3 – Spectral gap

Let be given L an irreducible Markov generator, reversible with respect to π (still on the finite set V).

We denote $(X(t))_{t \geq 0}$ an associated Markov process and

$$\forall t \geq 0 \qquad m_t := \mathcal{L}(X(t))$$

Our goal is to investigate quantitatively the convergence

$$m_t \xrightarrow[t \to +\infty]{} \pi \tag{11}$$

Here a nice consequence of reversibility:

<u>Lemma 21</u>

The operator L is diagonalizable, and except for the eigenvalue 0 (recall that its eigenspace consists of the constant functions), all the other eigenvalues are negative.

<u>Proof</u>

The self-adjointness of L

in $\mathbb{L}^2(\pi)$ implies that L is diagonalizable in $\mathbb{R}$. We took into account that $\mathbb{L}^2(\pi) = \mathcal{F}(V)$, since $\pi > 0$.

Consider $\lambda \in \mathbb{R}$ an eigenvalue of L, with φ a corresponding eigenfunction:

$$L\varphi = \lambda\varphi$$

We deduce

$$\forall t \geq 0 \qquad P_t[\varphi] = e^{\lambda t}\varphi$$

Since $\varphi \neq 0$, there exists $x_0 \in V$ such that $\varphi(x_0) \neq 0$.

If $\lambda > 0$, we deduce

$$P_t[\varphi](x_0) = e^{\lambda t}\varphi(x_0) \xrightarrow[t\to\infty]{} +\infty$$

but this is impossible, since

$$|P_t[\varphi](x_0)| \leq \|\varphi\|_\infty .$$

$\square$

This observation leads to (11):

Consider $(\lambda_i, \varphi_i)_{i \in [\![0,N]\!]}$ a

Spectral decomposition of $-L$ where

$(\lambda_i)_{i \in [\![0,N]\!]}$ is the family of eigenvalues

and $(\varphi_i)_{i \in [\![0,N]\!]}$ is a corresponding family

of eigenfunctions which constitutes an

orthonormal basis of $\mathbb{L}^2(\pi)$ (in

particular $|V| = N+1$):

$$\forall i \in [\![0,N]\!], \quad L\varphi_i = -\lambda_i \varphi_i$$

$$\forall i,j \in [\![0,N]\!], \quad \pi[\varphi_i \varphi_j] = \delta_{i,j}$$

Kronecker symbol $= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

We will always assume that

- $\lambda_0 = 0$ and $\varphi_0 = \mathbb{1}$

  (the only other normalized choice would have been $\varphi_0 = -\mathbb{1}$)

- $(\lambda_i)_i$ are ordered $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$

Decompose $f_0 := \frac{m_0}{\pi}$ in this basis

$$f_0 = \sum_{i=1}^{N} a_i \varphi_i$$

with $\forall i \in [0, N]$, $a_i = \pi[\rho_0 \varphi_i]$

$= m_0[\varphi_i]$

We get

$$\rho_t = P_t[\rho_0] = \sum_{i=1}^{N} a_i \, P_t[\varphi_i]$$

$$= \sum_{i=1}^{N} a_i e^{-\lambda_i t} \varphi_i \qquad (12)$$

$$\xrightarrow[t \to +\infty]{} a_0 \varphi_0 = a_0$$

Note that $a_0 = m_0[\varphi_0]$

$= 1$

so that $\rho_t \xrightarrow[t \to +\infty]{} 1$

This is equivalent to (11).

By taking the $L^2(\pi)$-norm of (12), we can get quantitative estimate on the convergence (11).

Let us slightly modify this argument to be able to extend it to other situations (non-reversibility and entropy).

First let us recall two ways of measuring the discrepancy between two probability measures.

Definitions 22

$\mu, \pi \in \mathcal{P}(V)$.

(i) Total variation distance

$$\|\mu - \pi\|_{TV} = \max_{A \subset V} \mu(A) - \pi(A)$$

$$= \frac{1}{2} \max_{\substack{f \in \mathcal{F}(V) \\ \|f\|_\infty = 1}} \mu[f] - \pi[f]$$

$$= \frac{1}{2} \sum_{x \in V} |\mu(x) - \pi(x)|$$

(ii) χ^2-discrepancy : if $\pi > 0$,

$$\chi^2(\mu | \pi) := \left\| \frac{\mu}{\pi} - 1 \right\|^2_{\ell^2(\pi)}$$

$$= \sum_{x \in V} \left(\frac{\mu}{\pi}(x) - 1 \right)^2 \pi(x)$$

A convention for the definition of $\chi^2(\mu, \pi)$ when π is not positive will be presented in the next lesson.

