COUPLINGS OF BROWNIAN MOTIONS WITH SET-VALUED DUAL PROCESSES ON RIEMANNIAN MANIFOLDS

MARC ARNAUDON, KOLÉHÈ COULIBALY-PASQUIER, AND LAURENT MICLO

ABSTRACT. The purpose of this paper is to construct a Brownian motion $(X_t)_{t\geqslant 0}$ taking values in a Riemannian manifold M, together with a compact set-valued process $(D_t)_{t\geqslant 0}$ such that, at least for small enough $\mathscr{F}^D$-stopping time $\tau > 0$ and conditioned by $\mathscr{F}^D_\tau$, the law of X_τ is the normalized Lebesgue measure on D_τ. This intertwining result is a generalization of Pitman theorem. We first construct regular intertwined processes related to Stokes' theorem. Then using several limiting procedures we construct synchronous intertwined, free intertwined, mirror intertwined processes. The local times of the Brownian motion on the (morphological) skeleton or the boundary of each D_t plays an important role. Several examples with moving intervals, discs, annulus, symmetric convex sets are investigated.

KEYWORDS. Brownian motions on Riemannian manifolds, intertwining relations, set-valued dual processes, couplings of primal and dual processes, stochastic mean curvature evolutions, boundary and skeleton local times, generalized Pitman theorem.

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1. INTRODUCTION AND MAIN RESULTS

Markov intertwinings were introduced by Rogers and Pitman [25] to give a direct proof of the famous relation between the Brownian motion and the Bessel-3 process due to Pitman [23]. These relations were next used by Yor and his coauthors (see e.g. [28, 6]) to get identities in law and by Diaconis and Fill [11] to construct strong stationary times. For a historical account of the subsequent development of the Markov intertwining technique, consult for instance Pal and Shkolnikov [22].

At an algebraic level, a **Markov intertwining relation** is a (directed) weak similar relation, from a Markov semi-group $(\bar{P}_t)_{t\geqslant 0}$ on a measurable state space $(\bar{M}, \bar{\mathcal{M}})$ to another Markov semi-group $(P_t)_{t\geqslant 0}$ on a measurable state space $(M, \mathcal{M})$, consisting of a Markov kernel (called the **link**) Λ from $(\bar{M}, \bar{\mathcal{M}})$ to $(M, \mathcal{M})$ such that

$$\forall\, t \geqslant 0, \qquad \bar{P}_t\Lambda \;=\; \Lambda P_t \tag{1.1}$$

in the sense of the composition of Markov kernels. Depending on non-degeneracy properties of Λ, such a relation is more or less strong. Especially when Markov semi-groups are described by their generators, (1.1) is often replaced by

$$\bar{L}\Lambda \;=\; \Lambda L \tag{1.2}$$

where $\bar{L}$ and L are respectively the generators of $(\bar{P}_t)_{t\geqslant 0}$ and $(P_t)_{t\geqslant 0}$. But then one has to be more careful with the meaning of generators (e.g. in the sense of martingale problems) and their domains, in particular the domains are transported via (1.2).

To be more useful from a probabilist point of view, it is convenient to convert (1.2) into a coupling between $(\bar{X}_t)_{t\geqslant 0}$ and $(X_t)_{t\geqslant 0}$, two Markov processes respectively associated to

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$\bar{L}$ and L (called the **dual** and **primal processes**), so that the following relations hold for the conditional laws:

$$\forall\, t \geqslant 0, \qquad \mathcal{L}(X_t|\bar{X}_{[0,t]}) \quad = \quad \Lambda(\bar{X}_t, \cdot). \tag{1.3}$$

In addition, one asks that $(\bar{X}_t)_{t\geqslant 0}$ can be constructed from $(X_t)_{t\geqslant 0}$ in an adapted way, meaning

$$\forall\, t \geqslant 0, \qquad \mathcal{L}(\bar{X}_{[0,t]}|X) \quad = \quad \mathcal{L}(\bar{X}_{[0,t]}|X_{[0,t]}). \tag{1.4}$$

Yor was wondering about such couplings between some piecewise linear Markov processes and squared Bessel processes, in order to simplify his approach to certain properties of the former processes similar to those of the latter, see the end of the introduction of [28].

Such couplings are crucial for the constructions of strong stationary times, as explained by Diaconis and Fill [11] in a discrete time and finite setting. More precisely, in this situation X is an ergodic Markov chain with invariant probability π and $\bar{X}$ is a Markov chain absorbed in a unique point. A strong stationary time τ for $(X_t)_{t\geqslant 0}$ is a finite stopping time for $(X_t)_{t\geqslant 0}$ (and some independent randomness) such that τ and X_τ are independent and X_τ is distributed according to π. Taking into account (1.3) and (1.4), one can see that the absorption time for $(\bar{X}_t)_{t\geqslant 0}$ is a strong stationary time for $(X_t)_{t\geqslant 0}$.

Strong stationary times are important for two reasons (cf. Diaconis and Fill [11]):

- They enable to sample *exactly* the invariant probability π, contrary to the usual approximations provided by Monte Carlo techniques.

- They provide a probabilistic alternative to functional analysis approaches for the quantitative investigation of convergence to equilibrium. More precisely, for any strong stationary time τ, we have

$$\forall\, t \geqslant 0, \qquad \mathfrak{s}(\mathcal{L}(X_t), \pi) \quad \leqslant \quad \mathbb{P}[\tau > t]$$

where the **separation discrepancy** $\mathfrak{s}(\mu, \pi)$ between two probability measures μ and π is defined by

$$\mathfrak{s}(\mu, \pi) \quad := \quad \operatorname*{ess\,sup}_{\pi} \left(1 - \frac{d\mu}{d\pi}\right)$$

(where $d\mu/d\pi$ is the Radon-Nikodym density). The separation discrepancy dominates the total variation norm and gives positivity properties of μ with respect to π. In the context of convergence to equilibrium, it is very difficult to estimate the discrepancy of $\mathfrak{s}(\mathcal{L}(X_t), \pi)$ via functional inequality techniques (see e.g. the book [5] of Bakry, Gentil and Ledoux).

In the objective of constructing strong stationary times via intertwining duality, there are particular dual processes $(\bar{X}_t)_{t\geqslant 0}$ which are taking values in $\mathcal{M}$, the set of measurable subsets of M, but in general $\bar{M}$ is only a subset of $\mathcal{M}$, consisting in some regular subsets. The absorption set is the whole set M. The heuristic goal of intertwining duality is then to construct random subsets $\bar{X}_t \subset M$ such that X_t is already at equilibrium in $\bar{X}_t$, for all $t \geqslant 0$, in such a way that $\bar{X}$ is itself Markovian and ends up covering the whole state space M.

In the diffusion context, set-valued intertwining dual processes started to be constructed in Fill and Lyzinski [13] and [19]. In [10], set-valued dual processes for diffusions on Riemannian manifolds were identified as stochastic perturbations of mean-curvature flows. But the coupling of primal and dual processes were not considered in [10] and this is our present goal, mainly for Brownian motions on Riemannian manifolds. As we will see, there are numerous ways to construct such couplings (this is true in more general contexts, see [20] for the diversity of such couplings in a finite framework), but none of them is immediate and they are related to fine geometric features of the evolving subsets, such as their skeletons. We are thus to consider synchronous intertwined, free intertwined, mirror set-valued intertwined dual processes.

The reader must be warned that, as it stands now in the context of multidimensional diffusions, the set-valued dual processes are not defined up to the absorption time (except in symmetric settings), and as a consequence the same will be true for our couplings, which will be defined only up to some positive stopping times. We hope to investigate this point in future works, to end the construction of strong stationary times for Brownian motion on compact Riemannian manifolds, which remains our remote motivation. Other motivations for the couplings of primal and dual processes in the context of diffusions can be found in Machida [17] and [20].

Let us now present more precise definitions. Here the state space M is a d-dimensional complete Riemannian manifold. Denote respectively by ρ, μ and $\underline{\mu}$, the Riemannian distance, the Lebesgue measure on M and the corresponding $(d-1)$-Hausdorff measure. The main objective of this paper is to construct couplings of primal diffusions processes with their set-valued dual intertwined processes. This will partially solve Conjecture 6 in [10] in the case of Brownian motion $(X_t)_{t\geqslant 0}$ and stochastic modified mean curvature flow $(D_t)_{t\geqslant 0}$ (which were generically denoted $(\bar{X}_t)_{t\geqslant 0}$ above). This conjecture says that an intertwined construction in the sense of Definition 1.1 is always possible.

Definition 1.1. Consider a Markov process $D = (D_t)_{t\in[0,\tau]}$, with values in compact subsets of M and continuous with respect to the Hausdorff topology, and where τ is an a.s. positive stopping time in the filtration $\mathscr{F}^D$ of D, serving as a lifetime for D. We say that a Brownian motion $X = (X_t)_{t\geqslant 0}$ in M and D are intertwined when for all bounded $\mathscr{F}^D$-stopping time τ' smaller than τ, conditioned on $\mathscr{F}^D_{\tau'}$, $X_{\tau'}$ has uniform law in $D_{\tau'}$ (and in particular $X_{\tau'} \in D_{\tau'}$). More generally, for any $\mathscr{F}^D$-stopping time $\widetilde{\tau}$ smaller than τ, we say that X and D are $\widetilde{\tau}$-intertwined when X and $(D_t)_{t\in[0,\widetilde{\tau}]}$ are intertwined.

This is a generic definition, below stronger topologies on subsets of M will be considered. Note that the above lifetime is not necessary the explosion time, i.e. the exit time from all compact sets for the considered topology. In the infinite dimensional state space of D, compactness does not seem an appropriate notion.

Also notice that $\widetilde{\tau}$-intertwining prevents $(X_t)_{t\geqslant 0}$ to have a lifetime smaller that $\widetilde{\tau}$. So we will never have to consider the lifetime of $(X_t)_{t\geqslant 0}$.

Our main results are Theorems 2.8, 2.12 3.5 and 4.1 presenting such joint constructions of the primal Brownian motion $(X_t)_{t\geqslant 0}$ and the dual domain-valued $(D_t)_{t\geqslant 0}$ processes. The coupling of Theorem 2.8 which is proved to be intertwined in Theorem 2.12, consists of the infinite-dimensional stochastic differential equation (2.11), based on a function $f : (x, D) \mapsto f(x, D)$ which is a deformation of the signed distance from $x \in M$ to the boundary of the domain D (see Assumption (2.2) for the precise requirements). Theorem 3.5 is obtained by specifying some functions f approximating the distance to boundary. Given the trajectory $(X_t)_{t\geqslant 0}$ of the Brownian motion, we construct the domain evolution $(D_t)_{t\geqslant 0}$ using the local time of $(X_t)_{t\geqslant 0}$ on the skeletons of $(D_t)_{t\geqslant 0}$ and the mean curvatures of the normal foliations of these domains (see (3.29)). Functions f approximating the null function lead to Theorem 4.1, where the prominent role is played by the local time at the boundary. This situation is in some sense opposite to the previous one, since the driving Brownian motion of $(D_t)_{t\geqslant 0}$ is now independent from $(X_t)_{t\geqslant 0}$, while it is as correlated as it can be in Theorem 3.5. These theoretical results are illustrated by the fundamental examples of Section 5. First we recover the intertwining relation between the real Brownian motion and the three-dimensional Bessel process. Next we deal with rotationally symmetric manifolds. Finally we present the application of our results to symmetric convex domains in the plane, even if the detailed proofs are deferred to [2].

To come back to our initial motivation, assume that $(X_t)_{t\geqslant 0}$ and $(D_t)_{t\geqslant 0}$ are intertwined, where the lifetime τ is the hitting/covering time by D of the whole state space M. If furthermore τ is finite (typically true when M is compact), then the Riemannian measure can be normalized into a probability (called the **uniform distribution**, which is invariant and reversible for the Brownian motion $(X_t)_{t\geqslant 0}$) and τ is a strong stationary time for $(X_t)_{t\geqslant 0}$. In this situation, the tail distributions of τ provide quantitative estimates for the speed of convergence of the Brownian motion toward equilibrium, in the separation sense. These estimates will need geometric ingredients such as Ricci bounds and it will be interesting to see how they will enter the game.

The needs for couplings between primal and dual processes of a Markovian intertwining relation is illustrated by [3], where strong stationary times τ_n are constructed for the n-dimensional sphere (when the subset-valued dual is starting from a singleton), satisfying

$$\mathbb{E}[\tau_n] \sim \frac{\ln(n)}{n}$$

and for any $r > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[\tau_n > (1+r)\frac{\ln(n)}{n}\right] = \lim_{n\to\infty} \mathbb{P}\left[\tau_n < (1-r)\frac{\ln(n)}{n}\right] = 0.$$

2. Intertwined dual processes: existence in connection with Stoke's formula

In this section we make a construction of intertwined processes $(X_t)_{t\geqslant 0}$ and $(D_t)_{t\geqslant 0}$ based on the Stokes' Formula (2.1) below. Consider a compact domain D in M with C^2 boundary. Let $f : D \to \mathbb{R}$ a C^2 function such that $\nabla f|_{\partial D} = N^D$ the normal inward vector on boundary. Then by Stoke's formula, for any C^2 function $g : D \to \mathbb{R}$,

$$-\int_{\partial D} g d\underline{\mu} = \int_{\partial D} g\langle \nabla f, -N^D\rangle \, d\underline{\mu} = \int_D g\Delta f \, d\mu + \int_D \langle \nabla g, \nabla f\rangle \, d\mu. \tag{2.1}$$

For $\alpha \in (0,1)$, denote by $\mathscr{D}^{2+\alpha}$ the set of compact connected subsets D of M with $C^{2+\alpha}$ boundary. It will be more convenient to work with this state space (endowed with its natural topology) than with the larger one considered in Definition 1.1. Let us even restrict it further:

We fix a point $o \in M$ for convenience.

Definition 2.1. For a given $\alpha \in (0,1)$, $\varepsilon > 0$, we denote by $\mathcal{F}^{\alpha,\varepsilon}$ the set of $D \in \mathscr{D}^{2+\alpha}$ such that

- $D \subset B(o, 1/\varepsilon)$ the Riemannian ball centered at o with radius $1/\varepsilon$;
- $\rho(\partial D, S(D)) \geqslant \varepsilon$, where $S = S(D)$ is the skeleton of D (see appendix A for details);
- $\rho(\partial D, S^{\text{out}}(D)) \geqslant \varepsilon$, where $S^{\text{out}}(D)$ is the outer skeleton of D, i.e. the skeleton of $(D)^c$;
- the coefficients of the α-Hölderianity of the second fundamental form of ∂D are bounded by $1/\epsilon$.

The set $\mathcal{F}^{\alpha,\varepsilon}$ will serve as the state space of the set-valued process $(\widetilde{D}_t)_{t\in[0,\tau_\varepsilon]}$ and $\tau_\varepsilon \in (0,+\infty]$ will be the exiting time from $\mathcal{F}^{\alpha,\varepsilon}$. This process will be a diffusion, i.e. a Markov process with continuous trajectories (for the topology inherited from $\mathscr{D}^{2+\alpha}$), and its generator $\widetilde{\mathscr{L}}$ will be defined later in (2.13). We extend the trajectory $(\widetilde{D}_t)_{t\in[0,\tau_\varepsilon]}$ by taking $\widetilde{D}_t = \widetilde{D}_{\tau_\varepsilon}$ for any $t > \tau_\varepsilon$. It amounts to imposing that $\widetilde{\mathscr{L}}$ vanishes outside $\mathcal{F}^{\alpha,\varepsilon}$. It is possible to define in the same way $(\widetilde{D}_t)_{t\in[0,\tau)}$ on $\mathcal{D}^{2+\alpha}$ (which coincides with $\cup_{\varepsilon>0}\mathcal{F}^{\alpha,\varepsilon}$), where τ is the exiting time from $\mathcal{D}^{2+\alpha}$. But it will be more convenient for us to work with a process with an infinite lifetime (to be able to apply Proposition D.3 in Appendix D) and whose set of values has a boundary which is well-separated from the skeleton.

Let $\beta \in \{0, \alpha\}$. For $D_0 \in \mathcal{D}^{2+\beta}$ and $\delta > 0$ small enough, a δ-neighborhood of D_0 is defined as follow:

$$\mathcal{V}_\delta^{2+\beta}(D_0) := \left\{\mathrm{int}(\exp_{\partial D_0}(f)), f \in C^{2+\beta}(\partial D_0), \|f\|_{C^{2+\beta}(\partial D_0)} < \delta\right\}, \tag{2.2}$$

where for $f \in C^{2+\beta}(\partial D_0)$

$$\exp_{\partial D_0}(f) := \left\{\exp_x(f(x)N^{D_0}(x)), x \in \partial D_0\right\}$$

($\exp$ being the exponential map in M), and $\mathrm{int}(\exp_{\partial D_0}(f))$ is the relatively compact open subset of M with boundary $\exp_{\partial D_0}(f)$. Let $\eta(\partial D_0) > 0$ be the maximal radius for a tubular neighborhood of ∂D_0 on which the signed distance to ∂D_0 is regular. Alternatively, $\eta(\partial D_0)$ is the distance between ∂D_0 and the union of inner and outer skeleton of D_0. Notice that $\delta < \eta(\partial D_0)$ garantees that all elements of $\mathcal{V}_\delta^{2+\beta}(D_0)$ are regular deformations of D_0. Also notice that all elements D of $\mathcal{F}^{\alpha,\varepsilon}$ have $\eta(\partial D) \geqslant \varepsilon$. The map which to $\left\{f \in C^{2+\beta}(\partial D_0), \|f\|_{C^{2+\beta}(\partial D_0)} < \delta\right\}$ associates $D \in \mathcal{V}_\delta^{2+\beta}(D_0)$ via (2.2) is one to one since for such a D, the corresponding function f is characterized by the fact that at a point z of ∂D which projects onto $\pi(z) \in \partial D_0$, $f(\pi(z))$ is the signed distance from ∂D_0 to z, positive when $z \in D_0$. This is a particular case of Theorem 1.5 in [7].

We identify two domains $D_1, D_2 \in \mathcal{V}_\delta^{2+\beta}(D_0)$ with the functions $f_1, f_2 \in C^{2+\beta}(\partial D_0)$ such that $D_1 = \mathrm{int}\{\exp_{\partial D_0}(f_1)\}$ and $D_2 = \mathrm{int}\{\exp_{\partial D_0}(f_2)\}$ and we define a local distance

$$d_{\beta,D_0}(D_1, D_2) := \|f_1 - f_2\|_{C^{2+\beta}(\partial D_0)}. \tag{2.3}$$

Assumption 2.2.

- The function

  $$f : M \times \mathcal{F}^{\alpha,\varepsilon} \to \mathbb{R}$$
  $$(x, D) \mapsto f(x, D) = f^D(x)$$

  is a $C^{2+\alpha}$ function in the two variables (the differential in D is in the sense of Fréchet with respect to the above local Banach structure defined by the distances $d_{\alpha,D}$). The functions f^D satisfy

  $$\left\|\nabla f^D\right\|_\infty \leqslant 1, \tag{2.4}$$

  and coincide with the signed distance to the boundary $\rho_{\partial D}^+$ (positive inside D and negative outside) in a neighbourhood of ∂D. The functions f^D have bounded Hessian, uniformly in $D \in \mathcal{F}^{\alpha,\varepsilon}$. Furthermore, we assume that the coefficients of the α-Hölderianity of $\mathrm{Hess} f^D$ are uniformly bounded over $\mathcal{F}^{\alpha,\varepsilon}$.
- There exists a positive integer m and a C^1 map

  $$\sigma_c : M \times \mathcal{F}^{\alpha,\varepsilon} \to \Gamma(TM \otimes (\mathbb{R}^m)^*)$$
  $$(x, D) \mapsto \sigma_c(x, D) = \sigma_c^D(x) \in L(\mathbb{R}^m, T_xM)$$

  where $\Gamma(TM \otimes (\mathbb{R}^m)^*)$ denotes the set of sections over M of $TM \otimes (\mathbb{R}^m)^*$ and $L(\mathbb{R}^m, T_xM)$ is the set of linear maps from $\mathbb{R}^m$ to T_xM, such that the linear map

  $$\begin{aligned} \sigma^D(x) : \mathbb{R} \times \mathbb{R}^m &\to T_xM \\ (w_0, w) &\mapsto w_0 \nabla f^D(x) + \sigma_c^D(x)(w) \end{aligned} \tag{2.5}$$

  satisfies

  $$\forall x \in D, \;\; \sigma^D(\sigma^D)^*(x) = \mathrm{Id}_{T_xM}. \tag{2.6}$$

□

Remark 2.3. The first condition of Assumption 2.2 implies that

$$\begin{aligned} \nabla f^D|_{\partial D} &= (\nabla \rho^+_{\partial D})|_{\partial D} (= N^D) \quad \text{and} \\ \Delta f^D|_{\partial D} &= (\Delta \rho^+_{\partial D})|_{\partial D} (= -h^D). \end{aligned} \tag{2.7}$$

where h^D stands for the mean curvature on ∂D: at $x \in \partial D$, $h^D(x)$ is the trace of the second fundamental form of ∂D, it can alternatively be described as the sum of the principal curvatures in 2-planes directions containing $N^D(x)$. The sign convention is that $h^D > 0$ when D is convex. It also implies that the functions f^D are uniformly Lipschitz and have uniformly bounded Laplacian. Also, for fixed $x \in \partial D$, varying D successively along a field K normal to the boundary ∂D and along N^D for the second derivative:

$$\begin{aligned} \langle \nabla f(x,\cdot), K\rangle(x) &= -\langle N^D(x), K(x)\rangle \quad \text{and} \\ \nabla df(x,\cdot)\left(N^D, N^D\right) &= 0 \end{aligned} \tag{2.8}$$

where $\nabla df(x,\cdot)$ is the Hessian of f in the second variable.

The second condition of Assumption 2.2 implies that for all $u \in T_xM$,

$$\|u\|^2 = \langle u, \nabla f^D(x)\rangle^2 + \sum_{i=1}^m \langle u, \sigma_{\mathrm{c}}^D(x)(e_i)\rangle^2 \tag{2.9}$$

for $e_1, \dots, e_m$ an orthonormal basis of $\mathbb{R}^m$. In particular, if $x \in \partial D$, taking $u = \nabla f^D(x) = N^D(x)$, we get since $\|N^D(x)\| = 1$:

$$0 = \langle \nabla f^D(x), \sigma_c^D(x)(e_i)\rangle, \quad i = 1, \dots m. \tag{2.10}$$

Proposition 2.4. *Assumption 2.2 can always be realized, with any $\alpha \in (0,1)$ and $\varepsilon > 0$.*

Proof. We begin with remarking that for $D \in \mathcal{F}^{\alpha,\varepsilon}$, $\rho(\partial D, S(D)) \geqslant \varepsilon$. In particular, the distance to ∂D is $C^{2+\alpha}$ on $D_\varepsilon := \{x \in M,\ \rho(x, \partial D) < \varepsilon\}$. Let h_ε be an odd smooth nondecreasing function from $\mathbb{R}$ to $\mathbb{R}_+$ such that $h_\varepsilon(r) = r$ for $r \in [0, \varepsilon/2]$, $h_\varepsilon(r) = (3/4)\varepsilon$ for $r \geqslant \varepsilon$ and $\|h'_\varepsilon\|_\infty \leqslant 1$. Then $f^D := h_\varepsilon \circ \rho^+_{\partial D}$ satisfies all the requirements of the first condition of Assumption 2.2. Then for constructing σ_c^D we proceed as in [4], Proposition 3.2 where $m+1, \nabla f^D, (\sigma_c^D(e_1), \dots, \sigma_c^D(e_m))$ here is denoted $m, \sigma_1, (\sigma_2, \dots, \sigma_m)$ there. The wanted regularity in D is easily checked. □

Let $(W_t)_{t\geqslant 0}$ and $(W_t^m)_{t\geqslant 0}$ two independent Brownian motions with values respectively in $\mathbb{R}$ and $\mathbb{R}^m$.

