The process R_+ is a Bessel process of dimension 3 starting from $\mu_+(I(0))$ and stopped when it hits $\mu_+((0,+\infty))$. In particular, ς_+ is finite a.s. and is the hitting time of $\mu_+((0,+\infty))$ by R_+. More precisely, we have:

- *for $n \in \mathbb{N}\backslash\{1\}$ or $I(0)$ of the form $(y_0,+\infty)$ for some $y_0 \in [0,+\infty)$, we have $\theta_+(\varsigma_+) < +\infty$ and the process I hits $(0,+\infty)$ in finite time (a.s.)*
- *for $n = 1$ and $I(0)$ not of the form $(y_0,+\infty)$ for some $y_0 \in [0,+\infty)$, we have $\theta_+(\varsigma_+) = +\infty$ and the process I does not hit $(0,+\infty)$ in finite time (a.s.).*

Thus hypoellipticity does not modify the Pitman property that the process of the volumes $(\mu_+(I(t)))_{t\geqslant 0}$ of the dual process is a stopped Bessel 3 process, up to a time change. The impact of hypoellipticity is to be found in the time change:

Proposition 4

Fix $(y,z) \in \mathcal{I}_+$ and consider the process I defined in Theorem 2 starting from (y,z). There are several behaviors for the time change θ_+ as t goes to 0_+:

- *If $(y,z) \neq (0,0)$, we have*

$$\theta_+(t) \quad \sim \quad \frac{t}{2(y^n \mu_+(y) + z^n \mu_+(z))^2}$$

- *If $(y,z) = (0,0)$, we have*

$$\theta_+(t) \quad \sim \quad \frac{1}{((2n-1)\ln(1/t))^{1/(2n-1)}}$$

Thus in the latter case, the volume $\mu_+[I(t)]$ begins by evolving very slowly, since the inverse function $\theta_+^{-1}(t)$ is negligible with respect to t, for $t \to 0_+$, the more so as the order n of hypoellipticity is large.

The interest of Λ_+ and $\mathfrak{L}_+$ is the **intertwining relation** $\mathfrak{L}_+\Lambda_+ = \Lambda_+ L$, in the sense that,

$$\forall\ (y,z) \in \mathcal{I}_+ \backslash \mathcal{S}_+,\ \forall\ f \in \mathcal{C}_{\mathrm{b}}^{\infty}(\mathbb{R}_+),$$

$$\mathfrak{L}_+[\Lambda_+[f]](y,z) \quad = \quad \Lambda_+[L[f]](y,z)$$

Let $\mathfrak{m}_0$ be a probability distribution on $\mathcal{I}_+$ and consider $m_0 \coloneqq \mathfrak{m}_0\Lambda_+$. There exists an coupling by intertwining of X with initial distribution m_0 and of I with initial distribution $\mathfrak{m}_0$ such that for any $t \geqslant 0$,

$$\begin{aligned} \mathcal{L}(X(t)|I[0,t]) &= \Lambda_+(I(t)) \\ \mathcal{L}(I[0,t]|X) &= \mathcal{L}(I[0,t]|X[0,t]) \end{aligned}$$

We deduce:

Corollary 5

As in Proposition 3, there are two situations:
- *for $n \in \mathbb{N}\backslash\{1\}$, whatever the initial distribution supported by $\mathbb{R}_+$, there exists a strong stationary time for X.*
- *for $n = 1$, for some initial distributions on $\mathbb{R}_+$ (in particular for any initial Dirac measure), a strong stationary time does not exist for X.*

These results are consequences of general considerations about intertwining relations between ergodic and absorbed Markov processes due to Diaconis and Fill [1990].

Plan

The investigation of the situation on $\mathbb{R}_-$ is similar, but with an important discrepancy at 0. Putting together the results on $\mathbb{R}_-$ and $\mathbb{R}_+$ will lead to Theorem 1. Introduce for $x < 0$,

$$\begin{aligned}
\sigma_-(x) &:= \exp\left(-\int_{-1}^{x} \frac{1}{u^{2n}}\, du\right) \\
&= \exp\left(\frac{x^{1-2n}+1}{2n-1}\right) \\
\mu_-(x) &:= \frac{1}{x^{2n}\sigma_-(x)} \\
&= \frac{1}{x^{2n}} \exp\left(-\frac{x^{1-2n}+1}{2n-1}\right)
\end{aligned}$$

Define the corresponding scale and speed measures on $\mathbb{R}_-$.

