2 – Semigroups

Here a Markov generator L on the finite set V is fixed, and X will stand for a solution to the corresponding martingale problem (denoted X_x if it starts from $x \in V$).

The associated $(P_t)_{t \geq 0}$ is called the semi-group generated by L.

The name comes from:

Lemma 4

We have

$$P_0 = Id \quad \text{(the identity matrix)}$$

$$\forall t, s \geq 0 \qquad P_{t+s} = P_t P_s$$

Proof

Let us recall that

$$\forall t \geq 0, \ \forall x, y \in V \qquad P_t(x, y) = \mathbb{P}[X_x(t) = y]$$

In particular, if $t = 0$,

$$P_0(x,y) = \mathbb{1}_{y=x}$$

i.e. $\quad P_0 = Id.$

We furthermore deduce from the Markov property

$$P_{t+1}(x,y) = \mathbb{E}[\, X_x(t+1) = y \,]$$

$$= \mathbb{E}\left[\mathbb{E}[\, X_x(t+1) = y \mid \mathcal{F}(1)]\right]$$

$$= \mathbb{E}[\, P_t(X_x(1), y)\,]$$

$$= \sum_{z \in V} P_1(x,z)\, P_t(z,y)$$

$$= (P_1 P_t)(x,y)$$

$\square$

While the above semi-group property is valid for any (time-homogeneous) Markov process, we can go further with our solution of a martingale problem:

<u>Lemma</u> 5

We have that $t \mapsto P_t$

is continuous and differentiable, with

$$\forall t \geq 0 \qquad \partial_t P_t = P_t L$$

It follows that

$$\forall t \geq 0 \qquad P_t = \exp(tL)$$

Proof

Consider $x \in V$ and $f \in \mathcal{F}(V)$.

Integrating

$$f(X_x(t)) = f(x) + \int_0^t L[f](X_x(s))\, ds + M^f(t)$$

we get

$$\mathbb{E}[f(X_x(t))] = f(x) + \mathbb{E}\left[\int_0^t L[f](X_x(s))\, ds\right]$$

$$= f(x) + \int_0^t \mathbb{E}[L f(X_x(s))]\, ds$$

where we used Fubini's theorem in the last equality (since the mapping

$\mathbb{R}_+ \times \Omega \ni (s, \omega) \mapsto L[f](X_x(s)(\omega))$ is

measurable with respect to the product sigma-field $\mathcal{B}(\mathbb{R}_+) \otimes \Sigma$

Borelian σ-field)

The last equality can be written

$$P_t[f](x) = f(x) + \int_0^t P_s[L[f]](x) \qquad (6)$$

Since $P_s[L[f]]$ is bounded

by $\|L[f]\|_\infty = \max\{|L[f](y)| : y \in V\}$

it appears that

$\mathbb{R}_+ \ni t \mapsto P_t[f](x)$ is continuous.

Taking $f = \mathbb{1}_{\{y\}}$, we get the

continuity of $\mathbb{R}_+ \ni t \mapsto P_t(x,y)$, i.e.

the continuity of $\mathbb{R}_+ \ni t \mapsto P_t$.

Replacing f by $L[f]$, we deduce

the continuity of

$$\mathbb{R}_+ \ni s \mapsto P_s[L[f]](x)$$

and coming back to (6), we

get the differenciability of

$$\mathbb{R}_+ \ni t \mapsto P_t[f](x), \quad \text{with}$$

$$\partial_t P_t[f](x) = P_t[L[f]](x)$$

Taking $f = \mathbb{1}_{\{y\}}$, this

writes

$$\partial_t P_t(x,y) = (P_t L)(x,y)$$

i.e. $\partial_t P_t = P_t L$

which itself implies

$$P_t = \exp(tL)$$

$\square$

<u>Definition 6</u>

A proba $\pi \in \mathcal{P}(V)$ is said to be invariant

(i) for $(P_t)_{t \geq 0}$ if

$$\forall t \geq 0, \quad \pi P_t = \pi$$

(ii) for L if

$$\pi L = 0$$

The terminology for (i) comes from the fact that if the initial law of X is π, then for any $t \geq 0$ $X(t) \sim \Pi$