Give the proof of the equivalent expressions.

The total variation is a distance on $\mathcal{P}(V)$ but the χ^2-discrepancy is not, because it is not symmetric in its arguments. Nevertheless, we have:

Lemma 23

$\forall \mu, \pi \in \mathcal{P}(V)$, with $\pi > 0$,

$$\|\mu - \pi\|_{TV} \leq \frac{1}{2}\sqrt{\chi^2(\mu \mid \pi)}$$

Proof

$$\sum_{x \in V} |\mu(x) - \pi(x)| = \sum_{x \in V} \left|\frac{\mu}{\pi}(x) - 1\right| \pi(x)$$

$$\leq \sqrt{\sum_{x \in V} \left(\frac{\mu}{\pi}(x) - 1\right)^2 \pi(x)}$$

by Cauchy-Schwartz' inequality

$\square$

Here our quantity of interest in this section:

Definition 24

The <u>spectral gap</u> $\lambda(L)$ is the smallest non-zero eigenvalue of $-L$.

It is given by the variational formula:

$$\lambda(L) = \min_{\substack{f \in \ell^2(\pi)\setminus\{0\} \\ \pi[f]=0}} \frac{\mathcal{E}_L(f,f)}{\pi[f^2]} \tag{13}$$

$$= \min_{f \in \ell^2(\pi)\setminus \mathrm{Vect}(\mathbb{1})} \frac{\mathcal{E}_L(f,f)}{\mathrm{Var}(f,\pi)} \tag{14}$$

where the <u>Dirichlet form</u> associated to L is the bilinear mapping:

$$\mathcal{F}(V) \ni (f,g) \longmapsto \mathcal{E}(f,g) := -\pi[f\, L[g]]$$

$$= \frac{1}{2} \sum_{x \in V} \pi(x) L(x,y) (f(y)-f(x))(g(y)-g(x)) \tag{15}$$

and where the <u>variance</u> of $f \in \mathcal{F}(V)$ with respect to π is

$$\mathrm{Var}(f,\pi) := \pi\left[(f - \pi(f))^2\right]$$

$$= \Pi[f^2] - \Pi(f)^2 \quad (16)$$

Let us check these classical assertions:

The variational formulation of $\lambda(L)$ states,

$$\lambda(L) = \min_{\substack{f \perp \text{eigenspace}(0) \\ f \neq 0}} \frac{\Pi[f(-L)f]}{\Pi[f^2]}$$

(the min is attained at eigenfunctions associated to $\lambda(L)$),

and we get (13) since

$$\text{eigenspace}(0) = \text{Vect}(\mathbb{1})$$

(14) is an easy consequence of the fact that for any $f \in \mathcal{F}(V)$ and any constant c

$$\mathcal{E}_L(f-c, f-c) = \mathcal{E}_L(f,f).$$

For (15), writes that for any $f, g \in \mathcal{F}(V)$

$$-\Pi[fLg] = -\sum_{x\in V} \pi(x) f(x) \sum_{y \in V} L(x,y)\, g(y)$$

$$= -\sum_{x\in V} \pi(x) f(x) \sum_{y\in V} L(x,y)(g(y) - g(x))$$

(recall that $\sum_{y \in V} L(x,y) = 0$)

$$= - \sum_{\substack{x \in V \\ y \in V}} \pi(x) L(x,y)\, f(x) \left(g(y) - g(x) \right)$$

reversibility

$$= - \sum_{\substack{x \in V \\ y \in V}} \pi(x) L(x,y)\, f(y) \left(g(x) - g(y) \right)$$

$$= \frac{1}{2} \sum_{\substack{x \in V \\ y \in V}} \pi(x) L(x,y) \left(f(y) - f(x) \right) \left(g(y) - g(x) \right)$$

Finally, (16) is a usual observation:

$$\pi\left[(f - \pi(f))^2 \right] = \pi\left[f^2 - 2 f \pi(f) + (\pi(f))^2 \right]$$

$$= \pi[f^2] - 2\pi[f]\pi(f) + (\pi(f))^2$$

$$= \pi[f^2] - \pi(f)^2$$

The variance can also be seen as the Dirichlet form associated to the Markov generator L_π given by :

$\forall x \neq y \quad L_\pi(x,y) = \pi(y)$

namely

$$L_\pi = \Pi - I$$

where Π is reinterpreted as the Markov kernel whose rows are equal to the probability measure π.