We compute that

$$
\begin{aligned}
\int_{-1}^{0} \sigma_-([x,0])\,\mu_-(x)dx &< +\infty \\
\int_{-1}^{0} \mu_-([x,0])\,\sigma_-(x)dx &= +\infty \\
\int_{-\infty}^{-1} \sigma_-((-\infty,x])\,\mu_-(x)dx &= +\infty \\
\int_{-\infty}^{-1} \mu_-((-\infty,x])\,\sigma_-(x)dx & \begin{cases} = +\infty & \text{, if } n=1, \\ < +\infty & \text{, if } n \in \mathbb{N}\backslash\{1\}. \end{cases}
\end{aligned}
$$

When X starts from an initial distribution supported by $\mathbb{R}_-$, 0 is an exit boundary (i.e. it is a.s. attained in finite time). Furthermore, depending on $n = 1$ or $n \in \mathbb{N}\backslash\{1\}$, $-\infty$ is an entrance or a natural boundary.

Evolving segment generator

Consider

$$\begin{aligned}
\mathcal{I}_- &:= \{(y,z) \,:\, y,z \in [-\infty,0),\, y \leqslant z\} \backslash \{(-\infty,-\infty)\} \\
\mathring{\mathcal{I}}_- &:= \{(y,z) \in (-\infty,0)^2 \,:\, y < z\} \\
\mathcal{S}_- &:= \{(y,y) \in \mathcal{I}_- \,:\, y \in (-\infty,0)\}
\end{aligned}$$

Again, the element $(y,z) \in \mathcal{I}_-$ should be interpreted as the compact interval $[y,z]$ in $[-\infty,0)$. Let Λ_- be the Markov kernel from $\mathcal{I}_-$ to $(-\infty,0)$ which is the restriction of Λ. Let $\mathfrak{L}_-$ be the diffusion generator on $\mathring{\mathcal{I}}_-$ given by

$$\begin{aligned}
\mathfrak{L}_- &:= (z^n\partial_z - y^n\partial_y)^2 + (ny^{2n-1}-1)\partial_y + (nz^{2n-1}-1)\partial_z \\
&\quad +2\frac{y^n\mu_-(y)+z^n\mu_-(z)}{\mu_-([y,z])}(z^n\partial_z - y^n\partial_y)
\end{aligned}$$

and complete this definition on $\{-\infty\} \times (-\infty,0)$ by

$$\mathfrak{L}_- := (z^n\partial_z)^2 + (nz^{2n-1}-1)\partial_z + 2\frac{z^{2n}\mu_-(z)}{\mu_-([0,z])}\partial_z$$

Evolving segment

With the domain $\mathcal{D}(\mathfrak{L}_-)$ defined similarly to $\mathcal{D}(\mathfrak{L}_+)$:

Theorem 6

For any probability distribution $\mathfrak{m}_0$ on $\mathcal{I}_-$, there is a unique (in law) continuous Markov process $I := (Y(t), Z(t))_{t\in[0,\tau_I)}$ whose initial distribution is $\mathfrak{m}_0$ and whose generator is $\mathfrak{L}_-$ in the sense of martingale problems.
The diagonal $\mathcal{S}_-$ is an entrance boundary for I: for any $t \in (0, \tau_I)$, we have $(Y(t), Z(t)) \notin \mathcal{S}_-$. The explosion time τ_I corresponds to the "hitting" time of 0 by Z, in the sense that $\lim_{t\to\tau_I-} Z(t) = 0$. It is a.s. finite.

Pitman's property

Introduce, with $\underline{\mu}_-$ the analogue on $\mathbb{R}_-$ of $\underline{\mu}_+$,

$$\varsigma_- \quad := \quad 2\int_0^{\tau_I} \underline{\mu}_-(\partial I(s))^2\, ds$$

and the **time change** $(\theta_-(t))_{t\in[0,\varsigma_-]}$ defined by

$$\forall\ t \in [0,\varsigma_-), \qquad 2\int_0^{\theta_-(t)} \underline{\mu}_-(\partial I(s))^2\, ds \quad = \quad t$$

and $\theta_-(\varsigma_-) := \lim_{t\to(\varsigma_-)_-} \theta_-(t)$.
We are interested in the process $R_- := (R_-(t))_{t\geqslant 0}$ given by

$$\forall\ t \geqslant 0, \qquad R_-(t) \quad := \quad \mu_-(I(\theta_-(t\wedge\varsigma_-)))$$

Corollary 7

We have $\varsigma_- = +\infty$, $\theta_-(+\infty) = \tau_I$ and the process R_- is a Bessel process of dimension 3 starting from $\mu_-(I(0))$.

