4 - Relative entropy

There is another way to measure the discrepancy between two probability measures, which is so popular that it has several names depending on the field (up to an additive term or/and change of sign): Kullback-Leibler divergence, Shannon entropy, Boltzmann entropy, ...

<u>Definition 33</u>

Consider $\mu, \pi \in \mathcal{P}(V)$.

The relative entropy of μ wrt π is

$$\mathrm{Ent}(\mu | \pi) = \begin{cases} \sum \ln\left(\frac{\mu}{\pi}(x)\right)\mu(x) & \text{if } \mu \ll \pi \\ +\infty & \text{otherwise} \end{cases}$$

The relative entropy is not a distance, as it is not symmetric in μ and π. It is similar in nature to the chi-square discrepancy, as shown by the following generalization.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex function vanishing only at 1.

The $\underline{\varphi\text{-entropy}}$ is defined by

$\forall \mu, \pi \in \mathcal{P}(V)$,

$$\mathrm{Ent}_\varphi(\mu|\pi) := \begin{cases} \sum_{x\in V} \varphi\left(\frac{\pi}{\mu}(x)\right) \pi(x) & \text{if } \mu \ll \pi \\ +\infty & \text{otherwise} \end{cases}$$

We have $\mathrm{Ent}_\varphi(\mu|\pi) \in [0, +\infty]$

and $\mathrm{Ent}_\varphi(\mu|\pi) = 0$ if and only if $\mu = \pi$. But it is not a distance, since, as the relative entropy, it can take the value $+\infty$.

- Let, $\forall r \geq 0$, $\varphi_0(r) := |r-1|$

Then $\forall \mu \ll \pi \in \mathcal{P}(V)$, $\mathrm{Ent}_{\varphi_0}(\mu|\pi) = 2\|\mu - \pi\|_{TV}$

(the only difference between Ent_{φ_0} and $2\|\cdot - \cdot\|_{TV}$ comes from the convention when $\mu \cancel{\ll} \pi$).

- Let, $\forall r \geq 0$ $\varphi_2(r) := (r-1)^2$

Then $\forall \mu, \pi \in \mathcal{P}(V), \pi > 0$ $\underbrace{\mathrm{Ent}_{\varphi_2}(\mu|\pi)}_{\text{extension to all } \mu, \pi \in \mathcal{P}(V)} = \chi^2(\mu, \pi)$

- Let, $\forall r \geq 0$, $\varphi_1(r) := r\ln r - (r-1)$

Then $\forall \mu, \pi \in \mathcal{P}(V)$, $\mathrm{Ent}_{\varphi_1}(\mu|\pi) = \mathrm{Ent}(\mu|\pi)$

Indeed it is sufficient to check this relation for $\mu << \pi$. Then

$$\sum_{x \in V} \left(\frac{\mu(x)}{\pi(x)} \ln \frac{\mu(x)}{\pi(x)} - \frac{\mu(x)}{\pi(x)} + 1 \right) \pi(x)$$

$$= \sum_{x \in V} \ln \frac{\mu(x)}{\pi(x)} \mu(x) \underbrace{- \sum_{x \in V} \mu(x) + \sum_{x \in V} \pi(x)}_{= -1+1=0}$$

Similarly to χ^2, the relative entropy controls the total variation:

<u>Lemma 34</u> (Pinsker's inequality)

$\forall \mu, \pi \in \mathcal{P}(V)$,

$$\| \mu - \pi \|_{tv} \leq \frac{1}{\sqrt{2}} \sqrt{\mathrm{Ent}(\mu | \pi)}$$

<u>Proof</u>

It is sufficient to prove this bound for $\mu << \pi$.

Denote then $f = \frac{\mu}{\pi}$ the density of μ / π.

We have seen that

$$\|\mu - \pi\|_{TV} = \frac{1}{2} \sum_{x \in V} |f(x) - 1| \, \pi(x)$$

$$\mathrm{Ent}(\mu | \pi) = \sum_{x \in V} \varphi_1(f(x)) \, \pi(x)$$

One can check the elementary bound:

$$\forall r \geq 0, \quad 3(r-1)^2 \leq (2r+4)(r \ln r - r + 1)$$

Thus Cauchy-Schwarz' inequality implies

$$\sum |f-1| \pi \leq \sum \sqrt{\frac{2f+4}{3}} \sqrt{\varphi_1(f)} \, \pi$$

$$\leq \underbrace{\sqrt{\sum \frac{2f+4}{3} \pi}}_{\frac{2}{3} + \frac{4}{3} = 2} \; \sqrt{\underbrace{\sum \varphi_1(f) \pi}_{\mathrm{Ent}(\mu|\pi)}}$$

