On representations of functions from $[0, 1]$ to $[0, 1]$ by random equimeasured subsets

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Abstract

Given a measurable function $u : [0, 1] \to [0, 1]$, a random Borelian subset E of $[0, 1]$ is constructed, so that (i) the Lebesgue measure of E is constant and (ii) for any $t \in [0, 1]$, $\mathbb{P}[t \in E] = u(t)$. An application to the existence of a random bang-bang controller in optimal control is given.

Keywords: Random representations of measurable functions, random Borelian subsets, equimeasured subsets, bang-bang controller, optimal control.

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1 Introduction

The goal of the paper is to give a representation of a measurable function $u : [0,1] \to [0,1]$ in terms of random Borelian subsets. The problem is of those that Patrick Cattiaux likes: *simplissime* to state and natural, but whose investigation requires some care and which leads to open extensions. It is motivated by a question in optimal control theory.

More precisely, let $\mathcal{E}$ be the set of Borelian subsets from $[0,1]$, and $\mathcal{P}$ the set of probability measures on $[0,1]$. We endow $\mathcal{E}$ with the sigma-field $\mathfrak{E}$ generated by the mappings

$$\mathcal{E} \ni A \quad \mapsto \quad \mu(A) \in [0,1]$$

for all $\mu \in \mathcal{P}$ (of course we would end up with the same $\mathfrak{E}$, should we replace $\mathcal{P}$ by the set of signed measures on $[0,1]$).

Formally, a **subset-valued random variable** is a measurable mapping from the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathcal{E}, \mathfrak{E})$. Such a mapping E is said to be **regular** when

$$\{(t,\omega) \in [0,1] \times \Omega \,:\, t \in E(\omega)\} \quad \in \quad \mathcal{E} \otimes \mathfrak{E}$$

The subset-valued random variable is said to be **equimeasured**, if there exists $c \in [0,1]$ such that

$$\forall\, \omega \in \Omega, \qquad \ell(E(\omega)) \quad = \quad c \tag{1}$$

where ℓ stands for the restriction of the Lebesgue measure on $[0,1]$. The constant c is then said to be the **equimeasure** of E.

We say that the subset-valued random variable E is a **random subset representation** of the measurable mapping $u : [0,1] \to [0,1]$ when

$$\forall\, t \in [0,1], \qquad \mathbb{P}[t \in E] \quad = \quad u(t) \tag{2}$$

In the sequel, the term representation will stand for random subset representation.

The main objective of this paper is to show the following result.

Theorem 1 *Any measurable mapping $u : [0,1] \to [0,1]$ can be represented by a regular equimeasured subset-valued random variable. The equimeasure of the latter is necessarily given by $\int_{[0,1]} u\, d\ell$.*

The last assertion is an immediate consequence of Fubini's theorem and is the reason for the regularity requirement. Indeed, let c be the equimeasure of a regular equimeasured subset-valued random variable E solving the representation problem for the measurable mapping u. Integrating (2), we get

$$\begin{aligned}
\int u\, d\ell &= \int \mathbb{E}\left[\mathbb{1}_{\{t\in E\}}\right] \ell(dt) \\
&= \mathbb{E}\left[\int \mathbb{1}_{\{t\in E\}}\, \ell(dt)\right] \\
&= \mathbb{E}[\ell(E)] \\
&= c
\end{aligned}$$

In view of the above considerations, it is natural to wonder if Theorem 1 is still true if the state space $[0,1]$ where u is defined is replaced by the hypercube $[0,1]^d$ endowed with the restriction of the multidimensional Lebesgue measure, for integer dimensions $d \geqslant 2$, and more generally on any probability space.

The plan of the paper is as follows. In the next section we deal successively with the cases where u is constant, u takes at most one non-zero value and u takes a countable set of values. Based on the existence of these representations, in Section 3 we first deduce a weak version of Theorem 1 for general functions u, but where (2) is only satisfied almost everywhere in $t \in [0,1]$, before extending to its above statement, with (2) true everywhere. To illustrate the fact that the representation is not unique, even in law, in Section 4 we give alternative constructions. In particular we will look for connected-subset random variables and bring to the fore a link with the absolute continuity of u. In the last section, we present an optimal control motivation, consisting in the existence of bang-bang controllers, that the above representations enable to solve.

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2 The function u takes at most a countable set of values

So here we deal with the case where $u = \sum_{k\in\mathbb{N}} a_k \mathbb{1}_{A_k}$ with given sequences $(a_k)_{k\in\mathbb{N}}$ of numbers from $[0,1]$ and $(A_k)_{k\in\mathbb{N}}$ of disjoints Borelian subsets from $\mathcal{E}$.

Instead of the state space $[0,1]$, from now on, we will only work with $[0,1)$, identified with the quotient space $\mathbb{R}/\mathbb{Z}$. Note that we can deduce Theorem 1 from the same statement but for measurable mappings $u : [0,1) \to [0,1]$. Indeed, let $\bar{u} : [0,1] \to [0,1]$ be a measurable mapping and consider u its restriction to $[0,1)$. Let E be a regular equimeasured subset-valued random variable representing u on $[0,1)$ (now with respect to the Lebesgue probability on $\mathbb{R}/\mathbb{Z}$, that will still be denoted ℓ). Let B be a Bernoulli random variable of parameter $\bar{u}(1)$, defined on the same underlying probability space as E (or on an enlargement of it, if it happened to be too small, note we don't specify any dependence relation between E and B). Define

$$\bar{E} \quad \coloneqq \quad \begin{cases} E \sqcup \{1\} & \text{, if } B = 1 \\ E & \text{, if } B = 0 \end{cases}$$

It is immediate to check that $\bar{E}$ is a regular equimeasured subset-valued random variable representing $\bar{u}$ on $[0,1]$.

