On set-valued intertwining duality for diffusion processes

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Based on joint works with
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Plan of the mini-lectures

1. **Strong stationary times for one-dimensional diffusions**
In particular we have seen that a positive recurrent elliptic diffusion on $\mathbb{R}$ admits a strong stationary time, whatever its initial distribution, if and only if $-\infty$ and $+\infty$ are entrance boundaries.

2. **Duality and hypoellipticity for one-dimensional diffusions**
It was shown that the convergence to equilibrium of hypo-elliptic diffusions on the circle can also be understood via intertwining relations.

3. **Stochastic evolutions of domains on manifolds**
We introduce stochastic modifications of mean curvature flows on manifolds and prove their existence at least for small times.

4. **Algebraic intertwining relations on manifolds**
We see how the previous evolutions serve as set-valued duals for diffusions on manifolds and present some of their properties.

Plan of the third lecture

Stochastic evolutions of domains on manifolds

1. Reminders of Riemannian geometry
2. Evolutions of domains
3. Doss-Sussmann method
4. Homogeneous situations
   - Euclidean spaces
   - Spherical spaces
   - Hyperbolic spaces

Plan

Riemannian metric

A Riemannian manifold M: a differentiable manifold endowed with smoothly-varying positive-definite inner products g on its tangent spaces. Write m for the dimension of M.
On any smooth coordinate chart $x := (x^i)_{i \in [\![m]\!]} : U \to \mathbb{R}^m$, we denote

$$g(x) \quad := \quad (g_{i,j}(x))_{i,j \in [\![m]\!]} \; := \; (g(x)[\partial_i, \partial_j])_{i,j \in [\![m]\!]}$$

where ∂_i is the vector field associated to the differentiation with respect to x^i. The inverse matrix $g(x)^{-1}$ is written $(g^{i,j}(x))_{i,j \in [\![m]\!]}$.
The scalar products lead to the Riemannian measure λ given in any chart by

$$\lambda(dx) \quad := \quad \sqrt{|g|(x)}dx^1 dx^2 \cdots dx^m$$

where $|g|(x)$ is the absolute value of the determinant of $g(x)$.
When the total weight of λ is finite, we normalize it into a probability measure.

Gradient

The scalar products also lead to the gradient operator: for any $f \in \mathcal{C}^\infty(M)$ and $x \in M$, $\nabla f(x)$ is the unique vector in the tangent space $T_x M$ such that

$$\forall\ v \in T_x M, \qquad df(x)[v] \;=\; \langle \nabla f(x), v \rangle_x$$

where $\langle \cdot, \cdot \rangle_x \coloneqq g(x)[\cdot, \cdot]$. In any chart, we have

$$\forall\ i \in [\![m]\!], \qquad (\nabla f)^i \;=\; \sum_{j \in [\![m]\!]} g^{i,j} \partial_j f$$

and in particular

$$\|\nabla f\|^2 \;\coloneqq\; \langle \nabla f, \nabla f \rangle \;=\; \sum_{i,j \in [\![m]\!]} g^{i,j} \partial_i f \partial_j f$$

Laplace-Beltrami operator

The Dirichlet form $\mathcal{E}$ is the bilinear form on $\mathcal{C}^\infty(M)$ given by

$$\forall\ f,g\in\mathcal{C}^\infty(M),\qquad \mathcal{E}(f,g)\ :=\ \int_M \langle\nabla f,\nabla g\rangle\, d\lambda$$

The Laplace-Beltrami operator acts on $\mathcal{C}^\infty(M)$ so that

$$\forall\ f,g\in\mathcal{C}^\infty(M),\qquad \int f\triangle g\, d\lambda\ =\ -\mathcal{E}(f,g)$$

In any chart

$$\triangle f\ =\ \frac{1}{\sqrt{|g|}}\sum_{i,j\in[\![m]\!]}\partial_i\left(\sqrt{|g|}g^{i,j}\partial_j f\right)$$

The operator $\frac{\triangle}{2}$ is a Markov generator whose associated diffusion is called the Brownian motion on M. When λ is a probability measure, it is an invariant (and even reversible) probability measure for this Brownian motion.

