

Long time behaviour of the Euler scheme associated to a SDE with memory.

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Euler Scheme with step $h > 0$

Let $X := (X_n)_{n \geq 0}$ be an \mathbb{R}^d -valued process such that

$$X_{n+1} = X_n + hb(X_n) + \sigma(X_n)\Delta_{n+1} \quad (1.1)$$

where $\Delta_{n+1} := Z_{(n+1)h} - Z_{nh}$ corresponds to the increments, assumed to be stationary and ergodic, of a Gaussian process (Z_t) .

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Example of noise

Fractional Brownian motion (fBm) increments with Hurst parameter $H \in (0, 1)$ denoted by $(B_t^H)_{t \in \mathbb{R}}$.

The fBm is a centered Gaussian process with stationary increments such that for all t, s ,

$$\mathbb{E}[(B_t^H - B_s^H)^2] = |t - s|^{2H}.$$

Moving average representation

Wold's decomposition theorem,

$$\forall n \in \mathbb{Z}, \quad \Delta_n = \sum_{k=0}^{+\infty} a_k \xi_{n-k} \quad (1.2)$$

with

$$\left\{ \begin{array}{l} (a_k)_{k \geq 0} \in \mathbb{R}^{\mathbb{N}} \text{ such that } a_0 \neq 0 \text{ and } \sum_{k=0}^{+\infty} a_k^2 < +\infty \\ (\xi_k)_{k \in \mathbb{Z}} \text{ an i.i.d sequence such that } \xi_1 \sim \mathcal{N}(0, I_d). \end{array} \right.$$

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Remarks

- ▷ Without loss of generality, we can take $a_0 = 1$. Otherwise we can go back to this case by setting $\tilde{\Delta}_n = \sum_{k=0}^{+\infty} \tilde{a}_k \xi_{n-k}$ with $\tilde{a}_k = a_k/a_0$.
- ▷ $\mathbb{E}[\Delta_n \Delta_{n+k}] = \sum_{i=0}^{+\infty} a_i a_{k+i}$

Tool : a Toeplitz type operator

Definition

Let \mathbf{T}_a be defined on $\ell_a(\mathbb{Z}^-, \mathbb{R}^d) := \left\{ w \in (\mathbb{R}^d)^{\mathbb{Z}^-} \mid \forall k \geq 0, \sum_{l=0}^{+\infty} a_l w_{-k-l} < +\infty \right\}$ by

$$\forall w \in \ell_a(\mathbb{Z}^-, \mathbb{R}^d), \quad \mathbf{T}_a(w) = \left(\sum_{l=0}^{+\infty} a_l w_{-k-l} \right)_{k \geq 0}. \quad (1.3)$$

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Remark : This operator links $(\Delta_n)_{n \in \mathbb{Z}}$ to the underlying noise process $(\xi_n)_{n \in \mathbb{Z}}$.

Proposition

Let \mathbf{T}_b be defined on $\ell_b(\mathbb{Z}^-, \mathbb{R}^d)$ with the following sequence $(b_k)_{k \geq 0}$

$$b_0 = \frac{1}{a_0} \quad \text{and} \quad \forall k \geq 1, \quad b_k = -\frac{1}{a_0} \sum_{l=1}^k a_l b_{k-l}. \quad (1.4)$$

Then, $\mathbf{T}_b = \mathbf{T}_a^{-1}$.

Let $\mathcal{X} := \mathbb{R}^d$ be the state space and $\mathcal{W} := (\mathbb{R}^d)^{\mathbb{Z}^-}$ be the noise space.

Equivalent system:

$$(X_{n+1}, (\Delta_{n+1+k})_{k \leq 0}) = \varphi((X_n, (\Delta_{n+k})_{k \leq 0}), \Delta_{n+1}) \quad (1.5)$$

where

$$\begin{aligned} \varphi : (\mathcal{X} \times \mathcal{W}) \times \mathbb{R}^d &\rightarrow \mathcal{X} \times \mathcal{W} \\ ((x, w), \delta) &\mapsto (x + hb(x) + \sigma(x)\delta, w \sqcup \delta). \end{aligned}$$

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Transition kernel: $\mathcal{Q} : \mathcal{X} \times \mathcal{W} \rightarrow \mathcal{M}_1(\mathcal{X} \times \mathcal{W})$