$$= \sqrt{2} \sqrt{\mathrm{Ent}(\mu|\pi)}$$

□

When $(\mu_t)_{t \geq 0}$ is the family of time-marginal laws of a Markov process associated to an irreducible and reversible generator L, it is then meaningful to estimate

$$\forall t \geq 0, \quad I_t := \mathrm{Ent}(m_t \mid \pi)$$

Lemma 35

We have

$$\forall t > 0, \quad \partial_t I_t = - \mathcal{E}_L(\ln f_t, f_t)$$

where $f_t = \frac{m_t}{\pi}$

Proof

Recall that as soon as $t > 0$, $m_t > 0$. Furthermore φ_1 is differentiable on $(0, +\infty)$ and we have

$$\forall r > 0, \quad \varphi_1'(r) = \frac{d}{dr}(r \ln r - r + 1)$$

$$= \ln(r)$$

We deduce for any $t > 0$,

$$\partial_t I_t = \sum_{x \in V} \ln f_t(x) \, \partial_t f_t(x) \, \pi(x)$$

$$= \sum_{x \in V} \ln f_t(x) \, L[f_t](x) \, \pi(x)$$

$$= \pi[\ln f_t \, L[f_t]]$$

$$= -\mathcal{E}_L(\ln f_T, f_T)$$

$\square$

It leads us to introduce the

Definition 36

The modified logarithmic Sobolev constant associated to L is

$$\tilde{\alpha}(L) := \inf_{f \in \mathcal{F}_+(V) \setminus \mathrm{Vect}(\mathbb{1})} \frac{\mathcal{E}_L(\ln f, f)}{\mathrm{Ent}_\pi(f)}$$

where $\mathcal{F}_+(V) = \{ f \in \mathcal{F}(V) : f > 0 \}$

and

$\forall f \in \mathcal{F}_+(V)$,

$$\mathrm{Ent}_\pi(f) = \sum_{x \in V} \pi[f] \varphi_1\left(\frac{f(x)}{\pi[f]} \right) \pi(x)$$

Equivalently, $\tilde{\alpha}(L)$ is the smallest constant $\tilde{\alpha} \geqslant 0$ such that

$\forall f \in \mathcal{F}_+(V)$,

$$\tilde{\alpha}\, \mathrm{Ent}_\pi(f) \leq \mathcal{E}_L(f, \ln f)$$

From Lemma 35, we deduce

the differential inequality

$$\forall t > 0, \quad \partial_t I_t \leq -\tilde{\alpha}(L) I_t$$

taking into account that for any

$$f = \frac{\mu}{\pi}, \qquad \mathrm{Ent}_\pi(f) = \mathrm{Ent}(\mu | \pi).$$

Since furthermore

$$\lim_{t \to 0_+} I_t = I_0$$

we get

Corollary 37

For any $t \geq 0$, we have

$$\mathrm{Ent}(m_t | \pi) \leq e^{-\tilde{\alpha}(L) t} \mathrm{Ent}(m_0 | \pi)$$

and

$$\| m_t - \pi \|_{tv} \leq e^{-\frac{\tilde{\alpha}(L)}{2} t} \sqrt{\frac{\mathrm{Ent}(m_0 | \pi)}{2}} \qquad (21)$$

One can be doubtful about this last bound: from (17') the rate $\frac{\tilde{\alpha}(L)}{2}$ cannot be better than $\lambda(L)$, i.e. we always have

$$\frac{\tilde{\alpha}(L)}{2} \leq \lambda(L) \qquad (21')$$

Worse: we did not check yet that $\tilde{\alpha}(L) > 0$, see Corollary 21 below.

The interest of (21) is in the factor $\sqrt{\mathrm{Ent}(\mu|\pi)}$ which can be much smaller than $\sqrt{\chi^2(\mu|\pi)}$.

Thus (21) should not be applied as $t \to +\infty$, but in some intermediate time scale, as it will be illustrated at the end of this lesson.