First π is reversible for L_π :

$\forall x \neq y \qquad \pi(x) L_\pi(x,y) = \pi(x)\pi(y)$

Next, for any $f \in \mathcal{F}(V)$

$$- \pi[f L_\pi[f]] = -\pi[f(\pi(f) - f)]$$

$$= -\pi[f]^2 + \pi[f^2]$$

$$= \mathrm{Var}(f, \pi).$$

Here is the first interest of the spectral gap:

<u>Proposition</u> 25

| We have for any initial $m_0 \in \mathcal{P}(V)$,

$$\forall t \geq 0 \quad \chi^2(m_t, \pi) \leq e^{-2\lambda(L)t} \chi^2(m_0, \pi) \tag{17}$$

As a consequence, we also have

$$\forall t \geq 0 \quad \| m_t - \pi \|_{tv} \leq \frac{1}{\sqrt{\pi_1}} e^{-\lambda(L)t}$$

with $\pi_1 = \min_V \pi$

<u>Proof</u>

According to Corollary 20, we have

$$\forall t \geq 0 \quad \chi^2(m_t, \pi) = \pi\Big[\Big(\underbrace{\frac{m_t}{\pi}}_{=: f_t} - 1\Big)^2\Big]$$

$$= \pi\Big[\Big(P_t\Big[\underbrace{\frac{m_0}{\pi}}_{=: f_0}\Big] - 1\Big)^2\Big].$$

We can differentiate:

$$\partial_t \chi^2(m_t, \pi) = 2\pi\Big[\big(P_t[f_0] - 1\big)\, \partial_t P_t[f_0]\Big]$$

$$= 2\pi\Big[\big(P_t[f_0] - 1\big) L\, P_t[f_0]\Big]$$

$$= -2\mathcal{E}(f_t, L f_t)$$

$$\leq -2\lambda(L)\, \mathrm{Var}(f_t, \pi)$$

$$= -2\,\lambda(L)\,\chi^2(m_t, \pi)$$

Gronwall's lemma leads to (17)

Using Lemma 23, we deduce

$$\forall t \geq 0, \quad \| m_t - \pi \|_{TV} \leq \frac{1}{2} e^{-\lambda(L)t} \sqrt{\chi_0(m_0, \pi)}$$

We have

$$\begin{aligned}
\chi_0(m_0, \pi) &= \mathrm{Var}(f_0, \pi) \\
&\leq \pi[f_0^2] \\
&= \sum_{x \in V} \pi(x) \frac{m_0^2(x)}{\pi^2(x)} \\
&\leq \frac{1}{\pi_1} \sum_{x \in V} m_0^2(x) \\
&\leq \frac{1}{\pi_1} \sum_{x \in V} m_0(x) = \frac{1}{\pi_1}
\end{aligned}$$

$\square$

This method can be extended to all irreducible Markov generators L:

Let π be its invariant measure

and denote

$$\tilde{L} = \frac{1}{2}(L + L^*)$$

(where L^* is the adjoint of L in $L^2(\pi)$)

the (additive) symmetrization of L.

Since $\tilde{L}$ is irreducible and π is reversible for $\tilde{L}$, we can consider the spectral gap $\lambda(\tilde{L}) > 0$.

Let us come back to the previous arguments:

$$\partial_t \chi^2(m_t, \pi) = \partial_t \operatorname{Var}(\underbrace{P_t^*[f_0]}_{=: f_t}, \pi)$$

$$= 2\pi[(f_t - 1)\partial_t f_t]$$

$$= 2\pi[(f_t - 1) L^* f_t]$$

$$= 2\pi[(f_t - 1) L^*(f_t - 1)]$$

$$= 2\pi[L(f_t - 1) \cdot (f_t - 1)]$$

$$= 2\pi[(f_t - 1)\tilde{L}[f_t - 1]]$$

$$\leq -2\lambda(\tilde{L})\, Var(f_t, \pi)$$

$$= -2\lambda(\tilde{L})\, \chi^2(m_t, \pi)$$

$$\Rightarrow \forall t \geq 0 \quad \chi^2(m_t, \pi) \leq e^{-2\lambda(\tilde{L})t} \chi^2(m_0, \pi)$$

$$\Rightarrow \forall t \geq 0 \quad \| m_t - \pi \|_{tv} \leq \frac{1}{\sqrt{\pi_1}}\, e^{-\lambda(\tilde{L})t}$$

In particular, we recover the well-known qualitative result:

<u>Corollary 26</u>

Let L be irreducible and π be its invariant probability.