Again the interest of Λ_- and $\mathfrak{L}_-$ is the intertwining relation $\mathfrak{L}_-\Lambda_- = \Lambda_- L$, in the sense that,

$$\forall\ \iota \in \mathcal{I}_-\setminus\mathcal{S}_-,\ \forall\ f \in \mathcal{C}_{\mathrm{b}}^{\infty}((-\infty,0)),\ \mathfrak{L}_-[\Lambda_-[f]](\iota) \quad = \quad \Lambda_-[L[f]](\iota)$$

We have the corresponding probabilist intertwining:

Theorem 8

Let $\mathfrak{m}_0$ be a probability distribution on $\mathcal{I}_-$ and consider $m_0 \coloneqq \mathfrak{m}_0\Lambda_-$. There exists a coupling of X with initial distribution m_0 and of I with initial distribution $\mathfrak{m}_0$ such that for any $t \geqslant 0$, we have on $\{\tau_I > t\}$,

$$\mathcal{L}(X(t)|I[0,t]) \quad = \quad \Lambda_-(I(t),\cdot)$$

Furthermore, the construction of I from X is adapted.

As a consequence of the Pitman's property, we have

$$\lim_{t\to\tau_I-} \mu_-(I(t)) = +\infty$$

so that in addition to $\lim_{t\to\tau_{I-}} Z(t) = 0$, we get

$$\lim_{t\to\tau_{I-}} X(t) = 0 \qquad (1)$$

In general, we do not have $\lim_{t\to(\tau_0)_-} Y(t) = 0$, e.g. if we started with $Y(0) = -\infty$, then $Y(t) = -\infty$ for all $t \in [0, \tau_0)$. Anyway, (1) enables to set $(Y(\tau_0), Z(\tau_0)) \coloneqq (0, 0)$ while preserving the validity of Theorem 8. This is the lack of left-continuity of the process I at time $\tau_0 = \tau_{I-}$. It suggests to replace the process I by the probability measure-valued Markov process $(\Lambda(I(t), \cdot))_{t\geqslant 0}$, which is continuous at τ_0, since it takes the value δ_0 at this time.

Next we extend the process I after time τ_0 as in Theorem 2, starting from $(0, 0)$. Theorem 1 follows from the merging of the previous results.

Plan

A hypoelliptic generator on the circle

Let a and b be two smooth functions on $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, such that a is non-negative, $\sqrt{a}$ is analytical and vanishes at most at a finite number of points, write $\mathfrak{N}$ for their set. Assume that for any $x \in \mathfrak{N}$, $b(x) \neq 0$. Consider on $\mathcal{C}^\infty(\mathbb{T})$ the Markov generator

$$L \quad := \quad a\partial^2 + b\partial$$

and let $X := (X(t))_{t\geqslant 0}$ be a corresponding diffusion process. The generator L is hypoelliptic and we are looking for the behavior in law of X for large times.

Let us write $\mathfrak{N} := \{\mathfrak{y}_k \ : \ k \in \mathbb{Z}_N\}$, where the representative points in $[0,1[$ satisfy $0 \leqslant \mathfrak{y}_0 < \mathfrak{y}_1 < \cdots < \mathfrak{y}_{N-1} < 1$ and where $N \in \mathbb{N}$. For $k \in \mathbb{Z}_N$, let $\mathbb{I}_k$ be the projection on $\mathbb{T}$ of the interval $(\mathfrak{y}_k, \mathfrak{y}_{k+1})$ (for $l = N-1$, it is the interval $(\mathfrak{y}_{N-1}, \mathfrak{y}_0 + 1)$), to which is added $\mathfrak{y}_k$ if $b(\mathfrak{y}_k) > 0$ and $\mathfrak{y}_{k+1}$ if $b(\mathfrak{y}_{k+1}) < 0$. Remark that $(\mathbb{I}_k)_{k\in\mathbb{Z}_N}$ forms a partition of $\mathbb{T}$. Denote for $k \in \mathbb{Z}_N$, μ_k the speed measure associated to the restriction of L to $\mathbb{I}_k$.

