Set-valued intertwining duality for multi-dimensional diffusions

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Based on joint works with
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Plan of the talk

1. Multi-dimensional diffusions
2. Stochastic perturbations of mean curvature flows
3. Coupling primal and dual diffusions
4. Planar convex-domain-valued duals
5. Cut-off
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Plan of the talk

Riemannian manifolds

Consider a (complete) Riemannian manifold V of dimension $n \geq 2$. The Laplace-Beltrami operator Δ associated to V is the generator of the Brownian motion $X := (X_t)_{t\geq 0}$ on V (up to speeding of time by 2). The generator Δ is reversible with respect to the Riemannian measure μ. When V is compact, μ can be normalized into a probability measure.

We would like to construct set-valued intertwining duals for X, or even more generally for hypo-elliptic diffusions on V. Among the long-term goals:

- to give another probabilistic proof of Hörmander's density theorems,
- to construct strong stationary times when V is compact,
- to provide probabilistic interpretations of curvature conditions for convergence.

Evolving domains

$\mathcal{D}$ will stand for a set of compact and connected subdomains of V with a smooth boundary and which coincide with the closure of their interior. Consider Λ the Markov kernel from $\mathcal{D}$ to V, corresponding to the conditioning of μ. Our general road map is:

- to find a Markov generator $\mathcal{G}$ on $\mathcal{D}$ intertwined with Δ through Λ (**algebraic intertwining relation**):

$$\mathcal{G}\Lambda \;=\; \Lambda\Delta \tag{1}$$

- To associate to $\mathcal{G}$, Markov evolutions of domains $(\mathfrak{D}_t)_{t\in[0,\zeta)}$, at least for a positive stopping time ζ (depending in particular on the starting $\mathfrak{D}_0$).
- To couple the evolutions X and $(\mathfrak{D}_t)_{t\in[0,\zeta)}$ to get a **probabilistic intertwining relation**: for any stopping time $T \leq \zeta$,

$$\begin{cases} \mathcal{L}(X_T|\mathfrak{D}_{[0,T]}) &=& \Lambda(\mathfrak{D}_T,\cdot) \\ \mathcal{L}(\mathfrak{D}_{[0,T]}|X) &=& \mathcal{L}(\mathfrak{D}_{[0,T]}|X_{[0,T]}) \end{cases}$$

Plan of the talk

Mean curvature flow

Let $D_0 \in \mathcal{D}$ be given and denote ∂D_0 its boundary. To any $y \in \partial D_0$, associate the **inward unitary normal vector** $\nu^{D_0}(y)$ and the **mean curvature** $\rho^{D_0}(y)$. They can be seen respectively as the gradient and the Beltrami-Laplacian of the signed distance to the boundary (positive inside D_0 and negative outside).
At least for small time $t \geq 0$, it is possible to make the domain evolve according to the **mean curvature flow**:

$$\forall\ y_t \in \partial D_t, \qquad \dot{y}_t \ = \ \rho^{D_t}(y_t)\nu^{D_t}(y_t)$$

The domains D_t have a tendency to round up and to shrink to a point in finite time.

Stochastic modification of the mean curvature flow

Modify the previous deterministic evolution into an infinite-dimensional **stochastic differential equation** on $(\mathfrak{D}_t)_{t\in[0,\zeta)}$: for any $t \in [0,\zeta)$ and $Y_t \in \partial\mathfrak{D}_t$,

$$dY_t \quad = \quad \left(\sqrt{2}dB_t + \left(\rho^{\mathfrak{D}_t}(Y_t) - 2\frac{\underline{\mu}(\partial\mathfrak{D}_t)}{\mu(\mathfrak{D}_t)}\right) dt\right) \nu^{\mathfrak{D}_t}(Y_t) \quad (2)$$

where $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion and $\underline{\mu}$ is $(n-1)$-dimensional Hausdorff measure. The global isoperimetric ratio $\underline{\mu}(\partial\mathfrak{D}_t)/\mu(\mathfrak{D}_t)$ counters the effect of the mean curvature and prevents the evolution to collapse to a singleton.