The equation we are interested in writes in Itô form for all $y \in \partial D_t$:

$$\left\{ \begin{array}{rcl} dX_t &=& \left(\nabla f^{D_t}(X_t)\, dW_t + \sigma_{\mathrm{c}}^{D_t}(X_t)\, dW_t^m\right) \\ d\partial D_t(y) &=& N^{D_t}(y)\left(dW_t + \left(\frac{1}{2} h^{D_t}(y) + \Delta f^{D_t}(X_t)\right) dt\right) \end{array} \right. \tag{2.11}$$

started at a compact domain D_0 with $C^{2+\alpha}$ boundary and X_0 such that $\mathscr{L}(X_0) = \mathscr{U}(D_0)$, where $\mathscr{U}(D_0)$ is the uniform probability measure on D_0. This assumption is essential and will be made all along the paper. The notation $d\partial D_t(y) = (d\partial D_t)(y)$ stands for an infinitesimal move of the boundary ∂D_t at point y and is rigorously presented in Appendix B, see (B.6). The second equation in (2.11) and (2.12) below are stochastic differential equations in $\mathscr{D}^{2+\alpha}$, and a geometric way to represent stochastic partial differential equations locally defined in $C^{\alpha,2+\alpha}([0,\infty) \times \partial D_0)$. Similar equations can be found in Appendix A.2 of [14].

In fact, as in Definition 2.1, the evolution equation (2.11) is implicitly considered only up to the a.s. positive exit time τ_ε of $\mathcal{F}^{\alpha,\varepsilon}$ for some fixed $\alpha \in (0,1)$, $\varepsilon > 0$, after which the process is assumed not to move.

In (2.11), the processes $(D_t)_{t\geqslant 0}$ and $(X_t)_{t\geqslant 0}$ are fully interacting, since the evolution of one of them depends on the other one. In particular, they are not Markovian by themselves in general.

Another subset-valued process $(\widetilde{D}_t)_{t\geqslant 0}$ will be interesting for our purposes. It is solution to the evolution equation

$$\forall\, t \leqslant \widetilde{\tau}_\epsilon,\ \forall y \in \partial\widetilde{D}_t,$$

$$d\partial\widetilde{D}_t(y) \quad = \quad N^{\widetilde{D}_t}(y)\left(d\widetilde{W}_t + \left(\frac{1}{2}h^{\widetilde{D}_t}(y) - \frac{\underline{\mu}^{\partial\widetilde{D}_t}(\partial\widetilde{D}_t)}{\mu(\widetilde{D}_t)}\right)dt\right), \tag{2.12}$$

where $(\widetilde{W}_t)_{t\geqslant 0}$ is a real-valued Brownian motion and where $\widetilde{\tau}_\epsilon$ is the exit time from $\mathcal{F}^{\alpha,\epsilon}$.

Notice that the equation for $(\widetilde{D}_t)_{t\geqslant 0}$ does no longer depend of $(X_t)_{t\geqslant 0}$, so if the solution is unique, $(\widetilde{D}_t)_{t\geqslant 0}$ will be Markovian. It is Equation (44) in [10] (up to a time scaling by 2). Theorem 40 of [10] (where (44) has been rewritten as (79)) proves local existence of a solution. The second term in the right of (2.12) is the one of mean curvature flow (with $\frac{1}{2}$ in front of it). The first term is a stochastic perturbation, uniform in the normal direction. Equation for stochastic front propagation in [14] has exactly these two terms. The last term in our equation is also uniform in the normal direction. It can be seen as a conditioning which prevents the solution to implode. One of the main goals of this article will be to prove that the solution to the second equation in (2.11) has same law as the solution to (2.12).

Theorem 2.5. *Fix $\alpha \in (0,1)$ and $\varepsilon > 0$. Then* (2.12) *admits a unique global solution. In particular the process $(\widetilde{D}_t)_{t\geqslant 0}$ is Markovian.*

Proof. The proof is a consequence of Theorem 22 in [10]. It can be found in Appendix C. □

To describe the generator $\widetilde{\mathscr{L}}$ of $(\widetilde{D}_t)_{t\geqslant 0}$ we must introduce the following notations. For any smooth function k on M, consider the mapping F_k on $\mathcal{D}^{2+\alpha}$ by

$$\forall\, D \in \mathcal{D}^{2+\alpha}, \qquad F_k(D) \quad := \quad \int_D k\, d\mu.$$

For any $k, g \in \mathcal{C}^\infty(M)$ and any $D \in \mathcal{D}^{2+\alpha}$, define

$$\widetilde{\mathscr{L}}[F_k](D) \quad := \quad \underline{\mu}^{\partial D}(k)\frac{\underline{\mu}^{\partial D}(\partial D)}{\mu(D)} - \frac{1}{2}\underline{\mu}^{\partial D}(\langle \nabla k, N^D\rangle), \tag{2.13}$$

$$\Gamma_{\widetilde{\mathscr{L}}}[F_k, F_g](D) \quad := \quad \int_{\partial D} k\, d\underline{\mu}\int_{\partial D} g\, d\underline{\mu}. \tag{2.14}$$

Next consider $\mathfrak{A}$ the algebra consisting of the functionals of the form $\mathfrak{F} := \mathfrak{f}(F_{k_1}, ..., F_{k_n})$, where $n \in \mathbb{Z}_+$, $k_1, ..., k_n \in \mathcal{C}^\infty(M)$ and $\mathfrak{f} : \mathcal{R} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ mapping, with $\mathcal{R}$ an open subset of $\mathbb{R}^n$ containing the image of $\mathcal{D}^{2+\alpha}$ by $(F_{k_1}, ..., F_{k_n})$. For such a functional $\mathfrak{F}$, define

$$\begin{aligned}\widetilde{\mathscr{L}}[\mathfrak{F}] \quad := \quad & \sum_{l=1}^n \partial_j\mathfrak{f}(F_{k_1}, ..., F_{k_n})\widetilde{\mathscr{L}}[F_{k_l}] \\ & + \sum_{j,l\in[\![1,n]\!]}^n \partial_{j,l}\mathfrak{f}(F_{k_1}, ..., F_{k_n})\Gamma_{\widetilde{\mathscr{L}}}[F_{k_j}, F_{k_l}].\end{aligned} \tag{2.15}$$

To two elements of $\mathfrak{A}$, $\mathfrak{F} := \mathfrak{f}(F_{k_1}, ..., F_{k_n})$ and $\mathfrak{G} := \mathfrak{g}(F_{g_1}, ..., F_{g_m})$, we also associate

$$\Gamma_{\widetilde{\mathscr{L}}}[\mathfrak{F}, \mathfrak{G}] \quad := \sum_{l\in[\![n]\!], j\in[\![m]\!]} \partial_l\mathfrak{f}(F_{k_1}, ..., F_{k_n})\partial_j\mathfrak{g}(F_{g_1}, ..., F_{g_m})\Gamma_{\widetilde{\mathscr{L}}}[F_{k_l}, F_{g_j}]. \tag{2.16}$$

Remark 2.6. To see that the above definitions are non-ambiguous, since a priori they could depend on the writing of $\mathfrak{F} \in \mathfrak{A}$ under the form $\mathfrak{f}(F_{k_1}, ..., F_{k_n})$ and similarly for $\mathfrak{G}$, see Remark 2 of [10]. More generally, the forms of (2.15) and (2.16) are consequences of the diffusion feature of $\widetilde{\mathscr{L}}$, for more on the subject, see e.g. the book of Bakry, Gentil and Ledoux [5].

Remark 2.7. In the above considerations, $\widetilde{\mathscr{L}}$ was defined on $\mathcal{D}^{2+\alpha}$, but from now on, $\widetilde{\mathscr{L}}$ will stand for the restriction of this generator to $\mathcal{F}^{\alpha,\varepsilon}$ and will be zero on $\mathcal{D}^{2+\alpha}\backslash\mathcal{F}^{\alpha,\varepsilon}$, in accordance with Definition 2.1. Similarly, all stochastic differential equations will be valid only up to the stopping time τ_ε (which was defined after Definition 2.1) or $\widetilde{\tau}_\varepsilon$ (defined after (2.12)).

The interest of Assumption 2.2 comes from the following result:

Theorem 2.8. *Let $(x, D) \mapsto f^D(x)$ and $(x, D) \mapsto \sigma_{\rm c}^D(x)$ satisfy Assumption 2.2. Then equation* (2.11) *has a solution $(X_t, D_t)_{t\geqslant 0}$ started at $D_0 \in \mathscr{F}^{\alpha,\varepsilon}$, $X_0 \sim \mathscr{U}(D_0)$.*

Proof. We begin to prove the existence of a diffusion with modified drift, and then we will get the result by change of probability. The modified equation writes

$$\begin{cases} d\partial D_t(y) &= N^{D_t}(y)\left(d\widehat{W}_t + \left(\frac{1}{2}h^{D_t}(y) - \frac{\underline{\mu}^{\partial D_t}(\partial D_t)}{\mu(D_t)}\right)dt\right); \\ dX_t &= \left(\nabla f^{D_t}(X_t)\left[d\widehat{W}_t - \left(\frac{\underline{\mu}^{\partial D_t}(\partial D_t)}{\mu(D_t)} + \Delta f^{D_t}(X_t)\right)dt\right]\right. \\ &\qquad \left. +\sigma_{\rm c}^{D_t}(X_t)\, dW_t^m\right) \end{cases} \tag{2.17}$$

for $(\widehat{W}_t)_{t\geqslant 0}$ and $(W_t^m)_{t\geqslant 0}$ independent Brownian motions. Notice that the first equation is the same as (2.12). Thus due to Theorem 2.5, $(D_t)_{t\geqslant 0}$ is a diffusion process with generator $\widetilde{\mathscr{L}}$. Then given $(D_t)_{t\geqslant 0}$, the equation for $(X_t)_{t\geqslant 0}$

$$\begin{aligned} dX_t = &\left(\nabla f^{D_t}(X_t)\left[d\widehat{W}_t - \left(\frac{\underline{\mu}^{\partial D_t}(\partial D_t)}{\mu(D_t)} + \Delta f^{D_t}(X_t)\right)dt\right]\right. \\ &\left. + \sigma_{\rm c}^{D_t}(X_t)\, dW_t^m\right) \end{aligned} \tag{2.18}$$

can also be solved, since the coefficients in front of $d\widehat{W}_t$ and dW_t^m are Lipschitz, $\sigma^D(\sigma^D)^*(x) = \mathrm{Id}_{T_xM}$ and Δf^D is bounded and uniformly Hölder continuous (due to Assumption 2.2). Notice that X_t remains in D_t, since when $X_t \in \partial D_t$, we have, using (2.10) which yields on boundary $\langle N^{D_t}(X_t), \sigma_{\rm c}^{D_t}(X_t)dW_t^m\rangle = 0$,

$$\begin{aligned} &d(\rho^+_{\partial D_t}(X_t)) \\ &= \langle \nabla\rho^+_{\partial D_t}, dX_t\rangle - \frac{1}{2}h^{D_t}(X_t)\, dt - \langle d\partial D_t(X_t), N^{D_t}(X_t)\rangle \\ &= \langle N^{D_t}(X_t), dX_t\rangle - \frac{1}{2}h^{D_t}(X_t)\, dt - \langle d\partial D_t(X_t), N^{D_t}(X_t)\rangle = 0 \end{aligned} \tag{2.19}$$

where we used (2.17) and (2.7). We also have no covariation since the martingale part of $d\partial D_t$ acts on the normal flow only, and any normal flow

$$r \mapsto D(r) := \{x \in M,\ \rho^+(x) \geqslant r\}$$

satisfies $\rho^+_{\partial D(r)}(x) = \rho^+_{\partial D(0)}(x) - r$ for $x \in D(0)$ and $|r|$ small, (see Appendix A).

Once we have a solution to (2.17), make by Girsanov theorem a change of probability such that $(W_t, W_t^m)_{t\geqslant 0}$ is a Brownian motion where

$$W_t := \widehat{W}_t - \int_0^t \left(\frac{\underline{\mu}^{\partial D_s}(\partial D_s)}{\mu(D_s)} + \Delta f^{D_s}(X_s) \right) ds. \tag{2.20}$$

We get a solution to (2.11) in the new probability. □

Proposition 2.9. *Let* $(D_t)_{t\geqslant 0}$ *satisfy*

$$d\partial D_t(y) = N^{D_t}(y) \left(dW_t + \left(\frac{1}{2} h^{D_t}(y) + b_t \right) dt \right), \quad \forall y \in \partial D_t \tag{2.21}$$

for some Brownian motion $(W_t)_{t\geqslant 0}$ *and some adapted locally bounded real-valued process* b_t*. Let* $\mu_t = \mu^{D_t}$ *be the Lebesgue measure on* D_t *and* $\bar{\mu}_t = \bar{\mu}^{D_t} = \mathscr{U}(D_t) = \dfrac{\mu^{D_t}}{\mu(D_t)}$*. Denote by* $\underline{\mu}_t = \underline{\mu}^{\partial D_t}$ *the Lebesgue measure on* ∂D_t *and* $\underline{\bar{\mu}}_t = \underline{\bar{\mu}}^{\partial D_t} = \dfrac{\underline{\mu}^{\partial D_t}}{\mu(D_t)}$*. Let* k *be a smooth function of* M*. Then*

$$d\mu_t(k) = -\underline{\mu}_t(k)\, dW_t - \frac{1}{2} \left(2b_t \underline{\mu}_t(k) + \underline{\mu}_t \left(\langle dk, N^{D_t} \rangle \right) \right) dt \tag{2.22}$$

and

$$\begin{aligned} d\bar{\mu}_t(k) &= \left(-\underline{\bar{\mu}}_t(k) + \bar{\mu}_t(k) \underline{\bar{\mu}}_t(\partial D_t) \right) dW_t - \frac{1}{2} \underline{\bar{\mu}}_t \left(\langle dk, N^{D_t} \rangle \right) dt \\ &\quad + \left(\underline{\bar{\mu}}_t(\partial D_t) + b_t \right) \left(-\underline{\bar{\mu}}_t(k) + \bar{\mu}_t(k) \underline{\bar{\mu}}_t(\partial D_t) \right) dt. \end{aligned} \tag{2.23}$$

In particular, if $b_t = -\underline{\bar{\mu}}_t(\partial D_t)$ *we get*

$$d\bar{\mu}_t(k) = \left(-\underline{\bar{\mu}}_t(k) + \bar{\mu}_t(k) \underline{\bar{\mu}}_t(\partial D_t) \right) dW_t - \frac{1}{2} \underline{\bar{\mu}}_t \left(\langle dk, N^{D_t} \rangle \right) dt. \tag{2.24}$$

Proof. Let us first work at fixed time $t \geqslant 0$. Denote $D = D_t$ and adopt the corresponding notations presented in Appendix A. For k a smooth function on M and $r \in \mathbb{R}$ sufficiently close to 0 so that $\partial D(r)$ (defined in (A.3) and (A.4)) is a smooth manifold without boundary, let

$$F(r, k) = \int_{D(r)} k\, d\mu. \tag{2.25}$$

We have

$$F(r, k) = \int_{\partial D} \left(\int_r^{\tau(y)} k\left(\psi(s)(y)\right) e^{-\int_0^s h^D(\psi(u)(y))\, du}\, ds \right) \underline{\mu}(dy) \tag{2.26}$$

with $\tau(y)$ the hitting time of $S(D)$ by the inward normal flow started at y (defined in (A.1)) and $\psi(s)(y) := \psi(0, s)(y) := \exp_y(sN_y)$ defined in (A.5). The mapping h^D is defined in (A.7) and is an extension of the mean curvature on the boundary ∂D: it corresponds to the mean curvature for the foliation induced by the $\partial D(r)$, $r \in \mathbb{R}$ sufficiently small. With this formulation we can differentiate with respect to r, to obtain

$$F'(r, k) = -\int_{\partial D} k(\psi(r, y)) e^{-\int_0^r h^D(\psi(s)(y)\, ds}\, \underline{\mu}(dy). \tag{2.27}$$

Differentiating again we get

$$F''(r, k) = -\int_{\partial D} \left(\langle dk, \partial_r \psi(r, y) \rangle - (kh)(\psi(r, y)) \right) e^{-\int_0^r h^D(\psi(s)(y))\, ds}\, \underline{\mu}(dy). \tag{2.28}$$

In particular,

$$F'(0,k) = -\underline{\mu}(k) \quad \text{and} \quad F''(0,k) = \underline{\mu}(kh - \langle dk, N\rangle). \tag{2.29}$$

This allows us to compute

$$d(F(W_t,k)) = F'(W_t,k)\,dW_t + \frac{1}{2}F''(W_t,k)\,dt \tag{2.30}$$

and then, since dW_t and $\langle d\partial D_t, N^{D_t}\rangle(\cdot)$ differ only by a finite variation process

$$d\mu_t(k) = \int_{\partial D_t} -k(y)\langle d\partial D_t(y), N^{D_t}(y)\rangle + \frac{1}{2}\left(kh^{D_t} - \langle dk, N^{D_t}\rangle\right)(y)\,\underline{\mu}_t(dy). \tag{2.31}$$

This yields

$$d\mu_t(k) = \int_{\partial D_t} k(y)\,(-dW_t - b_t\,dt) - \frac{1}{2}\langle dk, N^{D_t}\rangle(y)\,\underline{\mu}_t(dy)\,dt, \tag{2.32}$$

which gives (2.22). In particular, taking $k \equiv 1$ we obtain

$$d\mu(D_t) = \underline{\mu}_t(\partial D_t)\,(-dW_t - b_t\,dt)\,. \tag{2.33}$$

Now we can compute

$$\begin{aligned}
&d\bar{\mu}_t(k)\\
&= d\left(\frac{\mu_t(k)}{\mu(D_t)}\right)\\
&= \frac{1}{\mu(D_t)}d\mu_t(k) - \frac{\mu_t(k)}{\mu(D_t)^2}d\mu(D_t) + \frac{\mu_t(k)}{\mu(D_t)^3}d\left\langle \mu(D_.)\right\rangle_t - \frac{1}{\mu(D_t)^2}d\left\langle \mu_.(k), \mu(D_.)\right\rangle_t\\
&= \frac{1}{\mu(D_t)}d\mu_t(k) - \frac{\mu_t(k)}{\mu(D_t)^2}d\mu(D_t) + \frac{\mu_t(k)}{\mu(D_t)^3}\underline{\mu}(\partial D_t)^2 dt - \frac{1}{\mu(D_t)^2}\underline{\mu}_t(k)\underline{\mu}_t(\partial D_t)dt\\
&= -\underline{\bar{\mu}}_t(k)\,(dW_t + b_t\,dt) - \frac{1}{2}\underline{\bar{\mu}}_t(\langle dk, N^{D_t}\rangle)\,dt + \bar{\mu}_t(k)\underline{\bar{\mu}}_t(\partial D_t)(dW_t + b_t\,dt)\\
&+ \bar{\mu}_t(k)\underline{\bar{\mu}}_t(\partial D_t)^2\,dt - \underline{\bar{\mu}}_t(k)\underline{\bar{\mu}}_t(\partial D_t)\,dt.
\end{aligned}$$

This yields (2.23). ☐

Denote τ_ε the exiting time of $(D_t)_{t\geqslant 0}$ from $\mathcal{F}^{\alpha,\varepsilon}$. As in Definition 2.1 we stop $(X_t, D_t)_{t\geqslant 0}$ at τ_ε.

Proposition 2.10. *Any solution of equation* (2.11) *stopped at τ_ε is a Markov process solution to a martingale problem associated to a generator $\mathscr{L}$ acting in the following way: for any g, k smooth functions on M and*

$$F_k(D) := \int_D k\,d\mu, \tag{2.34}$$

we have for $(x, D) \in M \times \mathcal{F}^{\alpha,\varepsilon}$,

(2.35)

$$\begin{aligned}
\mathscr{L}(gF_k)(x,D) = &-g(x)\Delta f^D(x)\underline{\mu}^{\partial D}(k) - \frac{1}{2}g(x)\underline{\mu}^{\partial D}(\langle \nabla k, N^D\rangle) + \frac{1}{2}F_k(D)\Delta g(x)\\
&- \underline{\mu}^{\partial D}(k)\langle \nabla g, \nabla f^D\rangle(x).
\end{aligned}$$

Proof. From (2.11) and (2.22) with $b_t = \Delta f^{D_t}(X_t)$ we have

$$dF_k(D_t) = -\,\underline{\mu}^{\partial D_t}(k)\left(dW_t + \Delta f^{D_t}(X_t)\,dt\right) - \frac{1}{2}\underline{\mu}^{\partial D_t}\left(\langle \nabla k, N^{D_t}\rangle\right)\,dt. \tag{2.36}$$

This implies that

$$\mathscr{L}(F_k)(x,D) = -\underline{\mu}^{\partial D}(k)\Delta f^D(x) - \frac{1}{2}\underline{\mu}^{\partial D}\left(\langle \nabla k, N^D\rangle\right), \tag{2.37}$$

and the covariation of $g(X_t)$ and $F_k(D_t)$ is $\Gamma_{\mathscr{L}}[g,F_k](X_t,D_t)\,dt$ with

$$\Gamma_{\mathscr{L}}[g,F_k](x,D) = -\underline{\mu}^{\partial D}(k)\langle \nabla g, \nabla f^D\rangle(x). \tag{2.38}$$

Consequently, using

$$\mathscr{L}(gF_k)(x,D) = g(x)\mathscr{L}(F_k)(x,D) + F_k(D)\frac{1}{2}\Delta g(x) + \Gamma_{\mathscr{L}}[g,F_k](x,D) \tag{2.39}$$

we get (2.35). □

It is possible to extend the description of $\mathscr{L}$ to more general functions on $M \times \mathcal{F}^{\alpha,\varepsilon}$ (it vanishes on its complementary set), by replacing F_k in (2.35) by a mapping $\mathfrak{F}$ from $\mathfrak{A}$, as presented before Theorem 2.8.

Let $(\mathscr{P}_t)_{t\geqslant 0}$ be the Markovian semi-group associated to the processes $(X_t,D_t)_{t\geqslant 0}$ solution to (2.11) stopped at τ_ε. This semi-group is associated to $\mathscr{L}$ in the weak sense of martingale problems, as described in Appendix D.

