On set-valued intertwining duality for diffusion processes

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Based on joint works with
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Plan of the mini-lectures

1. **Strong stationary times for one-dimensional diffusions**

In particular we have seen that a positive recurrent elliptic diffusion on $\mathbb{R}$ admits a strong stationary time, whatever its initial distribution, if and only if $-\infty$ and $+\infty$ are entrance boundaries.

2. **Duality and hypoellipticity for one-dimensional diffusions**

It was shown that the convergence to equilibrium of hypo-elliptic diffusions on the circle can also be understood via intertwining relations.

3. **Stochastic evolutions of domains on manifolds**

We introduced stochastic modifications of mean curvature flows on manifolds and proved their existence at least for small times.

4. **Algebraic intertwining relations on manifolds**

We see how the previous evolutions serve as set-valued duals for diffusions on manifolds and present some of their properties.

Plan of the fourth lecture

Algebraic intertwining relations on manifolds

1. Generators
2. Intertwinings
3. Pitman's property
4. Planar large-time profile

Plan

Observables

To introduce the generator $\mathcal{L}$ associated to the domain-valued process $(D_t)_{t\in[0,\tau)}$ considered in the previous lecture, we modify some already met observables. Let μ be a smooth and positive function on M, the same symbol is used to denote the measure $d\mu := \mu d\lambda$ and we write $d\underline{\mu} := \mu d\sigma$.

- **Elementary observables**:

$$F_f \,:\, \mathcal{D} \ni D \quad \mapsto \quad F_f(D) := \int_D f\, d\mu$$

associated to the functions $f \in \mathcal{C}^\infty(M)$, the space of smooth mappings on M.

- **Composite observables**: the functionals of the form $\mathfrak{F} := \mathfrak{f}(F_{f_1}, ..., F_{f_n})$, where $n \in \mathbb{Z}_+$, $f_1, ..., f_n \in \mathcal{C}^\infty(M)$ and $\mathfrak{f} : \mathcal{R} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ mapping, with $\mathcal{R}$ an open subset of $\mathbb{R}^n$ containing the image of $\mathcal{D}$ by $(F_{f_1}, ..., F_{f_n})$.

Generator $\mathcal{L}$

On elementary observables and for any $D \in \mathcal{D}$, define

$$\mathcal{L}[F_f](D) := \int_{\partial D} \langle \nu_{\partial D}, \nabla f \rangle + \left(2\frac{\underline{\mu}(\partial D)}{\mu(D)} + \langle \nu_{\partial D}, \nabla \ln(\mu) + \zeta \rangle \right) f \, d\underline{\mu}$$

For the extension to composite observables, the **carré du champs** is also required:

$$\forall \ D \in \mathcal{D}, \quad \Gamma_{\mathcal{L}}[F_f, F_g](D) \ := \ 2 \left(\int_{\partial D} f \, d\underline{\mu} \right) \left(\int_{\partial D} g \, d\underline{\mu} \right)$$

Then on composite observables $\mathfrak{F}$ as above:

$$\begin{aligned} \mathcal{L}[\mathfrak{F}] \ := \ & \sum_{j \in [\![1,n]\!]} \partial_j \mathfrak{f}(F_{f_1}, \dots, F_{f_n}) \mathcal{L}[F_{f_j}] \\ & + \frac{1}{2} \sum_{k,l \in [\![1,n]\!]} \partial_{k,l} \mathfrak{f}(F_{f_1}, \dots, F_{f_n}) \Gamma_{\mathcal{L}}[F_{f_k}, F_{f_l}] \end{aligned}$$

(imposed by the continuity of the trajectories of $(D_t)_{t \geqslant 0}$).

Martingale problem

Define a corresponding process $(M_t^{\mathfrak{F}})_{t\in[0,\tau_D]}$ via

$$\forall\ t \in [0,\tau_D], \qquad M_t^{\mathfrak{F}} \ := \ \mathfrak{F}(D_t) - \mathfrak{F}(D_0) - \int_0^t \mathcal{L}[\mathfrak{F}](D_s)\,ds$$

Theorem 1

The law of the process $(D_t)_{t\in[0,\tau_D]}$ is a solution to the martingale problem associated to $\mathcal{L}$: for any $\mathfrak{F}$, the process $(M_t^{\mathfrak{F}})_{t\in[0,\tau_D]}$ is a martingale (in the filtration generated by the initial condition and the Brownian motion).