Theorem

Starting from any $\mathfrak{D}_0 \in \mathcal{D}$, it is possible to define $(\mathfrak{D}_t)_{t\in[0,\zeta)}$ for some lifetime $\zeta > 0$, solving (2) and whose generator $\mathcal{G}$ satisfies the intertwining relation (1).

The proof is based on an extension of the Doss-Sussman method to the infinite dimensional setting of $\mathcal{D}$.

Observables

To define rigorously the generator $\mathcal{G}$, we must have at our disposal "nice observables".

• **Elementary observables**:

$$F_f \ : \mathcal{D} \ni D \quad \mapsto \quad F_f(D) := \int_D f \, d\mu$$

associated to the functions $f \in \mathcal{C}^\infty(V)$, the space of smooth mappings on V.

• **Composite observables**: the functionals of the form $\mathfrak{F} := \mathfrak{f}(F_{f_1}, \dots, F_{f_n})$, where $n \in \mathbb{Z}_+$, $f_1, \dots, f_n \in \mathcal{C}^\infty(V)$ and $\mathfrak{f} \, : \, \mathcal{R} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ mapping, with $\mathcal{R}$ an open subset of $\mathbb{R}^n$ containing the image of $\mathcal{D}$ by $(F_{f_1}, \dots, F_{f_n})$.

Generator $\mathcal{G}$

On elementary observables:

$$\forall\ D \in \mathcal{D}, \quad \mathcal{G}[F_f](D) \ := \ \int_D G[f]\, d\mu + 2\frac{\mu(\partial D)}{\mu(D)} \int_{\partial D} f\, d\underline{\mu}$$

For the extension to composite observables, the **carré du champs** is also required:

$$\forall\ D \in \mathcal{D}, \quad \Gamma_{\mathcal{G}}[F_f, F_g](D) \ = \ \left(\int_{\partial D} f\, d\underline{\mu}\right)\left(\int_{\partial D} g\, d\underline{\mu}\right)$$

Then on composite observables $\mathfrak{F}$ as above:

$$\mathcal{G}[\mathfrak{F}] = \sum_{j \in [\![1,n]\!]} \partial_j \mathfrak{f}(F_{f_1}, \ldots, F_{f_n}) \mathcal{G}[F_{f_j}] + \sum_{k,l \in [\![1,n]\!]} \partial_{k,l} \mathfrak{f}(F_{f_1}, \ldots, F_{f_n}) \Gamma_{\mathcal{G}}[F_{f_k}, F_{f_l}]$$

(consequence of the continuity of the trajectories of $\mathfrak{D}$).

Euclidean spaces

When V has constant curvature, (2) can be solved for all times, even starting from a singleton $\{x_0\}$. In this situation D_t is a ball centered at x_0 and of radius R_t solving the following stochastic differential equations:

- Euclidean space $\mathbb{R}^n$ (null curvature):

$$dR_t \;=\; \sqrt{2}dB_t + \frac{n+1}{R_t}dt$$

(Bessel process of dimension $n+2$, up to scaling time by $1/2$).

Furthermore when $n = 2$, it can be proved that starting from any $D \in \mathcal{D}$, the normalized domain $\mathfrak{D}_t/\sqrt{\mu(\mathfrak{D}_t)}$ converges to the disk of diameter $1/\sqrt{\pi}$ for large times (under the restriction that (2) can be solved for any time $t \geq 0$).

Constant curvature

- Spherical space $\mathbb{S}^n$ (positive curvature=1):

$$dR_t = \sqrt{2}dB_t + \left(\frac{2\sin^{n-1}(R_t)}{\int_0^{R_t} \sin^{n-1}(z)\,dz} - (n-1)\cot(R_t) \right) dt$$

Enable to construct strong stationary times (leading to the cut-off phenomenon in separation with respect to the dimension).