A segment-valued intertwining dual process

Let $\mathcal{I}$ stand for the set of non-empty closed intervals from $\mathbb{T}$ which are included into one of the $\mathbb{I}_k$, for $k \in \mathbb{Z}_N$ and let $\mathcal{S} := \{\{x\} : x \in \mathbb{T}\}$. Define a Markov kernel Λ from $\mathcal{I}$ to $\mathbb{T}$ by

$$\forall\ \iota \in \mathcal{I},\ \forall\ A \in \mathcal{B}(\mathbb{T}),$$

$$\Lambda(\iota, A) \quad := \quad \begin{cases} \delta_x(A) & \text{, when } \iota = \{x\} \in \mathcal{S} \\ \frac{\mu_k(\iota \cap A)}{\mu_k(\iota)} & \text{, when } \iota \in \mathcal{I} \setminus \mathcal{S} \text{ with } \iota \subset \mathbb{I}_k \text{ and } k \in \mathbb{Z}_N \end{cases}$$

(in fact $0 < \mu_k(\iota) < +\infty$ for $\iota \in \mathcal{I} \setminus \mathcal{S}$ with $\iota \subset \mathbb{I}_k$ and $k \in \mathbb{Z}_N$).
Theorem 1 extends to this context:

Theorem 9

Let X be a diffusion on the circle whose generator is the hypoelliptic operator L. There exists a dual process I associated to X satisfying all the probabilistic intertwining relations of Theorem 1, with the above $\mathcal{I}$ and Λ.

Let $\mathbb{I}$ be one of the segments $\mathbb{I}_k$, for $k \in \mathbb{Z}_N$, up to an affine transformation we identify $\mathbb{I}$ with a segment of $\mathbb{R}$ whose interior is $(0,1)$.
The restriction on $\mathbb{I}$ of the generator L can be written

$$L \quad = \quad \frac{1}{\mu}\partial\left(\frac{1}{\sigma}\partial\right)$$

with the help of its scale and speed densities:

$$\forall\ x \in (0,1), \qquad \sigma(x) \quad := \quad \exp\left(-\int_{1/2}^{x} \frac{b(u)}{a(u)}\,du\right)$$

$$\mu(x) \quad := \quad \frac{1}{a(x)\sigma(x)}$$

Consider the corresponding scale and speed measures, still written σ and μ.

There are four possibilities for the status of the boundaries $\{0,1\}$ of $\mathbb{I}$, which are determined by the finiteness or not of the quantities

$$\begin{aligned}
\Sigma(0) &:= \int_0^{1/2} \sigma((0,u))\,\mu(u)du, & N(0) &:= \int_0^{1/2} \mu((0,u))\,\sigma(u)du,\\
\Sigma(1) &:= \int_{1/2}^{1} \sigma((u,1))\,\mu(u)du, & N(1) &:= \int_{1/2}^{1} \mu((u,1))\,\sigma(u)du
\end{aligned}$$

Redefine $\mathcal{I}$ the set of compact subsegments included in $\mathbb{I}$ and $\mathcal{S}$ the set of singletons from $\mathcal{I}$. Consider the Markov kernel Λ from $\mathcal{I}$ to $[0,1]$:

$$\forall\ [y,z]\in\mathcal{I}, \qquad \Lambda([y,z],\cdot) \;:=\; \begin{cases} \delta_y & \text{, if } y=z,\\ \frac{\mu([y,z]\cap\cdot)}{\mu([y,z])} & \text{, otherwise.}\end{cases}$$

Case (C1): $\mathbb{I} = [0,1]$

It corresponds to $b(0) > 0$ and $b(1) < 0$, and we compute that $\Sigma(0) = +\infty$, $N(0) < +\infty$, $\Sigma(1) = +\infty$ and $N(1) < +\infty$, so that 0 and 1 are entrance boundaries for X. It follows that under the initial condition $X(0) = x_0$, where x_0 is fixed in $[0,1]$, the process X stays forever in $[0,1]$ and more precisely in $(0,1)$ for positive times. We have $\lim_{x\to 0_+} \mu(x) = 0 = \lim_{x\to 1_-} \mu(x)$, so the measure μ has a finite weight over $\mathbb{I}$. It is also clear that μ is positive on $(0,1)$. Thus Λ is indeed well-defined. Furthermore X (restricted to $\mathbb{I}$) is reversible with respect to π, the normalization of μ into a probability measure.