2.1 The function u is constant

We start with the elementary situation where u is constant, namely $u = a\mathbb{1}_A$, with $A = [0,1)$ the whole state space and $a = c \coloneqq \int_{\mathbb{R}/\mathbb{Z}} u \, d\ell$.

Let π be the canonical projection of $\mathbb{R}$ onto $\mathbb{R}/\mathbb{Z}$. Let X be a random variable uniformly distributed on $[0,1)$ and consider E the regular subset-valued random variable defined by

$$E \quad \coloneqq \quad \pi([X, X+c))$$

which has equimeasure c, since $c \leqslant 1$.

Note that the law of E is invariant by shifts in $\mathbb{R}/\mathbb{Z}$. It follows that the quantity $\mathbb{P}[t \in E]$ does not depend on $t \in \mathbb{R}/\mathbb{Z}$, and the Fubini argument given after Theorem 1 shows that

$$\forall\, t \in \mathbb{R}/\mathbb{Z}, \qquad \mathbb{P}[t \in E] \;=\; c \;=\; u(t)$$

(this is also a consequence of the more general Lemma 4 in Section 4).

This ends the proof that Theorem 1 and its variant on $\mathbb{R}/\mathbb{Z}$ are satisfied by constant functions.

2.2 The function u takes at most one non-zero value

Let us now consider the case $u = a\mathbb{1}_A$ with $a \in [0,1]$ and A a Borelian subset of $\mathbb{R}/\mathbb{Z}$.

Denote by μ the measure admitting the density $\mathbb{1}_A$ with respect to the Lebesgue measure ℓ and consider its repartition function

$$\varphi \,:\, [0,1) \ni x \quad\mapsto\quad \int_{[0,x]} \mathbb{1}_A(s)\,\ell(ds) \in [0,\ell(A)]$$

It is well-known that the image of μ by φ is the Lebesgue measure on $[0,\ell(A)]$, let us call it ν.

Using scaling properties and the above existence of representation of constant functions (furthermore taking into account the observation made before Subsection 2.1), there exists a regular equimeasured subset-valued random variable $\widetilde{E}$ representing the constant a on $[0,\ell(A)]$. The equimeasure of $\widetilde{E}$ is then $a\ell(A)$.

Consider the regular subset-valued random variable E defined by

$$E \quad\coloneqq\quad A \cap \varphi^{-1}(\widetilde{E})$$

We compute that

$$\begin{aligned}
\ell(E) &= \mu(\varphi^{-1}(\widetilde{E})) \\
&= \nu(\widetilde{E}) \\
&= a\ell(A) \\
&= \int_{\mathbb{R}/\mathbb{Z}} a\mathbb{1}_A\,d\ell
\end{aligned}$$

namely E has equimeasure $\ell[u]$ as desired.

Furthermore we get for any $t \in [0,1)$,

$$\begin{aligned}
\mathbb{P}[t \in E] &= \mathbb{1}_A(t)\mathbb{P}[t \in \varphi^{-1}(\widetilde{E})] \\
&= \mathbb{1}_A(t)\mathbb{P}[\varphi(t) \in \widetilde{E}] \\
&= a\mathbb{1}_A(t) \\
&= u(t)
\end{aligned}$$

Thus Theorem 1 holds for functions taking at most one non-zero value.

2.3 The function u takes at most a countable set of values

We now come to the case where $u = \sum_{k\in\mathbb{N}} a_k\mathbb{1}_{A_k}$, with given sequences $(a_k)_{k\in\mathbb{N}}$ of numbers from $[0,1]$ and $(A_k)_{k\in\mathbb{N}}$ of disjoints Borelian subsets from $\mathcal{E}$.

From the previous subsection, for any $k \in \mathbb{N}$, we can find a regular equimeasured subset-valued random variable E_k representing $a_k\mathbb{1}_{A_k}$ on $[0,1)$, with furthermore $E_k \subset A_k$. Denote $(\Omega_k, \mathcal{F}_k, \mathbb{P}_k)$ the underlying probability space.

Consider the product space

$$(\Omega, \mathcal{F}, \mathbb{P}) \quad\coloneqq\quad \prod_{k\in\mathbb{N}} (\Omega_k, \mathcal{F}_k, \mathbb{P}_k)$$

and for any $\omega = (\omega_k)_{k\in\mathbb{N}} \in \Omega$, define

$$E(\omega) \quad\coloneqq\quad \bigsqcup_{k\in\mathbb{N}} E_k(\omega_k)$$

which is easily seen to be a regular subset-valued random variable.

Since the E_k, for $k \in \mathbb{N}$, are disjoint, we compute

$$\begin{aligned}\ell(E) &= \sum_{k\in\mathbb{N}} \ell(E_k) \\ &= \sum_{k\in\mathbb{N}} a_k \ell(A_k) \\ &= \ell[u]\end{aligned}$$

so that E is equimeasured with equimeasure $\ell[u]$ as wanted.