Reminder: consider V a vector field on $\mathcal{D}$, corresponding to the vector field $v_D \nu_{\partial D}$ on ∂D where v_D is a function from ∂D to $\mathbb{R}$. Introduce the deterministic evolution $(D_t)_t$ described by

$$dD_t = V(D_t)\, dt$$

We have

$$\begin{aligned} \frac{d}{dt} F_f(D_t) &= \int_{\partial D} \mu f v_{D_t}\, d\sigma \\ \left(\frac{d}{dt}\right)^2 F_f(D_t) &= \int_{\partial D} (\langle \nu_{\partial D_t}, \nabla(\mu f)\rangle + \kappa_{\partial D_t} \mu f) v_{D_t}\, d\sigma \end{aligned}$$

Heuristic proof

We deduce that for a stochastic evolution of the form

$$dD_t \quad = \quad \sqrt{2}V(D_t)\,dB_t + W(D_t)\,dt$$

we have

$$\begin{aligned} dF_f(D_t) \quad = \quad & \sqrt{2}\left(\int_{\partial D} \mu f v_{D_t}\,d\sigma\right) dB_t \\ & + \left(\int_{\partial D} (\langle \nu_{\partial D_t}, \nabla(\mu f)\rangle + \kappa_{\partial D_t}\mu f) v_{D_t}\,d\sigma\right) dt \\ & + \left(\int_{\partial D} \mu f w_{D_t}\,d\sigma\right) dt \end{aligned}$$

In particular with $v_D \equiv 1$ for any $D \in \mathcal{D}$ and

$$w_D \quad \coloneqq \quad 2\frac{\sigma(\partial D)}{\lambda(D)} + \langle \nu_{\partial D}, \zeta\rangle - \kappa_{\partial D}$$

we get

$$dF_f(D_t) \;=\; \sqrt{2}\left(\int_{\partial D_t} f\, d\underline{\mu}\right) dB_t + 2\frac{\sigma(\partial D_t)}{\lambda(D)}\left(\int_{\partial D_t} f\, d\underline{\mu}\right) dt$$
$$+\left(\int_{\partial D_t} \frac{1}{\mu}\langle \nu_{\partial D_t}, \nabla(\mu f)\rangle + \langle \nu_{\partial D_t}, \zeta\rangle f\, d\underline{\mu}\right) dt$$

Since the bounded variation term must coincide with $\mathcal{L}[F_f](D_t)\,dt$, we get the action of $\mathcal{L}$ on elementary observables. Concerning the carré du champs, use that

$$d\left\langle F_f(D_\cdot), F_g(D_\cdot)\right\rangle_t \;=\; \Gamma[F_f, F_g](D_t)\,dt$$

where the l.h.s. stands for the bracket of semi-martingales.

Plan

Elliptic diffusion

Consider an elliptic diffusion $(X(t))_{t\geqslant 0}$ on a differential manifold M. Let L be its generator, in a coordinate chart, it can be written

$$\sum_{i,j\in[\![m]\!]} a_{i,j}(x)\partial_{i,j} + \sum_{i\in[\![m]\!]} \widetilde{b}_i(x)\partial_i$$

where $(a_{i,j}(x))_{i,j\in[\![m]\!]}$ are symmetric positive definite matrices. Endowing M with the Riemannian structure whose matrices of scalar products are the inverse of the $(a_{i,j}(x))_{i,j\in[\![m]\!]}$, we get

$$L\cdot \quad = \quad \triangle\cdot + \langle b, \nabla\cdot\rangle$$

where b is a vector field on M. Assume that L leaves invariant the measure $\mu d\lambda$, with a smooth density $\mu > 0$. Denote $U := \ln(\mu)$. The μ-weighted Helmholtz decomposition says that $b = \nabla U + \beta$ with $\mathrm{div}(\mu\beta) = 0$.