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Uniqueness: We define $S\mu := \mathcal{L}((X_n^\mu)_{n \geq 0})$. Then, $\mu \simeq \nu \iff S\mu = S\nu \quad (*)$

(H_{poly}): The following conditions are satisfied,

- there exist $\rho, \beta > 0$ and $C_\rho, C_\beta > 0$ such that

$$\forall k \geq 0, \quad |a_k| \leq C_\rho (k+1)^{-\rho} \quad \text{and} \quad \forall k \geq 0, \quad |b_k| \leq C_\beta (k+1)^{-\beta}.$$

- there exist $\kappa \geq \rho + 1$ and $C_\kappa > 0$ such that

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(H_{b,σ}): $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is continuous, $\sigma : \mathbb{R}^d \rightarrow \mathcal{M}_d(\mathbb{R})$ is bounded, continuous and $\sigma^{-1} : x \mapsto \sigma(x)^{-1}$ is well defined and continuous. Moreover,

- $\exists C > 0$ such that $\forall x \in \mathcal{X}, \quad |b(x)| \leq C(1 + |x|)$
- $\exists \tilde{\beta} \in \mathbb{R}$ and $\tilde{\alpha} > 0$ such that $\forall x \in \mathcal{X}, \quad \langle x, b(x) \rangle \leq \tilde{\beta} - \tilde{\alpha}|x|^2$.

Theorem

Assume $(\mathbf{H}_{b,\sigma})$. Then,

- (i) there exists an invariant distribution μ_\star associated to (1.1).
- (ii) Assume $(\mathbf{H}_{\text{poly}})$ with $\rho, \beta > 1/2$ and $\rho + \beta > 3/2$. Then, uniqueness holds for μ_\star . Moreover, for every initial condition μ_0 such that $\int_{\mathcal{X}} |x| \Pi_{\mathcal{X}}^* \mu_0(dx) < +\infty$ and for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\|\mathcal{L}((X_{n+k}^{\mu_0})_{k \geq 0}) - S\mu_\star\|_{TV} \leq C_\varepsilon n^{-(v(\beta, \rho) - \varepsilon)}.$$

where the function v is given by

$$v(\beta, \rho) = \sup_{\alpha \in (\frac{1}{2} \vee (\frac{3}{2} - \beta), \rho)} \min\{1, 2(\rho - \alpha)\} (\min\{\alpha, \beta, \alpha + \beta - 1\} - 1/2).$$

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Example: fBm (with $H \in (0, 1/2)$)

Convergence to equilibrium : $n^{-(v_H - \varepsilon)}$ with

$$v_H = \begin{cases} H(1 - 2H) & \text{if } H \in (0, 1/4] \\ 1/8 & \text{if } H \in (1/4, 1/2) \end{cases}$$

Scheme of coupling: Let us consider (X^1, X^2) solution of the system:

$$\begin{cases} X_{n+1}^1 = X_n^1 + hb(X_n^1) + \sigma(X_n^1)\Delta_{n+1}^1 \\ X_{n+1}^2 = X_n^2 + hb(X_n^2) + \sigma(X_n^2)\Delta_{n+1}^2 \end{cases} \quad (3.1)$$

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We have

$$\|\mathcal{L}((X_{n+k}^1)_{k \geq 0}) - \mathcal{S}\mu_\star\|_{TV} \leq \mathbb{P}(\tau_\infty > n).$$

where $\tau_\infty := \inf\{n \geq 0 \mid X_k^1 = X_k^2, \forall k \geq n\}$.

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We define the sequence of r.v. $(g_n)_{n \in \mathbb{Z}}$ by

$$\forall n \in \mathbb{Z}, \quad \xi_{n+1}^1 = \xi_{n+1}^2 + g_n, \quad \text{hence} \quad g_n = 0 \quad \forall n < 0.$$

Step of the coupling procedure

- ▷ **Step 1** : Try to stick the positions at a given time with a “controlled cost”.