Let $(\widetilde{D}_t)_{t\geqslant 0}$ be a diffusion process with generator $\widetilde{\mathscr{L}}$ stopped outside $\mathcal{F}^{\alpha,\varepsilon}$, started at $\widetilde{D}_0 = D_0$ (due to Theorem 2.5, this process can be obtained as a solution to the evolution equation (2.12)), $\widetilde{\nu}_t$ its law at time t and let

$$\nu_t(dD,dx) := \widetilde{\nu}_t(dD)\mathscr{U}(D)(dx). \tag{2.40}$$

Proposition 2.11. *We have for all smooth functions g,k on M:*

$$\partial_t \nu_t(gF_k) = \nu_t(\mathscr{L}(gF_k)). \tag{2.41}$$

Proof. Integrating (2.35) in x with respect to the uniform law $\bar{\mu}^D := \mathscr{U}(D)$ in D yields

$$\begin{aligned} &-\bar{\mu}^D\left(g\Delta f^D\right)\underline{\mu}^{\partial D}(k) - \frac{1}{2}\bar{\mu}^D(g)\underline{\mu}^{\partial D}(\langle\nabla k, N^D\rangle)\\ &+\frac{1}{2}F_k(D)\bar{\mu}^D(\Delta g) - \underline{\mu}^{\partial D}(k)\bar{\mu}^D(\langle \nabla g,\nabla f^D\rangle). \end{aligned} \tag{2.42}$$

By Stokes theorem,

$$\bar{\mu}^D\left(g\Delta f^D + \langle\nabla g,\nabla f^D\rangle\right) = \underline{\bar{\mu}}^{\partial D}\left(g\langle \nabla f^D, -N^D\rangle\right) = -\underline{\bar{\mu}}^{\partial D}(g), \tag{2.43}$$

so the expression (2.42) writes

$$H(D) := \underline{\mu}^{\partial D}(k)\underline{\bar{\mu}}^{\partial D}(g) - \frac{1}{2}\bar{\mu}^D(g)\underline{\mu}^{\partial D}(\langle\nabla k,N^D\rangle) + \frac{1}{2}F_k(D)\bar{\mu}^D(\Delta g). \tag{2.44}$$

On the other hand

$$\nu_t(gF_k) = \widetilde{\nu}_t[\bar{\mu}^{D_t}[g]F_k] \tag{2.45}$$

which implies that

$$\partial_t\nu_t(gF_k) = \partial_t\widetilde{\nu}_t((\bar{\mu}^{D_t}(g)F_k) = \widetilde{\nu}_t\left(\widetilde{\mathscr{L}}\left(\bar{\mu}^{D_t}(g)F_k\right)\right). \tag{2.46}$$

By (2.24),

$$\widetilde{\mathscr{L}}\left(\bar{\mu}^{D_t}(g)\right) = -\frac{1}{2}\underline{\bar{\mu}}^{\partial D_t}(\langle\nabla g, N^{D_t}\rangle), \tag{2.47}$$

so, taking into account (2.14),

$$
\begin{aligned}
&\widetilde{\mathscr{L}}\left(\bar\mu^{D_t}(g)F_k\right)\\
&=\bar\mu^{D_t}(g)\widetilde{\mathscr{L}}(F_k)+F_k\widetilde{\mathscr{L}}\left(\bar\mu^{D_t}(g)\right)+\Gamma_{\widetilde{\mathscr{L}}}\left[\bar\mu^{D_t}(g),F_k\right]\\
&=\bar\mu^{D_t}(g)\Big\{\underline{\mu}^{\partial D_t}(k)\bar{\underline{\mu}}^{\partial D_t}(\partial D_t)-\frac12\underline{\mu}^{\partial D_t}(\langle\nabla k,N^{D_t}\rangle)\Big\}-\frac12\mu^{D_t}(k)\bar{\underline{\mu}}^{\partial D_t}(\langle\nabla g,N^{D_t}\rangle)\\
&-\left(-\bar{\underline{\mu}}^{\partial D_t}(g)+\bar\mu^{D_t}(g)\bar{\underline{\mu}}^{\partial D_t}(\partial D_t)\right)\underline{\mu}^{\partial D_t}(k)\\
&=-\frac12\bar\mu^{D_t}(g)\underline{\mu}^{\partial D_t}(\langle\nabla k,N^{D_t}\rangle)-\frac12\mu^{D_t}(k)\bar{\underline{\mu}}^{\partial D_t}(\langle\nabla g,N^{D_t}\rangle)+\bar{\underline{\mu}}^{\partial D_t}(g)\underline{\mu}^{\partial D_t}(k).
\end{aligned}
$$

But $\bar\mu^{D_t}(\Delta g)=-\bar{\underline{\mu}}^{\partial D_t}(\langle\nabla g,N^{D_t}\rangle)$ and $F_k(D_t)=\mu^{D_t}(k)$, so

$$H(D_t)=\widetilde{\mathscr{L}}\left(\bar\mu^{D_t}(g)F_k\right), \tag{2.48}$$

which together with (2.46) proves (2.41). □

Theorem 2.12. *Let $(x,D)\mapsto f^D(x)$ and $(x,D)\mapsto\sigma_{\rm c}^D(x)$ satisfy Assumption 2.2. Consider a solution $(X_t,D_t)_{t\geqslant0}$ to equation* (2.11) *started at $D_0\in\mathscr{F}^{\alpha,\varepsilon}$, $X_0\sim\mathscr{U}(D_0)$. Then for all $t\geqslant0$, (D_t,X_t) has law ν_t, implying that $(X_t)_{t\geqslant0}$ and $(D_t)_{t\geqslant0}$ are τ_ε-intertwined. Moreover $(D_t)_{t\geqslant0}$ is a diffusion with generator $\widetilde{\mathscr{L}}$.*

Proof. Let us now prove that for any $t\geqslant0$, $\mathscr{P}_t$ transports ν_0 into ν_t, where $(\mathscr{P}_t)_{t\geqslant0}$ is the semi-group introduced after the proof of Proposition 2.10. Consider the map

$$G(g,k,t)(s)=\nu_s\left(\mathscr{P}_{t-s}(gF_k)\right),\quad s\in[0,t]. \tag{2.49}$$

We compute

$$
\begin{aligned}
G(g,k,t)'(s)&=(\partial_s\nu_s)\left(\mathscr{P}_{t-s}(gF_k)\right)-\nu_s\left(\partial_t\mathscr{P}_{t-s}(gF_k)\right)\\
&=\nu_s\left(\mathscr{L}\mathscr{P}_{t-s}(gF_k)\right)-\nu_s\left(\mathscr{L}\mathscr{P}_{t-s}(gF_k)\right)=0
\end{aligned}
\tag{2.50}
$$

where we used Proposition 2.11 in the first term of the second line, and Proposition D.3 in Appendix D to justify the differentiations (as well as the fact that

$$\mathscr{L}\mathscr{P}_{t-s}(gF_k)=\mathscr{P}_{t-s}\mathscr{L}(gF_k)$$

is bounded to be able to use differentiation under the integral ν_s). So we get $G(g,k,t)(0)=G(g,k,t)(t)$ which rewrites as

$$\nu_0\mathscr{P}_t(gF_k)=\nu_t(gF_k), \tag{2.51}$$

More generally, by similar arguments, we can replace in this formula F_k by any mapping $\mathfrak{F}$ from $\mathfrak{A}$. This in turn implies that $\nu_0\mathscr{P}_t=\nu_t$.

To finish, by iteration, we see that if $X_0\sim\bar\mu^{D_0}$ then $(D_t)_{t\geqslant0}$ has the same finite time marginals as $(\widetilde D_t)_{t\geqslant0}$, proving that $(D_t)_{t\geqslant0}$ is a diffusion with generator $\widetilde{\mathscr{L}}$. □

3. Intertwined dual processes: a generalized Pitman theorem

In this section we will consider the case where f^D is the distance to boundary. It is not covered by Section 2 since distance to boundary is not smooth, it is singular on the skeleton of D. We will make an approximation of it, and then go to the limit in law.

Let $(\widetilde W_t)_{t\geqslant0}$ be a real-valued Brownian motion and $(\widetilde D_t)_{t\geqslant0}$ be the solution of (2.12) started at $\widetilde D_0$, with driving Brownian motion $(\widetilde W_t)_{t\geqslant0}$.et et et

Assumption 3.1. Fix $\alpha \in (0,1)$ and $\varepsilon > 0$. There exists a closed bounded subset $\widetilde{\mathcal{F}}^{\alpha,\varepsilon}$ of $\mathcal{F}^{\alpha,\varepsilon}$ in which the process $(\widetilde{D}_t)_{t\geqslant 0}$ a.s. takes its values, such that the map $D \mapsto S(D)$ is continuous from $\widetilde{\mathcal{F}}^{\alpha,\varepsilon}$ with the C^2 metric to $\mathcal{K}(M)$, the set of compact subsets of M endowed with the Hausdorff metric. Moreover Brownian motions with probability one never hit the singular part of $S(\tilde{D}_t)$.

Conjecture 3.2. *We conjecture that Assumption 3.1 is always realized, for any $\alpha \in (0,1)$, $\varepsilon > 0$, $\widetilde{D}_0 \in \mathcal{F}^{\alpha,\varepsilon}$.*

Notice that Theorem 1.1 in [1] proves the first part of the conjecture, i.e. the continuity of $D \mapsto S(D)$, in the case where $M = \mathbb{R}^d$ endowed with a possibly varying Riemannian metric. All examples in Section 5 together with the study of the motion of the skeleton in Appendix B make us believe that Conjecture 3.2 is true. In particular Subsection 5.4 provides a large class of examples in $\mathbb{R}^2$ which do not reduce to finite dimensional processes, some of them having infinite lifetime. They are characterized by the fact that the motion of skeleton can be explicitely described. The considered skeletons have sufficient number of symmetries. For simplicity we considered n-branches skeletons, but we could consider trees with as many ramifications as we want. We could also replace $\mathbb{R}^2$ by the hyperbolic plane or the two dimensional sphere, as well as dimension 2 by higher dimension. All these situations would furnish true infinite dimensional set-valued processes, some of them with completely describable skeleton. However a better knowledge of skeletons is necessary to solve the conjecture in the general situation. We believe that the process $(S(\tilde{D}_t))_{t\geqslant 0}$ takes its values in a set of regular stratified spaces, and that it has absolutely continuous variation in this space.

Let us begin with some preparatory results. To describe the approximation of $\rho(x,\partial D)$ we are interested in, let us introduce some notations.

• Let $(x,D) \mapsto \ell_\varepsilon(x,D) := (h_\varepsilon \circ \rho_{\partial D})(x)$ where $h_\varepsilon \equiv 1$ in $[0,\varepsilon/2]$, $h_\varepsilon \equiv 0$ in $[3\varepsilon/4,\infty)$ and h_ε is smooth and nonincreasing in $[0,\infty)$. When D is fixed by the context, we will denote $\ell_\varepsilon(x) := \ell_\varepsilon(x,D)$.

• For any $\delta \in (0,\varepsilon)$, let $\varphi_\delta : \mathbb{R}_+ \to \mathbb{R}$ be a nonnegative function with support in $[0,\delta]$, such that the mapping $\mathbb{R}^d \ni u \mapsto \varphi_\delta(|u|)$ is smooth and $\int_{\mathbb{R}^d} \varphi_\delta(|u|)\,du = 1$ (in the sequel, $|\cdot|$ will stand for the usual Euclidean norm or for the Riemannian norm on any tangent space of M, depending on the context) .

• Let g_δ be a smooth, 1-Lipschitz and odd function defined on $\mathbb{R}$, with $g_\delta(r) = r$ on $[0,\varepsilon/4]$, $0 \leqslant g_\delta(r) \leqslant r$ for any $r \geqslant 0$, and $g_\delta(r) = c_\delta r$ on $[3\varepsilon/8,\infty)$, for an appropriate constant $c_\delta \leqslant 1$ very close to 1 that will be defined below in (3.2). We write $\rho_\delta(x,\partial D) := g_\delta(\rho(x,\partial D))$.

The approximation of $\rho(x,\partial D)$ we choose is

$$f_\delta(x,D) = \ell_\varepsilon(x,D)\rho_\delta(x,\partial D) + (1-\ell_\varepsilon(x,D))\int_{T_xM} \varphi_\delta(|v|)\rho_\delta(\exp_x(v),\partial D)\,dv \tag{3.1}$$

(where dv stands for the Lebesgue measure on T_xM).

Define

$$e(\delta) \quad := \quad \sup\{|||(\nabla \exp)(u)]|||,\ x \in B(o,1/\varepsilon),\ u \in B_x(0,\delta) \subset T_xM\}$$

where $\nabla \exp(u) : T_xM \to T_{\exp_x(u)}M$ is the covariant derivative of $\exp$ with respect to the base point, $|||\cdot|||$ is the operator norm, when T_xM and $T_{\exp_x(u)}M$ are endowed with their Euclidean structures, and $B_x(0,\delta)$ is the open ball in T_xM with center 0 and radius δ. Recall that ε is fixed as in Assumption 3.1. The previously mentioned constant c_δ is given by

$$c_\delta \quad := \quad e^{-1}(\delta)\,(1-\delta\|\nabla_1\ell_\varepsilon\|_\infty)\,. \tag{3.2}$$

Notice that c_δ does not depend on D and is as close as we want to 1.

More precisely, we have

Lemma 3.3. *There exists two constants $C_1', C_1'' > 0$, depending only on ε, such that for $\delta > 0$ sufficiently small,*

$$\begin{aligned} 0 \leqslant e(\delta) - 1 &\leqslant C_1'\delta, \\ |c_\delta - 1| &\leqslant C_1''\delta. \end{aligned}$$

Proof. The inequalities of the first line are well-known properties of the exponential mapping. The second bound follows, since $\|\nabla_1 \ell_\varepsilon\|_\infty = \|h_\varepsilon'\|_\infty$ is independent of D (and of order $1/\varepsilon$). □

From the second bound, we can and will assume that the function g_δ is furthermore chosen so that $g_\delta(r)$ converges uniformly to r on compact sets of $\mathbb{R}_+$, as well as the corresponding derivatives up to order 2 as $\delta \searrow 0$. In addition, we choose $\delta > 0$ sufficiently small so that the map $(x, y) \mapsto \exp_x^{-1}(y)$ is well-defined and smooth in the δ-neighborhood the diagonal of $B(o, 1/\varepsilon) \times B(o, 1/\varepsilon)$. Then, for any $x \in M$, we can rewrite (3.1) under the form

$$\begin{aligned} f_\delta(x, D) &= \ell_\varepsilon(x, D)\rho_\delta(x, \partial D) \\ &\quad + (1 - \ell_\varepsilon(x, D)) \int_M \varphi_\delta(|\exp_x^{-1}(y)|)\rho_\delta(y, \partial D)\, J \exp_x^{-1}(y) dy, \end{aligned} \tag{3.3}$$

where $J \exp_x^{-1}$ is the absolute value of the determinant of the Jacobian of $\exp_x^{-1}(\cdot)$.

The interest of all these preparations is:

Proposition 3.4. *For all $\delta > 0$ sufficiently small, the function $(x, D) \mapsto f_\delta(x, D) := f_\delta^D(x)$ has the following properties*

- *f_δ satisfies the conditions of Assumption 2.2;*
- *there exists $C_1 > 0$ such that $\forall D \in \widetilde{\mathcal{F}}^{\alpha,\varepsilon}$ and $x \in D$, we have*

$$|f_\delta(x, D) - \rho(x, \partial D)| \leqslant C_1\delta; \tag{3.4}$$

- *the differential and the Hessian of f_δ with respect to the second variable D satisfy $\forall D \in \widetilde{\mathcal{F}}^{\alpha,\varepsilon}$, $\forall x \in D \backslash S(D)$, for all vector fields K normal to ∂D:*

$$\langle d_2 f_\delta(x, D), K\rangle \leqslant C_4\|K\|_\infty \quad \textit{and} \quad \|\nabla_2 d_2 f_\delta(x, D)\,(N_{\partial D}, N_{\partial D})\| \leqslant C_4 \tag{3.5}$$

for a C_4 not depending on x, D, δ. The second term is the second derivative along the inward normal flow on D.

Proof. We first prove $\|d_1 f_\delta(x, D)\| \leqslant 1$, d_1 denoting the differential with respect to the first or the x variable. For $x \in B(o, 1/\varepsilon)$ we have

$$\begin{aligned} d_1 f_\delta(x, D) =& \ell_\varepsilon(x, D) d_1\rho_\delta(x, \partial D) \\ &+ (1 - \ell_\varepsilon(x, D)) d_1 \left(\int_{T_xM} \varphi_\delta(|u|)\rho_\delta(\exp_x(u), \partial D)\, du \right) \\ &+ d_1\ell_\varepsilon(x, D) \int_{T_xM} \varphi_\delta(|u|) \left(\rho_\delta(x, \partial D) - \rho_\delta(\exp_x(u), \partial D)\right)\, du. \end{aligned} \tag{3.6}$$

Notice that if x' is close to x and $\imath_{x,x'} : T_xM \to T_{x'}M$ is the parallel transport along the minimal geodesic from x to x', then

$$\int_{T_{x'}M} \varphi_\delta(|u|)\rho_\delta(\exp_{x'}(u), \partial D)\, du = \int_{T_xM} \varphi_\delta(|u|)\rho_\delta(\exp_{x'}(\imath_{x,x'}(u), \partial D)\, du.$$

Taking the differential with respect to x' at $x'=x$ and using $\nabla_{x'}|_{x'=x}\imath_{x,x'}=0$ by definition of parallel transport yields

$$d_1\left(\int_{T_xM}\varphi_\delta(|u|)\rho_\delta(\exp_x(u),\partial D)\,du\right)=\int_{T_xM}\varphi_\delta(|u|)d_1\rho_\delta((\nabla\exp)(u),\partial D)\,du.$$

If $\rho(x,\partial D)\leqslant\varepsilon/2$ then $\ell_\varepsilon(x,D)=1$, $\nabla\ell_\varepsilon(x,D)=0$ and

$$\|d_1f_\delta(x,D)\|\leqslant\ell_\varepsilon(x,D)\|d_1\rho_\delta(x,\partial D)\|\leqslant 1.$$

If $\rho(x,\partial D)\geqslant\varepsilon/2$ then for $\delta\leqslant\varepsilon/8$, we have, for $u\in T_xM$ with $|u|\leqslant\delta$, $\rho(\exp_x(u),\partial D)\geqslant 3\varepsilon/8$. It follows

$$\begin{aligned}\|d_1f_\delta(x,D)\|\leqslant&\ell_\varepsilon(x)e^{-1}(\delta)\left(1-\delta\|d_1\ell_\varepsilon\|_\infty\right)\\&+(1-\ell_\varepsilon(x))\int_{T_xM}\varphi_\delta(|u|)c_d\|(\nabla\exp)(u)\|\,du\\&+\|d_1\ell_\varepsilon(x)\|_\infty\int_{T_xM}\varphi_\delta(|u|)\delta\,du\\&\leqslant 1.\end{aligned}$$

It is easily checked that the function f_δ satisfies the other properties of Assumption 2.2. Let us check that it also satisfies (3.4).

We have

(3.7)
$$f_\delta(x,D)-\rho_\delta(x,\partial D)=(1-\ell_\varepsilon(x,D))\int_{T_xM}\varphi_\delta(|u|)\left(\rho_\delta(\exp_x(u),\partial D)-\rho_\delta(x,\partial D)\right)\,du$$

which implies

$$|f_\delta(x,D)-\rho_\delta(x,\partial D)|\leqslant\delta.$$

On the other hand

$$|\rho(x,\partial D)-\rho_\delta(x,\partial D)|\leqslant(1-c_\delta)\max\left(\frac{2}{\epsilon},\frac{3\epsilon}{8}\right)\leqslant C_1'''\delta$$

for some constant $C_1'''>0$ (depending on ε). This yields (3.4) with $C_1:=1+C_1'''$.

For proving (3.5), we take a vector field $K(y)=k(y)N(y)$, $y\in\partial D$ and compute

$$\langle d_2\rho(x,\partial D),K\rangle=\langle -N(P(x)),K(P(x))\rangle=-k(P(x))\tag{3.8}$$

where $P(x)$ is the projection of x onto ∂D, and

$$\nabla_2d_2\rho(x,\partial D)\left(N_{\partial D},N_{\partial D}\right)=0.\tag{3.9}$$

Remarking that $\|d_2\ell_\varepsilon(x,D)\|$ is bounded by $\|h_\varepsilon'\|_\infty$, we get (3.5) via a straightforward computation. ☐

Theorem 3.5. *Fix $D_0=\tilde D_0\in\widetilde{\mathcal{F}}^{\alpha,\varepsilon}$ and let $X_0\sim\mathscr{U}(D_0)$. Under Assumption 3.1, there exists a pair $(X_t,D_t)_{t\geqslant 0}$ of τ_ε intertwined processes in the sense of Definition 1.1, such that the process $(D_t)_{t\geqslant0}$ satisfies*

$$\begin{aligned}d\partial D_t(y)=N^{D_t}(y)\Bigg(&\langle dX_t,N^{D_t}(X_t)\rangle+\left(\frac12h^{D_t}(y)-h^{D_t}(X_t)\mathbb{1}_{D_t\setminus S_t}(X_t)\right)\,dt\\&-2\sin(\theta^{S_t}(X_t))\,dL_t^{S_t}(X)\Bigg).\end{aligned}\tag{3.10}$$

Here $\theta^{S_t}(x)=\pi/2-\varphi^{S_t}(x)$, $\varphi^{S_t}(x)$ being the angle between the orthogonal line to S_t at x and any of the two minimal geodesics from ∂D_t to $x\in S_t$ (recall S_t is the regular skeleton *of D_t, see Appendix A). In other words $\theta^{S_t}(x)$ is the smallest angle between S_t and the geodesics. The process L^{S_t} is the local time of X_t at $S_t := S(D_t)$:*

$$L_t^{S_t}(X) = \lim_{\beta \searrow 0} \frac{1}{2\beta} \int_0^t 1_{\{X_s \in S_s^\beta\}} \, ds, \tag{3.11}$$

S_s^β being the thickening of the regular part of S_s in normal direction, of thickness β in both directions.