- Poincaré's model of hyperbolic space $\mathbb{H}^n$ (negative curvature=-1)

$$dR_t = \sqrt{2}dB_t + \left(\frac{2\sinh^{n-1}(R_t)}{\int_0^{R_t} \sinh^{n-1}(z)\,dz} - (n-1)\coth(R_t) \right) dt$$

Elliptic diffusions on manifolds

The above constructions can be extended to any elliptic second order differential generator G on a manifold V admitting an invariant measure μ. The definition of the generator $\mathcal{G}$ is exactly the same, but there is a difference in the description of the infinitesimal evolution of the boundaries.
The operator G induces on V a Riemannian structure so that $G = \triangle + b$, where b is a vector field. Write $\exp(U)$ the density of μ with respect to the Riemannian measure. Then b admits a (weighted Hodge) decomposition $\nabla U + \beta$. The s.d.e. (2) must be replaced by

$$dY_t = \left(\left(\rho^{\mathfrak{D}_t}(Y_t) - 2\frac{\mu(\partial \mathfrak{D}_t)}{\mu(\mathfrak{D}_t)} + \left\langle \beta - \nabla U, \nu^{\mathfrak{D}_t} \right\rangle (Y_t) \right) dt + \sqrt{2} dB_t \right) \nu^{\mathfrak{D}_t}(Y_t)$$

Pitman property

Up to the stopping time until which everything is well-defined, we always have:

Theorem

The volume process $(\mu(\mathfrak{D}_{\theta_t}))_t$ is a Bessel process of dimension 3, where the time change is given by

$$2\int_0^{\theta_t} (\underline{\mu}(\partial\mathfrak{D}_s))^2\, ds \quad = \quad t$$

The ubiquity of the Bessel-3 process suggests that hypoellipticity in general could be investigated in a similar probabilistic way.

Plan of the talk

Minimal coupling

Here is one way to get an intertwining coupling between X and $(\mathfrak{D}_t)_{t\in[0,\zeta)}$, given $\mathfrak{D}_0 \in \mathcal{D}$:

Theorem

There exists a pair $(X_t, \mathfrak{D}_t)_{t\in[0,\zeta)}$ *of probabilistically intertwined processes satisfying, for any* $t \in [0,\zeta)$ *and* $Y_t \in \partial\mathfrak{D}_t$,

$$dY_t \;=\; \left(dW_t + \frac{1}{2}\rho^{\mathfrak{D}_t}(Y_t)dt - dL_t^{\partial\mathfrak{D}_t}(X)\right)\nu^{\mathfrak{D}_t}(Y_t)$$

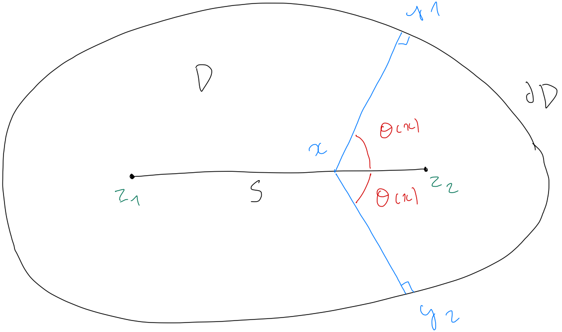
where $X := (X_t)_{t\geq 0}$ *is a* V*-valued Brownian motion started with the uniform (Riemannian) law on* $\mathfrak{D}_0$, $(W_t)_{t\geq 0}$ *is a real-valued Brownian motion independent of* X, $(L_t^{\partial\mathfrak{D}_t}(X))_{t\geq 0}$ *is the local time of* $(X_t)_{t\geq 0}$ *at the moving boundary* $(\partial\mathfrak{D}_t)_{t\geq 0}$.