Furthermore we have for any $t \in [0,1)$,

$$\begin{aligned}\mathbb{P}[t \in E] &= \sum_{k\in E_k} \mathbb{P}[t \in E_k] \\ &= \sum_{k\in E_k} \mathbb{P}_k[t \in E_k] \\ &= \sum_{k\in E_k} a_k \mathbb{1}_{A_k}(t) \\ &= u(t)\end{aligned}$$

Thus Theorem 1 holds for functions taking at most a countable set of values.

3 The general case

Theorem 1 is proven here in three steps, corresponding to the next subsections. First we write the function u as the sum of its "discrete" and "continuous" parts, which are related to the associated decomposition of the image of ℓ by u. Next we deal with the continuous component, since the discrete component has been treated in the previous section. But it only gives us the almost-everywhere version of Theorem 1. The final step consists in passing from almost-everywhere to everywhere.

3.1 Decomposition into discrete and continuous parts

Let R be the set of values $r \in [0,1]$ such that $\ell(u = r) > 0$. This set is at most countable. Define

$$\begin{aligned}D &:= \{x \in [0,1) \,:\, u(x) \in R\} \\ C &:= [0,1)\backslash D\end{aligned}$$

The discrete and continuous parts are simply given by

$$\begin{aligned}u_D &:= u\mathbb{1}_D \\ u_C &:= u\mathbb{1}_C\end{aligned}$$

and we have

$$u = u_D + u_C$$

From Subsection 2.3, we get the existence of a regular equimeasured subset-valued random variable E_D representing u_D on $[0,1)$, with $E_D \subset D$ and even $E_D \subset \{x \in [0,1) \,:\, u(x) \in R\backslash\{0\}\}$. The corresponding equimeasure is $\ell[u_D]$. Denote $(\Omega_D, \mathcal{F}_D, \mathbb{P}_D)$ the underlying probability space.

If we had a regular equimeasured subset-valued random variable E_C representing u_C on C, we could deduce a regular equimeasured subset-valued random variable E representing u on $[0,1)$ as in Subsection 2.3: denote $(\Omega_C, \mathcal{F}_C, \mathbb{P}_C)$ the underlying probability space of E_C, we take

$$\begin{aligned}(\Omega, \mathcal{F}, \mathbb{P}) &:= (\Omega_D, \mathcal{F}_D, \mathbb{P}_D) \otimes (\Omega_C, \mathcal{F}_C, \mathbb{P}_C) \\ E &= E_D \bigsqcup E_C\end{aligned}$$

Since E_D and E_C are disjoint, there is no difficulty in checking that the subset-valued random variable E has equimeasure $\ell[u_D + u_C] = \ell[u]$ and that it represents u.

The goal of the two next subsections is to construct E_C.

Remark 2 As in Subsection 2.2, we could try to reduce this quest of E_C to functions u without discrete part. Consider μ_C the measure admitting $\mathbb{1}_C$ as density with respect to ℓ, as well as the corresponding repartition function $\varphi_C : [0,1) \to [0, \ell(C)]$. Let ψ_C be the pseudo-inverse of φ_C:

$$\forall\, y \in [0, \ell(C)], \qquad \psi_C(y) \;\coloneqq\; \inf\{x \in [0,1) \,:\, \varphi_C(x) = y\}$$

(with the convention that $\psi_C(\ell(C)) = 1$ if the r.h.s. is empty for $y = \ell(C)$).

The function $\varphi_C \circ \psi_C$ is the identity mapping, at least on $[0, \ell(C))$. But in general $\psi_C \circ \varphi_C$ does not coincide everywhere with the identity mapping on C. Thus even if we knew how to find a equimeasured subset-valued random variable $\widetilde{E}_C$ representing $u \circ \psi_D$ on $[0, \ell(C))$, the equimeasured subset-valued random variable given by

$$E \;\coloneqq\; C \cap \varphi_C^{-1}(\widetilde{E}_C)$$

may not represent u_C, since for $t \in [0,1)$,

$$\begin{aligned} \mathbb{P}[t \in E] &= \mathbb{1}_C(t)\mathbb{P}[\varphi_C(t) \in \widetilde{E}_C] \\ &= u(\psi_C \circ \varphi_C(t)) \end{aligned}$$

This is the technical reason why in the next two subsections we will work directly on $[0,1)$, without using a transfer through the repartition function. □

3.2 The almost-everywhere version

Assume that u_C is not a.s. 0, since in this case the almost-everywhere version of Theorem 1 follows at once.

Our purpose here is to construct a sequence $(a_n)_{n\in\mathbb{N}}$ of positive numbers satisfying $\sum_{n\in\mathbb{N}} a_n = 1$ and a sequence of Borelian subsets $(A_n)_{n\in\mathbb{N}}$ included into C such that a.s.

$$u \;=\; \sum_{n\in\mathbb{N}} u_n$$

with

$$\forall\, n \in \mathbb{N}, \qquad u_n \;\coloneqq\; a_n \mathbb{1}_{A_n}$$

The construction is based on an iteration. So let us start with a_1 and A_1.

Define

$$c_C \;\coloneqq\; \int u_C \, d\ell$$

and note that $c_C \in (0, \ell(C))$.