Remark 3.6. Compared to Section 2 with f^D replaced by distance to boundary $\rho_{\partial D}$, we have outside the skeleton S^D

$$\nabla \rho_{\partial D}(x) = N^D(x) \quad \text{and} \quad \Delta \rho_{\partial D}(x) = -h^D(x) \tag{3.12}$$

and we will see that on the moving skeleton $S_t = S^{D_t}$:

$$\text{“}\Delta \rho_{\partial D_t}(X_t)\, dt\text{”} = -2 \sin(\theta^{S_t}(X_t))\, dL_t^{S_t}(X). \tag{3.13}$$

Proof. Under Assumption 3.1, Proposition 3.4 allows us to construct for each $\delta > 0$, intertwined processes $(X_t^\delta, D_t^\delta)_{t \geqslant 0}$ started at $(X_0^\delta, D_0^\delta) = (X_0, D_0)$, associated with the functions f_δ^D, stopped at τ_ε^δ, the exit time from $\widetilde{\mathcal{F}}^{\alpha,\varepsilon}$. We have from Equation (2.11)

$$d\partial D_t^\delta(y) = N^{D_t^\delta}(y) \left(dW_t^\delta + \left(\frac{1}{2} h^{D_t^\delta}(y) + \Delta f_\delta^{D_t^\delta}(X_t^\delta) \right) dt \right) \tag{3.14}$$

for some Brownian motion W_t^δ. On the other hand, from Proposition 2.11 and (2.1), $(D_t^\delta)_{t \geqslant 0}$ satisfies equation (2.12):

$$d\partial D_t^\delta(y) = N^{D_t^\delta}(y) \left(d\widetilde{W}_t^\delta + \left(\frac{1}{2} h^{D_t^\delta}(y) - \frac{\underline{\mu}^{\partial D_t^\delta}(\partial D_t^\delta)}{\mu(D_t^\delta)} \right) dt \right) \tag{3.15}$$

where $\widetilde{W}_t^\delta$ is the $\mathscr{F}_t^{D^\delta}$−Brownian motion

$$d\widetilde{W}_t^\delta = dW_t^\delta + \Delta f_\delta^{D_t^\delta}(X_t)\, dt + \frac{\underline{\mu}^{\partial D_t^\delta}(\partial D_t^\delta)}{\mu(D_t^\delta)}\, dt. \tag{3.16}$$

A remarkable fact about all $(X_t^\delta, D_t^\delta)_{t \geqslant 0}$ is that their marginals are constant in law: for the second marginal we use Proposition F.2 which states that the martingale problem associated to $\widetilde{\mathscr{L}}$ is well posed, and this implies uniqueness in law. Notice that also $((D_t^\delta)_{t \geqslant 0}, \tau_\varepsilon^\delta)$ is constant in law since τ_ε^δ is a functional of $(D_t^\delta)_{t \geqslant 0}$ independent of δ. As a consequence, the family

$$\left((X_t^\delta, D_t^\delta, W_t^\delta, \widetilde{W}_t^\delta, W_t^{\delta,m})_{t \geqslant 0}, \tau_\varepsilon^\delta \right) \tag{3.17}$$

is tight (in (3.17) the Brownian motions $(W_t^\delta)_{t \geqslant 0}$ and $(W_t^{\delta,m})_{t \geqslant 0}$ are the ones defined by equation (2.11)). Denote by

$$\left((X_t, D_t, W_t, \widetilde{W}_t, W_t^m)_{t \geqslant 0}, \tau_\varepsilon \right) \tag{3.18}$$

a limiting point. Let us prove the intertwining.

Using Proposition 2.11, for any smooth functions g and k on M, for any $t \geqslant 0$,

$$\begin{aligned}
\mathbb{E}[g(X_t^\delta) F_k(D_t^\delta)] &= \mathbb{E}[\mathbb{E}[g(X_t^\delta) F_k(D_t^\delta) | \mathscr{F}_t^{D^\delta}]] \\
&= \mathbb{E}[\mathscr{U}(D_t^\delta)(g) F_k(D_t^\delta)] \\
&= \mathbb{E}\left[\frac{F_g(D_t^\delta)}{F_1(D_t^\delta)} F_k(D_t^\delta) \right]
\end{aligned}$$

and passing to the limit yields the intertwining.

This property of $(D_t^\delta, \widetilde{W}_t^\delta)_{t\geqslant 0}$ being constant in law passes to the limit, and we have

$$d\partial D_t(y) = N^{D_t}(y)\left(d\widetilde{W}_t + \left(\frac{1}{2}h^{D_t}(y) - \frac{\mu^{\partial D_t}(\partial D_t)}{\mu(D_t)}\right)\, dt\right). \tag{3.19}$$

We need to work with real-valued processes: we have from (2.33), for all $\delta > 0$,

$$\int_0^t \frac{d\mu(D_s^\delta)}{\underline{\mu}(\partial D_s^\delta)} = -W_t^\delta - \int_0^t \Delta_1 f_\delta(X_s^\delta, D_s^\delta)\, ds. \tag{3.20}$$

This together with (3.16) yields

$$d\partial D_t^\delta(y) = N^{D_t^\delta}(y)\left(-\frac{d\mu(D_s^\delta)}{\underline{\mu}(\partial D_s^\delta)} + \frac{1}{2}h^{D_t^\delta}(y)\, dt\right). \tag{3.21}$$

Again by constantness in law:

$$d\partial D_t(y) = N^{D_t}(y)\left(-\frac{d\mu(D_s)}{\underline{\mu}(\partial D_s)} + \frac{1}{2}h^{D_t}(y)\, dt\right). \tag{3.22}$$

So to prove our result we only need to prove that

$$\int_0^t \frac{d\mu(D_s)}{\underline{\mu}(\partial D_s)} = -W_t + \int_0^t h^{D_s}(X_s)\, ds + \int_0^t 2\sin\left(\theta^{S_s}(X_s)\right)\, dL_s^{S_s}(X) \tag{3.23}$$

and that

$$W_t = \int_0^t \langle N^{D_s}(X_s), dX_s\rangle. \tag{3.24}$$

Let us prove (3.24). In all this paragraph we consider M as isometrically embedded in some Euclidean space. In particular we are allowed to integrate vectorial quantities. We use the fact that $dX_t^\delta \otimes dW_t^\delta$ converges in law to $dX_t \otimes dW_t$ (where $\otimes$ stands for bracket of semi-martingales). But $dX_t^\delta \otimes dW_t^\delta$ is equal to $\nabla_1 f_\delta(X_t^\delta, D_t^\delta)\, dt$. Then by Lemma G.1 applied to $\nabla_1 f_\delta(X_t^\delta, D_t^\delta)$ (which is uniformly bounded) and $U = \{(x, D),\ x \notin S(D)\}$ defined in (G.3) we see that the integral of $\nabla_1 f_\delta(X_t^\delta, D_t^\delta)\, dt$ converges to the one of $N^{D_t}(X_t)\, dt$. But almost surely $N^{D_t}(X_t)$ has norm 1 dt-a.e., implying that $dW_t = \langle N^{D_t}(X_t), dX_t\rangle$.

Let us now establish (3.23). It will be a consequence of the convergence as $\delta \to 0$ of $(f_\delta(X_t^\delta, D_t^\delta))_{t\geqslant 0}$ to $(\rho(X_t, \partial D_t)_{t\geqslant 0}$.

Write the Itô formula for $f_\delta(X_t^\delta, D_t^\delta)$:

(3.25)

$$\begin{aligned} d\left(f_\delta(X_t^\delta, D_t^\delta)\right) =& \langle d_1 f_\delta(X_t^\delta, D_t^\delta), dX_t^\delta\rangle + \frac{1}{2}\Delta_1 f_\delta(X_t^\delta, D_t^\delta)\, dt \\ &+ \langle d_2 f_\delta(X_t^\delta, D_t^\delta), d\partial D_t^\delta\rangle + \frac{1}{2}\nabla_2 d_2 f_\delta(X_t^\delta, D_t^\delta)(d\partial D_t^\delta, d\partial D_t^\delta)\, dt \\ &+ \langle \nabla_2 d_1 f_\delta(X_t^\delta, D_t^\delta), d\partial D_t^\delta \otimes dX_t^\delta\rangle. \end{aligned}$$

From Proposition 3.4, possibly by extracting a subsequence,

$$\left(f_\delta(X_t^\delta, D_t^\delta)\right)_{t\geqslant 0} \xrightarrow{\mathscr{L}} \left(\rho(X_t, \partial D_t)\right)_{t\geqslant 0}. \tag{3.26}$$

From (3.7) we get for $i=1,2$,

$$\begin{aligned}
&d_i f_\delta(x,D)-d_i\rho_\delta(x,\partial D)\\
&=-d_i\ell_\varepsilon(x,D)\int_{T_xM}\varphi_\delta(|u|)\left(\rho_\delta(\exp_x(u)),\partial D)-\rho_\delta(x,\partial D)\right)\,du\\
&+(1-\ell_\varepsilon(x,D))\int_{T_xM}\varphi_\delta(|u|)\left(d_i\rho_\delta(\exp_x(u)),\partial D)-d_i\rho_\delta(x,\partial D)\right)\,du.
\end{aligned}\tag{3.27}$$

From this we see that $d_1f_\delta(\cdot,D)$ converges, locally uniformly outside $S(D)$, to $d_1\rho(\cdot,\partial D)$ with respect to the distance d_0 of Appendix G. We obtain, with Lemma G.1, possibly by again extracting a subsequence, that

$$\left(\int_0^t\langle d_1f_\delta(X_s^\delta,D_s^\delta),dX_s^\delta\rangle\right)_{t\geqslant 0}\xrightarrow{\mathscr{L}}\left(\int_0^t\langle d_1\rho(X_s,\partial D_s),dX_s\rangle\right)_{t\geqslant 0}.\tag{3.28}$$

More precisely, we have a sequence of martingales converging in law to a martingale M_t which is a Brownian motion by Theorem 3 in [30]. For identifying the limiting martingale we use the convergence of $\langle d_1f_\delta(X_s^\delta,D_s^\delta),dX_s^\delta\rangle\otimes dX_s^\delta$ to $dM_s\otimes dX_s$ obtained again by Theorem 3 in [30] (here again we use an isometric embedding of M). But Lemma G.1 proves that the limit is equal to $\nabla_1\rho(X_s,\partial D_s)\,ds$, yielding (3.28).

Next we prove that

$$\left(\int_0^t\langle d_2f_\delta(X_s^\delta,D_s^\delta),d\partial D_s^\delta\rangle\right)_{t\geqslant 0}\xrightarrow{\mathscr{L}}\left(\int_0^t\langle d_2\rho(X_s,\partial D_s),d\partial D_s\rangle\right)_{t\geqslant 0}.\tag{3.29}$$

The argument is similar except that as we see with (3.14), the drift part of $d\partial D_s^\delta$ is not well controlled as X_t^δ approaches the skeleton. So one cannot proceed exactly the same way. But fortunately, for x outside a $3\varepsilon/4$-neighbourhood of ∂D and outside $S(D)$, we have

$$\begin{aligned}
&\langle d_2f_\delta(x,D),N|_{\partial D}\rangle\\
&=c_\delta\int_{T_xM}\varphi_\delta(|u|)\langle -N\left(P(\exp_x(u))\right),N\left(P(\exp_x(u))\right)\rangle\,du=-c_\delta
\end{aligned}\tag{3.30}$$

where c_δ is defined in (3.2). This together with (3.21) suggests to write

$$\begin{aligned}
\int_0^t\langle d_2f_\delta(X_s^\delta,D_s^\delta),d\partial D_s^\delta\rangle&=\left(\int_0^t\langle d_2f_\delta(X_s^\delta,D_s^\delta),d\partial D_s^\delta\rangle+c_\delta\int_0^t\langle N^{D_s^\delta},d\partial D_s^\delta\rangle\right)\\
&\quad-c_\delta\int_0^t\langle N^{D_s^\delta},d\partial D_s^\delta\rangle.
\end{aligned}$$

The second line clearly converges. The right hand side in the first line can be written

$$\int_0^t\tilde\ell_\varepsilon(X_s^\delta,D_s^\delta)\left\langle d_2f_\delta(X_s^\delta,D_s^\delta)+c_\delta N^{D_s^\delta},d\partial D_s^\delta\right\rangle\tag{3.31}$$

with $(x,D)\mapsto\tilde\ell_\varepsilon(x,D):=(\tilde h_\varepsilon\circ\rho_{\partial D})(x)$ where $\tilde h_\varepsilon\equiv 1$ in $[0,3\varepsilon/4]$, $\tilde h_\varepsilon\equiv 0$ in $[\varepsilon,\infty)$ and $\tilde h_\varepsilon$ is smooth and nonincreasing in $[0,\infty)$.

With this last integral we can proceed as for (3.28), after passing to the limit, and since $\lim_{\delta\to 0}c_\delta=1$, we get (3.30).

Similarly we obtain the two following convergences for the second derivatives.

$$\begin{aligned}
&\left(\int_0^t\nabla_2d_2f_\delta(X_s^\delta,D_s^\delta)(d\partial D_s^\delta,d\partial D_s^\delta)\right)_{t\geqslant 0}\\
&\xrightarrow{\mathscr{L}}\left(\int_0^t\nabla_2d_2\rho(X_s,\partial D_s)\left(N(P^{\partial D_s}(X_s),N(P^{\partial D_s}(X_s)\right)\,ds\right)_{t\geqslant 0}\equiv 0
\end{aligned}\tag{3.32}$$

where $P^{\partial D_s}(X_s)$ is the orthogonal projection of X_s on ∂D_s (which is defined ds-almost everywhere),

$$\begin{aligned}(3.33)\qquad &\left(\int_0^t \langle \nabla_2 d_1 f_\delta(X_s^\delta, D_s^\delta), d\partial D_t^\delta \otimes dX_t^\delta \rangle)\right)_{t\geqslant 0} \\ &\xrightarrow{\mathscr{L}} \left(\int_0^t \langle \nabla_2 d_1 \rho(X_s, \partial D_s), d\partial D_s \otimes dX_s\rangle\right)_{t\geqslant 0} \equiv 0\end{aligned}$$

since $d_1\rho(X_s, \partial D_s) = +\langle N^{D_s}(X_s), \cdot\rangle$ which implies that the covariant derivative in the second variable with respect to N^{D_s} is equal to 0. On the other hand, by Itô-Tanaka formula (see Proposition E.1 in Appendix E using that $\rho(x, \partial D)$ is almost everywhere the minimum of two smooth functions) together with Assumption 3.1 which allows to only consider the regular skeleton, together with Theorem B.1 which says that the latter has absolutely continuous variation (useful for the term $dL_t^{S_t}(X)$), we have

$$\begin{aligned}&d\left(\rho(X_t, \partial D_t)\right) \\ (3.34)\qquad &= \langle d_1\rho(X_t, \partial D_t), dX_t\rangle - \frac{1}{2} h^{D_t}(X_t) \mathbb{1}_{D_t\setminus S_t}(X_t)\, dt + \langle d_2\rho(X_t, \partial D_t), d\partial D_t\rangle \\ &+ 0 + 0 - \sin\left(\theta^{S_t}(X_t)\right)\, dL_t^{S_t}(X).\end{aligned}$$

Using (3.25), (3.26), (3.28), (3.29), (3.32), (3.33), (3.34) we obtain that

$$\begin{aligned}(3.35)\qquad &\left(\int_0^t \Delta_1 f_\delta(X_s^\delta, D_s^\delta)\, ds\right)_{t\geqslant 0} \\ &\xrightarrow{\mathscr{L}} \left(\int_0^t -h^{D_s}(X_s)\mathbb{1}_{D_s\setminus S_s}(X_s)\, ds - \int_0^t 2\sin\left(\theta^{S_s}(X_s)\right)\, dL_s^{S_s}(X)\right)_{t\geqslant 0}.\end{aligned}$$

It remains to pass in the limit as δ goes to zero in (3.20), to deduce (3.23). □

Remark 3.7. From (3.34), it can be deduced that

$$\begin{aligned}&d\left(\rho(X_t, \partial D_t)\right) \\ (3.36)\qquad &= \frac{1}{2}\left(h^{D_t}(X_t)\mathbb{1}_{D_t\setminus S_t}(X_t) - h^{D_t}\left(P^{\partial D_t}(X_t)\right)\right) dt + \sin\left(\theta^{S_t}(X_t)\right)\, dL_t^{S_t}(X).\end{aligned}$$

Indeed, (3.24) implies that

$$\langle d_1\rho(X_t, \partial D_t), dX_t\rangle \quad = \quad dW_t$$

and due to (3.29), we have

$$\begin{aligned}&\langle d_2\rho(X_t, \partial D_t), d\partial D_t\rangle \\ &\quad= \lim_{\delta\to 0} \langle d_2\rho(X_t^\delta, \partial D_t^\delta), d\partial D_t^\delta\rangle \\ &\quad= \lim_{\delta\to 0} -dW_t^\delta - \left(\Delta_1 f_\delta(P^{\partial D_t^\delta}(X_t^\delta), D_t^\delta) + \frac{1}{2} h^{D_t^\delta}(P^{\partial D_t^\delta}(X_t^\delta))\right) dt\end{aligned}$$

where we used (3.20) in conjunction with (3.21).

Taking into account (3.35), we identify the last limit with

$$-dW_t + \left(h^{D_t}(X_t)\mathbb{1}_{D_t\setminus S_t}(X_t) - \frac{1}{2} h(P^{\partial D_t}(X_t))\right) dt + 2\sin\left(\theta^{S_t}(X_t)\right)\, dL_t^{S_t}(X).$$

4. Intertwined dual processes: decoupling and reflection on boundary

In this section we consider another canonical and extremal situation, the case where f^D vanishes almost everywhere. More precisely, it is the limiting situation where f^D is constant outside a ε-neighbourhood of the boundary. This situation is completely opposite to the one of Section 3 where the coupling is maximal.

Theorem 4.1. *There exists a pair $(X_t, D_t)_{t\geqslant 0}$ of τ_ε-intertwined processes in the sense of Definition 1.1 satisfying*

$$d\partial D_t(y) = N^{D_t}(y)\left(dW_t + \frac{1}{2}h^{D_t}(y)dt - dL_t^{\partial D_t}(X)\right) \tag{4.1}$$

where $(X_t)_{t\geqslant 0}$ is a M-valued Brownian motion started at uniform law in D_0, $(W_t)_{t\geqslant 0}$ is a real-valued Brownian motion independent of X_t, $(L_t^{\partial D_t}(X))_{t\geqslant 0}$ is the local time of $(X_t)_{t\geqslant 0}$ on the moving boundary $(\partial D_t)_{t\geqslant 0}$.

Remark 4.2. Equation (4.1) can be considered as a limiting case of (2.11). Here Assumption 3.1 is not needed since the morphological skeleton of D does not play a role, and the map $D \mapsto \partial D$ is already sufficiently regular.

Proof. The proof is quite similar to the one of Theorem 3.5, but with another family of functions f_δ^D, namely $f_\delta^D := h_\delta \circ \rho_{\partial D}$ where h_δ is defined in the proof of Proposition 2.4: h_δ is a smooth nondecreasing function from $[0,\infty)$ to $\mathbb{R}_+$ such that $h_\delta(r) = r$ for $r \in [0, \delta/2]$, $h_\delta(r) = (3/4)\delta$ for $r \geqslant \delta$ and $\|h'_\delta\|_\infty \leqslant 1$. But here, as ε is fixed, we will let $\delta \searrow 0$. Again we construct for each $\delta > 0$, an intertwined processes $(X_t^\delta, D_t^\delta)_{t\geqslant 0}$ stopped at τ_ε^δ. Again all $(X_t^\delta, D_t^\delta)_{t\geqslant 0}$ are tight, and a limiting process $(X_t, D_t)_{t\geqslant 0}$ stopped at τ_ε provides an intertwining. The proof of (4.1) goes along the same lines as the one of (3.10). □

We end this section with another canonical construction, where the functions f_δ^D approximate $-\rho_{\partial D}$.

Theorem 4.3. *Under Assumption 3.1, there exists an intertwining $(X_t, D_t)_{t\geqslant 0}$ stopped at τ_ε, satisfying*

$$\begin{aligned} d\partial D_t(y) =&N^{D_t}(y)\Bigg(-\left\langle dX_t, N^{D_t}(X_t)\right\rangle + \left(\frac{1}{2}h^{D_t}(y) + h^{D_t}(X_t)\mathbb{1}_{D_t\backslash S_t}(X_t)\right)dt \\ &+ 2\sin(\theta^{S_t}(X_t))\, dL_t^{S_t}(X) - 2dL_t^{\partial D_t}(X)\Bigg). \end{aligned} \tag{4.2}$$

Proof. It is completely similar to the ones of Theorems 3.5 and 4.1. □

5. Some fundamental examples

5.1. **Real Brownian motion and three-dimensional Bessel process.** We come back to the case where $M = \mathbb{R}$. Assume that the Brownian motion X starts from 0 (to respect rigorously the above framework, X should start from the uniform distribution on $D_0 := [-\epsilon, \epsilon]$ and next we should let ϵ go to 0_+). Due to the invariance by symmetry of (3.10), for any $t > 0$, D_t remains a symmetric interval, let us write it $[-R_t, R_t]$. In this simple setting, we have $N^{D_t}(\cdot) = -\mathrm{sign}(\cdot)$ on $\mathbb{R}\backslash\{0\}$, $h^{D_t} = 0$ and $S_t = \{0\}$, for any $t > 0$. Thus (3.10) writes

$$dR_t = \mathrm{sign}(X_t)dX_t + 2dL_t \tag{5.1}$$

where $(L_t)_{t\geqslant 0}$ is the local time of X at 0. Namely we get that

$$\begin{aligned}\forall\, t \geqslant 0, \qquad R_t &= \int_0^t \operatorname{sign}(X_s)\, dX_s + 2L_t \\ &= |X_t| + L_t\end{aligned}$$

by Tanaka's formula. It is well-known that $(R_t)_{t\geqslant 0}$ is a Bessel process of dimension 3 (see e.g. Corollary 3.8 of Chapter 6 of Revuz and Yor [24]). In particular the signed distance $\rho^+_{\partial D_t}$ to ∂D_t (chosen to be positive inside D_t) is given by

$$\forall\, t \geqslant 0, \qquad \rho^+_{\partial D_t}(X_t) \;=\; \min(X_t + R_t, R_t - X_t).$$

But except at time $t = 0$, this quantity is always positive: a.s. X_t never touch the boundary of D_t for $t > 0$. Indeed, if for some $t > 0$ we have $|X_t| = R_t$, we deduce that $L_t = 0$, namely a contradiction, since $X_0 = 0$.

In particular, we see that the intertwining coupling we have constructed is different from the one proposed by Pitman [23], which is a.s. touching (the upper) boundary repeatedly. Instead we end up with the intertwining dual constructed in [20] via stochastic flows. It is mentioned there how to deduce the classical Pitman's dual, via Lévy's theorem.