Skeletons (1)

To define another coupling, we need some geometric notions.

Denote δ the Riemannian distance and $\delta_{\partial D}$ the distance to ∂D, for a given $D \in \mathcal{D}$.
The **(inward) morphological skeleton** of D: S' the set of points in D where $\delta_{\partial D}$ "does not behave well".
The **(inward) regular skeleton** of D: S the set of regular points of S': for $x \in S$ there are exactly two points $y_1 \neq y_2 \in \partial D$ with $\delta_{\partial D}(x) = \delta(x, y_1) = \delta(x, y_2)$, plus a non-degeneracy property of the distance to some neighborhoods of y_1 and y_2 respectively. For such $x \in S$, define $\theta^S(x)$ the common angle between S and the geodesic curves going from x to y_1 and y_2 respectively.
The set S is a codimension 1 submanifold of M and $S' \setminus S$ has Hausdorff dimension smaller than or equal to $d - 2$.

Skeletons (2)



$S' = [z_1, z_2]$, $S = (z_1, z_2)$

Skeletons (3)

We need to extend the inward unitary normal vector ν^D and the mean curvature ρ^D, a priori only given on ∂D. So consider for any $r \geq 0$,

$$D(r) \quad := \quad \{z \in D \setminus S',\ \delta_{\partial D}(z) \geq r\}$$

The set $D(r)$ is a (possibly empty) manifold with smooth boundary $\partial D(r)$ on which one can define an inward normal ν^D and the mean curvature ρ^D, which are thus now defined on $D \setminus S'$. Equivalently, in the interior of $D \setminus S'$, they are the gradient and the Beltrami-Laplacian of $\delta_{\partial D}$.

The family $(\partial D(r))_{r0}$ provides a **normal foliation** of $D \setminus S'$.

Local times

When given a "nice" $V \times \mathcal{D}$-valued process $(X_t, \mathfrak{D}_t)_{t\in[0,\zeta)}$, we denote $\mathfrak{S} := (\mathfrak{S}_t)_{t\in[0,\zeta)}$, where for any $t \in [0,\zeta)$, $\mathfrak{S}_t$ stands for the regular skeleton of $\mathfrak{D}_t$. The **local time** process $L^{\mathfrak{S}}(X) := (L_t^{\mathfrak{S}}(X))_{t\in[0,\zeta)}$ of X at the moving $\mathfrak{S}$ is given by

$$\forall\ t \in [0,\zeta), \qquad L_t^{\mathfrak{S}}(X) \quad := \quad \lim_{\beta\to 0_+} \frac{1}{2\beta} \int_0^t 1_{\{X_s \in \mathfrak{S}_s^\beta\}}\, ds,$$

where $\mathfrak{S}_s^\beta$ is the β-thickening of $\mathfrak{S}_s$ in both normal directions.

Maximal coupling

Under a technical assumption on the state space of domains $\mathcal{D}$, we have

Theorem

There exists a pair $(X_t, \mathfrak{D}_t)_{t\in[0,\zeta)}$ *of probabilistically intertwined processes, such that the process* $(\mathfrak{D}_t)_{t\in[0,\zeta)}$ *satisfies for any* $t \in [0, \zeta)$ *and* $Y_t \in \partial\mathfrak{D}_t$,

$$dY_t = \left(\left(\frac{1}{2}\rho^{\mathfrak{D}_t}(Y_t) - \rho^{\mathfrak{D}_t}(X_t)\mathbb{1}_{\mathfrak{D}_t\setminus\mathfrak{S}_t}(X_t)\right) dt + \left\langle dX_t, \nu^{\mathfrak{D}_t}(X_t)\right\rangle - 2\sin(\theta^{\mathfrak{S}_t}(X_t))\, dL_t^{\mathfrak{S}_t}(X)\right)\nu^{\mathfrak{D}_t}(Y_t)$$