We consider for a_1 the unique element of $(0,1)$ such that

$$\ell(u_C \geqslant a_1) \;=\; c_C$$

and

$$A_1 \;\coloneqq\; \{x \in C \,:\, u_C(x) \geqslant a_1\}$$

Note that $v_1 := u - u_1$, with $u_1 = a_1 \mathbb{1}_{A_1}$, is a measurable function taking its values in $[0,1]$ and satisfying that for any $r \in [0,1]$,

$$\ell(\{x \in C \,:\, v_1(x) = r\}) \quad = \quad 0$$

Assume that $a_1, ..., a_n$ and $A_1, ..., A_n$ have been constructed, so that

$$v_n \quad := \quad u_C - u_1 - u_2 - \cdots - u_n$$

is a measurable function taking its values in $[0,1]$ and satisfying that for any $r \in [0,1]$,

$$\ell(\{x \in C \,:\, v_n(x) = r\}) \quad = \quad 0$$

In particular, there exists a unique element a_{n+1} of $(0,1)$ such that

$$\ell(v_n \geqslant a_{n+1}) \quad = \quad c_C \tag{3}$$

and we take

$$\begin{aligned} A_{n+1} &:= \{x \in C \,:\, v_n(x) \geqslant a_{n+1}\} \\ u_{n+1} &:= a_{n+1} \mathbb{1}_{A_{n+1}} \end{aligned}$$

Note that the measurable function $v_{n+1} := u - u_1 - u_2 - \cdots - u_{n+1}$ takes its values in $[0,1]$ and satisfies that for any $r \in [0,1]$,

$$\ell(\{x \in C \,:\, v_{n+2}(x) = r\}) \quad = \quad 0$$

so that the iteration condition is valid.

By construction, the sequence of functions $(v_n)_{n \in \mathbb{Z}_+}$, with $v_0 = u_C$, is non-increasing and non-negative. In view of (3), which is valid for all $n \in \mathbb{Z}_+$, we deduce that the sequence $(a_n)_{n \in \mathbb{N}}$ is non-increasing. In addition, we can define the function v given on C by

$$\forall\, x \in A, \qquad v(x) \quad := \quad \lim_{n \to \infty} v_n(x)$$

The important observation about this function is:

Lemma 3 *The function v vanishes a.s. on C.*

Proof

We use an argument by contradiction. So assume that there exists $\epsilon > 0$ such that

$$\ell(v \geqslant \epsilon) \quad > \quad 0$$

We can then find a subset B of $\{v \geqslant \epsilon\}$ such that furthermore $\ell(B) \in (0, c_C)$, e.g. we can find such a set of the form $\{v \geqslant \epsilon, u \geqslant a\}$ for an appropriate choice of the value $a \in (0,1)$.

Since for any $n \in \mathbb{Z}_+$, we have $v_n \geqslant v$, we get from (3) that $a_{n+1} \geqslant \epsilon$ and that $B \subset A_{n+1}$. The bound

$$u_C(x) \quad \geqslant \quad u_1(x) + u_2(x) + \cdots + u_n(x)$$

then implies that for $x \in B$, $1 \geqslant u_C(x) \geqslant \epsilon n$ and we get a contradiction by letting n go to infinity. ■

It follows that a.s.,

$$u_C \quad = \quad \sum_{n \in \mathbb{N}} u_n$$

and by integration

$$\ell[u_C] \;=\; \sum_{n\in\mathbb{N}} \ell[u_n]$$

Since $\ell[u_C] = c_C$ and that for any $n \in \mathbb{N}$, we have $\ell[u_n] = a_n\ell(A_n) = a_n c_C$, we deduce

$$\sum_{n\in\mathbb{N}} a_n \;=\; 1$$

Consider then the subset-valued random variable $\widehat{E}_C$ taking the value A_n with probability a_n. It is equimeasured with equimeasure c_C. Furthermore for any $t \in C$, we have

$$\begin{aligned}
\mathbb{P}[t \in \widehat{E}_C] &= \sum_{n\in\mathbb{N}} a_n \mathbb{1}_{A_n}(t) \\
&= \sum_{n\in\mathbb{N}} a_n \mathbb{1}_{v_n(t)\geqslant a_n} \\
&= \sum_{n\in\mathbb{N}} (v_{n-1}(t) - v_n(t))\mathbb{1}_{v_n(t)\geqslant a_n} \\
&= \sum_{n\in\mathbb{N}} v_{n-1}(t) - v_n(t) \\
&= v_0(t) - v(t) \\
&= u(t)
\end{aligned}$$

where in the fourth equality we used that if $v_n(t) < a_n$, then $v_n(t) = v_{n+1}(t)$ and where the last equality holds a.s. in $t \in C$.

It follows that the subset-valued random variable $\widehat{E}_C$ is a.s. representing the function u_C on C. The arguments given in Subsection 3.1 then show that Theorem 1 holds a.s., in the sense that (2) is only satisfied a.s.

3.3 Going from almost-everywhere to everywhere

This the last step of the proof of Theorem 1.

With the notations of the previous subsection, consider the negligible set

$$\mathcal{N} \;\coloneqq\; \{x \in C \,:\, v(x) > 0\}$$

and the regular subset-valued random variable $\breve{E}_C$ defined by

$$\breve{E}_C \;\coloneqq\; \widehat{E}_C \cap (C\backslash\mathcal{N})$$

We check immediately that $\breve{E}_C$ has the same equimeasure as $\widehat{E}_C$, that it is included into $C\backslash\mathcal{N}$ and that it exactly represents the mapping $u_C\mathbb{1}_{C\backslash\mathcal{N}}$.