Here is an alternative approach. While Equation (5.1) is obtained from approximating $x \mapsto |r - x|$ outside an ε-neighbourhood of 0 when $D = [-r, r]$ by smooth functions f^D satisfying Assumption 2.2, we are able to recover Pitman theorem by rather approximating $x \mapsto -x$ in $D = [-r, r]$ outside the only ε-neighbourhood of $-r$. In the limit of (2.11) as ϵ goes to zero, on the one hand we have

$$\mathbb{1}_{\{X_t \neq R_t\}} dR_t = dX_t, \tag{5.2}$$

on the other hand we have $X_t + R_t \geqslant 0$, so that $X_t + R_t$ is the solution to the Skorohod problem associated to $2X_t$. We get

$$R_t + X_t = 2X_t - 2\min_{0\leqslant s\leqslant t} X_s \tag{5.3}$$

which is equivalent to

$$R_t = X_t - 2\min_{0\leqslant s\leqslant t} X_s. \tag{5.4}$$

The answer to the question: what would be a symmetric construction with local time at the two ends of $[-R_t, R_t]$ is given by Theorem 4.3. We obtained intertwined processes with

$$R_t = -\int_0^t \operatorname{sign}(X_s)\, dX_s - 2L_t^0(X) + 2L_t^0(R - X) + 2L_t^0(R + X). \tag{5.5}$$

5.2. **Brownian motion and disks in rotationally symmetric manifolds.** This is the simplest example since the skeleton is never hit by the Brownian motion. Consider a complete d-dimensional manifold with $d \geqslant 2$, rotationally symmetric around a point $o \in M$. Denote by (r, Θ) polar coordinates with $r(x) = \rho(o, x)$ and

$$ds^2 = dr^2 + f^2(r)\, d\Theta^2 \tag{5.6}$$

the metric in polar coordinates. Then the radial Laplacian is

$$\Delta_r = \frac{\partial^2}{(\partial r)^2} + b(r)\frac{\partial}{\partial r} \quad \text{with} \quad b = (d-1)(\ln f)'. \tag{5.7}$$

We will investigate set-valued processes $(D_t = B(o, R_t))_{t\geqslant 0}$ where $B(o, r)$ is the open geodesic ball centered at o, with radius r. The skeleton of $B(o, R_t)$ is the point o.

Let $(X_t)_{t\geqslant 0}$ be a Brownian motion in M satisfying $X_0 \sim \mathscr{U}(D_0)$ for some $D_0 = B(o, r_0)$. Denote by $\rho_t := r(X_t)$ the radial part of X_t. Then

$$d\rho_t = d\beta_t + \frac{1}{2}b(\rho_t)\,dt, \qquad \rho_0 \sim \mathscr{U}^f((0, r_0)) \tag{5.8}$$

where $(\beta_t)_{t\geqslant 0}$ is a real Brownian motion and

$$\mathscr{U}^f(dr) := \frac{f(r)}{\int_0^{r_0} f(s)\,ds}\,dr. \tag{5.9}$$

The evolution equation (3.10) for D_t shows by symmetry that for all $t \geqslant 0$, $D_t = B(0, R_t)$ for some real-valued process $(R_t)_{t\geqslant 0}$. Moreover it writes

$$\begin{aligned} d\rho_t &= d\beta_t + \frac{1}{2}b(\rho_t)\,dt \\ dR_t &= d\beta_t + \left[-\frac{1}{2}b(R_t) + b(\rho_t)\right]\,dt. \end{aligned} \tag{5.10}$$

Proposition 5.1. *The system of equations* (5.10) *has a solution up to explosion time of* $(R_t)_t$

$$\tau^D := \inf\{t \geqslant 0, R_t \notin (0, \infty)\}, \tag{5.11}$$

which satisfies for all $t < \tau^D$*,*

$$0 < \rho_t < R_t. \tag{5.12}$$

The corresponding set-valued process $(D_t = B(o, R_t))_{t\geqslant 0}$ *is solution to equation* (3.10)*, and in particular, for all* $\mathscr{F}^D$*-stopping time* τ*,*

$$\mathscr{L}(X_\tau|\mathscr{F}_\tau^D) = \mathscr{U}(D_\tau) \quad \textit{as well as} \quad \mathscr{L}(\rho_\tau|\mathscr{F}_\tau^D) = \mathscr{U}^f((0, R_\tau)). \tag{5.13}$$

Proof. We only have to check (5.12). By (5.10),

$$d(R_t - \rho_t) = \frac{1}{2}\left[b(\rho_t) - b(R_t)\right]\,dt, \tag{5.14}$$

which vanishes on $\{R_t = \rho_t\}$, and since b is smooth, if $\rho_0 < R_0$, then $\rho_t < R_t$ for all times. □

5.3. **Brownian motion and annulus in 2-dimensional rotationally symmetric manifolds.** Let M be a complete 2-dimensional Riemannian manifold, rotationally symmetric around a point $o \in M$. Denote by (r, θ) polar coordinates with $r(x) = \rho(o, x)$ and

$$ds^2 = dr^2 + f^2(r)\,d\theta^2 \tag{5.15}$$

the metric in polar coordinates. Then the radial Laplacian is

$$\Delta_r = \frac{\partial^2}{(\partial r)^2} + b(r)\frac{\partial}{\partial r} \quad \text{with} \quad b = (\ln f)'. \tag{5.16}$$

If $0 \leqslant r^- \leqslant r^+$, let

$$A(r^-, r^+) := \{x \in M,\ r^- \leqslant r(x) \leqslant r^+\} \quad \text{if} \quad r^- < r^+, \quad A(r^-, r^+) := \varnothing, \tag{5.17}$$

the closed annulus delimited by the radius r^- and r^+.

In the following we will investigate set-valued processes $D_t = A(R_t^-, R_t^+)$. The skeleton of $A(R_t^-, R_t^+)$ is the circle

$$S_t = C(o, R_t^0) \quad \text{with} \quad R_t^0 := \frac{1}{2}(R_t^- + R_t^+). \tag{5.18}$$

Let X_t be a Brownian motion in M satisfying $X_0 \sim \mathscr{U}(D_0)$ for some $D_0 = A(r_0^-, r_0^+)$. Denote by $\rho_t := r(X_t)$ the radial part of X_t. Then

$$d\rho_t = d\beta_t + \frac{1}{2}b(\rho_t)\,dt, \qquad \rho_0 \sim \mathscr{U}^f((r_0^-, r_0^+)) \tag{5.19}$$

where $(\beta_t)_{t\geqslant 0}$ is a real Brownian motion and

$$\mathscr{U}^f((r_0^-, r_0^+))(dr) := \frac{f(r)}{\int_{r_0^-}^{r_0^+} f(s)\,ds}\,dr. \tag{5.20}$$

The evolution equation (3.10) for $(D_t)_{t\geqslant 0}$ shows by symmetry that for all $t \geqslant 0$, $D_t = A(R_t^-, R_t^+)$ for some real-valued processes $R_t^- \leqslant R_t^+$. Moreover it writes

$$\begin{aligned} d\rho_t &= \operatorname{sign}(\rho_t - R_t^0)\,dW_t + \frac{1}{2}b(\rho_t)\,dt \\ dR_t^+ &= dW_t + \left[-\frac{1}{2}b(R_t^+) + \operatorname{sign}(\rho_t - R_t^0)b(\rho_t)\right]dt + 2L_t^{R_t^0}(\rho) \\ dR_t^- &= -dW_t + \left[-\frac{1}{2}b(R_t^-) - \operatorname{sign}(\rho_t - R_t^0)b(\rho_t)\right]dt - 2L_t^{R_t^0}(\rho) \\ R_t^0 &= \frac{1}{2}\left(R_t^- + R_t^+\right) \end{aligned} \tag{5.21}$$

and these equations imply

$$dR_t^0 = -\frac{1}{4}\left[b(R_t^+) + b(R_t^-)\right]dt. \tag{5.22}$$

Proposition 5.2. *The system of equations* (5.21) *has a solution up to explosion time*

$$\tau^D := \inf\{t \geqslant 0, (R_t^-, R_t^+) \notin (0,\infty)^2\}, \tag{5.23}$$

which satisfies for all $t < \tau^D$,

$$R_t^- \leqslant \rho_t \leqslant R_t^+. \tag{5.24}$$

The corresponding set-valued process $(D_t = A(R_t^-, R_t^+))_{t\geqslant 0}$ is solution to equation (3.10)*, and in particular, for all $\mathscr{F}^D$-stopping time τ,*

$$\mathscr{L}(X_\tau | \mathscr{F}_\tau^D) = \mathscr{U}(D_\tau) \quad \textit{as well as} \quad \mathscr{L}(\rho_\tau | \mathscr{F}_\tau^D) = \mathscr{U}^f((R_\tau^-, R_\tau^+)). \tag{5.25}$$

Proof. Fix $\varepsilon > 0$ and $\alpha \in (0,1)$. We will first solve the system of equations until the exit time τ_ε and then let $\varepsilon \searrow 0$. Let us construct functions $f_\delta^D(x)$ which satisfies equation (3.1). It will be easier here because there is no need of functions ℓ_ε and g_δ.

For $\delta \in (0,\varepsilon)$, let $\varphi_\delta : \mathbb{R} \to \mathbb{R}$ be the function with support equal to $[-\delta/2, \delta/2]$, satisfying for $-\delta/2 < r < \delta/2$:

$$\varphi_\delta(r) := \frac{1}{c(\delta)}\exp\left(-\frac{1}{\left(\frac{\delta}{2}\right)^2 - r^2}\right) \quad \text{with} \quad c(\delta) := \int_{-\delta/2}^{\delta/2}\exp\left(-\frac{1}{\left(\frac{\delta}{2}\right)^2 - s^2}\right)ds, \tag{5.26}$$

and let

$$\begin{aligned} &\operatorname{sign}_\delta : \mathbb{R} \to \mathbb{R} \\ &r \mapsto -1 + 2\int_{-\infty}^r \varphi_\delta(s)\,ds. \end{aligned} \tag{5.27}$$

The functions φ_δ and $\operatorname{sign}_\delta$ are both smooth and Lipschitz, and they respectively approximate δ_0 and sign. For $0 < r^- < r^+$ satisfying $r^+ - r^- \geqslant 2\varepsilon$, defining $r^0 := \dfrac{1}{2}(r^- + r^+)$, for

$x \in A(r^-, r^+)$ let

$$f^{A(r^-,r^+)}(x) = f(x, r^-, r^+) = g(r(x))$$
$$\text{with} \quad g(r) = g(r, r^-, r^+) = \int_{r^-}^{r} -\mathrm{sign}_\delta(s - r^0)\, ds. \tag{5.28}$$

Clearly $f(x, r^-, r^+)$ is 1-Lipschitz in the first variable. A computation shows that

$$\partial_{r^+} g(r, r^-, r^+) = \int_{-\varepsilon}^{r^0} \varphi_\delta(v)\, dv \quad \text{and} \quad \partial_{r^-} g(r, r^-, r^+) = -\int_{r-r^0}^{\varepsilon} \varphi_\delta(v)\, dv \tag{5.29}$$

showing that g and f are 1-Lipschitz. Then the vector $N := N_{\partial A(r^-,r^+)}$ is equal to

$$-\mathbb{1}_{\{r(x)=r^+\}}\partial_{r^+} + \mathbb{1}_{\{r(x)=r^-\}}\partial_{r^-}$$

so that

$$\langle \nabla f, N\rangle \equiv 1 \quad \text{and} \quad \nabla df(N, N) \equiv 0. \tag{5.30}$$

This yields an elementary proof of the properties of Proposition 3.4. We can use Theorem 3.5 to solve equation (5.21) until the stopping time τ_ε.

We are left to prove that $\tau_\varepsilon \nearrow \tau^D$ a.s. as $\varepsilon \searrow 0$. This is a direct consequence of the fact that the volume of $A(R_t^-, R_t^+)$ is a time changed Bessel process of dimension 3 (by [10] Theorem 5), proving that $A(R_t^-, R_t^+)$ cannot collapse onto its skeleton. □

Remark 5.3. After the hitting time of 0 by R_t^-, the processes can continue to evolve under the regime of Section 5.2.

We recover from Proposition 5.2 a result from [19] stating that $([R_t^-, R_t^+])_{t\geqslant 0}$ is an intertwining dual process for the real diffusion $(\rho_t)_{t\geqslant 0}$. In particular, we deduce that if $(\rho_t)_{t\geqslant 0}$ is positive recurrent and if $+\infty$ is an entrance boundary, then $([R_t^-, R_t^+])_{t\geqslant 0}$ reaches $[0, +\infty]$ in finite time and this finite time is a strong stationary time for $(\rho_t)_{t\geqslant 0}$, see [19] for more details.

5.4. **Brownian motion and symmetric convex sets in $\mathbb{R}^2$.** In this section we take $M = \mathbb{R}^2$ endowed with the Euclidean metric. For any integer $n \geqslant 2$, let G_n the group of isometries of $\mathbb{R}^2$ generated by the rotation of angle $\frac{2\pi}{n}$ and the symmetry with respect to the horizontal axis. Consider a smooth strictly convex bounded set $\widetilde{D}_0 \subset M$ with smooth boundary, stable by the action of G_n. Let us investigate the evolution of $(\widetilde{D}_t)_{t\geqslant 0}$ solution to (2.12). Notice that it is the first example where we really have to deal with infinite dimensional processes. By conservation of the convexity by the normal and mean curvature flows, $\widetilde{D}_t$ will stay convex. It will also stay symmetric. All the results of this subsection are proved in [2].

Proposition 5.4. *Assume that its skeleton has the form $\widetilde{S}_0 = G_n\widetilde{H}_0$, $\widetilde{H}_0$ being an horizontal interval $\widetilde{H}_0 = [0, \widetilde{x}_0] \times \{0\}$ for some $\widetilde{x}_0 > 0$ (an example of such a set when $n = 2$ is the interior of an ellipse, the skeleton being the interval between the two foci). The skeleton of $\widetilde{D}_t$ always takes the form $\widetilde{S}_t = G_n\widetilde{H}_t$ with $\widetilde{H}_t = [0, \widetilde{x}_t] \times \{0\}$ an horizontal interval.*

The right endpoint $(\widetilde{x}_t, 0)$ in the horizontal axis of the skeleton $\widetilde{S}_t$ satisfies

$$\frac{d\widetilde{x}_t}{dt} = \frac{\rho^2((\widetilde{x}_t, 0), \widetilde{y}_t)}{2}(h^{\widetilde{D}_t})''(\widetilde{y}_t), \tag{5.31}$$

$\widetilde{y}_t$ being the point of $\partial\widetilde{D}_t$ in the horizontal line with the greatest abscissa, and the second derivative being calculated with curvilinear coordinates on $\partial\widetilde{D}_t$. Notice that $(h^{\widetilde{D}_t})''(\widetilde{y}_t) \leqslant 0$, proving that the process $S(\widetilde{D}_t)$ is monotonly decreasing.

Let us return to the general situation of G_n-symmetric $(\widetilde{D}_t)_{t\geqslant 0}$. The investigation of the lifetime of the solution to (2.12) is not easy. In [2] we prove that the lifetime is the time when $\widetilde{D}_t$ meets its skeleton $\widetilde{S}_t$. We have no example where $\tilde{D}_t$ meets its skeleton $\tilde{S}_t$ in finite time. The next proposition yields examples where the lifetime is infinite, together with nice properties related to the symmetry group G_n.

Proposition 5.5. (1) *The process* $\left(\dfrac{\mu^{\partial\tilde{D}_t}(\partial\widetilde{D}_t)}{\mu(\widetilde{D}_t)}\right)_{0\leqslant t<\tilde{\tau}}$ *is a supermartingale.*

(2) *Define the entropy* $\widetilde{\mathrm{Ent}}_t$ *as the integral of* $\rho_t\log\rho_t$ *with respect to the curvilinear abscissa in* $\partial\tilde{D}_t$*,* ρ_t *being the curvature of* $\partial\tilde{D}_t$*. Assume* $\tilde{S}_0$ *is* G_n*-symmetric with* $n\geqslant 3$*. Then the entropy process* $\left(\widetilde{\mathrm{Ent}}_t\right)_{0\leqslant t<\tilde{\tau}}$ *is a supermartingale.*

(3) *Assume* $\tilde{S}_0$ *is* G_n*-symmetric with* $n\geqslant 7$*. Then* $\tilde{\tau}=\infty$ *a.s. Consequently, when* S_0 *is* G_n*-symmetric with* $n\geqslant 7$*, Equation* (4.1) *for the decoupled* $(X_t,D_t)_{t\geqslant 0}$ *provides an intertwining with infinite lifetime. If moreover the skeleton* S_0 *of* D_0 *has the form* $\widetilde{S}_0=G_n\widetilde{H}_0$*,* $\widetilde{H}_0$ *being an horizontal interval* $\widetilde{H}_0=[0,x_0]\times\{0\}$ *for some* $x_0>0$*, Equation* (3.10) *for the full coupled* $(X_t,D_t)_{t\geqslant 0}$ *provides an intertwining with infinite lifetime.*

APPENDIX A. AN INTEGRATION BY PARTS ON DOMAINS WITH BOUNDARY

Our goal here is to obtain an extension of Stokes's formula on a domain with a smooth boundary, for functions which degenerate on the skeleton. We take the opportunity to recall this notion, as well as related geometric concepts.

Let M be a d-dimensional Riemannian manifold and $D\subset M$ a compact and connected domain with smooth boundary ∂D. For $y\in\partial D$, let $N(y)$ be the inward normal vector. Denote by S' the inward (morphological) skeleton of D: S' is the set of points in D such that (i) the distance to ∂D is not smooth and (ii) there are points around them where the distance to ∂D is smooth with a non vanishing gradient. Denote

$$\tau(y)=\inf\{t>0,\ \exp_y(tN(y))\in S'\}. \tag{A.1}$$

Let S be the set of regular points of S', which we can describe as follows: if $x\in S$, then there exists a unique couple (y_1,y_2) of distinct points from ∂D such that

$$x=\exp_{y_1}\left(\tau(y_1)N(y_1)\right)=\exp_{y_2}\left(\tau(y_2)N(y_2)\right). \tag{A.2}$$

We have $\tau(y_1)=\tau(y_2)$, and for $i=1,2$, the differential at $(\tau(y_i),y_i)$ of the map $\mathbb{R}_+\times\partial D\ni(t,y)\mapsto\exp_y(tN(y))$ is nondegenerate. The set S is a codimension 1 submanifold of M and $S'\backslash S$ has Hausdorff dimension smaller than or equal to $d-2$. It is the union of the focal set which is the set of points $x=\exp_y(\tau(y)N(y))$ such that $(t,y')\mapsto\exp_{y'}(tN(y'))$ is degenerate at $(\tau(y),y)$, and the union of the sets defined like S but with strictly more than two points y_1, y_2, y_3,... For $r\geqslant 0$, let

$$D(r)=\{z\in D\backslash S',\ \rho_{\partial D}(z)\geqslant r\} \tag{A.3}$$

where ρ is the Riemannian distance. The set $D(r)$ is a (possibly empty) manifold with smooth boundary $\partial D(r)$ on which one can define an inward normal $N(y)$ and an orientation by parallel transporting oriented basis of ∂D along normal geodesics. So we have for all $y\in D\backslash S'$: $N(y)=\nabla\rho_{\partial D}(y)$.

We will also need the sets $D(r)$ for all $r\in\mathbb{R}$. We will let for $r<0$

$$D(r)=\{z\in M,\ \rho^+_{\partial D}(z)\geqslant r\} \tag{A.4}$$

where $\rho^+_{\partial D}$ is the signed distance to ∂D, positive inside D, negative outside D.

Define for $s, t \in \mathbb{R}$

$$\begin{aligned} \psi(s,t) : \partial D(s) &\to \partial D(t) \\ y &\mapsto \exp_y\left((t-s)N(y)\right) \end{aligned} \tag{A.5}$$

and $\psi(t) = \psi(0,t)$. We will indifferently write $\psi(t)(x) = \psi(t,x)$. The function $\psi(s,t)$ is not defined for all points of $\partial D(s)$ because we ask $\psi(s,t)(y) \in \partial D(t)$, nor is $N(\cdot)$. However for $|s|$ and $|t|$ small it is a map, defined for all $y \in \partial D(s)$, and is is also a diffeomorphism with inverse $\psi(t,s)$.

We have for $0 \leqslant s \leqslant t$, $\psi(t) = \psi(s,t) \circ \psi(s)$, which implies

$$\det T\psi(t) = \det T\psi(s,t) \times \det T\psi(s). \tag{A.6}$$

Notice that thanks to the orientation of the sets $\partial D(r)$ we get an orientation of $D \backslash S'$ by adding N as first vector to oriented basis, consequently $\det T\psi$ is well defined and always positive. It is well-known that

$$\left.\frac{d}{dt}\right|_{t=s} \det T\psi(s,t)(y) = -h(y) \tag{A.7}$$

where $h(y)$ is the inward mean curvature of $\partial D(s)$ (the minus sign of the r.h.s. of (A.7) insures that h is non-negative on $\partial D(s)$ when $D(s)$ is convex). This together with (A.6) yields

$$\left.\frac{d}{dt}\right|_{t=s} \det T\psi(t)(y) = -h\left(\psi(s)(y)\right) \det T\psi(s)(y) \tag{A.8}$$

and consequently, using $\psi(0) = \mathrm{id}$ and $\det T\psi(0) \equiv 1$,

$$\det T\psi(t)(y) = \exp\left(\int_0^t -h\left(\psi(s)(y)\right)\, ds\right). \tag{A.9}$$

Denote by μ the volume measure of D and by $\underline{\mu}$ the volume measures of the manifolds $\partial D(s)$ and of S. Then

$$\mu(D) = \int_0^\infty \underline{\mu}\left(\partial D(r)\right)\, dr. \tag{A.10}$$

But for $r \geqslant 0$

$$\underline{\mu}\left(\partial D(r)\right) = \int_{\partial D} \det T\psi(r)(y)\, \underline{\mu}(dy) \tag{A.11}$$

with convention $\det T\psi(r)(y) = 0$ if $r \geqslant \tau(y)$. We get

$$\underline{\mu}\left(\partial D(r)\right) = \int_{\partial D} \exp\left(-\int_0^r h(\psi(s)(y))\, ds\right) 1_{\{r<\tau(y)\}}\, \underline{\mu}(dy) \tag{A.12}$$

which yields with (A.10)

$$\mu\left(D\right) = \int_{\partial D} \left(\int_0^{\tau(y)} \exp\left(-\int_0^r h(\psi(s,y))\, ds\right) dr\right) \underline{\mu}(dy). \tag{A.13}$$

More generally, for a measurable function $g : D \to \mathbb{R}$ bounded below,

$$\int_D g\, d\mu = \int_{\partial D} \left(\int_0^{\tau(y)} g\left(\psi(r,y)\right) \exp\left(-\int_0^r h(\psi(s,y))\, ds\right) dr\right) \underline{\mu}(dy). \tag{A.14}$$

Applying this formula to the function gh which we assume to be bounded below or integrable, we get by integration by parts

$$\begin{aligned}\int_D gh\,d\mu &= \int_{\partial D}\left(\int_0^{\tau(y)} -g\left(\psi(r,y)\right)\frac{d}{dr}\exp\left(-\int_0^r h(\psi(s,y))\,ds\right)\,dr\right)\underline{\mu}(dy)\\ &= \int_{\partial D}\left[-g\left(\psi(r,y)\right)\exp\left(-\int_0^r h(\psi(s,y))\,ds\right)\right]_0^{\tau(y)}\underline{\mu}(dy)\\ &+\int_{\partial D}\left(\int_0^{\tau(y)}\langle dg,N\rangle\left(\psi(r,y)\right)\exp\left(-\int_0^r h(\psi(s,y))\,ds\right)\,dr\right)\underline{\mu}(dy)\\ &= \int_{\partial D} g(y)\,\underline{\mu}(dy) - \int_{\partial D} g(\psi(\tau(y),y))e^{-\int_0^{\tau(y)} h(\psi(u,y))\,du}\,\underline{\mu}(dy)\\ &+\int_D \langle dg,N\rangle\,d\mu.\end{aligned}$$