Another construction

Under the same assumption on $\mathcal{D}$, we also have

Theorem

There exists a pair $(X_t, \mathfrak{D}_t)_{t\in[0,\zeta)}$ of probabilistically intertwined processes, such that the process $(\mathfrak{D}_t)_{t\in[0,\zeta)}$ satisfies for any $t \in [0,\zeta)$ and $Y_t \in \partial\mathfrak{D}_t$,

$$dY_t = \left(\left(\frac{1}{2}\rho^{\mathfrak{D}_t}(Y_t) + \rho^{\mathfrak{D}_t}(X_t)\mathbb{1}_{\mathfrak{D}_t\setminus\mathfrak{S}_t}(X_t) \right) dt - \left\langle dX_t, \nu^{\mathfrak{D}_t}(X_t) \right\rangle \right.$$

$$\left. +2\sin(\theta^{\mathfrak{S}_t}(X_t))\, dL_t^{\mathfrak{S}_t}(X) - 2dL_t^{\partial\mathfrak{D}_t}(X) \right) \nu^{\mathfrak{D}_t}(Y_t)$$

The three preceding theorems follow as limit cases of a more general construction based on certain functionals defined on $V \times \mathcal{D}$, which are deformations of the signed distance from a point to the boundary of the domain.

Plan of the talk

Algebraic planar convex-domain-valued duals

To go further, let us consider the particular case of $V := \mathbb{R}^2$ with $\mathfrak{D}_0$ strictly convex, situation which has been well-studied for the mean curvature flow, see [Gage and Hamilton, 1986]. Assume in addition that $\mathfrak{D}_0$ is symmetric with respect to the action of G_N, the group generated by the reflexion with respect to the x-axis and the rotation of angle $2\pi/N$. It provides us with an example of a true infinite-dimensional algebraic set-valued intertwining dual defined for all times:

Theorem

Assume that $N \geq 7$, then the solution to (2) can be defined with lifetime $\zeta = +\infty$ and remains G_N-symmetric.

This result extends to the first presented coupling:

Theorem

Assume that $N \geq 7$, then the minimal coupling can be defined with lifetime $\zeta = +\infty$.

We were only able to define the second coupling for all times under an additional hypothesis:

Theorem

Assume that $N \geq 7$ and that the initial morphological skeleton $\mathfrak{S}'_0 = G_N H_0$, where H_0 is an horizontal segment $[0, x_0] \times \{0\}$, with $x_0 > 0$. then the maximal coupling can be defined with lifetime $\zeta = +\infty$.

G_N-symmetry (1)

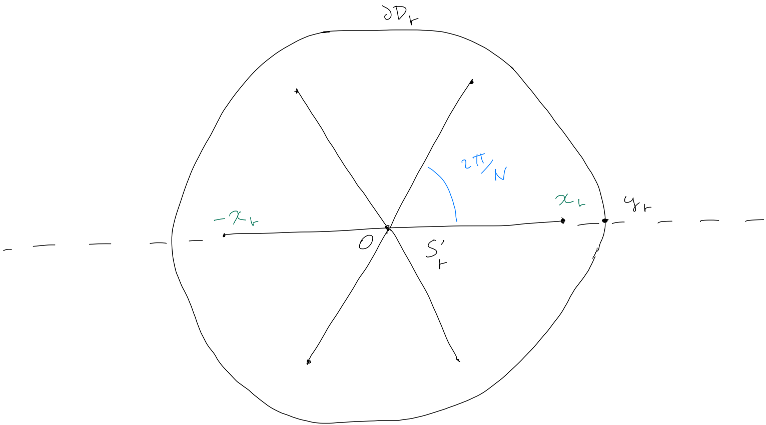
Under the assumption that $\mathfrak{S}'_0 = G_N H_0$, where H_0 is an horizontal segment $[0, x_0] \times \{0\}$, with $x_0 > 0$, one can be more precise about the largest possible lifetime ζ. The algebraic intertwining dual solution of (2) preserves the form of the morphological skeleton: at any time $t \in [0, \zeta)$, $\mathfrak{S}'_t = G_N H_t$ where $H_t = [0, x_t] \times \{0\}$, with $x_t > 0$.