Before, we must present some comparisons of ergodic constants such as $\lambda(L)$ and $\tilde{\alpha}(L)$, but also with the following one:

<u>Definition 38</u>

The logarithmic Sobolev constant associated to L is defined by

$$\alpha(L) := \inf_{f \in \mathcal{F}_+(V) \setminus \mathrm{Vect}(\mathbb{1})} \frac{\mathcal{E}_L(\sqrt{f}, \sqrt{f})}{\mathrm{Ent}_\pi(f)} \qquad (22)$$

This constant $\alpha(L)$ comes from the investigation of hypercontractivity,

Write a short introduction ?

initially in continuous infinite dimensional frameworks.

Its interest for us is that it is simpler to approximate than its "modified" version.

Here is a first comparison:

Lemma 39

We have

$$\tilde{\alpha}(L) \geq 2\alpha(L)$$

Proof

The elementary bound

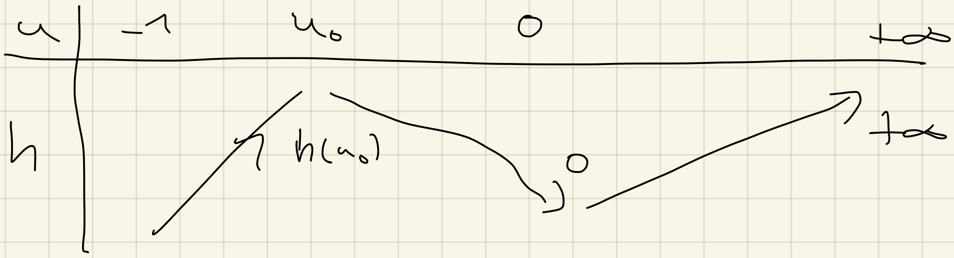
$$\forall\, t \geq 0, \quad \forall\, s \geq -t$$

$$(t+s)\ln(t+s) \geq t\ln t + (1+\ln(t))s + \left(\sqrt{t+s} - \sqrt{t}\right)^2 \qquad (22)$$

is a refinement of the convexity of the mapping $\mathbb{R}_+ \ni t \longmapsto t\ln(t)$

It is shown by posing $u = s/t$ and considering the function $h(u)$ obtained by substracting the r.h.s. to the l.h.s.

The variation table of h is



for a certain value $u_0 \in (-1, 0)$.

Consider $f \in \mathcal{F}_+(V)$ and $x, y \in V$ and apply (23) to $t+s := f(y)$ and $t := f(x)$, to get

$$f(y)\left(\ln f(y) - \ln f(x)\right) \geq f(y) - f(x) + \left(\sqrt{f(y)} - \sqrt{f(x)}\right)^2$$

Multiply by $\pi(x) L(x, y)$ and sum over $y, x \in V$ to get

$$\pi\left[f\, L[\ln f]\right] \geq \pi[L f] + 2\,\mathcal{E}(\sqrt{f}, \sqrt{f})$$

$$= 2\,\mathcal{E}_L(\sqrt{f}, \sqrt{f})$$

namely

$$\mathcal{E}_L(f, \ln f) \geq 2\,\mathcal{E}_L(\sqrt{f}, \sqrt{f})$$

which leads to the desired bound by minimizing over $f \in \mathcal{F}_+(V)$. $\square$

Let us consider the particular case of the generator L_π where $\pi \in \mathcal{P}(V)$, $\pi > 0$ is given

Lemma 40

We have

$$\alpha(L_\pi) \geq \frac{(1 - \sqrt{\pi_1})^2}{\ln \frac{1}{\pi_1} - 1 + \pi_1}$$

$=: a_{\pi_1}$

Proof

It is based on the elementary bound:

$\forall A > 1, \quad \forall r \in [0, A],$

$$(\sqrt{r} - 1)^2 \geq \frac{(\sqrt{A} - 1)^2}{A \ln A - A + 1} \left(r \ln r - r + 1 \right) \tag{24}$$

Indeed for any $a < 1/2$, consider the mapping

$$\varphi_a : \mathbb{R}_+ \ni r \mapsto (\sqrt{r} - 1)^2 - a(r \ln r - r + 1)$$

We compute

$$\forall r > 0, \quad \varphi_a''(r) = \frac{1}{r}\left(\frac{1}{2\sqrt{r}} - a\right)$$

We deduce the variation table

r		1		$\frac{1}{(2a)^2}$	
φ''	$+$			0	$-$
φ'	$\nearrow$	0	$\nearrow$		$\searrow$
φ	$\searrow$	0	$\nearrow$		$\searrow$

Thus taking $a < 1/2$ such that $\varphi_a(A) = 0$, for $A > 1$, we get that $\varphi_a \geq 0$ on $[0, A]$, this amounts to (24).