Define the map

$$\begin{aligned}\varphi : \partial D &\to S'\\ y &\mapsto \psi(\tau(y),y).\end{aligned}\tag{A.15}$$

For $z = \psi(\tau(y_i),y_i) \in S$ $(i = 1,2)$ define $\theta(z) \in (0,\pi/2]$ the angle between the vector $N(\psi(\tau(y_i)-,y_i))$ and the skeleton S. In the sequel we assume that $\theta(z) \neq \pi/2$ (the case $\theta(z) = \pi/2$ is simpler to deal with and Proposition A.1 is always valid). Notice that this angle does not depend on i, this is a consequence of $z \in S$ staying at the same distance to y_1 and y_2 by infinitesimal variation. For later use, let also $\theta(z) = 0$ when $z \in S'\backslash S$. Let us prove that for $z = \psi(\tau(y_i),y_i)) \in S$,

$$\det T\psi(\tau(y_i),y_i) = \sin\theta(\varphi(y_i))\det T\varphi(y_i),\qquad i = 1,2.\tag{A.16}$$

Set $y = y_1$. Let $e_1 = N(y)$, $e_1^S = N(\psi(\tau(y)-,y))$, $N^S(z)$ the normal to S at z such that $\langle N^S(z), e_1^S\rangle > 0$, let $e'' = (e_3,\dots,e_d)$ be a family of orthonormal normalized vectors in $T_y\partial D$ such that letting $e_2 = \dfrac{\nabla\tau(y)}{\|\nabla\tau(y)\|}$ (we have $\nabla\tau(y) \neq 0$, since $\theta(z) \neq \pi/2$), $e' := (e_2, e'')$ is an orthonormal basis of $T_y\partial D$, let $(e^S)'' = (e_3^S,\dots,e_d^S)$ be an orthonormal basis of $T_y\varphi(\mathrm{Vect}(e''))$, let e_2^S such that $(e^S)' := (e_2^S,\dots,e_d^S)$ is an orthonormal basis of T_zS. Finally let $e_2^\theta \in T_zM$ be such that $\langle e_2^\theta, N(z)\rangle < 0$ (e_2^θ and $N^S(z)$ are not orthogonal, since $\theta(z) \neq \pi/2$) and $(e_1^S, e_2^\theta, (e^S)'')$ is an orthonormal basis of T_zM. Figure 1 shows the configuration of $e_1^S, N^S(z), e_2^S$ and e_2^θ on an example of dimension 2. In the sequel we will denote for instance $T\varphi(e') = \begin{pmatrix}T\varphi(e_2)\\ \vdots\\ T\varphi(e_d)\end{pmatrix}$, so that $\langle T\varphi(e'),(e^S)'\rangle$ will be the matrix of all scalar products. We have

$$\begin{aligned}&\langle T\varphi(e'),(e^S)'\rangle\\ &= \langle d\tau, e'\rangle\langle\partial_t\psi(\tau(y),y),(e^S)'\rangle + \langle T\psi(e'),(e^S)'\rangle\\ &= \begin{pmatrix}\langle d\tau,e_2\rangle\langle\partial_t\psi,e_2^S\rangle + \langle T\psi(e_2),e_2^S\rangle & \langle T\psi(e_2),(e^S)''\rangle\\ \langle d\tau,e''\rangle\langle\partial_t\psi,e_2^S\rangle + \langle T\psi(e''),e_2^S\rangle & \langle T\psi(e''),(e^S)''\rangle\end{pmatrix}.\end{aligned}$$

Let us simplify and make more explicit this expression. We have $\langle d\tau, e''\rangle = 0$. Also $e_2^\theta \perp (e^S)''$ and $e_2^S \perp (e^S)''$ so $e_2^S \in \mathrm{Vect}(e_1^S, e_2^\theta)$ and more precisely

$$e_2^S = \cos(\theta(z))e_1^S + \sin(\theta(z))e_2^\theta.\tag{A.17}$$

On the other hand $T\psi(e') \perp e_1^S$ which implies

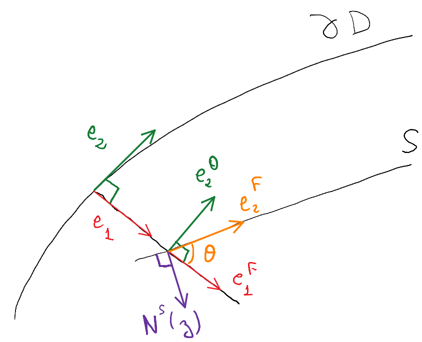

FIGURE 1. The vectors $e_1^S, N^S(z), e_2^S$ and e_2^θ

$$\langle T\psi(e'), e_2^S\rangle = \sin(\theta(z))\langle T\psi(e'), e_2^\theta\rangle. \tag{A.18}$$

Also $\langle \partial_t\psi, e_2^S\rangle = \cos(\theta(z))$. We arrive at

$$\begin{aligned}
&\det\langle T\varphi(e'), (e^S)'\rangle \\
&= \sin\theta(z)\det\begin{pmatrix} \langle T\psi(e_2), e_2^\theta\rangle & \langle T\psi(e''), e_2^\theta\rangle \\ \langle T\psi(e_2), (e^S)''\rangle & \langle T\psi(e''), (e^S)''\rangle \end{pmatrix} \\
&+ \cos\theta(z)\det\begin{pmatrix} \langle d\tau, e_2\rangle & 0 \\ \langle T\psi(e_2), (e^S)''\rangle & \langle T\psi(e''), (e^S)''\rangle \end{pmatrix} \\
&= \sin\theta(z)\det T\psi + \cos\theta(z)\langle d\tau, e_2\rangle \det\langle T\psi(e''), (e^S)''\rangle.
\end{aligned} \tag{A.19}$$

For the last equation we used the fact that $\det T\psi = \det\langle T\psi(e'), (e_2^\theta, (e^S)'')\rangle$, since e' and $(e_2^\theta, (e^S)'')$ are orthonormal bases. Note that by definition, $\langle T\psi(e''), e_2^\theta\rangle = 0$, so we also get $\det T\psi = \det\langle T\psi(e''), (e^S)''\rangle \times \langle T\psi(e_2), e_2^\theta\rangle$. On the other hand, we have

$$\langle d\tau, e_2\rangle = \langle T\psi(e_2), e_2^\theta\rangle \cot\theta(z). \tag{A.20}$$

Indeed, note that

$$\begin{aligned}
0 &= \left\langle T\varphi(e_2), N^S\right\rangle \\
&= \langle d\tau, e_2\rangle\left\langle e_1^S, N^S\right\rangle + \left\langle T\psi(e_2), N^S\right\rangle \\
&= \langle d\tau, e_2\rangle\sin(\theta(z)) - \cos(\theta(z))\left\langle T\psi(e_2), e_2^\theta\right\rangle
\end{aligned}$$

where the last term is obtained by taking into account that $T\psi(e_2)$ is parallel to e_2^θ. This is the change of length of the geodesic needed to stay in S. We obtain

$$\begin{aligned}
\det T\varphi &= \sin\theta(z)\det T\psi + \cos\theta(z)\cot\theta(z)\det T\psi \\
&= \frac{\sin^2\theta(z) + \cos^2\theta(z)}{\sin\theta(z)}\det T\psi.
\end{aligned}$$

This yields (A.16).

We arrived at

$$\begin{aligned}\int_D gh\,d\mu = &\int_{\partial D} g(y)\,\underline{\mu}(dy) - \int_{\partial D} g(\psi(\tau(y),y))\det T\psi(\tau(y),y)\,\underline{\mu}(dy)\\ &+\int_D \langle dg, N\rangle\,d\mu.\end{aligned} \tag{A.21}$$

this yields with (A.16)

$$\begin{aligned}\int_D gh\,d\mu = &\int_{\partial D} g(y)\,\underline{\mu}(dy) - \int_{\partial D} g(\varphi(y))\sin\theta(\varphi(y))\det T\varphi(y)\,\underline{\mu}(dy)\\ &+\int_D \langle dg, N\rangle\,d\mu.\end{aligned} \tag{A.22}$$

Using the change of variable $y \mapsto \varphi(y)$ and the fact that all $z \in S$ is equal to $\varphi(y_i)$, $i = 1, 2$, we obtain the key formula

Proposition A.1. *With the above notations, for any smooth function g defined on D such that gh is integrable or bounded below, we have:*

$$\int_D gh\,d\mu = \int_{\partial D} g(y)\,\underline{\mu}(dy) - 2\int_S g(z)\sin\theta(z)\,\underline{\mu}(dz) + \int_D \langle dg, N\rangle\,d\mu. \tag{A.23}$$

Appendix B. Moving sets

In this section we describe how to move a domain with smooth boundary by deformation of its boundary. We also investigate the deformation of its skeleton. The deformation we will consider will have a general absolutely continuous finite variation part, together with a very specific martingale part and singular finite variation part. First we introduce some notation.

For a domain D_0 with smooth boundary ∂D_0 and $\alpha > 0$, define the map $\psi = \psi^{D_0}$ by

$$\begin{aligned}\psi : (-\alpha,\alpha)\times\partial D_0 &\to M\\ (s,y) &\mapsto \exp_y\left(sN(y)\right).\end{aligned} \tag{B.1}$$

Here $N = N^{D_0}$ is the inward normal defined in Section A. We take α sufficiently small so that ψ is a diffeomorphism on its range which we will call $D_{0,\alpha}$. Consider a moving domain $t \mapsto D_t$ started at D_0. We assume that the deformation is sufficiently regular so that for all $t \geqslant 0$, we can write D_t as

$$D_t = \{\psi([f_t(y),\tau_{D_0}(y)],y),\quad y \in \partial D_0\} \tag{B.2}$$

with $\tau_{D_0}(y)$ defined in (A.1), $\psi([f_t(y),\tau_{D_0}(y)],y) := \{\psi(s,y),\ s \in [f_t(y),\tau_{D_0}(y)]\}$, and $t \mapsto f_t(y)$ a semimartingale with values in $(-\alpha,\alpha)$, smoothly depending on y. In particular, the skeleton S_0' of D_0 satisfies $S_0' \subset D_t$. In other words, D_t is the union of rays $\psi([f_t(y),\tau_{D_0}(y)],y)$ orthogonal to ∂D_0 at y (notice that all $\psi([f_t(y),\tau_{D_0}(y)),y)$ are disjoint). Alternatively, D_t is also the interior of the set $\exp_{\partial D_0}(f)$ described in (2.2) with f_t instead of f. Also, in the special case where the real valued semimartingale $t \mapsto f_t(y) = f_t$ does not depend on y, then we have

$$D_t = D_0(f_t) \tag{B.3}$$

where $D_0(r)$ is defined in (A.4). In this situation, the skeleton is not moving, at least as long as ∂D_t remains smooth (i.e. until ∂D_t hits the inner skeleton S_0' or the outer skeleton of D_0), and $t \mapsto f_t$ can be allowed to be a semimartingale with singular continuous drift.

When $t \mapsto f_t(y)$ depends on y the situation is more complicated and we like to use a more convenient and intrinsic description of the motion of D_t. More precisely, we will describe it by the motion of its boundary via semimartingales $(Y_t(y))_{t\geqslant 0}$ indexed by $y \in \partial D_0$, satisfying $Y_0(y) = y$ and the Itô equation in manifold with respect to the Levi Civita connection ∇

$$dY_t(y) = d^\nabla Y_t(y) = N^{D_t}(Y_t(y))\left(H^{D_t}(Y_t(y))\,dt + dz_t\right) \tag{B.4}$$

where H^{D_t} is a smooth function on ∂D_t (which later on will be chosen to be $h^{D_t}/2$, where h^{D_t} is the mean curvature of ∂D_t) and $(z_t)_{t\geqslant 0}$ is a real valued continuous semimartingale. Recall that fomally $d^\nabla Y_t(y)$ is a vector which writes in local coordinates $(y^1, \ldots, y^d)$ with the Christoffel symbols $\Gamma^i_{j,k}$:

$$d^\nabla Y_t(y) = \left(dY_t^i(y) + \frac{1}{2}\Gamma^i_{j,k}(Y_t(y))\,d\langle Y_t^j(y), Y_t^k(y)\rangle\right) D_i(Y_t(y)) \tag{B.5}$$

where $D_i(Y_t(y))$ is the vector $\dfrac{\partial}{\partial y^i}$ taken at point $Y_t(y)$. Since the semimartingale $(z_t)_{t\geqslant 0}$ does not depend on y, the Itô equation is equivalent to the Stratonovich one: indeed, using (B.3) the Itô to Stratonovich convertion term is

$$\frac{1}{2}\nabla_{N^{D_t}(Y_t(y))dz_t} N^{D_t}(\cdot)dz_t = \frac{1}{2}\nabla_{N^{D_t}(Y_t(y))} N^{D_t}(\cdot)d\langle z, z\rangle_t = 0$$

since $N^{D_t}(Y_t(y))$ is the speed at time $a = 0$ of the geodesic $a \mapsto \psi^{D_t}(a, Y_t(y))$.

We assume that Equation (B.4) has a strong solution for all times, possibly by stopping it, and that a.s. for all times the map $y' \mapsto Y_t(y')$ is a diffeomorphism from ∂D_0 to ∂D_t. Since $dY_t(y)$ represents the motion of ∂D_t, writing $Y_t(y') = y$ and using the diffeomorphism property, equation (B.4) rewrites as

$$d\partial D_t(y) := dY_t(y') = N^{D_t}(y)\left(H^{D_t}(y)\,dt + dz_t\right). \tag{B.6}$$

In other words, our equations are driven by two vector fields $(H^D(y)N^D(y))_{y\in\partial D}$ and $(N^D(y))_{y\in\partial D}$, and the stochastic part is in front of the second one. All the set-valued processes considered in this paper satisfy this assumption.

We can obtain the random functions $f_t : \partial D_0 \to \mathbb{R}$ from the semimartingales $(Y_t(y'))_{t\geqslant 0}$ with the following procedure. The orthogonal projection $\pi_t : \partial D_t \to \partial D_0$ is a diffeomorphism, and by definition of ψ, we have

$$Y_t(y') = \psi\left(f_t(\pi_t(Y_t(y'))), \pi_t(Y_t(y'))\right), \tag{B.7}$$

yielding

$$f_t(\pi_t(Y_t(y')) = \left(\psi^{-1}\right)_1 (Y_t(y')) \tag{B.8}$$

with $\left(\psi^{-1}\right)_1$ the first coordinate of ψ^{-1}. Writing $y = \pi_t(Y_t(y'))$ and using the diffeomorphism properties, we get

$$f_t(y) = \left(\psi^{-1}\right)_1 (\pi_t^{-1}(y)). \tag{B.9}$$

Consequently, the real-valued semimartingale $(f_t(y))_{t\geqslant 0}$ solves the Stratonovitch equation

$$\circ df_t(y) = T\left(\psi^{-1}\right)_1 (\circ d\pi_t^{-1}(y)) \tag{B.10}$$

which is impossible to work with. This is why we will always consider the formulation (B.6).

Let us now investigate the motion of the skeleton S_t under this motion of D_t. First we remark that by local inversion theorem, at regular points of the skeleton, the variation in Stratonovich sense is linear and the sum of all variations of the concerned point at the boundary. As we already remarked, the motion dz_t does not change S_t, so this together with the linearity just mentioned implies that we have a finite variation of the skeleton.

Recall the situation of (A.2) in Section A. We consider a domain D, $x \in S$, y_1, y_2 the two elements of ∂D such that $\exp_{y_1}(\tau(y_1)N(y_1)) = \exp_{y_2}(\tau(y_2)N(y_2))$, with $\tau(y_1) = \tau(y_2)$.

For $i = 1, 2$, we will consider a variation of the minimal geodesic from y_i to x, represented by a Jacobi field J_i satisfying $J_i(0) \in T_{y_i}M$, $J_1(1) = J_2(1) \in T_xM$,

$$J_i(0) = \lambda_i N(y_i) + J_i^{\perp}(0), \quad J_i'(0) = \lambda_i' N(y_i) + (J_i^{\perp})'(0), \tag{B.11}$$

with $J_i^{\perp}$ orthogonal to $N(y_i)$. The motion of S corresponding to the motion of y_1 and y_2 will be represented by $J_1(1)$. Since S has a boundary, the observation of the orthogonal part to S of $J_1(1)$ is not sufficient.

Let γ_i be the projection on M of J_i. It is the geodesic in time 1 from y_i to x (as usual in the computations of Jacobi fields, the speed is not normalized). Denote $N_i(x) = \dot\gamma_i(1)/\|\dot\gamma_i(1)\|$. Recall that the angle between $N_i(x)$ and T_xS is $\theta(x) \in (0, \pi/2]$. We will also let

$$N_1^S(x) = \frac{1}{2\sin\theta(x)}(N_1(x) - N_2(x)). \tag{B.12}$$

Figure 2 shows the configuration of the points x, y_1, y_2 and the vectors $N_1(x)$, $N_2(x)$, $N_1^S(x)$. The vector $N_1^S(x)$ is is the normal vector to S at point x, in the same side as $N_1(x)$. We will

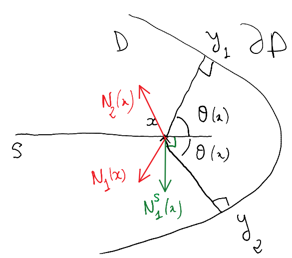

FIGURE 2. The points x, y_1, y_2 and the vectors $N_1(x), N_2(x), N_1^S(x)$

consider variations of geodesics with same final value:

$$J_1(1) = J_2(1) = \lambda N_1^S(x) + J_1^T(1) \tag{B.13}$$

for some $\lambda \in \mathbb{R}$, where $J_1^T(1) \in T_xS$. Writing $\lambda N_1^S(x) = \frac{\lambda}{2\sin\theta(x)}(N_1(x) - N_2(x))$ we have

$$\begin{aligned} \langle J_1(1), N_1(x)\rangle &= \frac{\lambda}{2\sin\theta(x)}\,(1 - \cos(2\theta(x)) + \langle J_1^T(1), N_1(x)\rangle \\ &= \lambda\sin\theta(x) + \langle J_1^T(1), N_1(x)\rangle \end{aligned} \tag{B.14}$$

and

$$\begin{aligned} \langle J_1(1), N_2(x)\rangle &= -\frac{\lambda}{2\sin\theta(x)}\,(1 - \cos(2\theta(x)) + \langle J_1^T(1), N_2(x)\rangle \\ &= -\lambda\sin\theta(x) + \langle J_1^T(1), N_2(x)\rangle. \end{aligned} \tag{B.15}$$

On the other hand we require that the variation of length of the two geodesics are the same. This writes as

$$\langle J_1(1), N_1(x)\rangle - \langle J_1(0), N(y_1)\rangle = \langle J_2(1), N_2(x)\rangle - \langle J_2(0), N(y_2)\rangle \tag{B.16}$$

or

$$\lambda \sin\theta(x) + \langle J_1^T(1), N_1(x)\rangle - \lambda_1 = -\lambda \sin\theta(x) + \langle J_1^T(1), N_2(x)\rangle - \lambda_2, \tag{B.17}$$

which finally, with $\langle J_1^T(1), N_1(x) - N_2(x)\rangle = 0$, yields $\lambda = \dfrac{\lambda_1 - \lambda_2}{2\sin\theta(x)}$, so the normal variation of S is given by

$$\langle J_1(1), N_1^S(x)\rangle N_1^S(x) = \frac{\lambda_1 - \lambda_2}{2\sin\theta(x)} N_1^S(x). \tag{B.18}$$

Next we will compute the tangential displacement $J^T(1)$ of x in S. As we will see later, we will only need a Jacobi field J_1 such that $J_1^\perp(0)$ and $(J_1^\perp)'(0)$ are known and

$$J_1(0) = \lambda_1 N(y_1), \text{ i.e. } \quad J_1^\perp(0) = 0. \tag{B.19}$$

So we know $J_1^\perp(1)$: and

$$J_1^\perp(1) = J\left(1, 0, (J_1^\perp)'(0)\right) \tag{B.20}$$

where $J(1, u, v)$ is the value at time 1 of the Jacobi field J with $J(0) = u$ and $J'(0) = v$. From

$$\begin{aligned} J_1(1) &= J_1^T(1) + \langle J_1(1), N_1^S(x)\rangle N_1^S(x) \\ J_1(1) &= J_1^\perp(1) + \langle J_1(1), N_1(x)\rangle N_1(x) \end{aligned} \tag{B.21}$$

we get

$$J_1^T(1) = J_1^\perp(1) + \langle J_1(1), N_1(x)\rangle N_1(x) - \langle J_1(1), N_1^S(x)\rangle N_1^S(x). \tag{B.22}$$

On the other hand we have

$$\begin{aligned} \langle J_1(1), N_2(x)\rangle &= \langle J_1^\perp(1), N_2(x)\rangle + \langle J_1(1), N_1(x)\rangle\langle N_1(x), N_2(x)\rangle \\ \langle J_1(1), N_2(x)\rangle &= \langle J_1(1), N_1(x)\rangle - (\lambda_1 - \lambda_2) \end{aligned} \tag{B.23}$$

where the second equation is a direct consequence of (B.18). Substracting the second equation to the first one yields

$$(1 - \cos(2\theta(x)))\langle J_1(1), N_1(x)\rangle = \langle J_1^\perp(1), N_2(x)\rangle + \lambda_1 - \lambda_2. \tag{B.24}$$

Replacing $\langle J_1(1), N_1(x)\rangle$ in (B.22) and after simplification, using (B.12) and (B.18), we finally obtain the horizontal displacement

(B.25)

$$(J_1^T)(1) = J_1^\perp(1) + \frac{1}{4\sin^2\theta(x)}\left(2\langle J_1^\perp(1), N_2(x)\rangle N_1(x) + (\lambda_1 - \lambda_2)(N_1(x) + N_2(x))\right).$$

We are now in position to write the motion of the skeleton S_t when the motion of the boundary is given by (B.6). For $x \in S_t$ with corresponding points y_1 and y_2 in ∂D_t,

$$dS_t^\perp(x) = \frac{1}{2\sin\theta^{S_t}(x)}\left(H^{D_t}(y_1) - H^{D_t}(y_2)\right) N_1^{S_t}(x)\, dt \tag{B.26}$$

which has finite variation. Observe that, as already mentioned, the term dz_t disappears.

Here we wrote $dS_t^\perp(x)$ for the normal variation of the regular skeleton. But as we already remarked, since S_t is not a closed manifold, it can expand via the motion of its boundary. So we have to investigate the horizontal motion $dS^T(x)$.

Notice that $J_1^\perp)'(0)$ is the perpendicular part of the time derivative of the speed at y_1 of the geodesic in time 1 from y_1 to x. So from equation (B.6) we deduce the rotation

$$(J_1^\perp)'(0)\,dt = \rho_S(y_1)\nabla_t N^{D_t}(y_1) = -\rho_S(y_1)\nabla H^{D_t}(y_1)\,dt. \tag{B.27}$$

(in the r.h.s. the gradient corresponds to the tangential gradient on ∂D_t, recall that H^{D_t} is only defined on this hypersurface).