Whichever the initial law $m_0 \in \mathcal{P}(V)$,

$$\lim_{t \to +\infty} m_t = \pi$$

<u>Remark</u>

In the reversible case, $\lambda(L)$ is the optimal rate, in the sense that

$$\lambda(L) = \min_{m_0 \in \mathcal{P}(V)\setminus\{\pi\}} -\lim_{t\to+\infty} \frac{1}{2t} \ln \chi^2(m_t, \pi)$$

we can have in the non-reversible case

$$\lambda(\tilde{L}) < \min_{m_0 \in \mathcal{P}(V)\setminus\{\pi\}} -\lim_{t\to+\infty} \frac{1}{2t} \ln \chi^2(m_t, \pi)$$

(to be checked on a finite kinetic model)

Coming back to (12) and considering

$$\forall x \in V, \; m_0(x) = \pi(x)\left(1 - \frac{\varphi_1(x)}{\|\varphi_1\|_\infty}\right)$$

where φ_1 is an eigenfunction associated to the spectral gap,

we show

$$\lambda(L) = \min_{m_0 \in \mathcal{P}(V)\setminus\{\pi\}} -\lim_{t\to+\infty} \frac{1}{t} \ln \|m_t - \pi\|_{tv} \qquad (17')$$

Let us present a very classical

Example 27

Continuous random walk on $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$

with $N \in \mathbb{N} \setminus \{1\}$.

The usual generator is given by

$$\forall x \neq y \quad L_N(x,y) = \begin{cases} 1 & \text{if } |y-x| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Consider for $n \in \mathbb{Z}_N$

$$\forall x \in \mathbb{Z}_N, \quad \varphi_n(x) = e^{\frac{2\pi i n x}{N}}$$

(where $i^2 = -1$)

$$L_N[\varphi_n](x) = \left(e^{\frac{2\pi i n}{N}} + e^{-\frac{2\pi i n}{N}} - 2\right) \varphi_n(x)$$

$$= 2\left(\cos\left(\frac{2\pi n}{N}\right) - 1\right) \varphi_n(x)$$

To come back to real functions writes

$$\forall n \in \mathbb{Z}_N, \quad \varphi_n(x) = \cos\left(\frac{2\pi n}{N} x\right) + i \sin\left(\frac{2\pi n}{N} x\right)$$

It follows that for $-L_N$:

eigenvalue 0 — multiplicity $[1]$

$\forall n \in [\![1, \lfloor \frac{N-1}{2} \rfloor]\!]$ — $2\left(1 - \cos\left(\frac{2\pi n}{N}\right)\right)$ — 2

if N is even:

eigenvalue 2 — 1

In particular

$$\lambda(L_N) = 2\left(1 - \cos\left(\frac{2\pi}{N}\right)\right)$$

$$\underset{N\to\infty}{\sim} \frac{2}{2}\left(\frac{2\pi}{N}\right)^2 = \frac{4\pi^2}{N^2}$$

It suggests a time of order N^2 for the associated Markov process to reach equilibrium, but see Example 2 below (with $d=1$) for a more precise statement.

Before presenting this other very classical example, let us show the important property of stability of the spectral gap by tensorization:

The irreducible and reversible generator L still being given, for $d \in \mathbb{N}$, consider on V^d the generator

$$L^{(d)} = \sum_{n=1}^{d} L_n \qquad (18)$$

acting on the n^{th} factor of V^N

i.e. $\forall \quad x := (x_n)_{n\in[\![d]\!]} \neq y := (y_n)_{n\in[\![d]\!]} \in V^d$

$$L_n(x,y) = \begin{cases} L(x_n, y_n) & , \text{ if } x_m = y_m \\ & \quad \forall m \in [\![d]\!] \setminus \{n\} \\ 0 & , \text{ otherwise} \end{cases}$$

$L^{(d)}$ is a Markov generator under which the coordinates $(X_n(t))_{t \geq 0}$, for $n \in [\![d]\!]$, of $(X^{(d)}(t))_{t \geq 0}$, evolve independently as Markov processes of generator L (under the assumption that initially the $X_n(0)$ are independent).