By definition, we have

$$\alpha(L_\pi) = \inf_{f \in \mathcal{F}_+(V) \setminus \mathrm{Vect}(\mathbb{1})} \frac{\mathcal{E}_{L_\pi}(\sqrt{f}, \sqrt{f})}{\mathrm{Ent}_\pi(f)}$$

$$= \inf_{\substack{f \in \mathcal{F}_+(V) \setminus \{\mathbb{1}\} \\ \pi[f] = 1}} \frac{\mathcal{E}_{L_\pi}(\sqrt{f}, \sqrt{f})}{\mathrm{Ent}_\pi(f)}$$

Since both the numerator and denominator are 1-homogeneous.

For $f \in \mathcal{F}_+(V)$, with $\pi(f) = 1$, we have

$$\forall x \in V \qquad f(x) \leq \frac{1}{\pi(x)} \leq \frac{1}{\pi_*}$$

Applying (24) with $r = f(y)$

, for $f \in \mathcal{F}_+(V)$, $\pi(f)=1$, and $x \in V$

and integrating with respect to π

in x, we get

$$\underbrace{\pi\left[\left(\sqrt{f}-1\right)^2\right]}_{\substack{\shortparallel \\ \mathrm{Var}(\sqrt{f},\pi) \\ \shortparallel \\ \mathcal{E}_{L_\pi}(\sqrt{f},\sqrt{f})}} \geqslant \underbrace{\frac{\left(\sqrt{1/\pi_1}-1\right)^2}{\frac{1}{\pi_1}\ln\frac{1}{\pi_1}-\frac{1}{\pi_1}+1}}_{\frac{\left(1-\sqrt{\pi_1}\right)^2}{\ln 1/\pi_1 - 1 + \pi_1}} \mathrm{Ent}_\pi(f)$$

which implies the wanted bound.

$\square$

<u>Remark</u>

The true value of $\alpha(L_\pi)$ has been obtained by more involved computations:

$$\alpha(L_\pi) = \frac{1-2\pi_1}{\ln\left((1-\pi_1)/\pi_1\right)}$$

But our bound is not bad as $\pi_1 \to 0_+$:

$$a_{\pi_1} := \frac{(1-\sqrt{\pi_1})^2}{\ln 1/\pi_1 - 1 + \pi_1} \sim \frac{1}{\ln 1/\pi_1} \sim \alpha(L_\pi)$$

(But for $\pi_1 = 1/2$, $a_{1/2} \simeq 0{,}444$

while $\alpha(L_{\underset{(1/2,1/2)}{\overset{\shortparallel}{\pi}}}) = 1/2 = \frac{\lambda(L_\pi)}{2}$)

Corollary 41

For any L irreducible and reversible with respect to π, we have

$$\tilde{\alpha}(L) \geq 2\alpha(L) \geq 2a_{\pi_1} \lambda(L)$$

In particular both $\tilde{\alpha}(L)$ and $\alpha(L)$ are positive

Proof

It is sufficient to show the bound

$$\alpha(L) \geq a_\pi \lambda(L)$$

Consider $f \in \mathcal{F}_+(V)$, we have

$$\mathcal{E}_L(\sqrt{f}, \sqrt{f}) \geq \lambda(L) \operatorname{Var}(\sqrt{f}, \pi)$$

$$\geq \lambda(L)\, a_{\pi_\lambda} \operatorname{Ent}(f, \pi)$$

according to lemma 40.

$\square$

Both the modified log-Sob and the log-Sob constants enjoy the tensorization property.

Let us show it for $\tilde{\alpha}(L)$, leaving the $\alpha(L)$ case as an exercise. (write it!)

Consider $d \geq 2$ finite spaces $V_1, \ldots, V_d$ endowed with irreducible Markov generators $L_1, \ldots, L_d$, reversible with respect to $\pi_1, \ldots, \pi_d$.

Define on $V = V_1 \times \cdots \times V_d$

$L = L_1^{(1)} + \cdots + L_d^{(d)}$, which is irred. and reversible $/\pi := \pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_d$

where $L_k^{(k)}$ acts on the factor V_k as L_k, leaving the other factors free of motion.

<u>Theorem 42</u>

We have
$$\tilde{\alpha}(L) = \min_{1 \leq k \leq d} \tilde{\alpha}(L_k)$$

<u>Proof</u>

By iteration, it is sufficient to consider the case $d = 2$.