We conclude that the horizontal displacement of x is $J_1^T(1)\,dt$

$$\begin{aligned} J_1^T(1)\,dt = J_1^\perp(1)\,dt + \frac{1}{4\sin^2\theta^{S_t}(x)} \Bigg(& 2\langle J_1^\perp(1), N_2^{D_t}(x)\rangle N_1^{D_t}(x) \\ & + (H^{D_t}(y_1) - H^{D_t}(y_2))(N_1^{D_t}(x) + N_2^{D_t}(x))\Bigg)\,dt \end{aligned} \tag{B.28}$$

where $J_1^\perp(1) = J(1, 0, -\rho_S(y_1)\nabla H^{D_t}(y_1))$. Again the processus z_t does not play a role.

To summarize, we have the following result for the evolution of S_t:

Theorem B.1. *When D_t evolves as* (B.6)

$$d\partial D_t(y) = N^{D_t}(y)(H^{D_t}(y)\,dt + dz_t), \tag{B.29}$$

the regular skeleton S_t has the normal evolution (B.26)

$$dS_t^\perp(x) = \frac{H^{D_t}(y_1) - H^{D_t}(y_2)}{4\sin^2\theta^{S_t}(x)} \left(N_1^{D_t}(x) - N_2^{D_t}(x)\right)\,dt \tag{B.30}$$

and the tangential evolution (B.28) *which can be rewritten as*

$$\begin{aligned} & dS_t^T(x) \\ & = p_S(J_1^\perp(1))\,dt \\ & + \left(-\frac{\langle J_1^\perp(1), N_1^S(x)\rangle}{2\sin\theta^{S_t}(x)} + \frac{H^{D_t}(y_1) - H^{D_t}(y_2)}{4\sin^2\theta^{S_t}(x)}\right)(N_1^{D_t}(x) + N_2^{D_t}(x))\,dt \end{aligned} \tag{B.31}$$

where p_S denotes the orthogonal projection on TS, $J_1^\perp(1) = J(1, 0, -\rho_S(y_1)\nabla H^{D_t}(y_1))$, and y_1, y_2 are defined in Figure 2.

Remark B.2. The points y_1 and y_2 do not play the same role in Theorem B.1. As formula (B.30) is symmetric in y_1 and y_2, formula (B.31) is not. The reason is that if we assume the motion of y_1 to be normal to the boundary ∂D_t and to have speed given by (B.29), the motion of y_2 has no reason to be normal to the boundary: $J_2^\perp(0)$ does not vanish.

APPENDIX C. DOSS-SUSSMAN REPRESENTATION OF ITÔ'S EQUATION (2.12)

In this section we adapt the results of [10] to our notations. Let the stochastic mean curvature flow be a solution of :

$$\forall\, t \in [0, \tau),\ \forall\, y \in C_t, \qquad d\partial D_t(y) \;=\; \left(dW_t + \frac{1}{2}h^{D_t}(y)dt\right) N^{D_t}(y) \tag{C.1}$$

where $C_t := \partial D_t$, starting at D_0. Notice that contrarily to [10] we don't have a term $\sqrt{2}$ in front of the Brownian motion, this explains the fact that we have put a normalization factor $\frac{1}{2}$ in front of the mean curvature term.

Let ∂G_t be a solution of

$$\left\{ \begin{array}{rcl} G_0 & = & D_0 \\ \forall\, t \in [0, \tilde{\epsilon}),\ \forall\, x \in \partial G_t, \qquad \partial_t x & = & \alpha_{\partial G_t, -W_t}(x) N^{G_t}(x) \end{array} \right. \tag{C.2}$$

for some $\widetilde{\epsilon} > 0$ small enough, where α is defined by

$$\forall\, r > 0,\ \forall\, D \in \mathcal{D}_r,\ \forall\, x \in C, \qquad \alpha_{C,r}(x) \ := \ \frac{1}{2} h^{\Psi(C,r)}(\psi_{C,r}(x)) \tag{C.3}$$

and $\Psi(C, r)$ is the normal (exterior) flow starting at C at time r (c.f. Chapter 3 and 4 of [10] for notations).

Similarly to the proof of Theorem 17 from [10], we show that $D_t = \Psi(G_t, -W_t)$ is a solution of the stopped martingale problem associated to the generator $(\mathcal{D}, \widetilde{\mathcal{L}})$ where for $f \in C^\infty(M)$ and $\mathbb{F}_f(D) = \int_D f\, d\mu$, $\nu = -N$ is the exterior normal

$$\widetilde{\mathcal{L}}\mathbb{F}_f(D) := \frac{1}{2} \int_{\partial D} \langle \nabla f, \nu \rangle\, d\underline{\mu} = \mathbb{F}_{\frac{1}{2}\Delta f}(D).$$

Recall that the equation (C.2), is in fact a quasiparabolic equation with coefficients that depend on trajectory of the Brownian motion (the meaning is trajectory by trajectory). Similarly to Section 4.1 from [10], we show that the solution of (C.2) have a regularity $C^{1+\frac{\alpha}{2}, 2+\alpha}$, for all $\alpha < 1$.

Proposition C.1. *Let ∂G_t be a solution of* (C.2)*. Then $\partial D_t = \Psi(\partial G_t, -W_t)$ is a solution of* (C.1) *in the Itô sense.*

Proof. Let $x \in \Psi(\partial G_t, -W_t)$, we have :

$$\begin{aligned} & d\Psi(\partial G_t, -W_t)(x) = \\ & = T_1\Psi_{(\partial G_t, -W_t)}(\frac{d}{dt}\partial G_t)(\Psi^{-1}(\partial G_t, -W_t)(x)\, dt \\ & - \nu^{\Psi(\partial G_t, W_t)}(x) dW_t \\ & = \left(dW_t + \frac{1}{2} h^{\Psi(\partial G_t, -W_t)}(x) dt \right) N^{\Psi(\partial G_t, -W_t)}(x), \end{aligned} \tag{C.4}$$

where in the first equality we use the Itô formula, the fact that $t \mapsto \partial G_t$ is of class $C^{1+\frac{\alpha}{2}}$, $\frac{d^2}{d^2 r}\Psi(x, r) = 0$, and in the second equality we used Lemma 13 in [10], i.e. ∂D_t is a solution in the Itô form :

$$\left\{ \begin{array}{rcl} d\partial D_t(x) & = & (dW_t + \frac{1}{2} h^{\partial D_t}(x) dt) N^{\partial D_t}(x) \\ x & \in & \partial D_t. \end{array} \right. \tag{C.5}$$

□

Proposition C.2. *Conversely, if ∂D_t is a solution of* (C.5) *then $\partial G_t = \Psi(\partial D_t, W_t)$ is a solution of* (C.2)*.*

Proof. Let $x \in \partial\Psi(\partial D_t, W_t)$

$$\begin{aligned} & d\Psi(\partial D_t, W_t)(x) \\ & = T_1\Psi_{(\partial D_t, W_t)}(\circ d\partial D_t)(x) + \nu^{\Psi(\partial D_t, W_t)}(x) dW_t \\ & = T_1\Psi_{(\partial D_t, W_t)}((dW_t + \frac{1}{2} h^{\partial D_t} dt) N^{\partial D_t})(x) \\ & - N^{\Psi(\partial D_t, W_t)}(x) dW_t \\ & = \left(\frac{1}{2} h^{\partial D_t}(\Psi^{-1}(\partial D_t, W_t)(x)) N^{\partial G_t}(x) dt \right) \\ & = \frac{1}{2} h^{\Psi(\partial G_t, -W_t)}(\Psi(\partial G_t, -W_t)(x)) N^{\partial G_t}(x) dt \end{aligned} \tag{C.6}$$

where we use that in this case, the Stratonovich differential is equal to the Itô's one (c.f. Appendix B), i.e. $\circ d\partial D_t(x) = d\partial D_t$, and $\frac{d^2}{d^2 r}\Psi(x, r) = 0$. So ∂G_t is a solution of (C.2). □

By the uniqueness of the solution of (C.2) (c.f. Theorem 22 in [10]) and the fact that it is adapted to the filtration of B we deduce that the solution of (C.5) is unique and is a strong solution. Similarly we have the uniqueness of the solution of

$$d\partial D_t(x) = \left(dW_t + \frac{1}{2}h^{\partial D_t}(x)dt - \frac{\mu(\partial D_t)}{\mu(D_t)}dt\right) N^{\partial D_t}(x).$$

Moreover, since we could also make a change of time in the Itô equation, Equation (2.12) has a unique strong solution.

Appendix D. Weak semi-group theory in the martingale problem sense

This theory has been developed in several books, see for instance Stroock and Varadhan [27] or Ethier and Kurtz [12]. Here we present a minimal version suitable for our purposes.

Let V be a measurable state space and consider Ω a set of trajectories from $\mathbb{R}_+$ to V. The canonical coordinates on Ω are denoted by the X_t, for $t \geqslant 0$: for $\omega \in \Omega$, $X_t(\omega)$ is the position at time t of ω. The set Ω is endowed with the sigma-field generated by the X_t, for $t \geqslant 0$. Our first assumption is that the mapping

$$\Omega \times \mathbb{R}_+ \ni (\omega, t) \quad \mapsto \quad X_t(\omega) \in V$$

is measurable, which usually means that "Ω is not too big".

For $t \geqslant 0$, we define

$$\mathcal{F}_t \quad := \quad \sigma(X_s \,:\, s \in [0,t]).$$

For $t \geqslant 0$, we will also need the time shift Θ_t associating to any $\omega \in \Omega$ the trajectory $\Theta_t(\omega)$ defined by

$$\forall\, s \geqslant 0, \qquad X_s(\Theta_t(\omega)) \quad = \quad X_{s+t}(\omega).$$

We assume that $\Theta_t(\Omega) \subset \Omega$.

A given family $\mathbb{P} := (\mathbb{P}_x)_{x\in V}$ of probability measures on Ω is said to be **Markovian** if for any $x \in V$ and any $t \geqslant 0$, the image by Θ_t of $\mathbb{P}_x$ conditioned by $\mathcal{F}_t$ is $\mathbb{P}_{X_t}$. In particular, it is assumed that $\mathbb{P}$ has the regularity of a Markov kernel from V to Ω.

From now on, we suppose that a Markovian family $\mathbb{P}$ is given. Let $\mathcal{B}$ be the space of bounded and measurable functions defined on V. The **semi-group** $P := (P_t)_{t\geqslant 0}$ associated to $\mathbb{P}$ is the family of operators acting on $\mathcal{B}$ via

$$\forall\, t \geqslant 0,\, \forall\, f \in \mathcal{B},\, \forall\, x \in V, \qquad P_t[f](x) \quad := \quad \mathbb{E}_x[f(X_t)].$$

The Markovianity of $\mathbb{P}$ implies at once the semi-group property

$$\forall\, s,t \geqslant 0, \qquad P_tP_s \quad = \quad P_{t+s}$$

and in particular the elements of P commute.

A subclass of "regular" functions that will be important for our purposes is $\mathcal{R}$ defined as

$$\mathcal{R} \quad := \quad \left\{ f \in \mathcal{B} \,:\, \forall\, x \in V,\, \lim_{t\to 0_+} P_t[f](x) \;=\; f(x) \right\}.$$

Exceptionally in the above limit, we assumed that $t \geqslant 0$ (i.e. not only that $t > 0$), so that by definition, for any $f \in \mathcal{R}$ and $x \in V$, $P_0[f](x) = f(x)$.

Let us observe that $\mathcal{R}$ is left stable by the semi-group:

Lemma D.1. *For any $t \geqslant 0$, we have $P_t[\mathcal{R}] \subset \mathcal{R}$. Thus for any given $f \in \mathcal{R}$ and $x \in V$, the mapping*

$$\mathbb{R}_+ \ni t \quad \mapsto \quad P_t[f](x)$$

is right continuous.

Proof. Indeed, fix $t \geqslant 0$ and $f \in \mathcal{R}$, we have for any $x \in V$ and $s \geqslant 0$,

$$\begin{aligned} P_s[P_t[f]](x) &= P_t[P_s[f]](x) \\ &= \mathbb{E}_x[P_s[f](X_t)]]. \end{aligned}$$

We have for any $s \geqslant 0$, $\|P_s[f]\|_\infty \leqslant \|f\|_\infty$ (where $\|\cdot\|_\infty$ stands for the supremum norm on $\mathcal{B}$) and since $f \in \mathcal{R}$, we get everywhere

$$\lim_{s\to 0_+} P_s[f](X_t) = f(X_t).$$

Dominated convergence implies that

$$\begin{aligned} \lim_{s\to 0_+} \mathbb{E}_x[P_s[f](X_t)]] &= \mathbb{E}_x[f(X_t)] \\ &= P_t[f] \end{aligned}$$

as desired. □

The **generator** L associated to P is the operator

$$L : \mathcal{D}(L) \to \mathcal{R}$$

defined in the following way: the space $\mathcal{D}(L)$ is the set of functions $f \in \mathcal{R}$ for which there exists a function $g \in \mathcal{R}$ such that the process $M^{f,g} := (M_t^{f,g})_{t\geqslant 0}$ defined by

$$\forall\, t \geqslant 0, \qquad M_t^{f,g} := f(X_t) - f(X_0) - \int_0^t g(X_s)\, ds$$

is a martingale under $\mathbb{P}_x$, for all $x \in V$.

Let us remark that g is then uniquely determined. Indeed, we have for any $x \in V$ and $t \geqslant 0$,

$$\mathbb{E}_x[f(X_t)] - \mathbb{E}[f(X_0)] - \mathbb{E}\left[\int_0^t g(X_s)\, ds\right] = 0.$$

Using Fubini's lemma (applicable due to our measurability requirement on Ω) and taking into account the definition of P, we get

$$P_t[f](x) - P_0[f](x) - \int_0^t P_s[g](x)\, ds = 0$$

namely, recalling that we required that $g \in \mathcal{R}$,

$$\begin{aligned} g &= P_0[g] \\ &= \lim_{t\to 0_+} \frac{1}{t}\int_0^t P_s[g](x)\, ds \\ &= \lim_{t\to 0_+} \frac{P_t[f](x) - f(x)}{t} \end{aligned} \qquad \text{(D.1)}$$

(we came back to the usual convention that $t > 0$ in the above limit) and as a by-product, we are assured of the existence of the latter limit.

We define $L[f] := g$ and $M^f := M^{f,g}$.

The differentiation property (D.1) can be extended into

Lemma D.2. *For any $f \in \mathcal{D}(L)$, $x \in V$ and $t \geqslant 0$, we have*

$$\partial_t P_t[f](x) = P_t[L[f]](x) \qquad \text{(D.2)}$$

Proof. For any $f \in \mathcal{D}(L)$, $x \in V$ and $t, s \geqslant 0$, we have

$$\begin{aligned} \mathbb{E}_x\left[M^f_{t+s} - M^f_t\right] &= \mathbb{E}_x\left[\mathbb{E}_x\left[M^f_{t+s} - M^f_t|\mathcal{F}_t\right]\right] \\ &= 0. \end{aligned}$$

We compute that

$$M^f_{t+s} - M^f_t = f(X_{t+s}) - f(X_t) - \int_t^{t+s} L[f](X_u)\,du$$

so that

$$\mathbb{E}_x\left[M^f_{t+s} - M^f_t\right] = P_{t+s}[f](x) - P_t[f](x) - \int_0^s P_{t+u}[L[f]](x)\,du.$$

Since $L[f] \in \mathcal{R}$, the mapping $[0, s] \ni u \mapsto P_{t+u}[L[f]](x)$ is right continuous, according to Lemma D.1, and the same argument as in (D.1) enables to conclude to (D.2). □

We can now come to the main goal of this appendix:

Proposition D.3. *For any $t \geqslant 0$, $\mathcal{D}(L)$ is stable by P_t and on $\mathcal{D}(L)$ we have $LP_t = P_tL$.*

Proof. Fix $f \in \mathcal{D}(L)$ and $x \in V$, the assertion of the lemma amounts to checking that the process $N := (N_s)_{s\geqslant 0}$ defined by

$$(N_s)_{s\geqslant 0} := \left(P_t[f](X_s) - P_t[f](X_0) - \int_0^s P_t[L[f]](X_u)\,du\right)_{s\geqslant 0}$$

is a martingale under $\mathbb{P}_x$. Consider $s' \geqslant s \geqslant 0$, we have to prove that

$$\mathbb{E}_x[N_{s'} - N_s|\mathcal{F}_s] = 0. \tag{D.3}$$

The l.h.s. is equal to

$$\begin{aligned} &\mathbb{E}_x\left[P_t[f](X_{s'}) - P_t[f](X_s) - \int_s^{s'} P_t[L[f]](X_u)\,du\Big|\mathcal{F}_s\right] \\ &= \mathbb{E}_x\left[P_t[f](X_{s'-s}\circ\Theta_s) - P_t[f](X_0\circ\Theta_s) - \int_0^{s'-s} P_t[L[f]](X_u\circ\Theta_s)\,du\Big|\mathcal{F}_s\right] \\ &= \mathbb{E}_y\left[P_t[f](X_{s'-s}) - P_t[f](X_0) - \int_0^{s'-s} P_t[L[f]](X_u)\,du\right] \end{aligned}$$

where $y = X_s$. By Fubini's lemma, the previous r.h.s. can be written

$$\begin{aligned} &\mathbb{E}_y\left[P_t[f](X_{s'-s})\right] - \mathbb{E}_y\left[P_t[f](X_0)\right] - \int_0^{s'-s} \mathbb{E}_y[P_t[L[f]](X_u)]\,du \\ &= P_{t+s'-s}[f](y) - P_t[f](y) - \int_0^{s'-s} P_{t+u}[L[f]](y)\,du. \end{aligned}$$

Taking into account (D.2), the last integral is equal to

$$\int_0^{s'-s} \partial_u P_{t+u}[f](y)\,du = P_{t+s'-s}[f](y) - P_t[f](y)$$

which ends the proof of (D.3). □

The advantage of the above approach is that it is quite sable by optional stopping, as it is the case for martingales. Let us succinctly give a simple example in the spirit of Section 2.

Assume that in the above framework, V is a metric space, endowed with its Borelian measurable structure, and that Ω is the set of continuous trajectories $\mathcal{C}(\mathbb{R}_+, V)$. Furthermore, we suppose that P is **Fellerian**, in the sense that it preserves $\mathcal{C}_\mathrm{b}(V)$, the set of bounded and continuous real functions on V.

Let be given $A \subset V$ a closed set. We consider τ the hitting time of A:

$$\tau \quad := \quad \inf\{t \geqslant 0 \,:\, X_t \in A\} \;\in\; \mathbb{R}_+ \sqcup \{+\infty\}.$$

Define the "new" process $\widetilde{X} := (\widetilde{X}_t)_{t\geqslant 0}$ via

$$\forall\, t \geqslant 0, \qquad \widetilde{X}_t \quad := \quad X_{t\wedge\tau}$$

and for $x \in V$, let $\widetilde{\mathbb{P}}_x$ be the image of $\mathbb{P}_x$ by $\widetilde{X}$, it is still a probability measure on $\mathcal{C}(\mathbb{R}_+, V)$. All notions corresponding to $\widetilde{\mathbb{P}} := (\widetilde{\mathbb{P}}_x)_{x\in V}$, which is still a Markovian family, receive a tilde. It appears without difficulty that $\widetilde{\mathcal{R}}$ is the set of functions $\widetilde{f} \in \mathcal{B}$ such that there exists $f \in \mathcal{R}$ with $\widetilde{f}$ coinciding with f on $V\backslash A$. The domain $\mathcal{D}(\widetilde{L})$ is the set of $\widetilde{f} \in \widetilde{\mathcal{R}}$ such that there exists $f \in \mathcal{D}(L)$ with $\widetilde{f}$ coinciding with f on $V\backslash A$. In addition, we have

$$\forall\, x \in V, \qquad \widetilde{L}[\widetilde{f}](x) \quad = \quad \begin{cases} L[f](x) & \text{, when } x \notin A \\ 0 & \text{, when } x \in A. \end{cases}$$

This expression does not depend on the choice of f, due to the fact that $\mathbb{P}$ is a diffusion, i.e. that $\Omega = \mathcal{C}(\mathbb{R}_+, V)$, which implies that L is a local operator (see for instance Theorem 7.29 of Schilling and Partzsch [26], they are working with Euclidean spaces, but the result can be extended to metric spaces).

According to (D.2) and Proposition D.3, we get

$$\forall\, \widetilde{f} \in \mathcal{D}(\widetilde{L}),\, \forall\, x \in V,\, \forall\, t \geqslant 0 \qquad \partial_t \widetilde{P}_t[\widetilde{f}](x) \quad = \quad \widetilde{P}_t[\widetilde{L}[\widetilde{f}]](x) \;=\; \widetilde{L}[\widetilde{P}_t[\widetilde{f}]](x).$$

Such relations are not so obvious if we had chosen to work in a Banach setting (cf. e.g. the book of Yosida [29]), considering for instance semi-groups acting on the space $\mathcal{C}_\mathrm{b}(V)$ (endowed with the supremum norm), since in general $\widetilde{L}$ would not naturally take values in $\mathcal{C}_\mathrm{b}(V)$.

Appendix E. An Itô-Tanaka formula

Let M be a d-dimensional Riemannian manifold and $D \subset M$ a compact and connected domain with C^2 boundary ∂D, and S be the regular skeleton of D, and $\rho^+_{\partial D}$ the signed distance to ∂D, which is positive inside D and negative outside D. The notations will be the same as in Appendix A.

Proposition E.1. *Let X_t a Brownian motion in M. We have the following Itô-Tanaka formula :*

$$d\rho^+_{\partial D}(X_t) = \langle N^D(X_t), dX_t\rangle - \frac{1}{2}h^D(X_t)dt - \sin\left(\theta^S(X_t)\right)\, dL^S_t(X),$$

in the above formula, $N^D(x) = \nabla\rho^+_{\partial D}(x)$ and $-h^D(x) = \Delta\rho^+_{\partial D}(x)$ for $x \notin S$, and define to be 0 *elsewhere, $L^S_t(X)$ is the local time defined as in* (3.11).