Indeed writes X_Π to say that

$\mathcal{L}(X(0)) = \Pi$. Conditioning by

$\mathcal{F}(0)$, we get for any $f \in \mathcal{F}(V)$

$$
\begin{aligned}
&\mathbb{E}[f(X_\Pi(t))] \\
&= \mathbb{E}\big[\mathbb{E}[f(X_\Pi(t)) \mid \mathcal{F}(0)]\big] \\
&= \mathbb{E}[P_t[f](X_0)] \\
&= \sum_{x \in V} \Pi(x) P_t[f](x) \\
&= (\Pi P_t)[f] \\
&= \Pi[f]
\end{aligned}
$$

It follows that $X_\Pi(t) \sim \Pi$.

<u>Lemma 7</u>

Both definitions are equivalent

if $(P_t)_{t \geq 0} = (e^{tL})_{t \geq 0}$

Proof

(i) $\Rightarrow$ (ii)

Differentiating (i) in $t \geq 0$, we get

$$\partial_t \pi P_t = 0$$

$$\parallel$$

$$\pi \partial_t P_t = \pi P_t L$$

So taking $t = 0$, we get $\pi L = 0$

(ii) $\Rightarrow$ (i)

$$\partial_t \pi P_t = \partial_t \pi e^{tL}$$

$$= \pi L e^{tL}$$

$$= \underbrace{(\pi L)}_{0} P_t$$

$$= 0$$

So πP_t is constant in t

and $\pi P_t = \pi P_0 = \pi$

$\square$

Lemma 8

In our finite state space framework,

there always exists an invariant probability measure

<u>Proof</u>

For $n \in \mathbb{N}$, consider, with $x \in V$ fixed,

$$\mu_n(\cdot) = \frac{1}{n} \int_0^n P_s(x, \cdot)\, ds$$

which is a probability measure.

By compactness of $\mathcal{P}(V)$, we can extract a converging subsequence $(\mu_{n_m})_{m \in \mathbb{N}}$:

$$\mu_{n_m} \xrightarrow[m \to \infty]{} \Pi \in \mathcal{P}(V)$$

Let us check that Π is invariant.

For any $t \geq 0$ and $f \in \mathcal{F}(V)$,

$$\mu_{n_m}[P_t[f]] = \frac{1}{n_m} \int_0^{n_m} P_s[P_t[f]](x)\, ds$$

$$= \frac{1}{n_m} \int_0^{n_m} P_{t+s}[f](x)\, ds$$

$$= \mu_{n_m}[f] + \frac{1}{n_m} \int_0^r P_{n_m+s}[f](x) - P_s[f](x)\,ds$$

$$\xrightarrow[m \to \infty]{} \Pi[f]$$

since

$$\left| \frac{1}{n_m} \int_0^r P_{n_m+s}[f](x) - P_s[f](x)\,ds \right|$$

$$\leq \frac{1}{n_m} 2\|f\|_\infty r \xrightarrow[m \to \infty]{} 0$$

Thus

$$\Pi P_r[f] = \lim_{m \to \infty} \mu_{n_m}[P_r[f]]$$

$$= \Pi[f]$$

and we get the invariance of Π, since this is true $\forall f \in \mathcal{F}(V)$. $\square$

(remark: it is slightly simpler to use the invariance for L).

To get a sufficient condition for the uniqueness of the invariant measure, we introduce irreducibility:

Definitions 9

(i) A path in V is any finite sequence $p = (p(k))_{k \in [\![0, \ell]\!]}$ of elements of V, with $\ell \in \mathbb{Z}_+$ called the length of p. The path p is said to start at $p(0)$ and to end at $p(\ell)$.

(ii) A L-path is a path $p = (p(k))_{k \in [\![0, \ell]\!]}$ such that $\forall k \in [\![\ell]\!] \; (=: [\![1, \ell]\!])$,

$$L(p(k-1), p(k)) > 0$$

(iii) L is said to be irreducible if $\forall x \neq y$, there exists a L-path starting at x and ending at y.