Consider any $f \in \mathcal{F}_+(V)$ with $\pi(f)$, we want to check that

$$a \, \mathrm{Ent}_\pi(f) \leq \mathcal{E}_L(f, \ln f)$$

where $a = \tilde{\alpha}(L_1) \wedge \tilde{\alpha}(L_2)$

Write μ the probability measure admitting f as density / π.

Let us decompose μ in

$$\forall (x_1,x_2)\in V, \quad \mu(x_1,x_2) = \underbrace{\mu_2(x_2|x_1)}_{\substack{\text{conditional}\\ \text{probability}\\ \text{of } x_2 \text{ knowing}\\ x_1}} \underbrace{\mu_1(x_1)}_{\substack{\text{marginal}\\ \text{of } \mu\\ \text{on } V_1}}$$

We have

$$\mathrm{Ent}_\pi(f) = \sum_{(x_1,x_2)\in V} \ln \frac{\mu_2(x_2|x_1)\mu_1(x_1)}{\pi_2(x_2)\,\pi_1(x_1)} \,\mu_2(x_2|x_1)\mu_1(x_1)$$

$$= \sum_{(x_1,x_2)\in V} \ln \frac{\mu_2(x_2|x_1)}{\pi_2(x_2)} \,\mu_2(x_2|x_1)\mu_1(x_1)$$

$$+ \sum_{(x_1,x_2)\in V} \ln \frac{\mu_1(x_1)}{\pi_1(x_1)} \,\mu_2(x_2|x_1)\,\mu_1(x_1)$$

$$= \sum_{x_1\in V} \mathrm{Ent}\left(\mu_2(\cdot|x_1)\,|\,\pi_2\right)\mu_1(x_1)$$

$$+ \sum_{x_1\in V} \ln \frac{\mu_1(x_1)}{\pi_1(x_1)} \,\mu_1(x_1)$$

$$= \sum_{x_1\in V} \mathrm{Ent}_{\pi_2}\left(f_2(\cdot|x_1)\right)\mu_1(x_1)$$

$$+ \mathrm{Ent}_{\pi_1}(f_1) \qquad (25)$$

where for all $(x_1, x_2) \in V$,

$$f_2(x_2 | x_1) = \frac{\mu_2(x_2 | x_1)}{\pi_2(x_2)}$$

$$f_1(x_1) = \frac{\mu_1(x_1)}{\pi_1(x_1)}$$

Let us first consider the part $L_2^{(2)}$ of L:

$$\mathcal{E}_{L_2^{(2)}}(\ln f, f) =$$

$$-\sum_{x, y \in V} f(x)\left(\ln f(y) - \ln f(x)\right) \pi(x) L_2^{(2)}(x, y)$$

$$= \sum_{\substack{(x_1, x_2) \in V \\ y_2 \in V}} f(x_1, x_2)\left(\ln f(x_1, y_2) - \ln f(x_1, x_2)\right) \pi(x_1)\pi(x_2) L_2(x_2, y_2)$$

$$= \sum_{x_1 \in V_1} \mathcal{E}_{L_2}\left(f_2(\cdot | x_1), \ln f_2(\cdot | x_1)\right) f_1(x_1)\pi(x_1)$$

$$\geq \tilde{\alpha}(L_2) \sum_{x_1 \in V_1} \mathrm{Ent}_{\pi_2}\left(f_2(\cdot | x_1)\right) \mu_1(x_1)$$

Which is the first part in (25).

Coming to $L_1^{(1)}$, we get

Symmetrically,

$$\mathcal{E}_{L_1^{(1)}}(f,f) \geq \tilde{\alpha}(L_1) \sum_{x_2 \in V_2} \mathrm{Ent}_{\pi_1}(f_1(\cdot \mid x_2))\, \mu_2(x_2)$$

Comparing with the second part in (2C), it remains to show that

$$\sum_{x_2 \in V_2} \mathrm{Ent}_{\pi_1}(f_1(\cdot \mid x_2))\, \mu_2(x_2) \geq \mathrm{Ent}_{\pi_1}(f_1) \qquad (26)$$

Using Jensen's inequality we get

$$\sum_{x_2 \in V_2} \mathrm{Ent}_{\pi_1}(f_1(\cdot \mid x_2))\, \mu_2(x_2)$$

$$= \sum_{x_2 \in V_2} \pi_1\left[\varphi_1(f_1(\cdot \mid x_2)\right]\mu_2(x_2)$$

$$\geq \pi_1\left[\varphi_1\left(\sum_{x_2 \in V_2} f_1(\cdot \mid x_2)\,\mu_2(x_2)\right)\right]$$

since φ_1 is convex.