Lemma 28

We have

$$\lambda(L^{(d)}) = \lambda(L)$$

Proof

$(\lambda_i, \varphi_i)_{i \in I}$ corresponding to (with $I = [\![0, |V|-1]\!]$)

the (ordered) spectral decomposition of $-L$,

we check that for any $i_1, \dots, i_N \in I$

the function

$$\varphi_{i_1,\dots,i_d} := \varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_d}$$

is an eigenfunction for $L^{(d)}$ and

$$-L^{(N)} \varphi_{i_1,\dots,i_d} = (\lambda_{i_1} + \lambda_{i_2} + \dots + \lambda_{i_d}) \varphi_{i_1,\dots,i_d}$$

It follows that the

smallest non zero eigenvalue of $L^{(d)}$

is λ_1 whose eigenspace is at

least of dimension d as it contains

for all $m \in [\![d]\!]$

$$x = (x_n)_{n \in [\![d]\!]} \longmapsto \varphi_1(x_m)$$

$$\Big(= \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \underset{\substack{\uparrow \\ m\text{th place}}}{\varphi_1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \Big)$$

$\square$

<u>Remark</u>: The above result extents to non-identical but still independent factors. The global spectral gap is then the minimum of the spectral gaps of the factors.

Example 29

Let us come back to Example 27 and consider for $d \in \mathbb{N}$, $L_N^{(d)}$ defined in (18), with $L := L_N$.

We get

$$\lambda(L_N^{(d)}) = \lambda(L_N) \sim \frac{4\pi^2}{N^2}.$$

We could think that the convergence to equilibrium does not depend on the dimension d, but it is not completely true!

Indeed, we have the bound

$$\forall t \geq 0 \quad \| m_t - \Pi^{(d)} \|_{L^2} \quad \frac{1}{\sqrt{\Pi^{(d)}_{N,1}}} e^{-\lambda(L)t}$$

Here $\Pi^{(d)}_{N,1} = \dfrac{1}{N^d}$, so the r.h.s. is mainly

$$\frac{1}{N^{d/2}} e^{-\frac{4\pi^2}{N^2} t} = e^{\frac{d \ln(N)}{2} - \frac{4\pi^2}{N^2} t}$$

$$\leq \varepsilon \quad \text{given}$$

For $t \geq \frac{N^2}{4\pi^2}\left(\frac{d}{2}\ln(N) + \ln 1/\varepsilon\right)$ (19)

We will show how to improve the dependence in d in the next lesson.

But it is quite rare that we can explicitly compute the spectrum of a generator.

Indeed, for $|V| \geq 6$, the eigenvalues (once 0 is removed) of "generic" irreducible and reversible Markov generators are the roots of generic polynomials of degree 5.

It is often preferable to have reasonable approximations of the spectral gap.

3.1 Paths arguments

The following approach is also called Poincaré's method.

Let be given, for any $x \neq y \in V$, a L-path $p_{x,y}$ of length $\ell_{x,y}$.

$p_{x,y}$ will also be seen as the set of its oriented transitions:

$$p_{x,y} \simeq \{ (p_{x,y}(k-1), p_{x,y}(k)) : k \in [\![\ell_{x,y}]\!] \}$$

For any oriented edge $(u,v) \in \vec{E} = \{(u',v') \in V^2 : L(u',v') > 0\}$

denote

$$B(u,v) = \{(x,y) \in V^2 : x \neq y \text{ and } (u,v) \in P_{x,y}\}$$

$$A(u,v) = \frac{1}{\pi(u) L(u,v)} \sum_{(x,y) \in B(u,v)} l_{x,y} \, \pi(x) \, \pi(y)$$

$$A(L) = \max\{A(u,v), (u,v) \in \vec{E}\}$$

Proposition 30

We have

$$\lambda(L) \geq \frac{1}{A(L)}$$

Proof

By definition, $\lambda(L)$ is the largest $\lambda \geq 0$ such that

$$\forall f \in \mathcal{F}(V), \quad \lambda \operatorname{Var}(f, \pi) \leq \mathcal{E}_L(f,f)$$

i.e.

$$\lambda \sum_{x \neq y \in V} (f(y) - f(x))^2 \pi(x) \pi(y) \leq \sum_{x \neq y \in V} (f(y) - f(x))^2 \pi(x) L(x,y) \qquad (20)$$