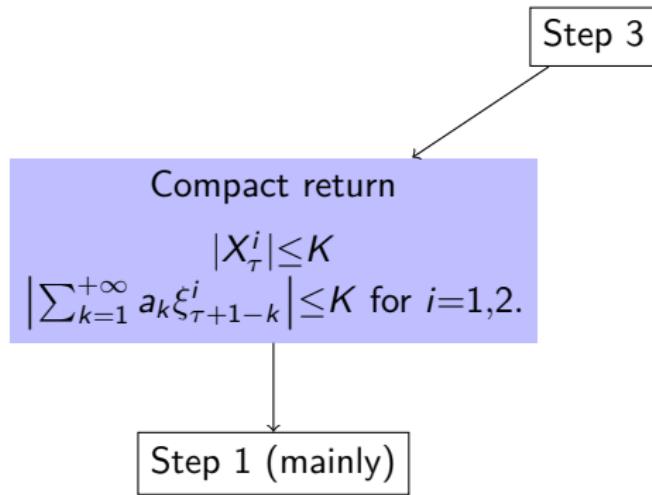
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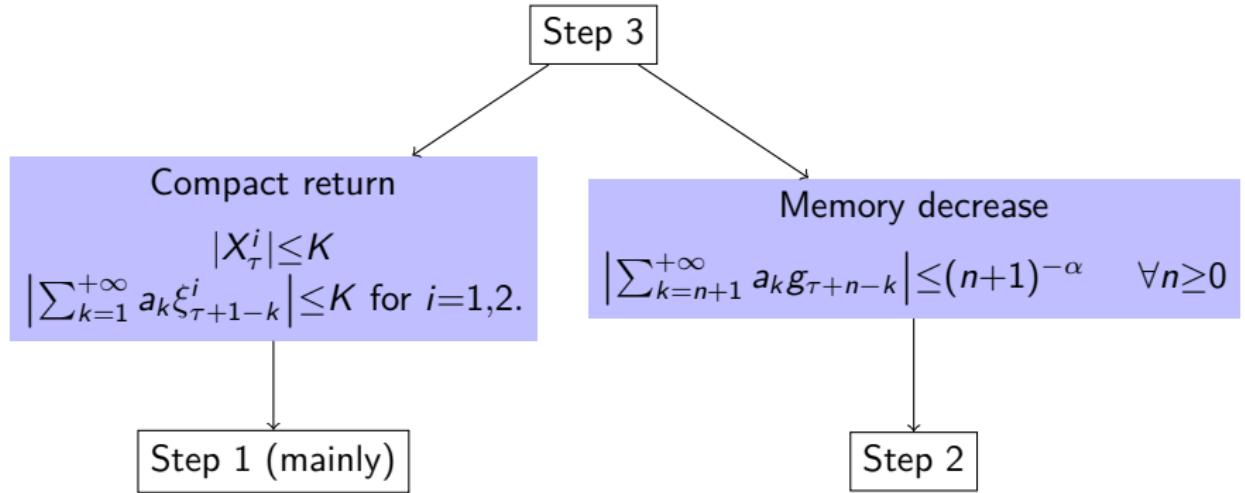
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- ▷ **Step 2** : Try to keep the paths fastened together (specific to non-Markov process).

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- ▷ **Step 2** : Try to keep the paths fastened together (specific to non-Markov process).
- ▷ **Step 3** : If Step 2 fails, impose $g_n = 0$ and wait long enough in order to allow Step 1 to be realized with a “controlled cost” and with a positive probability.

Step 3





At a given time $(\tau + 1)$, we want to build $(\xi_{\tau+1}^1, \xi_{\tau+1}^2)$ in order to get
 $X_{\tau+1}^1 = X_{\tau+1}^2$, i.e.

$$X_\tau^1 + hb(X_\tau^1) + \sigma(X_\tau^1) \sum_{k=0}^{+\infty} a_k \xi_{\tau+1-k}^1 = X_\tau^2 + hb(X_\tau^2) + \sigma(X_\tau^2) \sum_{k=0}^{+\infty} a_k \xi_{\tau+1-k}^2$$

$$\iff \xi_{\tau+1}^2 = \Lambda_{\mathbf{X}}(\xi_{\tau+1}^1) \text{ where } \mathbf{X} = \left(X_\tau^1, X_\tau^2, \sum_{k=1}^{+\infty} a_k \xi_{\tau+1-k}^1, \sum_{k=1}^{+\infty} a_k \xi_{\tau+1-k}^2 \right) \quad (3.2)$$

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Coupling Lemma to build $(\xi_{\tau+1}^1, \xi_{\tau+1}^2)$:

- Ensure (3.2) with positive probability.
- $|\xi_{\tau+1}^1 - \xi_{\tau+1}^2| \leq M_K$ a.s.