Here is a characterization of irreducibility:

Proposition 10

The following assertions are equivalent:

(i) L is irreducible

(ii) $\exists t > 0$; all entries of P_t are > 0

(iii) $\forall t > 0$: ______

Proof

Consider $a = \max \{ L(x), x \in V \} + 1$

and

$$K = \frac{L + a \, Id}{a}$$

Then K is a Markov kernel:

$$\forall x, y \in V, \qquad K(x,y) \geq 0$$

$$\forall x \in V \qquad \sum_y K(x,y) = 1 .$$

From $L = a(K - Id)$, we deduce for any $t \geq 0$,

$$
\begin{aligned}
P_t &= \exp(tL) \\
&= \exp(atK - atI) \\
&= e^{-at} \exp(atK)
\end{aligned}
$$

$$= e^{-at} \sum_{n \geq 0} \frac{(at)^n}{n!} K^n$$

i.e. $\forall\, x, y \in V$

$$P_t(x,y) = e^{-at} \sum_{n \geq 0} \frac{(at)^n}{n!} K^n(x,y)$$

In particular, since $a > 0$, for $t > 0$ and $x \neq y$,

$$P_t(x,y) > 0 \iff \exists\, l \in \mathbb{N} : K^l(x,y) > 0 \qquad (7)$$

and since the r.h.s. does not depend on $t > 0$, we get (ii) $\iff$ (iii).

Furthermore the r.h.s. of (7) is easily seen to be equivalent to

$\exists$ L-path going from x to y

(note that $L(x,y) > 0 \iff K(x,y) > 0$).

We thus get the equivalence of (ii) with (i). $\square$

The interest of irreducibility is:

<u>Theorem 11</u>

Assume that L is irreducible.

Then the invariant probability measure of L is unique and it gives a positive weight to all points of V.

<u>Remark</u>: The reverse is also true, but less useful.

Let us decompose the proof of Theorem 11 in several elementary steps.

<u>Lemma 12</u>

Assume that L is irreducible. Then any invariant probability for L is positive

<u>Proof</u>

Let π be an invariant probability for L, and $x_0 \in V$ be such that $\pi(x_0) > 0$.

For any $x \in V$,

$$\pi(x) = \sum_{y \in V} \pi(y) P_1(y,x)$$

$$\geq \underbrace{\pi(x_0)}_{>0} \underbrace{P_1(x_0,x)}_{>0} > 0$$

$\square$

Note that 0 is always an eigenvalue of L, since

$$L[\mathbb{1}] = 0 = 0 \cdot \mathbb{1}$$

Lemma 13

Assume that L is irreducible. Then the eigenvalue 0 of L is simple and its eigenspace consists of the constant functions, i.e. $\mathrm{Vect}(\mathbb{1})$.

Proof

By contradiction, assume that φ is a non-constant function satisfying

$$L[\varphi] = 0$$

The discrete strong maximum principle will shows this is not possible.

Indeed, since $P_1 = \exp(L)$, we deduce

$$P_1[\varphi] = e^0 \varphi = \varphi$$

namely

$$\forall x \in V, \quad \sum_{y \in V} \underbrace{P_1(x,y)}_{>0} \big(\varphi(y) - \varphi(x)\big) = 0 \tag{8}$$

Choose $x \in V$ such that $\varphi(x) = \max_V \varphi$, so that $\forall y \in V \quad \varphi(y) \leq \varphi(x)$.

We deduce from (8) that we

must have

$$\forall y \in V \quad \varphi(y) = \varphi(x),$$

i.e. $\varphi \in \mathrm{Vect}(\mathbb{1})$

$\square$

Lemma 14

Fix an invariant probability π for L. The adjoint L^* of L in $\ell^2(\pi)$ is an irreducible Markov generator.

Proof

Fix $x, y \in V$, we have

$$\pi\left[\mathbb{1}_{\{x\}} L[\mathbb{1}_{\{y\}}]\right] = \pi\left[\mathbb{1}_{\{y\}} L^*[\mathbb{1}_{\{x\}}]\right]$$

which writes

$$\pi(x) L(x,y) = \pi(y) L^*(y,x)$$

i.e., since $\pi > 0$

$$L^*(y,x) = \frac{\pi(x)}{\pi(y)} L(x,y)$$

It follows that

$$\forall x \neq y, \quad L^*(y,x) \geq 0$$

and for any $y \in V$

$$\sum_x L^*(y,x) = \frac{1}{\pi(y)} \underbrace{\sum_x \pi(x) L(x,y)}_{\pi L(y) = 0}$$

For the irreducibility, reverse paths. $\square$

<u>Remark</u>: It can be checked that conversely for irreducible L and for any probability measure $\pi > 0$,

L^* is a Markov generator

$\Longrightarrow$ π is invariant for L.