We are going to construct a regular subset-valued random variable $\breve{E}_C$, included into $\mathcal{N}$ and thus with equimeasure 0, and which is a representation of $u_C\mathbb{1}_{\mathcal{N}}$ on C. It will then follow that

$$E_C \;\coloneqq\; \breve{E}_C \bigsqcup \breve{E}_C$$

is a regular equimeasured subset-valued random variable representing $u_C = u_C\mathbb{1}_{C\backslash\mathcal{N}} + u_C\mathbb{1}_{\mathcal{N}}$ on C.

It is the random variable needed in Subsection 3.1 to end the proof of Theorem 1.

To construct $\breve{E}_C$, consider

$$\begin{aligned}
\forall\ n, k \in \mathbb{Z}_+, \qquad A_{n,k} &\coloneqq \{x \in \mathcal{N} \,:\, k2^{-n} < u_C(x) \leqslant (k+1)2^{-n}\} \\
\forall\ n \in \mathbb{Z}_+, \qquad w_n &\coloneqq \sum_{k\in\mathbb{Z}_+} \frac{k}{2^n}\mathbb{1}_{A_{n,k}}
\end{aligned}$$

so that we have on C,

$$u_C \mathbb{1}_{\mathcal{N}} = \lim_{n\to\infty} w_n$$

Since $w_0 = 0$ and $w_{n+1} - w_n \geqslant 0$ for any $n \in \mathbb{Z}_+$, we can rewrite this equality as

$$\begin{aligned} u_C \mathbb{1}_{\mathcal{N}} &= \sum_{n\in\mathbb{Z}_+} w_{n+1} - w_n \\ &= \sum_{n\in\mathbb{Z}_+} \frac{1}{2^{n+1}} \mathbb{1}_{B_n} \end{aligned}$$

with

$$\forall\, n \in \mathbb{Z}_+, \qquad B_n := \bigsqcup_{k\in\mathbb{Z}_+} A_{n,k} \backslash A_{n+1,2k}$$

It remains to choose the negligible set B_n with probability 2^{-n-1}, for any $n \in \mathbb{Z}_+$, to get the wanted regular subset-valued random variable $\breve{E}_C$.

4 Some particular cases

In general the representation given by Theorem 1 is not unique in law, and our goal in this section is to illustrate this feature by presenting alternative approaches. Note that the procedure described in the three previous constructions leads to subset-valued equimeasured random variables with typically unbounded numbers of connected components. Below we look for subset-valued equimeasured random variables with very few connected components and see a relation with absolute continuity assumptions on u.

Let's take the problem in reverse: we're going to construct a certain family of set-valued random variables and deduce the corresponding functions u.

Fix $c \in (0,1)$, that will end up playing the role of the equimeasure. Let X be a random variable with values in $[0,1)$, its distribution is denoted by μ. Recall that π is the canonical projection of $\mathbb{R}$ onto $\mathbb{R}/\mathbb{Z}$, identified with $[0,1)$. As in Section 2, we're interested in

$$E := \pi([X, X+c))$$

This subset-valued random variable E is regular and equimeasured with equimeasure c.

Let us compute the fonction $u : [0,1) \to [0,1]$ represented by E:

Lemma 4 *For any fixed $t \in [0,1)$, we have*

$$\mathbb{P}[t \in E] = F(t) - F(t-c) + 1 - F(1-c+t)$$

where $F : \mathbb{R} \to [0,1]$ is the distribution function of μ.

Proof

Note that

$$E = [X, (X+c) \wedge 1) \bigsqcup [0, (X+c-1)_+[$$

We deduce that for $t \in [0,1)$ fixed,

$$\begin{aligned}
\{t \in E\} &= \{X \leqslant t < (X+c)\wedge 1\} \sqcup \{t < (X+c-1)_+\} \\
&= \{X \leqslant 1-c,\, X \leqslant t < (X+c)\wedge 1\} \sqcup \{X \leqslant 1-c,\, t < (X+c-1)_+\} \\
&\quad \sqcup \{X > 1-c,\, X \leqslant t < (X+c)\wedge 1\} \sqcup \{X > 1-c,\, t < (X+c-1)_+\} \\
&= \{X \leqslant 1-c,\, X \leqslant t < X+c\} \sqcup \{X > 1-c,\, X \leqslant t < 1\} \sqcup \{X > 1-c,\, t < X+c-1\} \\
&= \{t-c < X \leqslant (1-c)\wedge t\} \sqcup \{1-c < X \leqslant t\} \sqcup \{t+1-c < X\} \\
&= \{t-c < X \leqslant t\} \sqcup \{t+1-c < X\}
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbb{P}[t \in E] &= \mathbb{P}[t-c < X \leqslant t] + \mathbb{P}[t+1-c < X] \\
&= F(t) - F(t-c) + 1 - F(1+t-c)
\end{aligned}$$

by definition of the distribution function. ■

Note that $t-c$ and $t-c+1$ cannot both belong to $[0,1)$. Given that $F(r) = 0$ for $r < 0$ and $F(r) = 1$ for $r \geqslant 1$, we deduce that for all $t \in [0,1)$,

$$\mathbb{P}[t \in E] = \begin{cases} F(t) - F(t-c+1) + 1 & \text{, if } t < c \\ F(t) - F(t-c) & \text{, if } t \geqslant c \end{cases}$$

Let $u_{c,F}$ be the function defined on $[0,1)$ by the r.h.s.