Proof. The formula is a consequence of the Itô formula outside the skeleton. Since the non regular part of the skeleton has Hausdorff dimension smaller than or equal to $d-2$, it is not visited by the Brownian motion. So we only focus on the regular skeleton. For all $x \in S$, the distance to the boundary is the minimum of two C^2 functions f, g defined on some neighborhood U of x in M. The function f (resp. g) is the distance function to a piece of ∂D containing y_1 (resp. y_2) as in (A.2). We have locally,

$$\rho^+_{\partial D} = f \wedge g = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|.$$

Using Itô formula and Tanaka formula we have

$$\begin{aligned} d\rho^+_{\partial D}(X_t) &= \frac{1}{2}\Big(\frac{1}{2}\Delta(f+g)(X_t)dt + \langle \nabla(f+g)(X_t), dX_t\rangle\Big) \\ &\quad - \frac{1}{2}\Big(\mathrm{sign}((f-g)(X_t))d((f-g)(X_t)) + dL^{0,+}_t((f-g)(X_\cdot))\Big), \end{aligned}$$

where $L^{0,+}_t((f-g)(X_\cdot)) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_0^t \mathbb{1}_{[0,\varepsilon]}((f-g)(X_s))d\langle (f-g)(X), (f-g)(X)\rangle_s$. Since locally $S = \{f-g=0\}$ and $\mu(S) = 0$, we have

$$d\rho^+_{\partial D}(X_t) = \frac{1}{2}\mathbb{1}_{X_t\notin S}\Delta\rho^+_{\partial D}(X_t)dt + \mathbb{1}_{X_t\notin S}\langle \nabla\rho^+_{\partial D}(X_t), dX_t\rangle - \frac{1}{2}dL^{0,+}_t((f-g)(X_\cdot)).$$

After changing the role of f and g we get

$$d\rho^+_{\partial D}(X_t) = \frac{1}{2}\mathbb{1}_{X_t\notin S}\Delta\rho^+_{\partial D}(X_t)dt + \mathbb{1}_{X_t\notin S}\langle \nabla\rho^+_{\partial D}(X_t), dX_t\rangle - \frac{1}{2}dL^{0}_t((f-g)(X_\cdot)), \tag{E.1}$$

where

$$L^0_t((f-g)(X_\cdot)) = \lim_{\varepsilon\to 0^+}\int_0^t \frac{1}{2\varepsilon}\mathbb{1}_{[-\varepsilon,\varepsilon]}((f-g)(X_s))\|\nabla(f-g)\|^2(X_s)\, ds.$$

In Appendix A it is shown that for $x \in S$, $\|\nabla(f-g)(x)\| = 2\sin\big(\theta^S(x)\big)$.

Using the flow $\frac{d}{dt}\gamma(t) = -\frac{\nabla(f-g)(\gamma(t))}{\|\nabla(f-g)(\gamma(t))\|^2}$ that starts at $y \in U$, we get

$$\{y \in M,\ \text{s.t. } |f-g|(y) \leqslant \varepsilon\} \subset \{y\in M,\ \text{s.t. } |d_S(y)| \leqslant \frac{\varepsilon}{2\sin\left(\theta^S(\gamma(g(y)))\right)} + o(\varepsilon)\},$$

where d_S is the distance to S. On the other hand, using the minimal geodesic from S to $y \in U$ we get

$$\{y\in M,\ \text{s.t. } |d_S(y)| \leqslant \varepsilon\} \subset \{y \in M,\ \text{s.t. } |f-g|(y) \leqslant 2\varepsilon \sin\big(\theta^S(P^S(y))\big) + o(\varepsilon)\}.$$

Hence

$$dL^0_t((f-g)(X_\cdot)) = 2\sin\big(\theta^S(X_t)\big)\, L^S_t(X_\cdot).$$

Together with (E.1), this yield the Proposition. □

Appendix F. Uniqueness in law of $\tilde{\mathcal{L}}$ diffusion

Let us consider the following generator $\widehat{\mathscr{L}}$ of a stochastic modified mean curvature flow. The action of this generator and its carré du champs on elementary observables are defined as follows. For any smooth function k on M, consider the mapping F_k on $\mathcal{D}^{2+\alpha}$ defined by

$$\forall\ D \in \mathcal{D}^{2+\alpha}, \qquad F_k(D) \quad := \quad \int_D k\, d\mu.$$

For any $k, g \in \mathcal{C}^\infty(M)$ and any $D \in \mathcal{D}^{2+\alpha}$,

$$\left\{\begin{array}{rcl} \widehat{\mathscr{L}}[F_k](D) &:=& -\frac{1}{2}\underline{\mu}^{\partial D}(\langle \nabla k, N^D\rangle) = F_{\frac{1}{2}\Delta k}(D) \\ \Gamma_{\widehat{\mathscr{L}}}[F_k, F_g](D) &:=& \int_{\partial D} k\, d\underline{\mu} \int_{\partial D} g\, d\underline{\mu}. \end{array}\right. \tag{F.1}$$

Note that $\widehat{\mathscr{L}}$ has the same carré du champs as the carré du champs associated to $\widetilde{\mathscr{L}}$. From now the generator $\widehat{\mathscr{L}}$ is defined as in (2.15).

Proposition F.1. *The martingale problem associated $\widehat{\mathscr{L}}$ is well-posed.*

Proof. We have already shown the existence result in [10], so it remains to prove the uniqueness in law. Let us first consider the two-dimensional Euclidean case, namely $M = \mathbb{R}^2$. For all $\lambda \in \mathbb{R}$ and for any function $k_\lambda \in \mathrm{vect}(e^{\lambda x}, e^{\lambda y})$ we have $\frac{1}{2}\Delta k_\lambda(x,y) = \frac{\lambda^2}{2} k_\lambda(x,y)$. Let $f_\lambda((x,y),D) := k_\lambda(x,y)F_{k_\lambda}(D)$, for $(x,y) \in \mathbb{R}^2$ and $D \in \mathcal{D}^{2+\alpha}$. This function satisfies the following property:

$$\begin{aligned}
\widehat{\mathscr{L}} f_\lambda((x,y),D) &= k_\lambda(x,y)\widehat{\mathscr{L}} F_{k_\lambda}(D) \\
&= k_\lambda(x,y) F_{\frac{1}{2}\Delta k_\lambda}(D) \\
&= k_\lambda(x,y) F_{\frac{\lambda^2}{2} k_\lambda}(D) \\
&= \frac{\lambda^2}{2} k_\lambda(x,y) F_{k_\lambda}(D) \\
&= \frac{1}{2}\Delta k_\lambda(x,y) F_{k_\lambda}(D) \\
&= \frac{1}{2}\Delta f_\lambda((x,y),D).
\end{aligned}$$

Let $(X_t)_{t\geqslant 0}$ be a $\mathbb{R}^2$-valued Brownian motion that starts at $X_0 = (x_1,x_2) \in \mathbb{R}^2$ and $(\hat{D}_t)_{t\geqslant 0}$ a $\widehat{\mathscr{L}}$ diffusion that starts at D_0 independent of $(X_t)_{t\geqslant 0}$. Even if we stop the diffusion, we can assume that its lifetime is infinite and we add indicators as described in Appendix D. For all $0 \leqslant s \leqslant t$, we have

$$df_\lambda(X_{t-s},\hat{D}_s) \overset{m}{=} -\frac{1}{2}\Delta f_\lambda(X_{t-s},\hat{D}_s)ds + \widehat{\mathscr{L}} f_\lambda(X_{t-s},\hat{D}_s)ds \overset{m}{=} 0.$$

Hence for all $\lambda \in \mathbb{R}$ we have

$$\mathbb{E}[f_\lambda(X_t,D_0)] = \mathbb{E}[f_\lambda(X_0,\hat{D}_t)]. \tag{F.2}$$

Since the left hand side of the above equation does not depend on the $\widehat{\mathscr{L}}$ diffusion, we get that for any $\widehat{\mathscr{L}}$ diffusion $(\tilde{D}_t)_{t\geqslant 0}$ that starts at D_0 :

$$\mathbb{E}[f_\lambda(X_0,\hat{D}_t)] = \mathbb{E}[f_\lambda(X_0,\tilde{D}_t)],$$

and so

$$\mathbb{E}[F_{k_\lambda}(D_t)] = \mathbb{E}[F_{k_\lambda}(\tilde{D}_t))].$$

In order to apply Theorem 4.2 of [12], we have to show that the above equation characterizes the law of the one-dimensional distribution, i.e. we have to show that (F_{k_λ}) is separating in the space of probability measures on $\mathcal{D}^{2+\alpha}$. This is equivalent to separate domains. Let $A, B \in \mathcal{D}^{2+\alpha}$ such that $F_{k_\lambda}(A) = F_{k_\lambda}(B)$ for all $\lambda \in \mathbb{R}$ and $k_\lambda \in \langle e^{\lambda x}, e^{\lambda y}\rangle$, we have for all λ:

$$\int_A k_\lambda(x,y)d\mu = \int_B k_\lambda(x,y)d\mu.$$

After successive derivations in λ and evaluation at $\lambda = 0$, we get for all $n \in \mathbb{N}$

$$\int_A x^n d\mu = \int_B x^n d\mu,$$

$$\int_A y^n d\mu = \int_B y^n d\mu.$$

The above computations could be done also for $\tilde{k}_{\lambda_1,\lambda_2} = e^{\lambda_1 x+\lambda_2 y}$, since $\frac{1}{2}\Delta\tilde{k}_{\lambda_1,\lambda_2} = \frac{\lambda_1^2+\lambda_2^2}{2}\tilde{k}_{\lambda_1,\lambda_2}$, and after derivations in λ_1, λ_2 and evaluating at $(0,0)$ we get that for all $n, m \in \mathbb{N}$:

$$\int_A x^n y^m d\mu = \int_B x^n y^m d\mu,$$

hence, using the boundary regularity, we get $A = B$.

We could also apply Stone-Weierstrass' theorem to the function algebra generated by the mappings $(x,y) \mapsto e^{\lambda_1 x}$ and $(x,y) \mapsto e^{\lambda_2 y}$.

The proof is the same for all Euclidean spaces.

If M is a compact manifold let

$$f_{\lambda_i}(X,D) := k_{\lambda_i}(X)F_{k_{\lambda_i}}(D),$$

where λ_i is an eigenvalue of $\frac{1}{2}\Delta$ and k_i is the associated eigenfunction (respectively the Neumann eigenvalue). By the same computation as above (F.2) is also valid for the boundary reflecting Brownian motion), to get the conclusion we have to show that $(F_{k_{\lambda_i}})_i$ separates domains. Since $(k_{\lambda_i})_i$ is an orthonormal basis of $L^2(\mu)$ we get that if $A, B \in \mathcal{D}^{2+\alpha}$ be such that for all i,

$$F_{k_{\lambda_i}}(A) = F_{k_{\lambda_i}}(B)$$

i.e $\langle \mathbb{1}_A, k_{\lambda_i}\rangle_{L^2} = \langle \mathbb{1}_B, k_{\lambda_i}\rangle_{L^2}$, then $\mathbb{1}_A \overset{L^2}{=} \mathbb{1}_B$ hence $A = B$.

For the complete manifold M, let Ω_k be an exhaustion of M with a regular boundary such that $D_0 \subset \Omega_k$, and stop the $\widehat{\mathscr{L}}$ diffusion when it hit Ω_k^c and use the above result for the manifold with boundary Ω_k, we get the result by localization.

□

Proposition F.2. *The martingale problem associated to $\mathscr{L}$ is well-posed.*

Proof. Let D_t be a $\mathscr{L}$ diffusion that starts at D_0, defined on $(\Omega, \mathcal{F}^D, \mathbb{Q})$. We first recall that there exist an enlargement of the probability space such that it carries a one dimensional Brownian motion B such that for all $k \in C^\infty(M)$

$$F_k(D_t) = F_k(D_0) + \int_0^t \mathscr{L}[F_k](D_s)\,ds + \int_0^t \sqrt{\Gamma_{\mathscr{L}}[F_k,F_k]}(D_s)\,dB_s \tag{F.3}$$

where $\sqrt{\Gamma_{\mathscr{L}}[F_k,F_k]}(D) := \int_{\partial D} k\,d\sigma$, this is actually Proposition 53 in [10]. Note that this procedure of enlargement (Theorem 1.7 chapter V in [24]) could be done by gluing the same independent Brownian motion for each $(\Omega, \mathcal{F}^D, \mathbb{Q})$. We denote by $(\tilde{\Omega}, \tilde{\mathcal{F}}^D, \tilde{\mathbb{Q}})$ the enlarged probability space. Since $\mathscr{L}$ is an h-transform of $\widehat{\mathscr{L}}$ namely

$$\mathscr{L}[F_k] = \widehat{\mathscr{L}}[F_k] + \frac{\Gamma_{\widehat{\mathscr{L}}}(F_1,F_k)}{F_1},$$

equation (F.3) becomes in a differential form

$$dF_k(D_t) - \widehat{\mathscr{L}}[F_k](D_t)dt = (\int_{\partial D} k\,d\sigma)\big(dB_t + \frac{\mu^{\partial D_t}(\partial D_t)}{\mu(D_t)}dt\big). \tag{F.4}$$

Let

$$M_t = e^{-\int_0^t \langle \frac{\mu^{\partial D_s}(\partial D_s)}{\mu(D_s)}, dB_s\rangle - \frac{1}{2}\int_0^t \left(\frac{\mu^{\partial D_s}(\partial D_s)}{\mu(D_s)}\right)^2 ds},$$

$$\mathbb{P}_{|\mathcal{F}_t} = M_t \tilde{\mathbb{Q}}_{|\mathcal{F}_t}.$$

Using Girsanov transform, D_t is solution of the $\widehat{\mathscr{L}}$ martingale problem on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}^D, \mathbb{P})$. Since $\tilde{\mathbb{Q}} = M^{-1}\mathbb{P}$ we get the uniqueness in law of the $\mathscr{L}$ diffusion by Proposition F.1. □

APPENDIX G. CONVERGENCE IN LAW: A KEY LEMMA

This Appendix is devoted to the adaptation to some domain-valued sequences of processes, of Lemma 4 in [30], which states stability of some time integrals under convergence in law.

Lemma G.1. *Let $\tilde{\mathscr{F}} := \tilde{\mathscr{F}}^{\alpha,\varepsilon}$. We endow the set of continuous paths $\mathscr{C}\left([0,\infty), M \times \tilde{\mathscr{F}}\right)$ with the two dissimilarity measures d_β, $\beta \in \{0, \alpha\}$, defined as:*

$$d_\beta\left((x^1, D^1), (x^2, D^2)\right) = \sup_{t\geqslant 0} \rho(x^1(t), x^2(t)) + \sup_{t\geqslant 0} d_{\beta,\tilde{\mathscr{F}}}(D^1(t), D^2(t)), \tag{G.1}$$

where for two domains D and D'

$$d_{\beta,\tilde{\mathscr{F}}}(D, D') = \begin{cases} d_{\beta,D}(D, D') \wedge d_{\beta,D'}(D', D) \wedge \varepsilon & \text{if } H(D, D') < \varepsilon \\ \varepsilon & \text{otherwise.} \end{cases} \tag{G.2}$$

Here $H(D, D')$ is the Hausdorff distance between D and D' and the distance $d_{\beta,D}$ is defined in (2.3).

Let $(X_t^n, D_t^n, \tau_\varepsilon^n)_{t\geqslant 0} := (X_t^{\delta_n}, D_t^{\delta_n}, \tau_\varepsilon^{\delta_n})_{t\geqslant 0}$ a subsequence of (3.17) *converging in law to the limit defined in* (3.18) *for the product of d_α and the Euclidean distance in $\mathbb{R}_+$.*

Let $f_n : (x, D) \mapsto f_n(x, D)$ and $f : (x, D) \mapsto f(x, D)$ be maps on $M \times \tilde{\mathscr{F}}$ with values in some Euclidean space, and U an open set in $M \times \tilde{\mathscr{F}}$ for d_0. Assume that:

(i) *the random variables $\int_0^\infty |f_n(X_s^n, D_s^n)|^p \, ds$ are uniformly bounded in probability for some $p > 1$,*
(ii) *in the open set U, the functions f_n converge locally uniformly to f with respect to d_0, and are d_0-continuous,*
(iii) *for a.e. $t \geqslant 0$, $(X_t, D_t) \in U$.*

Then $\left(X_t^n, D_t^n, \int_0^t f_n(X_s^n, D_s^n)\, ds\right)_{t\geqslant 0}$ converges in law to $\left(X_t, D_t, \int_0^t f(X_s, D_s)\, ds\right)_{t\geqslant 0}$ for $(d_\alpha, |\cdot|)$.

Remark G.2. In the applications we will always take

$$U = \left\{(x, D) \in M \times \tilde{\mathscr{F}},\ x \in D\backslash S(D)\right\}, \tag{G.3}$$

which is easily seen to be d_0-open thanks to Assumption 3.1 on $\tilde{\mathscr{F}}$.

Proof. We will follow the proof of Lemma 4 in [30], but with several differences due to infinite dimensional spaces. Set for $n \in \mathbb{N}$, $t \geqslant 0$,

$$A_t^n := \int_0^t f_n(X_s^n, D_s^n)\, ds, \quad A_t := \int_0^t f(X_s, D_s)\, ds. \tag{G.4}$$

Condition (i) implies that the processes A^n are tight. To get the conclusion il is sufficient to show that all the converging subsequences have the same limit. So assume that

$$(X_t^n, D_t^n, A_t^n)_{t\geqslant 0} \xrightarrow{\mathscr{L}} (X_t, D_t, a_t)_{t\geqslant 0}\,. \tag{G.5}$$

and let us prove that $(a_t)_{t\geqslant 0} = (A_t)_{t\geqslant 0}$. By Skorohod theorem we may realize all processes

$$(X_t^n, D_t^n, A_t^n, X_t, D_t, a_t)_{t\geqslant 0} \tag{G.6}$$

on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$ in such a way that

$$(Z_t^n)_{t\geqslant 0} := (X_t^n, D_t^n, A_t^n)_{t\geqslant 0} \xrightarrow{\text{a.s.}} (X_t, D_t, a_t)_{t\geqslant 0} =: (Z_t)_{t\geqslant 0}. \tag{G.7}$$

This means that $Z_t^n \to Z_t$ a.s. uniformly in $t \geqslant 0$.

Fix $\omega \in \Omega$. Let $t > 0$ be such that $(X_t(\omega), D_t(\omega)) \in U$. For some $\varepsilon' > 0$ we have $(X_s(\omega), D_s(\omega)) \in U$ for all $s \in [t-\varepsilon', t+\varepsilon']$. The set

$$S := \left\{(X_s(\omega), D_s(\omega)), \quad s \in [t-\varepsilon', t+\varepsilon']\right\} \tag{G.8}$$

is d_α-compact in $M \times \tilde{\mathscr{F}}$, so it has a d_α-neighbourhood V included in U of the form

$$V = \left\{(x, D) \in M \times \tilde{\mathscr{F}}, \quad d_\alpha\left((x, D), S\right) \leqslant \varepsilon''\right\}. \tag{G.9}$$

for some small enough $\varepsilon'' > 0$. For n sufficiently large, $(X_s^n(\omega), D_s^n(\omega)) \in V$ for all $s \in [t-\varepsilon', t+\varepsilon']$. On the other hand V is bounded for the distance d_α. This implies by Arzela-Ascoli theorem that it is compact for the distance d_0. We have the two following facts, the first one being an assumption on the f_n and f, the second one being a consequence of the d_0-compactness of V

(a) $f_n \to f$ as $n \to \infty$ uniformly in (V, d_0);
(b) f is uniformly continuous in (V, d_0).

Then

$$\begin{aligned}
&\sup_{s\in[t-\varepsilon,t+\varepsilon]} |f_n(X_s^n(\omega), D_s^n(\omega)) - f(X_s(\omega), D_s(\omega))| \\
\leqslant &\sup_{s\in[t-\varepsilon,t+\varepsilon]} |f_n(X_s^n(\omega), D_s^n(\omega)) - f(X_s^n(\omega), D_s^n(\omega))| \\
&+ \sup_{s\in[t-\varepsilon,t+\varepsilon]} |f(X_s^n(\omega), D_s^n(\omega)) - f(X_s(\omega), D_s(\omega))|.
\end{aligned}$$

Both terms in the right converge to 0, the first one by (a) and the second one by (b). So we have by (G.7) and the above calculation

$$\left\{\begin{array}{rl}
(A_s^n(\omega))_{s\in[t-\varepsilon,t+\varepsilon]} & \to (a_s(\omega))_{s\in[t-\varepsilon,t+\varepsilon]} \\
((A_s^n(\omega))' = f_n(X_s^n(\omega), D_s^n(\omega)))_{s\in[t-\varepsilon,t+\varepsilon]} & \to (f(X_s(\omega), D_s(\omega)))_{s\in[t-\varepsilon,t+\varepsilon]}
\end{array}\right. \tag{G.10}$$

both uniformly in $s \in [t-\varepsilon, t+\varepsilon]$. This implies that $a_s(\omega)$ is differentiable in $(t-\varepsilon, t+\varepsilon)$ with derivative $f(X_s(\omega), D_s(\omega))$ and in particular at t.

We have that for all $t \geqslant 0$, $(X_t(\omega), D_t(\omega)) \in U$ a.s.. So for all $t \geqslant 0$,

$$\frac{d}{dt} a_t(\omega) = f(X_t(\omega), D_t(\omega)) \quad \text{a.s.}. \tag{G.11}$$

This implies that ω a.s.

$$\frac{d}{dt} a_t(\omega) = f(X_t(\omega), D_t(\omega)) \quad \text{for a.e. } t. \tag{G.12}$$

On the other hand we know by [18] Theorem 10 that $(a_t)_{t\geqslant 0}$ is absolutely continuous :

$$a_t(\omega) = \int_0^t \ell_s(\omega)\, ds. \tag{G.13}$$

By Lebesgue theorem, ω a.s., for a.e. $t \geqslant 0$

$$\lim_{\varepsilon\searrow 0} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} |\ell_s(\omega) - \ell_t(\omega)|\, ds = 0. \tag{G.14}$$

Equalities (G.12) and (G.13) imply that ω a.s.

$$\lim_{\varepsilon\searrow 0} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \ell_s(\omega)\, ds = f(X_t(\omega), D_t(\omega)) \quad \text{for a.e. } t. \tag{G.15}$$

On the other hand

$$\left|\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\ell_s(\omega)-\ell_t(\omega)\,ds\right|\leqslant\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}|\ell_s(\omega)-\ell_t(\omega)|\,ds$$

so (G.14) implies that ω a.s. for a.e. $t\geqslant 0$

$$\lim_{\varepsilon\searrow 0}\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\ell_s(\omega)\,ds=\ell_t(\omega). \tag{G.16}$$

Consequently, using (G.12) and (G.16), we get ω a.s. for a.e. $t\geqslant 0$

$$\ell_t(\omega)=f(X_t(\omega),D_t(\omega)). \tag{G.17}$$

Integrating we get ω-a.s. for all $t\geqslant 0$

$$a_t(\omega)=A_t(\omega)=\int_0^t f(X_s(\omega),D_s(\omega))\,ds. \tag{G.18}$$

This together with (G.4) proves the lemma. □

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UNIV. BORDEAUX, CNRS, BORDEAUX INP,
INSTITUT DE MATHÉMATIQUES DE BORDEAUX, UMR 5251, F. 33405, TALENCE, FRANCE
Email address: marc.arnaudon@math.u-bordeaux.fr

INSTITUT ÉLIE CARTAN DE LORRAINE, UMR 7502
UNIVERSITÉ DE LORRAINE AND CNRS
Email address: kolehe.coulibaly@univ-lorraine.fr

INSTITUT DE MATHÉMATIQUES DE TOULOUSE, UMR 5219,
TOULOUSE SCHOOL OF ECONOMICS, UMR 5314,
CNRS AND UNIVERSITÉ DE TOULOUSE
Email address: laurent.miclo@math.cnrs.fr