Example of a surface C in $\mathbb{R}^3$. Consider $x \in C$ and $\nu_C(x)$ a unit normal vector to C at x. Let P a plane of $\mathbb{R}^3$ containing x "as well as" the vector $\nu_C(x)$. Locally around x, $C \cap P$ is a curve and let $(x(u))_u$ be a parametrization at unit speed, with $x(0) = x$. Let $T(u) := \frac{dx(u)}{du}$ be the speed vector. Denote $h(x, P) \in \mathbb{R}$ the curvature at 0, which is such that $\frac{d}{du}T(u)|_{u=0} = h(x, P)\nu_C(x)$. Parametrise such planes P by the angle $\theta \in [0, 2\pi)$ with a fixed element of T_xC. The mean curvature at x of C is

$$\kappa_C(x) \;:=\; \frac{1}{2\pi}\int_0^{2\pi} h(x, P_\theta)\, d\theta$$

Euler's theorem asserts that

$$\kappa_C(x) \;=\; \frac{h_1 + h_2}{2}$$

where h_1 and h_2 are the principal curvatures at x of C.

This geometric approach extends to any hypersurface C of $\mathbb{R}^{m+1}$, the mean curvature $\kappa_C(x)$ at $x \in C$ being a mean of the principal curvatures, and more generally, with more care, to any hypersurface of Riemannian manifolds.

Here are other points of view, but they correspond to the sum of the principal curvatures (nevertheless we still call it the "mean" curvature).

A more analytical point of view. Let C be an hypersurface of a Riemannian manifold. Consider $x \in C$ and $\nu_C(x)$ a unit normal vector. Define in a neighborhood of x the signed distance $\widetilde{\rho}$ to C, positive in the direction of $\nu_C(x)$. The "mean curvature" $\kappa_C(x)$ is defined as $\triangle\widetilde{\rho}(x)$. In the same spirit, $\nu_C(x) = \nabla\widetilde{\rho}(x)$.

The variational point of view is more convenient for our purposes. Consider D a compact domain (connected and coinciding with the closure of its interior) whose boundary $C := \partial D$ is smooth. At each $y \in C$, $\nu_C(y)$ stands for the unit outward normal vector.
Let v be a vector field on M. We use it to let the domain $D_0 = D$ evolves into $(D_t)_t$:

$$\forall\ y \in \partial D_t, \qquad \dot{y} \;=\; v(y)$$

For any $f \in \mathcal{C}^\infty(M)$, consider the mapping F_f defined on any compact domain D via

$$F_f(D) \;=\; \int_D f\, d\lambda$$

Then we have

$$\frac{d}{dt}F_f(D_t) \quad = \quad \int_{\partial D_t} f \left\langle \nu_{\partial D_t}, v \right\rangle \, d\sigma$$

where σ is the $(m-1)$-Hausdorff measure corresponding to λ. and differentiating once more in time:

$$\left(\frac{d}{dt}\right)^2 F_f(D_t) \quad = \quad \int_{\partial D_t} \left(\left\langle \nu_{\partial D_t}, \nabla f \right\rangle + f \kappa_{\partial D_t} \right) \left\langle \nu_{\partial D_t}, v \right\rangle \, d\sigma$$

where for any $y \in \partial D_t$, $\kappa_{\partial D_t}(y)$ is the "mean" curvature at y of ∂D_t.

Plan

The vector field v can be assumed to (regularly) depend on the domain itself: the set of nice domains can be seen as an infinite-dimensional manifold and "above" any such domain D the tangent space corresponds to vector fields defined on ∂D.

Classical example: mean curvature flow $(D_t)_t$ starting from D_0:

$$\forall\ y \in \partial D_t, \qquad \dot{y} \;=\; -\kappa_{\partial D_t}(y)\nu_{\partial D_t}(y)$$

This evolution may not be defined for all times due to the emergence of singularities.
Even if no singularity appears, the evolution collapses in finite time to a point. E.g. in $\mathbb{R}^m$ starting from ∂D_0 the circle centred at 0 of radius $r_0 > 0$, at time t, ∂D_t is the circle centred at 0 of radius r_t satisfying $\dot{r} = -(m-1)/r$, and the evolution implodes at time $r_0^2/(2m-2)$, since we get

$$r_t^2 - r_0^2 \;=\; -2(m-1)t$$

Mean curvature flow (2)