Let $\mathcal{F}$ be the set of functions on $[0,1)$ that are non-negative, non-decreasing, càdlàg (i.e. right-hand continuous with left-hand limits) and that verify $F(1-) = 1$. This is the set of distribution functions for random variables with values in $[0,1)$. We may wonder what is the set $\mathcal{U}_c$ consisting of $u_{c,F}$ when F goes through the set $\mathcal{F}$. It would also interesting to describe $\mathcal{U} := \bigsqcup_{c\in(0,1)} \mathcal{U}_c$, namely the set of functions which can be represented by connected-subset-valued equimeasured random variables (in $\mathbb{R}/\mathbb{Z}$).

As a first step in this direction, here we only investigate the set $\mathcal{U}_{1/2}$. To state our result, we denote by $\mathrm{Var}(f, I)$ the total variation of a function f over the interval I. If $\mathrm{Var}(f, I) < +\infty$, for any r inside I, the left-hand limit $f(r-)$ exists and we note $\delta[f](r) := f(r) - f(r-)$ the (possible) jump of f at r.

We also need $\mathcal{G}$, which is the set of functions $u : [0,1) \to [0,1]$ that are càdlàg and satisfy

$$\forall\, t \in [0,1/2), \qquad u(t) + u(t+1/2) = 1 \tag{4}$$

$$\mathrm{Var}(u, [1/2,1)) \leqslant 1 - |\delta[f](1/2)| \tag{5}$$

Proposition 5 *We have*

$$\mathcal{U}_{1/2} = \mathcal{G}$$

Proof

Consider $u \in \mathcal{U}_{1/2}$, which is therefore of the form

$$\forall\, t \in [0,1), \qquad u(t) = \begin{cases} F(t) - F(t+1/2) + 1 & \text{, if } t < 1/2 \\ F(t) - F(t-1/2) & \text{, if } t \geqslant 1/2 \end{cases} \tag{6}$$

for a certain $F \in \mathcal{F}$. This function u is clearly càdlàg, since F is.

For $t \in [0,1/2)$, we have $-1 \leqslant F(t) - F(t+1/2) \leqslant 0$, so that $u(t) \in [0,1]$. For $t \in [1/2,1)$, we have $0 \leqslant F(t) - F(t-1/2) \leqslant 1$, so we also have $u(t) \in [0,1]$. Moreover, for all $t \in [0,1/2)$,

$$u(t) + u(t+1/2) = F(t) - F(t+1/2) + 1 + F(t+1/2) - F(t) = 1$$

Finally, note that the total variation of the restriction of F to $[1/2, 1)$ is $F(1-)-F(1/2) = 1-F(1/2)$ and that the total variation of the non-decreasing function $[1/2, 1) \ni t \mapsto F(t-1/2)$ is $F(1/2-)-F(0)$. It follows that the total variation of u on $[1/2, 1)$ is less than $1 - \delta[F](1/2) - F(0)$.

We deduce from (6) that

$$\begin{aligned} u(1/2-) &= F(1/2-) - F(1-) + 1 \ = F(1/2-) \\ u(1/2) &= F(1/2) - F(0) \end{aligned}$$

hence $\delta[u](1/2) = \delta[F](1/2) - F(0)$ and on the one hand

$$\begin{aligned} \mathrm{Var}(u, [1/2, 1)) &\leqslant 1 - \delta[F](1/2) - F(0) \\ &= 1 - \delta[u](1/2) - 2F(0) \\ &\leqslant 1 - \delta[u](1/2) \end{aligned}$$

and on the other hand

$$\begin{aligned} \mathrm{Var}(u, [1/2, 1)) &\leqslant 1 - \delta[F](1/2) - F(0) \\ &= 1 - \delta[u](1/2) - 2\delta[F](1/2) \\ &\leqslant 1 + \delta[u](1/2) \end{aligned}$$

Putting these two inequalities together, we obtain

$$\mathrm{Var}(u, [1/2, 1)) \leqslant 1 - |\delta[u](1/2)|$$

thus showing that $u \in \mathcal{G}$.

Conversely, consider a function $u \in \mathcal{G}$.

Since the total variation of u on $[1/2, 1)$ is finite, Jordan's decomposition allows us to write

$$\forall\, t \in [1/2, 1), \qquad u(t) - u(1/2) = G(t) - H(t)$$

where G and H are defined on $[1/2, 1)$, càdlàg, non-decreasing, satisfying $G(1/2) = 0 = H(1/2)$ and

$$\mathrm{Var}(u, [1/2, 1)) = \mathrm{Var}(G, [1/2, 1)) + \mathrm{Var}(H, [1/2, 1))$$

Note that

$$\begin{aligned} \mathrm{Var}(G, [1/2, 1)) &= G(1-) - G(1/2) = G(1-) \\ \mathrm{Var}(H, [1/2, 1)) &= H(1-) - H(1/2) = H(1-) \end{aligned}$$

so that

$$G(1-) + H(1-) = \mathrm{Var}(u, [1/2, 1)) \tag{7}$$

Since we also have

$$u(1-) - u(1/2) = G(1-) - H(1-)$$

we deduce

$$\begin{aligned} 2G(1-) &= \mathrm{Var}(u, [1/2, 1)) + u(1-) - u(1/2) \\ &= \mathrm{Var}(u, [1/2, 1)) + 1 - u(1/2-) - u(1/2) \end{aligned} \tag{8}$$

where

$$u(1-) + u(1/2-) = 1$$

It follows that

$$
\begin{aligned}
1-G(1-)-u(1/2) &= 1-\frac{\mathrm{Var}(u,[1/2,1))+1-u(1/2-)-u(1/2)}{2}-u(1/2) \\
&= \frac{1-\mathrm{Var}(u,[1/2,1))-\delta[u](1/2)}{2} \\
&\geqslant 0
\end{aligned} \tag{9}
$$