For any $x_1 \in V_1$, we have

$$\sum_{x_2 \in V} f_1(x_1 \mid x_2)\, \mu_2(x_2) = \sum_{x_2 \in V} \frac{\mu_1(x_1 \mid x_2)\,\mu_2(x_2)}{\pi_1(x_1)}$$

$$= \frac{\mu_1(x_1)}{\pi_1(x_1)} = f_1(x_1)$$

Thus (26) follows.

$\square$

Let us come back to Example 29.

Since π is uniform over $\mathbb{Z}_N^d$,

we have $\pi_1 = \frac{1}{N^d}$.

Proposition 41 then implies as $N \to \infty$

$$\tilde{\alpha}(L_N^{(d)}) \gtrsim \frac{2}{d \ln(N)} \lambda(L_N^{(d)})$$

$$\sim \frac{2}{d \ln(N)} \frac{4\pi^2}{N^2}$$

It is thus better to use

Theorem 42:

$$\tilde{\alpha}(L_N^{(1)}) = \tilde{\alpha}(L_N)$$

$$\geqslant \frac{2}{\ln(N)} \lambda(L_N)$$

$$\sim \frac{2}{\ln(N)} \frac{4\pi^2}{N^2} \qquad (27)$$

We have for any $t \geq 0$,

$$\| m_t - \pi \|_{tv} \leq e^{-\tilde{\alpha}(L_N^{(1)})t} \sqrt{\frac{\mathrm{Ent}(m_0 \mid \pi)}{2}}$$

using Jensen, the worse case is when m_0 is a Dirac mass at some $x_0 \in V$

put that directly in (21)

$$\mathrm{Ent}(\delta_{x_0} \mid \pi) = \ln \frac{1}{\pi(x_0)} \leq \ln \frac{1}{\pi_1}$$

Let be given $\varepsilon > 0$.

We will have $\| m_t - \pi \|_{tv} \leq \varepsilon$

as soon as

$$\sqrt{\frac{\ln 1/\pi_1}{2}}\, e^{-\tilde{\alpha}(L_N^{(d)})t} \leq \varepsilon$$

and using (27) it will satisfied for

some $t \sim \frac{N^2 \ln(N)}{8\pi^2} \left(\frac{1}{2} \ln \frac{\ln(N^1)}{2} + \ln 1/\varepsilon \right)$

for fixed $\varepsilon > 0$, as $N, d \to \infty$

$$\sim \frac{N^2 \ln(N)}{16 \pi^2} \left(\ln d + \ln(\ln(N)) \right)$$

which is much better in term of $d \gg 1$ than the bound obtained using the spectral gap.

Remark

In Corollary 4.1, the bound

$$\tilde{\alpha}(L) \geq 2\, a_{\pi_1}\, \lambda(L)$$

can be far from optimal.

Indeed let us come back to L_π, for $\pi > 0$ on V.

we compute that the spectral decomposition of $-L_\pi$ is

eigenvalue	multiplicity	eigenspace
0	$[1]$	$\mathrm{Vect}(\mathbb{1})$
1	$[\lvert V \rvert - 1]$	$\{ f \in \mathcal{F}(V) : \pi[f] = 0 \}$

so that $\alpha(L_\pi) = 1$

Furthermore, $\forall f \in \mathcal{F}_+(V)$ with $\pi[f]=1$, we have

$$\mathcal{E}_{L_\pi}(f, \ln f) = \sum_{x,y \in V} \pi(x)\pi(y)\, f(x)\left(\ln f(x) - \ln f(y)\right)$$

$$= \sum_{x \in V} \pi(x) f(x) \ln f(x) \sum_{y \in V} \pi(y)$$

$$- \sum_{x \in V} \pi(x) f(x) \sum_{y \in V} \pi(y) \ln f(y)$$

$$= \mathrm{Ent}_\pi(f) - \pi[f]\, \pi[\ln f]$$

Note that by Jensen

$$\pi[\ln f] \leq \ln \pi[f] = 0$$

So $\mathcal{E}_{L_\pi}(f, \ln f) \geq \mathrm{Ent}_\pi(f)$

and $\tilde{\alpha}(L_\pi) \geq 1$

But $a_{\pi_1} \xrightarrow[\pi_1 \to 0]{} +\infty$.