Stoke's theorem

Stokes theorem asserts that for any vector field ξ and $D \in \mathcal{D}$, we have

$$\int_D \mathrm{div}(\xi)\, d\lambda \quad = \quad \int_{\partial D} \langle \nu_{\partial D}, \xi \rangle \, d\sigma$$

We deduce:

Theorem 2

Take $\zeta = \beta - \nabla U$, we get for any $f \in \mathcal{C}^\infty(M)$ and any $D \in \mathcal{D}$,

$$\mathcal{L}[F_f](D) := \int_D L[f]\, d\mu + 2\frac{\underline{\mu}(\partial D)}{\mu(D)} \int_{\partial D} f \, d\underline{\mu}$$

It amounts to see that with the chosen vector field ζ, for any $f \in \mathcal{C}^\infty(M)$ and any $D \in \mathcal{D}$,

$$\int_D L[f]\, d\mu \quad = \quad \int_{\partial D} \langle \nu_{\partial D_t}, \nabla f \rangle + \langle \nu_{\partial D}, \nabla \ln(\mu) + \zeta \rangle f \, d\underline{\mu}$$

Write

$$\begin{aligned} L[f] &= \triangle f + \langle \nabla U + \beta, \nabla f \rangle \\ &= \frac{1}{\mu}\mathrm{div}(\mu \nabla f) + \langle \beta, \nabla f \rangle \\ &= \frac{1}{\mu}\mathrm{div}(\mu \nabla f) + \frac{1}{\mu}\langle \mu\beta, \nabla f \rangle \\ &= \frac{1}{\mu}\mathrm{div}(\mu \nabla f) + \frac{1}{\mu}\mathrm{div}(\mu f \beta) \\ &= \frac{1}{\mu}\mathrm{div}(\mu(f\beta + \nabla f)) \end{aligned}$$

We deduce

$$\begin{aligned}\int_D L[f]\,d\mu &= \int_D \operatorname{div}(\mu(f\beta + \nabla f))\,d\lambda \\ &= \int_{\partial D} \langle \nu_{\partial D}, \mu(f\beta + \nabla f)\rangle\,d\sigma \\ &= \int_{\partial D} \langle \nu_{\partial D}, f\beta + \nabla f\rangle\,d\underline{\mu} \\ &= \int_{\partial D} \langle \nu_{\partial D_t}, \nabla f\rangle + \langle \nu_{\partial D}, \nabla \ln(\mu) + \zeta\rangle\, f\,d\underline{\mu}\end{aligned}$$

as wanted.

Intertwining

Consider the usual link Λ from $\mathcal{D}$ to M given by

$$\forall\ f \in \mathcal{C}^\infty(M), \qquad \Lambda[f](x) \;=\; \frac{1}{\mu(D)} \int_D f\, d\mu \;=\; \frac{F_f(D)}{F_{\mathbb{1}}(D)}$$

Theorem 3

For any $f \in \mathcal{C}^\infty(V)$, we have

$$\forall\ D \in \mathcal{D}, \qquad \mathfrak{L}[\Lambda[f]](D) \;=\; \Lambda[L[f]](D)$$

Consider $\mathcal{R} := \{(x,y) \in \mathbb{R}^2 \,:\, y > 0\}$ and the mapping

$$\mathfrak{f} : \mathcal{R} \ni (x,y) \quad \mapsto \quad \frac{x}{y}$$

For any $f \in \mathcal{C}^\infty(V)$, we have $\Lambda[f] = \mathfrak{f}(F_f, F_{\mathbb{1}})$, so that $\Lambda[f] \in \mathfrak{D}$. It follows that

$$\mathfrak{L}[\Lambda[f]] \;=\; \frac{1}{F_{\mathbb{1}}}\mathfrak{L}[F_f] - \frac{F_f}{F_{\mathbb{1}}^2}\mathfrak{L}[F_{\mathbb{1}}] - \frac{1}{F_{\mathbb{1}}^2}\Gamma_{\mathfrak{L}}[F_f, F_{\mathbb{1}}] + \frac{F_f}{F_{\mathbb{1}}^3}\Gamma_{\mathfrak{L}}[F_{\mathbb{1}}, F_{\mathbb{1}}]$$

which can be rewritten under the form

$$\begin{aligned} F_{\mathbb{1}}\mathfrak{L}[\Lambda[f]] \;&=\; \mathfrak{L}[F_f] - \frac{1}{F_{\mathbb{1}}}\Gamma_{\mathfrak{L}}[F_f, F_{\mathbb{1}}] \\ &\quad + F_f\left(\frac{1}{F_{\mathbb{1}}^2}\Gamma_{\mathfrak{L}}[F_{\mathbb{1}}, F_{\mathbb{1}}] - \frac{1}{F_{\mathbb{1}}}\mathfrak{L}[F_{\mathbb{1}}]\right) \end{aligned}$$