Replacing (8) in (7), we obtain

$$
\begin{aligned}
H(1-) &= \mathrm{Var}(u,[1/2,1))-\frac{\mathrm{Var}(u,[1/2,1))+1-u(1/2-)-u(1/2)}{2} \\
&= \frac{\mathrm{Var}(u,[1/2,1))-1+u(1/2-)+u(1/2)}{2} \\
&= \frac{\mathrm{Var}(u,[1/2,1))-1-\delta[u](1/2)}{2}+u(1/2) \\
&\leqslant u(1/2)
\end{aligned} \tag{10}
$$

Consider the function F given on $[0,1)$ by

$$
\forall\ t\in[0,1),\qquad F(t) \ :=\ \begin{cases} H(t+1/2)+1-G(1-)-u(1/2) & \text{, if } t<1/2 \\ G(t)+1-G(1-) & \text{, if } t\geqslant 1/2 \end{cases} \tag{11}
$$

In particular, for $t\in[1/2,1)$,

$$
\begin{aligned}
F(t)-F(t-1/2) &= G(t)+1-G(1-)-(H(t)+1-G(1-)-u(1/2)) \\
&= G(t)-H(t)+u(1/2) \\
&= u(t)-u(1/2)+u(1/2) \\
&= u(t)
\end{aligned}
$$

and for $t\in[0,1/2)$,

$$
\begin{aligned}
F(t)-F(t+1/2)+1 &= 1+H(t+1/2)+1-G(1-)-u(1/2)-(G(t+1/2)+1-G(1-)) \\
&= 1-(G(t+1/2)-H(t+1/2))-u(1/2) \\
&= 1-u(t+1/2)+u(1/2)-u(1/2) \\
&= u(t)
\end{aligned}
$$

Thus (6) is satisfied.

It remains to show that $F\in\mathcal{F}$.

• By definition, F is clearly càdlàg because G and H are (and because the two cases considered in (11) are for $t<1/2$ and $t\geqslant 1/2$, it would have been more problematic if it had been $t\leqslant 1/2$ and $t>1/2$).

• To see that F is non-decreasing, since G and H are non-decreasing, we need only check that $F(1/2-)\leqslant F(1/2)$, which comes, via (10), from

$$
\begin{aligned}
F(1/2) &= 1-G(1-) \\
F(1/2-) &= H(1-)+1-G(1-)-u(1/2)
\end{aligned}
$$

• To check that $F(0)\geqslant 0$, we calculate

$$
\begin{aligned}
F(0) &= H(1/2)+1-G(1-)-u(1/2) \\
&= 1-G(1-)-u(1/2) \\
&\geqslant 0
\end{aligned}
$$

from (9).

• To check that $F(1-) = 1$, we calculate

$$F(1-) \quad = \quad G(1-) + 1 - G(1-) \ = 1$$

■

Remark 6 Note that

$$\mathrm{Var}(u, [1/2, 1)) + |\delta[f](1/2)| \quad = \quad \lim_{\epsilon \to 0_+} \mathrm{Var}(u, [1/2 - \epsilon, 1))$$

so that the term on the left can be written as $\mathrm{Var}(u, [1/2-, 1))$. The condition (4) can then be simplified into

$$\mathrm{Var}(u, [1/2-, 1)) \quad \leqslant \quad 1$$

□

Let's give a consequence of Proposition 5 :

Corollary 7 *The function $u : [0, 1) \ni t \mapsto t$ can be represented by a subset-valued equimeasured random variable whose numbers of connected component are at most three.*

Proof

Consider first the function $\widetilde{u}$ given by

$$\forall\, r \in [0, 1), \qquad \widetilde{u}(r) \quad := \quad \begin{cases} r & , \text{if } r \in [0, 1/2) \\ \frac{3}{2} - r & , \text{if } r \in [1/2, 1) \end{cases}$$

For all $r \in [0, 1/2)$, we have

$$\begin{aligned} \widetilde{u}(t) + \widetilde{u}(t + 1/2) \quad &= \quad r + \frac{3}{2} - \left(r + \frac{1}{2}\right) \\ &= \quad 1 \end{aligned}$$

so that (4) is satisfied.

It also appears that $\mathrm{Var}(\widetilde{u}, [1/2-, 1)) = 1$, so (5) is also verified. It follows that there exists a connected-subset-valued equimeasured random variable $\widetilde{E}$ representing the function $\widetilde{u}$. It can be checked that it corresponds to $c = 1/2$ and X uniform on $[0, 1/2)$.

Consider the mapping $\widehat{\varphi} : [0, 1) \to [0, 1)$ defined by

$$\forall\, r \in [0, 1), \qquad \widehat{\varphi}(r) \quad := \quad \begin{cases} r & , \text{if } r \in [0, 1/2] \\ \frac{3}{2} - r & , \text{if } r \in (1/2, 1) \end{cases}$$

(it differs from $\widetilde{u}$ only in $1/2$).