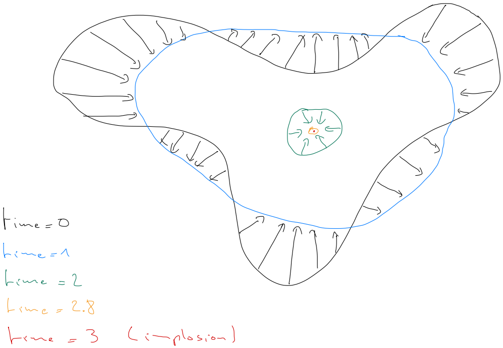

Figure: Example of a mean curvature flow

Let $\mathcal{D}$ be the set of compact domains whose boundary is smooth.
We are interested in stochastic processes $(D_t)_{t\in[0,\tau_D]}$ taking values in $\mathcal{D}$, where τ_D is the explosion time, i.e. the time where the process is exiting from $\mathcal{D}$.
We consider evolution of the form

$$\begin{aligned} &\forall\ t\in[0,\tau_D),\ \forall\ Y_t\in\partial D_t,\\ dY_t\ &=\left(\sqrt{2}dB_t+\left(2\frac{\sigma(\partial D_t)}{\lambda(D_t)}+\langle\nu_{\partial D_t},\zeta\rangle(Y_t)-\kappa_{\partial D_t}(Y_t)\right)dt\right)\nu_{\partial D_t}(Y_t)\end{aligned}$$

where

- $(B_t)_{t\geqslant 0}$ is a standard Brownian motion,
- ζ is a vector field on M.

The isoperimetric term $2\sigma(\partial D_t)/\lambda(D_t)$ counterbalances the mean curvature: when D_t is ball of radius r, the former is $2m/r$ and the latter $(m-1)/r$. The process $(D_t)_{t\in[0,\tau_D]}$ has rather a tendency to expand.

The previous evolution can be seen as an infinite dimensional stochastic differential equation on $\mathcal{D}$:

$$dD_t = \sqrt{2}V(D_t)\,dB_t + W(D_t)\,dt$$

where V and W are the vector fields on $\mathcal{D}$. For any $D \in \mathcal{D}$, $V(D)$ and $W(D)$ correspond respectively to the vector fields on ∂D given by

$$\begin{aligned} \partial D \ni y &\mapsto \nu_{\partial D}(y) \\ \partial D \ni y &\mapsto \left(2\frac{\sigma(\partial D)}{\lambda(D)} + \langle \nu_{\partial D}, \zeta\rangle(y) - \kappa_{\partial D}(y)\right)\nu_{\partial D}(y) \end{aligned}$$

the latter being equivalent to

$$\partial D \ni y \mapsto \zeta(y) + \left(2\frac{\sigma(\partial D)}{\lambda(D)} - \kappa_{\partial D}(y)\right)\nu_{\partial D}(y)$$

up to diffeomorphisms of ∂D.

It appears that the previous stochastic differential equation is equivalent to its Stratonovitch formulation.
Note that the flow associated to V is easy to describe for small times. Namely given $D_0 \in \mathcal{D}$, consider the evolution

$$dD_t = V(D_t)\, dt$$

We have for all $t > 0$ small enough,

$$D_t = \{x \in M \,:\, \rho(x, D_0) \leqslant t\}$$

where ρ is the Riemannian distance, and for $t < 0$ small enough

$$D_t = \{x \in D_0 \,:\, \rho(x, M \backslash D_0) \leqslant -t\}$$

It suggests to resort to Doss-Sussmann method to get local existence of a solution.