We compute, for any $D \in \mathcal{D}$,

$$\begin{aligned}\mathfrak{L}[F_{\mathbb{1}}](D) &= \int_D L[\mathbb{1}]\,d\mu + 2\frac{\underline{\mu}(\partial D)}{\mu(D)}\int_{\partial D}\mathbb{1}\,d\underline{\mu} \\ &= 2\frac{\underline{\mu}(\partial D)^2}{\mu(D)}\end{aligned}$$

Furthermore, remark that

$$\begin{aligned}\Gamma_{\mathfrak{L}}[F_{\mathbb{1}},F_{\mathbb{1}}](D) &= 2\left(\int_{\partial D}\mathbb{1}\,d\underline{\mu}\right)^2 \\ &= 2\underline{\mu}(\partial D)^2\end{aligned}$$

Proof of Theorem 3 (3)

so taking into account that $F_{\mathbb{1}}(D) = \mu(D)$, we get

$$\frac{1}{F_{\mathbb{1}}^2}\Gamma_{\mathfrak{L}}[F_{\mathbb{1}}, F_{\mathbb{1}}] - \frac{1}{F_{\mathbb{1}}}\mathfrak{L}[F_{\mathbb{1}}] \quad = \quad 0$$

Thus, we have

$$\begin{aligned} F_{\mathbb{1}}\mathfrak{L}[\Lambda[f]](D) &= \mathfrak{L}[F_f](D) - \frac{1}{F_{\mathbb{1}}}\Gamma_{\mathfrak{L}}[F_f, F_{\mathbb{1}}](D) \\ &= \int_D L[f]\, d\mu + 2\frac{\underline{\mu}(\partial D)}{\mu(D)}\int f\, d\underline{\mu} - \frac{2\underline{\mu}(\partial D)}{\mu(D)}\int_{\partial D} f\, d\underline{\mu} \\ &= \int_D L[f]\, d\mu \end{aligned}$$

and we conclude to the announced intertwining relation

$$\mathfrak{L}[\Lambda[f]] \quad = \quad \frac{F_{L[f]}}{F_{\mathbb{1}}} \quad = \quad \Lambda[L[f]]$$

Plan

Pitman's property

Let $(D_t)_{t\in[0,\tau_D)}$ be a $\mathcal{D}$-valued Markov process as in the previous section. Consider

$$\varsigma \;\; := \;\; 2\int_0^{\tau_D} (\underline{\mu}(\partial D_s))^2\, ds \;\in (0,+\infty]$$

and the time change $(\theta_t)_{t\in[0,\varsigma]}$ defined by

$$\forall\; t \in [0,\varsigma], \qquad 2\int_0^{\theta_t} (\underline{\mu}(\partial D_s))^2\, ds \;\; = \;\; t$$

Theorem 4

The process $(\mu(D_{\theta_{t\wedge\varsigma}}))_{t\geqslant 0}$ is a (possibly stopped) Bessel process of dimension 3.

Let a test function $\mathfrak{f} \in \mathcal{C}^\infty(\mathbb{R}_+)$ be given and consider the process $(S_t)_{t\in[0,\tau_D)}$ defined by

$$\forall\ t \in [0,\tau_D), \qquad \begin{aligned} S_t &:= \mathfrak{f}(\mu(D_t)) \\ &= \mathfrak{f}(F_{\mathbb{1}}(D_t)) \end{aligned}$$

From the martingale problem formulation, there exists a local martingale $(M_t)_{t\in[0,\tau_D)}$ such that for all $t \in [0,\tau_D)$,

$$S_t = S_0 + \int_0^t \mathfrak{L}[\mathfrak{f} \circ F_{\mathbb{1}}](D_s)\, ds + M_t$$

By definition of $\mathfrak{L}$, we have

$$\mathfrak{L}[\mathfrak{f} \circ F_{\mathbb{1}}](D) \quad = \quad \mathfrak{f}'(F_{\mathbb{1}})\mathfrak{L}[F_{\mathbb{1}}] + \frac{1}{2}\mathfrak{f}''(F_{\mathbb{1}})\Gamma_{\mathfrak{L}}[F_{\mathbb{1}}, F_{\mathbb{1}}]$$