As in the first lecture, we construct a $\mathcal{I}$-valued process $(I(t))_{t\geqslant 0}$ so that Theorem 1 is valid. Furthermore, the covering time

$$\tau \quad := \quad \inf\{t \geqslant 0 \,:\, I(t) = [0,1]\}$$

is finite a.s. and is a strong stationary time for X (restricted to $\mathbb{I}$).

Case (C2): $\mathbb{I} = [0, 1)$

It corresponds to $b(0) > 0$ and $b(1) > 0$, we get that $\Sigma(0) = +\infty$, $N(0) < +\infty$, $\Sigma(1) < +\infty$ and $N(1) = +\infty$, so that 0 is an entrance boundary and 1 an exit boundary for X. It follows that under the initial condition $X(0) = x_0$, where x_0 is fixed in $[0, 1)$, the process X ends up exiting $[0, 1)$ by hitting 1 in finite time, say at $\tau \coloneqq \inf\{t \geqslant 0 : X(t) = 1\}$. We have $\lim_{x \to 0_+} \mu(x) = 0$ (but $\lim_{x \to 1_-} \mu(x) = +\infty$), so any compact segment included into $\mathbb{I}$ has a finite weight, which is positive if it is not reduced to a singleton. Again the Markov kernel Λ is well-defined.

We construct a $\mathcal{I}$-valued intertwined dual process $I \coloneqq ([Y(t), Z(t)])_{t \in [0, \tau)}$, so that Theorem 8 is valid. We have a.s.

$$\lim_{t \to \tau_-} Z(t) \quad = \quad 1$$

and the natural way to extend I after time τ is to define $I(\tau) = \{1\}$ and to let I start from there into the corresponding segment.

The case (C3), $\mathbb{I} = (0, 1]$, is symmetrically treated.

Case (C4): $\mathbb{I} = (0,1)$

It corresponds to $b(0) < 0$ and $b(1) > 0$, we get that $\Sigma(0) < +\infty$, $N(0) = +\infty$, $\Sigma(1) < +\infty$ and $N(1) = +\infty$, so that 0 and 1 are exit boundaries for X. It follows that under the initial condition $X(0) = x_0$, where x_0 is fixed in $(0,1)$, the process X ends up exiting $(0,1)$ by hitting 0 or 1 in finite time, say $\tau_X := \inf\{t \geqslant 0 \,:\, X(t) \in \{0,1\}\}$. Any compact segment included into $\mathbb{I}$ has a finite weight, which is positive (except if it is a singleton), so the Markov kernel Λ is well-defined.

We construct a $\mathcal{I}$-valued dual process $I := ([Y(t), Z(t)])_{t\in[0,\tau_I)}$, where $\tau_I > 0$ is the explosion time, so that Theorem 8 is valid. It can be proven that

$$\lim_{t\to\tau_I-} Y(t) \;=\; 0 \quad \text{or} \quad \lim_{t\to\tau_I-} Z(t) \;=\; 1$$

so that $\tau_I = \tau_X$.

To define I at its explosion time τ_I, it is always possible to look at $X(\tau_I) \in \{0, 1\}$, and to set $I(\tau_I) = \{X(\tau_I)\}$. Immediately after τ_I, X and I will evolve in the segment containing $\{X(\tau_I)\}$.

But in this case we believe in

Conjecture 10 It should hold that

$$\lim_{t \to \tau_I -} \Lambda(I(t), \cdot) \quad = \quad \mathcal{L}(X(\tau_I))$$

□

Then one can extend the intertwining relation by resorting to probability measure-valued duals: we should consider at time τ_I the probability $\mathbb{P}[X(\tau_I) = 0]\delta_0 + \mathbb{P}[X(\tau_I) = 1]\delta_1$ or the disconnected set $\{0, 1\}$ (alias $\{\mathfrak{y}_k, \mathfrak{y}_{k+1}\}$).
In symmetric situations, the above conjecture can be proven and disconnected subset-valued duals do enable to construct smaller strong stationary times than those coming from segment-valued duals.