Plan

Let us explain this method in the context of Euclidean diffusions. In $\mathbb{R}^n$ consider the Stratonovitch s.d.e.

$$dZ(t) \quad = \quad V(Z(t)) \circ dB(t) + W(Z(t))\, dt$$

where V and W are two bounded and smooth vector fields in $\mathbb{R}^n$. Then $Z(t)$ can be computed without resorting to stochastic integrals.
For all $z \in \mathbb{R}^n$ and all time $t \in \mathbb{R}$, solve the ordinary differential equations

$$\begin{cases} F(0,z) & = & z \\ \partial_t F(t,z) & = & V(F(t,z)) \end{cases}$$

Consider a second o.d.e. in $\mathbb{R}^n$, only for $t \geqslant 0$,

$$\begin{cases} G(0) & = & Z(0) \\ \partial_t G(t) & = & (DF(B(t), G(t)))^{-1} W(F(B(t), G(t))) \end{cases}$$

where $DF(t, z)$ is the Jacobian matrix of $F(t, z)$ as a function of the spatial variable $z \in \mathbb{R}^n$. We will denote $\partial F(t, z)$ for the differentiation with respect to the temporal variable $t \in \mathbb{R}$. Then we have

$$\forall\ t \geqslant 0, \qquad Z(t) \;=\; F(B(t), G(t))$$

Indeed, using Stratonovitch calculus, we have

$$\begin{aligned} dZ(t) &= \partial F(B(t), G(t)) \circ dB(t) + DF(B(t), G(t))\, dG(t) \\ &= V(F(B(t), G(t)) \circ dB(t) \\ &\quad + DF(B(t), G(t))(DF(B(t), G(t)))^{-1} W(F(B(t), G(t)))\, dt \\ &= V(F(B(t), G(t)) \circ dB(t) + W(F(B(t), G(t)))\, dt \end{aligned}$$

Parabolic differential equations

Coming back to domain valued processes, we are led to investigate modified mean curvature flows, such as

$$\forall\ y \in \partial D_t, \quad \dot{y} \;=\; \left(2\frac{\sigma(\partial D_t)}{\lambda(D_t)} + \langle \nu_{\partial D_t}, \zeta \rangle (y) - \kappa_{\partial D_t}(y)\right) \nu_{\partial D_t}(y)$$

In a tubular neighborhood of the embedded manifold ∂D_0, for small $t \geqslant 0$, ∂D_t can be parametrized by

$$\partial D_0 \ni y \quad \mapsto \quad \exp_y(f_t(y)\nu_{\partial D_0}(y))$$

where $\exp_y$ is the exponential mapping from the tangent space $T_y M$ to M and f_t is a smooth function defined on ∂D_0.
The mapping $(t, y) \mapsto f_t(y)$ satisfies a quasi-linear parabolic equations on ∂D_0, for which we can apply classical p.d.e. results about existence, uniqueness and regularity.

Plan

The Doss-Sussmann method only provides a solution for times smaller than a positive and random time. There are instances where a solution can be constructed for all times.

This is the situation of M with constant curvature, $\zeta = 0$ and D_0 a ball $B(0, R_0)$, for any fixed $0 \in M$. We are particularly interested in the case where $R_0 = 0$, i.e. when the domain-valued process starts from the singleton $\{0\}$. Then there is a solution taking the form of balls $B(0, R_t)$ for all $t \geqslant 0$, where $(R_t)_{t \geqslant 0}$ is $\mathbb{R}_+$-valued process. We are essentially back to the one-dimensional situation.

For $t \geqslant 0$ such that $B(0, R_t)$ is a proper ball of M, the mean curvature $\kappa_{\partial B(0,R_t)}(y)$ is independent of $y \in \partial B(0, R_t)$ by symmetry, call it $\kappa_{\partial B(0,R_t)}$. We get the autonomous one-dimensional evolution

$$\begin{aligned} dR_t &= \sqrt{2}dB_t + \left(2\frac{\sigma(\partial B(0,R_t))}{\lambda(\partial B(0,R_t))} - \kappa_{\partial B(0,R_t)}\right) dt \\ &= \sqrt{2}dB_t + \left(2\,\frac{d}{dr}\ln(\lambda(B(0,r)))\Big|_{r=R_t} - \kappa_{\partial B(0,R_t)}\right) dt \end{aligned}$$

To compute $\kappa_{\partial B(0,R_t)}$, recall that for any smooth function $f : M \to \mathbb{R}$ and any $r > 0$,

$$\frac{d}{dr}\int_{\partial B(0,r)} f\,d\sigma = \int_{\partial B(0,r)} \langle \nabla f, \nu\rangle\,d\sigma + \int_{\partial B(0,r)} f\kappa_{\partial B(0,r)}\,d\sigma$$

Considering $f \equiv 1$, by symmetry we get

$$\forall\ r > 0, \qquad \kappa_{\partial B(0,r)} \;=\; \frac{d}{dr}\ln(\sigma(\partial B(0,r)))$$

so that

$$dR_t \;=\; \sqrt{2}dB_t + U'(R_t)\,dt$$

with

$$U(r) \;:=\; \ln\left(\frac{\lambda(B(0,r))^2}{\sigma(\partial B(0,r))}\right)$$

(as long as $B(0,r)$ is a proper ball of M).