Consider the subset-valued random variable $\widehat{E}$ defined by

$$\widehat{E} \quad := \quad \widehat{\varphi}^{-1}(\widetilde{E})$$

Since φ leaves ℓ invariant, $\widehat{E}$ has equimeasure $1/2$. Furthermore, noting that $\widehat{\varphi}$ is an involution, we compute that for all $t \in [0, 1)$,

$$\begin{aligned} \mathbb{P}[t \in \widehat{E}] \quad &= \quad \mathbb{P}[\widehat{\varphi}(t) \in \widetilde{E}] \\ &= \quad \widetilde{u}(\widehat{\varphi}(t)) \\ &= \quad \begin{cases} t & , \text{if } t \neq 1/2 \\ 1 & , \text{if } t = 1/2 \end{cases} \end{aligned}$$

This is not quite the identity function we want. To get it, let's also consider the mapping $\breve{\varphi}$: $[0, 1) \to [0, 1)$ defined by

$$\forall\ r \in [0,1), \qquad \breve{\varphi}(r) \ \coloneqq \ \begin{cases} r & , \text{if } r \in [0, 1/2) \\ 0 & , \text{if } r = 1/2 \\ \frac{3}{2} - r & , \text{if } r \in (1/2, 1) \end{cases}$$

and the subset-valued random variable $\breve{E}$ defined by

$$\breve{E} \ \coloneqq \ \breve{\varphi}^{-1}(\widetilde{E})$$

As above, $\breve{E}$ has equimeasure 1/2 and we compute that for all $t \in [0, 1)$,

$$\begin{aligned} \mathbb{P}[t \in \breve{E}] &= \mathbb{P}[\breve{\varphi}(t) \in \widetilde{E}] \\ &= \widetilde{u}(\breve{\varphi}(t)) \\ &= \begin{cases} t & , \text{if } t \neq 1/2 \\ 0 & , \text{if } t = 1/2 \end{cases} \end{aligned}$$

Consider two realizations of $\widehat{E}$ and $\breve{E}$, then choose E as one of them with probability $(1/2, 1/2)$. It is a representation of the identity function u, since we have $u = (\widetilde{u} \circ \widehat{\varphi} + \widetilde{u} \circ \breve{\varphi})/2$.

Since $\widehat{E}$ has at most two connected components and $\breve{E}$ at most three, we get that E has also at most three connected components. ■

5 Application to control theory

Consider the following control problem. General introductions to optimal control theory are provided by the books of Fleming and Rishel [1] and Liberzon [2].

Given the measurable function $f : [0,1] \times [0,1] \to [0,1]$, we are interested in the evolution $(x(t))_{t\in[0,1]}$ directed by

$$\begin{cases} & x(0) = 0 \\ \forall\ t \in [0,1], & \dot{x}(t) = f(x(t), u(t)) \end{cases}$$

where $u : [0,1] \to [0,1]$ is a measurable control. The cost of u is

$$C(u) \ \coloneqq \ \int_0^1 \phi(t, u(t))\, dt$$

where $\phi : [0,1] \times [0,1] \to \mathbb{R}_+$ is a measurable and bounded function.

A traditional problem $(\mathcal{P}_1)$ is to find a control u_* with minimal cost compatible with the objective $x(1) = c$, where $c \in [0,1]$ is given. Depending on f and c, it may be that no control meets the requirement $x(1) = c$ or that a minimal solution is not unique.

A stochastic version of this problem allows u to be random, while imposing further restrictions on its form. The interpretation is that the control is exercised by a population of agents, each of them only being able to act in a restricted modality. For instance, let us only allow bang-bang controls, namely measurable functions u taking values in $\{0, 1\}$. Such a control is the indicator function of a measurable subset of $[0,1]$. Thus we can identify the set of controls with $\mathcal{E}$, still endowed with its sigma-field $\mathfrak{E}$. We are led to look for regular subset-valued random variables E such that the random evolution $(x(t))_{t\in[0,1]}$ directed by

$$\begin{cases} & x(0) = 0 \\ \forall\ t \in [0,1], & \dot{x}(t) = f(x(t), \mathbb{1}_E(t)) \end{cases}$$

ends up with the deterministic condition $x(1) = c$, and whose cost defined by

$$C(E) \coloneqq \int_0^1 \phi(t, \mathbb{E}[\mathbb{1}_E(t)])\, dt$$

is minimal. This problem amounts to find the optimal weight each control should be given.

Consider the particular case where:

- The function f is given by

$$\forall\, x, u \in [0,1], \qquad f(x,u) \;=\; u$$

- Problem $(\mathcal{P}_1)$ admits a unique solution u_*. A toy example is when u_* is a priori given (satisfying $\int_0^1 u_*(t)\, dt = c$) and

$$\forall\, t, u \in [0,1], \qquad \phi(t,u) \;\coloneqq\; (u - u_*(t))^2$$

From the first point, we deduce that the condition $x(1) = c$ coincides with (1). From the second point, we see that a regular subset-valued random variables E will be optimal if it satisfies (2) with u replaced by u_*.

It was our initial motivation to investigate the existence problem mentioned in the introduction.

References

[1] Wendell H. Fleming and Raymond W. Rishel. *Deterministic and stochastic optimal control.* Springer-Verlag, Berlin-New York, 1975. Applications of Mathematics, No. 1.

[2] Daniel Liberzon. *Calculus of variations and optimal control theory.* Princeton University Press, Princeton, NJ, 2012. A concise introduction.

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