We can now come to:

<u>Proof of Theorem 11</u>

Let π and $\tilde{\pi}$ be two invariant probability measures for L.

The equation $\tilde{\pi} L = 0$ means

$$\forall f \in \mathcal{F}(V) \quad \tilde{\pi}[Lf] = 0$$

$$\text{i.e.} \quad \pi\left[\frac{\tilde{\pi}}{\pi} Lf\right] = 0$$

(since $\pi > 0$, the function

$$\frac{\tilde{\pi}}{\pi} : V \ni x \mapsto \frac{\tilde{\pi}(x)}{\pi(x)} \text{ is well-defined })$$

$$\Longleftrightarrow \quad \pi\left[f \, L^*\left[\frac{\tilde{\pi}}{\pi}\right]\right] = 0$$

(where L^* is the adjoint of L in $\mathbb{L}^2(\pi)$).

Since $\pi > 0$ and this is true $\forall f \in \mathcal{F}(V)$, we deduce

$$L^*\left[\frac{\tilde{\pi}}{\pi}\right] = 0$$

But L^* is an irreducible Markov

generator, so $\frac{\tilde{\Pi}}{\Pi}$ must be constant, say equal to $c > 0$.

$$1 = \Pi\left[\frac{\tilde{\Pi}}{\Pi}\right] = c, \quad \text{so } \tilde{\Pi} = \Pi.$$

□

The adjoint L^* of L in $\mathbb{L}^2(\Pi)$ has played a role in the above arguments. It can be interpreted as the Markov generator of the time-reversed process at equilibrium in the sense made more precise by the following

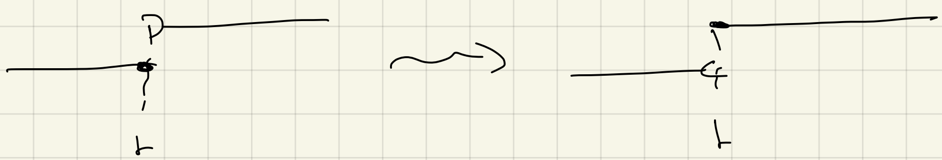
<u>Remark</u>

When Π is invariant for L, it is possible to define X_Π on the whole time domain $\mathbb{R}$: $(X_\Pi(t))_{t\in\mathbb{R}}$.

Consider $(\tilde{X}^*_\Pi(t))_{t\geq 0}$ the time-reversed process

$$\forall t \geq 0, \quad \tilde{X}^*_\Pi(t) = X_\Pi(-t)$$

The trajectories of $\tilde{X}^*_\Pi$ are làgcàd, but make the necessary modifications at the jump times to transform them in càdlàg trajectories



Denote $X^*_\pi = \big(\underbrace{\tilde{X}^* (t+)}_{\text{right limit at } t} \big)_{t \geq 0}$

the obtained process.

Then X^*_π is a solution to the martingale problem associated to L^* starting with π.

$\square$

For us, L^* will be interesting in another respect:

Assume that π is a positive invariant probability measure for L (without necessarily assuming L irreducible).

Let L^* be the dual of L in $\mathbb{L}^2(\pi)$.

Let $m_0 \in \mathcal{P}(V)$ be given and denote

$$\forall t \geq 0 \qquad m_t = m_0 P_t$$

$$= \mathcal{L}(X_{m_0}(t))$$

and define $f_t = \dfrac{m_t}{\pi} \in \mathcal{F}(V)$

(seen as a Radon-Nykodim density)

<u>Proposition 15</u>

We have

$$\forall t \geq 0, \qquad f_t = P^*_t [f_0]$$

with $P^*_t = \exp(t L^*)$.