We investigate separately the three cases of constant null, positive and negative curvatures.

Euclidean spaces

Consider the Euclidean space $\mathbb{R}^n$, with $n \in \mathbb{N}\backslash\{1\}$. The volume $\lambda(B(0,r))$ is proportional to r^n, so

$$\forall\ r > 0, \qquad \frac{d}{dr}\ln(\lambda(B(0,r)) \;=\; \frac{n}{r}$$

Using the geometric definition of the mean curvature, we get that

$$\forall\ r > 0, \qquad \kappa_{\partial B(0,r)} \;=\; \frac{n-1}{r}$$

so that

$$dR_t \;=\; \sqrt{2}dB_t + \frac{n+1}{R_t}dt$$

It appears that $(R_{t/2})_{t\geqslant 0}$ is a Bessel process of dimension $n+2$. In particular we can start with $R_0 = 0$, since 0 is an entrance boundary.

We consider the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, with $n \in \mathbb{N}$. Without loss of generality, we can assume that 0 is the point $(1, 0, 0, .., 0)$ from $\mathbb{R}^{n+1}$. For any $r \in [0, \pi]$, $B(0, r)$ is the closed cap centered at 0 of radius r. In particular, we have $B(0, 0) = \{0\}$ and $B(0, \pi) = \mathbb{S}^n$. The projection of λ on the first coordinate of $\mathbb{R}^{n+1}$ is the measure $Z_n^{-1}(1 - x^2)^{n/2-1}\mathbb{1}_{[-1,1]}(x)\, dx$, where the renormalising factor is given by the Wallis integral

$$
\begin{aligned}
Z_n &= \int_{-1}^{1} (1 - x^2)^{n/2-1}\, dx \\
&= \int_0^{\pi} \sin^{n-1}(u)\, du \\
&= \sqrt{\pi}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}
\end{aligned}
$$

We deduce that for any $r \in (0, \pi)$,

$$\begin{aligned} \lambda(B(0,r)) &= Z_n^{-1} \int_0^r \sin^{n-1}(u)\, du \\ \sigma(\partial B(0,r)) &= Z_n^{-1} \sin^{n-1}(r) \end{aligned}$$

so that

$$dR_t = \sqrt{2} dB_t + \left(\frac{2 \sin^{n-1}(R_t)}{\int_0^{R_t} \sin^{n-1}(z)\, dz} - (n-1)\cot(R_t) \right) dt$$

As $r \to 0_+$, we have again

$$\frac{2 \sin^{n-1}(r)}{\int_0^{rt} \sin^{n-1}(z)\, dz} - (n-1)\cot(r) \sim \frac{n+1}{r}$$

and this enables us to see that 0 is an entrance boundary for $(R_t)_{t \in [0,\tau)}$ and we can let this process start from 0.

Hyperbolic spaces

Consider the Poincaré's ball model of the hyperbolic space $\mathbb{H}^n$ of dimension $n \in \mathbb{N}\backslash\{1\}$. The situation is formally similar to the previous one, replacing the trigonometric functions by their hyperbolic analogues. Up to a factor, we have

$$\begin{aligned} \lambda(B(0,r)) &= \int_0^r \sinh^{n-1}(u)\,du && (1) \\ \sigma(\partial B(0,r)) &= \sinh^{n-1}(r) && (2) \end{aligned}$$

We deduce

$$dR_t = \sqrt{2}dB_t + \left(\frac{2\sinh^{n-1}(R_t)}{\int_0^{R_t} \sinh^{n-1}(z)\,dz} - (n-1)\coth(R_t)\right)dt$$

Again we can let it start from 0

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