Proof of Theorem 9

Consider the segments $\mathbb{I}_k$, for $k \in \mathbb{Z}_N$, as the vertices of an oriented graph whose edges are as follows: there is an edge from $\mathbb{I}_k$ to $\mathbb{I}_{k+1}$, if $\mathfrak{y}_{k+1} \in \mathbb{I}_{k+1}$ and an edge from $\mathbb{I}_{k+1}$ to $\mathbb{I}_k$, if $\mathfrak{y}_{k+1} \in \mathbb{I}_k$. Except when the segments are all of type (C2), or all of type (C3), following the oriented edges, one goes from segments of type (C4) or **springs** to segments of type (C1) or **sinks**, after possibly visiting a successive sequence of segments of type (C2), turning anti-clockwise, or a successive sequence of segments of type (C3), turning clockwise. In particular, it appears that the number of springs is the number of sinks. Inside each segment, the dual process is constructed according to its type. Putting them all together, we get all the requirements on the dual process I presented in Theorem 1.

Strong stationary times?

Assuming the drift b does not take a fixed sign on $\mathfrak{N}$, for large times, the process X converges in law, the process I converges a.s. and the limit law of X is $\mathbb{E}[\Lambda(I(+\infty), \cdot)]$, where $I(+\infty) := \lim_{t\to+\infty} I(t)$ belongs to $\mathcal{I}_\infty$ the set of the $\mathbb{I}_k$, $k \in \mathbb{Z}_N$, of type (C1). Define

$$\tau \quad := \quad \inf\{t \geqslant 0 \,:\, I(t) \in \mathcal{I}_\infty\}$$

When $\mathcal{I}_\infty$ is a singleton, i.e. there is a unique sink, X has a unique invariant probability, the normalization of the speed measure on the element of $\mathcal{I}_\infty$. In this situation τ is a strong stationary time. Otherwise, in general, τ is not independent from $X(\tau)$, but the latter is distributed according to an invariant probability associated to X (a convex combinaison of the normalizations of the speed measures on the sinks), which depends on its initial condition.

Plan

Another dual process

From now on, we consider the situation where b has a fixed sign on $\mathfrak{N}$. The previous process I does not converge a.s. since it appears that $I(T_n) = \{X(T_n)\}$ for all $n \in \mathbb{N}$, where $(T_n)_{n\in\mathbb{N}}$ is the unbounded increasing sequence of random times $t \geqslant 0$ such that $X(t) \in \mathfrak{N}$. Thus the dual process I is not helpful to understand the convergence in law of X. Another dual process $\widetilde{I}$ should be considered.

Now, X admits a unique invariant probability measure π absolutely continuous with respect to the Lebesgue measure. The support of π is $\mathbb{T}$ but its density vanishes on $\mathfrak{N}$. Consider $\widetilde{\mathcal{I}}$ the set of non-empty closed intervals from $\mathbb{T}$ and define a Markov kernel $\widetilde{\Lambda}$ from $\widetilde{\mathcal{I}}$ to $\mathbb{T}$ by

$$\forall\ \iota \in \widetilde{\mathcal{I}},\ \forall\ A \in \mathcal{B}(\mathbb{T}),$$

$$\widetilde{\Lambda}(\iota, A) \quad := \quad \begin{cases} \delta_x(A) & ,\text{ when } \iota = \{x\} \in \mathcal{S}, \\ \frac{\pi(\iota\cap A)}{\pi(\iota)} & ,\text{ when } \iota \in \widetilde{\mathcal{I}}\backslash\mathcal{S}. \end{cases}$$

Strong stationary time

Theorem 11

Assume b has a constant sign over $\mathfrak{N}$. There exists a dual process $\widetilde{I} := (\widetilde{I}(t))_{t\geqslant 0}$ associated to X taking values in $\widetilde{\mathcal{I}}$ and satisfying all the statements of Theorem 1, with $\mathcal{I}$ and Λ replaced by $\widetilde{\mathcal{I}}$ and $\widetilde{\Lambda}$. Furthermore $\widetilde{\mathcal{I}}$ converges in finite time to $\mathbb{T}$.

The classical arguments of Diaconis and Fill [1990] enable to conclude that

$$\tau \quad := \quad \inf\{t \geqslant 0 \,:\, I(t) = \mathbb{T}\}$$

is strong stationary time for X.