The right endpoint x-coordinate x_t of the skeleton $\mathfrak{S}'_t$ satisfies

$$\frac{dx_t}{dt} = \frac{\delta^2((x_t, 0), y_t)}{2}(\rho^{\mathfrak{D}_t})''(y_t),$$

y_t being the point of $\partial\mathfrak{D}_t$ in the horizontal line with the greatest abscissa, and the second derivative being calculated with curvilinear coordinates on $\partial\mathfrak{D}_t$. Notice that $(\rho^{\mathfrak{D}_t})''(y_t) \leq 0$, proving that the process $\mathfrak{S}_t$ is non-increasing.

G_N-symmetry (2)



Example with $N = 6$

G_N-symmetry (3)

It can be shown that the maximal lifetime is the time when $\partial\mathfrak{D}_t$ meets its skeleton $\mathfrak{S}'_t$. We have no example where $\partial\mathfrak{D}_t$ meets its skeleton $\mathfrak{S}'_t$ in finite time.

An example of an initial set $\mathfrak{D}_0$ with $N = 2$ is an ellipse, the skeleton being the interval between the two foci $(-x_0, 0)$ and $(x_0, 0)$. In this case, for any $t \geq 0$, $\mathcal{S}'_t = [-x_t, x_t] \times \{0\}$ and $\mathcal{S}_t = (-x_t, x_t) \times \{0\}$. When $N \geq 3$, the point $(0, 0)$ does not belong to the regular skeleton.

Concerning the proof of the three previous theorems, the investigation of the entropy of the (positive) mean curvature on the boundary with respect to the uniform distribution plays an important role, via Cage's inequality, as in the classical mean curvature flow. For instance we can show that this entropy is a supermartingale in the case of a G_N-symmetry with $N \geq 3$.

Plan of the talk

High dimensional spheres

Let us come back to the Brownian motion $X_n := (X_n(t))_{t\geq 0}$ on spherical spaces $\mathbb{S}^n$, starting from a point $x_0 \in \mathbb{S}^n$, $n \geq 1$. We saw that it exists an intertwining dual process taking values in balls centered at x_0. The first time τ_n this dual covers $\mathbb{S}^n$ is a sharp strong stationary time, so that we have

$$\forall\ t \geq 0, \qquad \mathfrak{s}(\mathcal{L}(X_n(t)), \mathcal{U}_n) \;=\; \mathbb{P}[\tau_n > t]$$

where $\mathcal{U}_n$ is the uniform probability measure on $\mathbb{S}^n$.
Note that τ_n is hitting time of π by the radius process starting from 0. Investigating this one-dimensional diffusion leads to the cut-off phenomenon in separation: for any $r > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[\tau_n > (1+r)\frac{\ln(n)}{n}\right] = 0 \text{ and } \lim_{n\to\infty} \mathbb{P}\left[\tau_n < (1-r)\frac{\ln(n)}{n}\right] = 0$$

High dimensional rotationally symmetric compact manifolds

The previous considerations can be extended to other families of Riemannian manifolds.
For $n \geq 2$, let V_f^n be the product manifold $[0, L] \times \mathbb{S}^{n-1}/\sim$, where $(r_1, \theta_1) \sim (r_2, \theta_2)$ if $(r_1, \theta_1) = (r_2, \theta_2)$ or $r_1 = r_2 = 0$ or $r_1 = r_2 = L$, endowed with the warping product metric

$$ds^2 = dr \otimes dr + f^2(r) d\theta \otimes d\theta$$

$d\theta \otimes d\theta$ is the standard metric on the sphere $\mathbb{S}^{n-1}$ and $f : [0, L] \to \mathbb{R}_+$ is a regular function satisfying:

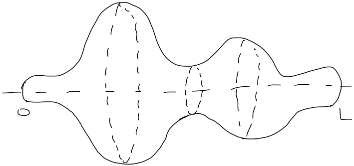
$$\begin{cases} \forall\ s \in [0, L], & f(L - s) = f(s), \\ \forall\ s \in [0, L/2), & f'(s) > 0, \\ \forall\ s \in [0, L] \setminus \{L/2\}, & f''(s) \leq 0, \\ f^{(2k)}(0) = 0, \forall\ k \in \mathbb{Z}_+ & \end{cases}$$

But f will not necessarily be C^2 at $L/2$: we assume there exist $\alpha \in (-1, +\infty)$ and $C > 0$ such that for $h \neq 0$ small enough,

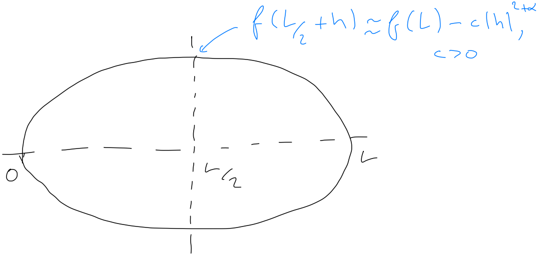
$$f''(L/2 - h) \quad = \quad -C|h|^\alpha + o(|h|^\alpha)$$

In pictures

general f :



our f :



Cut-off

Denote $X_n := (X_n(t))_{t\geq 0}$ a Brownian motion on V_f^n, starting from $\widetilde{0} \sim \{0\} \times \mathbb{S}^{n-1}$. It converges in large time toward the uniform (Riemannian) distribution $\mathcal{U}_n$ on V_f^n. More quantitatively:

Theorem

- *if* $\alpha \in (-1, 0)$ *then* $(X_n)_{n\in\mathbb{N}\setminus\{1\}}$ *has a cut-off in separation at time* C_1/n, *with* $C_1 = 2\int_0^{L/2} \frac{f(s)}{f'(s)}$,
- *if* $\alpha = 0$ *then* $(X_n)_{n\in\mathbb{N}\setminus\{1\}}$ *has a cut-off in separation at time* $C_2 \ln(n)/n$, *with* $C_2 = \frac{f(L/2)}{C}$,
- *if* $\alpha > 0$ *then* $(X_n)_{n\in\mathbb{N}\setminus\{1\}}$ *has no cut-off in separation.*

Recall that the cut-off in separation at some times $(t_n)_{n\in\mathbb{N}\setminus\{1\}}$ means that for any $r > 0$,

$$\lim_{n\to\infty} \mathfrak{s}(X_n(e^{-r}t_n), \mathcal{U}_n) = 0 \text{ and } \lim_{n\to\infty} \mathfrak{s}(X_n(e^{r}t_n), \mathcal{U}_n) = 1$$

Getting the proof started

Again there exist intertwining dual processes taking values in balls centered at $\widetilde{0}$. The first time τ_n such dual processes cover V_f^n are sharp strong stationary times. Furthermore the radius processes $(R(t))_{t\in[0,\tau_n]}$ of these balls satisfy $R(0) = 0$ and

$$\forall\ t \in (0, \tau_n), \qquad dR(t) \ = \ \sqrt{2}dB(t) + b_n(R(t))dt$$

where $(B(t))_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}$ and the mapping b_n is given by

$$\forall\ r \in (0, L), \qquad b_n(r) \ := \ 2\frac{f^{n-1}(r)}{\int_0^r f^{n-1}(u)\, du} - (n-1)\frac{f'(r)}{f(r)}$$

We have

$$\tau_n \ = \ \inf\{t \geq 0 \ : \ R(t) = L\}$$

and we are thus led back to the study of a one-dimensional diffusion.

Plan of the talk

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