In particular, $t \longmapsto f_t$ is differentiable and

$$\forall t \geq 0 \quad \partial_t f_t = L^*[f_t]$$

<u>Proof</u>

For any $g \in \mathcal{F}(V)$, we have, on one hand,

$$\begin{aligned} m_t[g] &= (m_0 P_t)[g] \\ &= m_0[P_t[g]] \\ &= \pi[f_0 \; P_t[g]] \\ &= \pi[P_t^*[f_0] \, g] \end{aligned}$$

On the other hand,

$$m_t[g] = \pi[f_t \, g]$$

Comparing these equalities and taking into account that $\pi > 0$ and that they are valid $\forall g \in \mathcal{F}(V)$, we deduce

$$\begin{aligned} f_t &= P_t^*(f_0) \\ &= \exp(t L^*) \, f_0 \end{aligned}$$

Thus $\quad \partial_t f_t = L^* \exp(t L^*) f_0$

$$= L^*[f_t]$$

$\square$

But in the sequel we will consider a more restricted situation.

Definition 16

A probability measure π is said to be *reversible* for L if

$$\forall x,y \in V, \qquad \pi(x)L(x,y) = \pi(y)L(y,x) \tag{9}$$

(local balance equation).

Summing (9) over $y \in V$, we get

$$\pi L(x) = \sum_{y\in V} \pi(y)L(y,x)$$

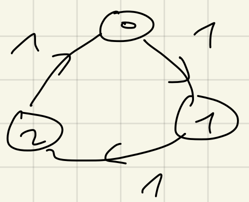
$$= \sum_{y\in V} \pi(x)L(x,y)$$

$$= \pi(x)\sum_{y\in V} L(x,y) = 0$$

So a reversible probability measure for L is also invariant.

The converse is not true in general:

Example



$$L = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

The uniform probability on $\{0,1,2\}$ is invariant but not reversible.

Here is a characterization of reversibility in a functional analysis spirit.

Lemma 17

π is reversible for L if and only if L is self-adjoint in $\ell^2(\pi)$

Proof

L is self-adjoint in $\ell^2(\pi)$ if and only if

$$\forall f, g \in \mathcal{F}(V), \quad \pi[f\, L[g]] = \pi[g\, L[f]] \tag{10}$$

Taking $f = \mathbb{1}_x$ and $g = \mathbb{1}_y$ with $x, y \in V$, we recover (9)

Conversely, writing

$$f = \sum_{x \in V} f(x)\, \mathbb{1}_x$$

$$g = \sum_{y \in V} g(y)\, \mathbb{1}_y$$

and expanding in (10), we get that $(9) \Rightarrow (10)$. $\square$

Kolmogorov provided a criterion enabling us to check the existence of a reversible measure:

For any L-path $p := (p(k))_{k \in [\![0, \ell]\!]}$, define

$$L_+(p) := \prod_{k \in [\![\ell]\!]} L(p(k-1), p(k))$$

$$L_-(p) := \prod_{k \in [\![\ell]\!]} L(p(k), p(k-1))$$

For $x \in V$, denote $\mathcal{C}(x)$ the set of L-paths starting and ending at x (also called x-cycles)

<u>Theorem 18</u> (Kolmogorov's criterion)

Let L be an irreducible Markov generator. There is equivalence between

(i) The invariant probability measure of L is reversible

(ii) $\forall x \in V$, $\forall p \in \mathcal{C}(x)$,

$$L_+(p) = L_-(p)$$

(iii) $\exists x \in V, \forall p \in \mathcal{C}(x),$

$$L_+(p) = L_-(p)$$

The proof is left as an exercise.

Recall that a tree on V is a connected un-oriented graph (V, E), where E is the set of edges, such that $|E| = |V| - 1$.

We say that L is supported by a tree (V, E) if

$$\forall x \neq y \in V, \quad L(x,y) > 0 \iff \{x,y\} \in E$$

un-oriented edge between x and y

From Kolmogorov's criterion, we deduce:

<u>Corollary 19</u>

If the generator L is supported

by a tree, then it is irreducible and its invariant probability is reversible.

Examples of Markov generators supported by trees are birth-and-death generators, which are those on $V = [\![0, N]\!]$, with $N \in \mathbb{N}$ and satisfying

$$L(x,y) > 0 \iff |x-y| = 1.$$

Another consequence of reversibility is:

<u>Corollary 20</u>

With the notations of Proposition 15, we have

$$\forall t \geq 0, \quad \begin{cases} f_t = P_t[f_0] \\ \partial_t f_t = L[f_t] \end{cases}$$