Assume e.g. $b > 0$ on $\mathfrak{N}$.
Fix some $k \in \mathbb{Z}_N$ and on $(\mathfrak{y}_k, \mathfrak{y}_{k+1})$, consider the equation

$$(a\eta_k)' = b\eta_k - 1$$

where a and b are still the coefficients of L. This decomposition enables to write

$$L = \frac{1}{\eta_k}\partial(a\eta_k\partial) + \frac{1}{\eta_k}\partial$$

Still denoting η_k the measure on $(\mathfrak{y}_k, \mathfrak{y}_{k+1})$ of density η_k, we deduce that for any $f \in \mathcal{C}^\infty([\mathfrak{y}_k, \mathfrak{y}_{k+1}])$, we have

$$\begin{aligned} \eta_k[L[f]] &= [a\eta_k f']_{\mathfrak{y}_k}^{\mathfrak{y}_{k+1}} - [f]_{\mathfrak{y}_k}^{\mathfrak{y}_{k+1}} \\ &= -(f(\mathfrak{y}_{k+1}) - f(\mathfrak{y}_k)) \end{aligned}$$

as soon as

$$\lim_{x \to \mathfrak{y}_{k+}} a(x)\eta_k(x) = 0 = \lim_{x \to \mathfrak{y}_{k+1-}} a(x)\eta_k(x)$$

The general solution of the above equation in η_k is given by

$$\begin{aligned}\eta_k(x) &= \frac{1}{a(x)}\left(p\int_{[\mathfrak{y}_k,x]}\exp\left(\int_{[u,x]}\frac{b}{a}(v)\,dv\right)du\right.\\ &\quad\left.+q\int_{[x,\mathfrak{y}_{k+1}]}\exp\left(-\int_{[x,u]}\frac{b}{a}(v)\,dv\right)du\right)\end{aligned}$$

with p, q any constant such that $-p+q=1$. For the previous convergences to hold (and even $\eta_k(\mathfrak{y}_k)=0=\eta_k(\mathfrak{y}_{k+1})$), we must take $p=0$ and thus consider

$$\begin{aligned}\eta_k(x) &= \frac{1}{a(x)}\int_{[x,\mathfrak{y}_{k+1}]}\exp\left(-\int_{[x,u]}\frac{b}{a}(v)\,dv\right)du\\ &\geqslant 0\end{aligned}$$

Define η on $\mathbb{T}$ coinciding with η_k on $\mathbb{I}_k$ for all $k \in \mathbb{Z}_N$. Again denote η the measure admitting η as density. It is continuous (and vanish on $\mathfrak{N}$), so that $\eta(\mathbb{T}) < +\infty$. Furthermore we have for any $f \in \mathcal{C}^\infty(\mathbb{T})$,

$$\eta[L[f]] \;=\; -\sum_{k\in\mathbb{Z}_N} f(\mathfrak{y}_{k+1}) - f(\mathfrak{y}_k) \;=\; 0$$

namely η is invariant for L. The invariant probability π is just the normalization of η into a probability measure. It no longer corresponds to the concatenation of the speed measures on the $\mathbb{I}_k$, for $k \in \mathbb{Z}_N$.

The intertwining dual process

Evolution of the dual process $\widetilde{I} := (\widetilde{Y}, \widetilde{Z})$. Assume that $X(0) = x_0 \in \mathbb{I}_k = [\mathfrak{y}_k, \mathfrak{y}_{k+1})$, for some $k \in \mathbb{Z}_N$. Begin by defining $(\widetilde{Y}(t), \widetilde{Z}(t))_{t\in[0,\tau_1)}$, with $\lim_{t\to 0_+} \widetilde{Y}(t) = x_0 = \lim_{t\to 0_+} \widetilde{Z}(t)$, as the solution of the s.d.e.

$$
\begin{cases}
d\widetilde{Y}(t) = \Big(a'(\widetilde{Y}(t)) - b(\widetilde{Y}(t)) + \frac{2}{\eta(\widetilde{Y}(t))} \\
\qquad -2\frac{\sqrt{a(\widetilde{Y}(t))}\eta(\widetilde{Y}(t)) + \sqrt{a(\widetilde{Z}(t))}\eta(\widetilde{Z}(t))}{\eta([\widetilde{Y}(t),\widetilde{Z}(t)])}\sqrt{a(\widetilde{Y}(t))}\Big)dt \\
\qquad -\sqrt{2a(\widetilde{Y}(t))}\, dB(t) \\
d\widetilde{Z}(t) = \Big(a'(\widetilde{Z}(t)) - b(\widetilde{Z}(t)) + \frac{2}{\eta(\widetilde{Z}(t))} \\
\qquad +2\frac{\sqrt{a(\widetilde{Y}(t))}\eta(\widetilde{Y}(t)) + \sqrt{a(\widetilde{Z}(t))}\eta(\widetilde{Z}(t))}{\eta([\widetilde{Y}(t),\widetilde{Z}(t)])}\sqrt{a(\widetilde{Z}(t))}\Big)dt \\
\qquad +\sqrt{2a(\widetilde{Z}(t))}\, dB(t),
\end{cases}
$$