Fix $x \neq y \in V$ and write

$$\left(f(y) - f(x) \right)^2 = \left(\sum_{k \in [\ell_{x,y}]} f(p_{x,y}(k-1)) \; f(p_{x,y}(k)) \right)^2$$

$$\leq \ell_{x,y} \sum_{k \in [\ell_{x,y}]} \left(f(p_{x,y}(k-1)) - f(p_{x,y}(k)) \right)^2$$

$$= \ell_{x,y} \sum_{(u,v) \in p_{x,y}} \left(f(v) - f(u) \right)^2$$

It follows that

$$\sum_{x \neq y} \left(f(y) - f(x) \right)^2 \pi(x) \pi(y)$$

$$\leq \sum_{x \neq y} \ell_{x,y} \pi(x) \pi(y) \sum_{(u,v) \in p_{x,y}} \left(f(v) - f(u) \right)^2$$

$$= \sum_{(u,v) \in \vec{E}} \left(f(v) - f(u) \right)^2 \sum_{(x,y) \in B(u,v)} \ell_{x,y} \pi(x) \pi(y)$$

$$= \sum_{(u,v) \in \vec{E}} \left(f(v) - f(u) \right)^2 A(u,v) \pi(u) L(u,v)$$

$$\leq A(L) \sum_{(u,v) \in \vec{E}} \left(f(v) - f(u) \right)^2 \pi(u) L(u,v)$$

$$= A(L) \sum_{u \neq v} \left(f(v) - f(u) \right)^2 \pi(u) L(u,v)$$

so (20) is satisfied with $\lambda = \dfrac{1}{A(L)}$ $\square$

Remarks

(a) It can be shown that in general the above bound does not provide an equality, even for the best possible choices of the paths.

(b) For the application to simulated annealing (see Lesson 5), the path method will provide the optimal exponential rate at small temperature.

(c) The above method admits a lot of variants:

- weights can be introduced in the Cauchy-Schwarz' inequality
- the paths can be random

etc.

3.2 Cheeger inequality

This is another geometric approach to the spectral gap, that was first developped in the context of compact

Riemannian manifolds for the spectral gap of the Beltrami-Laplacian.

For any $\emptyset \neq A \subset V$, define

$$j(A) := \frac{\pi[\mathbb{1}_A L[\mathbb{1}_{A^c}]]}{\pi(A)}$$

This is a kind of isoperimetric ratio:

$$\pi[\mathbb{1}_A L \mathbb{1}_{A^c}] = \sum_{\substack{x \in A \\ y \notin A}} \pi(x) L(x,y)$$

is a measure of the frontier between A and A^c, seen as the set $\{\{x,y\} : x \in A, y \notin A\}$.

Consider the Cheeger constant

$$c(L) = \min_{0 < \pi(A) \leq 1/2} j(A)$$

which corresponds to partition V into two parts communicating as little as possible.

Theorem 31

|

We have
$$\frac{\iota(L)^2}{8|L|} \leq \lambda(L) \leq \iota(L)$$
with $|L| = \max\{L(x), x \in V\}$

Proof to be written !

Remark

Theorem 31 admits several variants, one of them is homogeneous and states that
$$\frac{\iota(L)}{2(|V|-1)} \leq \lambda(L) \leq \iota(L)$$

3.3 Hardy's inequality

This approximation is valid only for finite segments (but there is an extension to trees, finite or not).

Nevertheless it is quite sharp, as it is valid up to a factor 8.

So consider on $V = [0, N]$ an irreducible birth-and-death generator. For $x \in [0, N-1]$, denote $b(x) = L(x, x+1)$, the birth rate and $d(x) = L(x, x-1)$ the death rate.

For $x \in [\![N-1]\!]$ introduce

$$B_+(x) = \max_{N \geq y > x} \left(\sum_{z=x+1}^{y} \frac{1}{\pi(z) d(z)} \right) \sum_{z' \geq y} \pi(z')$$

$$B_-(x) = \max_{0 \leq y < x} \left(\sum_{z=y}^{x-1} \frac{1}{\pi(z) b(z)} \right) \sum_{z' \leq y} \pi(z')$$

$$B(L) = \min_{x \in [\![N-1]\!]} \left(B_+(x) \vee B_-(x) \right)$$

Theorem 32

We have

$$\frac{1}{4B(L)} \leq \lambda(L) \leq \frac{2}{B(L)}$$

Proof to be written!

Remark

The same inequalities hold if $B(L)$ is replaced by

$$B_+(m) \vee B_-(m)$$

where $m \in [\![N-1]\!]$ is a median of π.