Proof of Theorem 3

We have already computed that for any $D \in \mathcal{D}$,

$$\begin{aligned} \mathfrak{L}[F_{\mathbb{1}}](D) &= 2\frac{\underline{\mu}(\partial D)^2}{\mu(D)} \\ \Gamma_{\mathfrak{L}}[F_{\mathbb{1}}, F_{\mathbb{1}}](D) &= 2\underline{\mu}(\partial D)^2 \end{aligned}$$

so that

$$\begin{aligned} \mathfrak{L}[\mathfrak{f} \circ F_{\mathbb{1}}](D) &= \underline{\mu}(\partial D)^2 \left(\mathfrak{f}''(F_{\mathbb{1}}) + 2\frac{\mathfrak{f}'(F_{\mathbb{1}})}{F_{\mathbb{1}}} \right)(D) \\ &= 2\underline{\mu}(\partial D)^2 K[\mathfrak{f}](F_{\mathbb{1}}(D)) \end{aligned}$$

where

$$\forall\ x \in (0, +\infty), \qquad K \ := \tfrac{1}{2}\partial_x^2 + \tfrac{1}{x}\partial_x$$

This is the generator of the Bessel process of dimension 3 on $\mathbb{R}_+$. Thus we obtain, for all $t \in [0, \tau_D)$,

$$S_t = S_0 + 2\int_0^t \underline{\mu}(\partial D_s)^2 K[\mathfrak{f}](\mu(D_s))\, ds + M_t$$

It leads us to introduce the time change described above and

$$\forall\ t \in [0, \varsigma), \qquad R_t \ := \ \mu(D_{\theta(t)})$$

to get that $(R_t)_{t\in[0,\varsigma)}$ is a stopped continuous solution to the martingale problem associated to the generator K. It follows that $(R_t)_{t\in[0,\varsigma)}$ is a stopped Bessel process of dimension 3.

Plan

On the plane

Consider the Laplacian $L = \triangle$ on the Euclidean plane $\mathbb{R}^2$ and let $\mathcal{L}$ be the associated diffusion generator on $\mathcal{D}$.

Theorem 5

Let $(D_t)_{t\geqslant 0}$ be a solution to the martingale problem associated to $\mathfrak{L}$ defined for all times. Then we have a.s. in the Hausdorff metric,

$$\lim_{t\to+\infty} \frac{D_t}{\sqrt{\lambda(D_t)}} \quad = \quad B(0, 1/\sqrt{\pi})$$

where $B(0, 1/\sqrt{\pi})$ is the Euclidean ball centered at 0 of radius $1/\sqrt{\pi}$.

Curiously the proof of this natural result is quite complicated, requiring an enrichment of the observables (including integrals on the boundary), Bonnesen's inequality: for any $D \in \mathcal{D}$,

$$\pi^2(R(D) - r(D))^2 \leqslant \sigma(\partial D)^2 - 4\pi\lambda(D)$$

where $R(D)$ and $r(D)$ are respectively the radius of the incircle and the circumcircle of D, as well as a new isoperimetric stability result: as soon as $\sigma(\partial D)^2 - 4\pi\lambda(D) \leqslant \lambda(D)/\pi$, we have

$$\|\mathfrak{b}(\partial D) - \mathfrak{b}(D)\| \leqslant c\lambda(D)^{1/4}(\sigma(\partial D)^2 - 4\pi\lambda(D))^{1/4}$$

where $c > 0$ is a universal constant and $\mathfrak{b}(\cdot)$ stands for the barycenter.

These inequalities are only valid in dimension 2.

Denote $\widetilde{D}_t := D_t/\sqrt{\lambda(D_t)}$. The starting point is to study the evolution of

$$\sigma(\partial\widetilde{D}_t)^2 - 4\pi\lambda(\widetilde{D}_t) \quad = \quad \frac{\sigma(C_t)^2 - 4\pi\lambda(D_t)}{\lambda(D_t)}$$

It happens that the numerator is non-increasing and that the denominator goes to $+\infty$ for large times.
Thus via Bonnesen's inequality, it remains to see that the barycenter of $\widetilde{D}_t$ goes to 0. The following critical phenomenon is important in this respect:

$$\int_0^{+\infty} \frac{1}{\lambda(D_t)^a}\, ds < +\infty \quad \Leftrightarrow \quad a > 1$$

Question

How could the previous result on the large-time profil of the $\mathcal{D}$-valued process not be true in Euclidean spaces of any dimension?

References

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