for $t \in (0, \tau_1)$, where τ_1 is the first time either $\widetilde{Y}$ hits $\mathfrak{y}_k$ or $\widetilde{Z}$ hits $\mathfrak{y}_{k+1}$, and where $(B(t))_{t\geqslant 0}$ is a standard Brownian motion.

First, assume that $\widetilde{Y}(\tau_1) = \mathfrak{y}_k$. Extend the process $(\widetilde{Y}, \widetilde{Z})$ after time τ_1 by letting $\widetilde{Y}(t) = \mathfrak{y}_k$, for all $t \geqslant \tau_1$, and by solving for $\widetilde{Z}$ the s.d.e., for $t \in [\tau_1, \tau_2)$,

$$\begin{aligned} d\widetilde{Z}(t) \quad &= \quad \Big(a'(\widetilde{Z}(t)) - b(\widetilde{Z}(t)) + \frac{2}{\eta(\widetilde{Z}(t))} \\ &\qquad + 2\frac{\eta(\widetilde{Z}(t))}{\eta([\mathfrak{y}_k, \widetilde{Z}(t)])} a(\widetilde{Z}(t)) \Big) dt + \sqrt{2a(\widetilde{Z}(t))}\, dB(t) \end{aligned}$$

where τ_2 is the first time after τ_1 that $\widetilde{Z}$ hits $\mathfrak{y}_{k+1}$. This time is a.s. finite, because $\mathfrak{y}_{k+1}$ is an exit boundary for $\widetilde{Z}$ (as well as for X) on $[\mathfrak{y}_k, \mathfrak{y}_{k+1})$. Next for $t \in [\tau_2, \tau_3)$, we ask that $\widetilde{Z}$ solves the same s.d.e., where τ_3 is the first time after τ_2 that $\widetilde{Z}$ hits $\mathfrak{y}_{k+2}$. This time is a.s. finite, because $\mathfrak{y}_{k+2}$ is an exit boundary for $\widetilde{Z}$ on $[\mathfrak{y}_{k+1}, \mathfrak{y}_{k+2})$. We keep solving this equation until $\widetilde{Z}$ ends up hitting $\mathfrak{y}_k$, say at time τ, which is also a.s. finite. After τ, we take $\widetilde{I}$ to be equal to $\mathbb{T}$.

When $\widetilde{Z}(\tau_1) = \mathfrak{y}_{k+1}$, we also impose that $\widetilde{Y}(\tau_1) = \mathfrak{y}_{k+1}$. Extend the process $(\widetilde{Y}, \widetilde{Z})$ after time τ_1 by letting $\widetilde{Y}(t) = \mathfrak{y}_{k+1}$, for all $t \geqslant \tau_1$, and by proceeding as above.

Since the generator of $\widetilde{I} := (\widetilde{Y}, \widetilde{Z})$ is intertwined with L through $\widetilde{\Lambda}$, we construct a coupling of $\widetilde{I}$ with the diffusion X, so that

$$\begin{array}{rrcl} & \widetilde{I}(0) & = & \{X(0)\} \\ \forall\ t \geqslant 0, & \mathcal{L}(X(t)|\widetilde{I}[0,t]) & = & \widetilde{\Lambda}(\widetilde{I}(t), \cdot) \end{array}$$

As announced, τ is a strong stationary time for X.
Thus in large time, X converges toward π in separation and in total variation.

References

- Persi Diaconis and James Allen Fill. Strong stationary times via a new form of duality. *Ann. Probab.*, 18(4):1483–1522, 1990.
- Lars Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.
- Samuel Karlin and Howard M. Taylor. *A second course in stochastic processes*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1981.
- Laurent Miclo. Duality and hypoellipticity: one-dimensional case studies. *Electron. J. Probab.*, 22:Paper No. 91, 32, 2017.