Keep the paths fastened : $X_{n+1}^1 = X_{n+1}^2 \quad \forall n \geq \tau + 1$, i.e.

$$X_n^1 + hb(X_n^1) + \sigma(X_n^1) \sum_{k=0}^{+\infty} a_k \xi_{n+1-k}^1 = X_n^1 + hb(X_n^1) + \sigma(X_n^1) \sum_{k=0}^{+\infty} a_k \xi_{n+1-k}^2$$

$$\begin{aligned} &\iff \forall n \geq \tau + 1, \quad \xi_{n+1}^1 - \xi_{n+1}^2 = g_n^{(s)} = - \sum_{k=1}^{+\infty} a_k \textcolor{red}{g_{n-k}} \\ &\iff \forall n \geq 1, \quad g_{\tau+n}^{(s)} = - \sum_{k=1}^n a_k g_{\tau+n-k}^{(s)} - \sum_{k=n+1}^{+\infty} a_k \textcolor{blue}{g_{\tau+n-k}}. \end{aligned} \quad (3.3)$$

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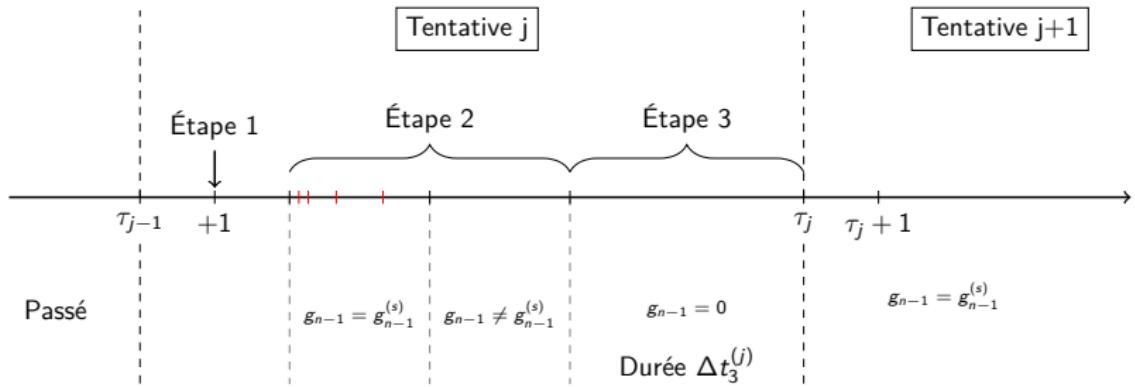
Coupling Lemma to build $((\xi_{\tau+n+1}^1, \xi_{\tau+n+1}^2))_{n \in [\![1, T]\!]}$:

- Ensure (3.3) with controlled positive probability.
- $\|(g_{\tau+n})_{n \in [\![1, T]\!]} \|$ a.s. controlled.

Aim : Determine for which value of $p > 0$ we can control $\mathbb{E}[\tau_\infty^p]$ since:

$$\mathbb{P}(\tau_\infty > n) \leq \frac{\mathbb{E}[\tau_\infty^p]}{n^p}$$

where $\tau_\infty := \inf\{n \geq 0 \mid X_k^1 = X_k^2, \forall k \geq n\}$.



Étape 1

À un instant $(n+1)$, on veut construire $(\xi_{n+1}^1, \xi_{n+1}^2)$ pour que $X_{n+1}^1 = X_{n+1}^2$, i.e.

$$F\left(X_n^1, \xi_{n+1}^1 + \sum_{k=1}^{+\infty} a_k \xi_{n+1-k}^1\right) = F\left(X_n^2, \xi_{n+1}^2 + \sum_{k=1}^{+\infty} a_k \xi_{n+1-k}^2\right)$$

Lemme 1 (inspiré de version continue J.Fontbona & F.Panloup)

Soit $K > 0$ et $\mu := \mathcal{N}(0, I_d)$. Sous **(H₂)**, il existe $\tilde{K} > 0$, tel que pour tout $(x, x', y, y') \in B(0, K)^4$, on peut construire (Z_1, Z_2) tel que

- (i) $\mathcal{L}(Z_1) = \mathcal{L}(Z_2) = \mu$,
- (ii) il existe $\delta_{\tilde{K}} > 0$ tel que

$$\mathbb{P}(F(x, Z_1 + y) = F(x', Z_2 + y')) \geq \delta_{\tilde{K}} > 0 \quad (3.4)$$

- (iii) il existe $M_K > 0$ tel que

$$\mathbb{P}(|Z_2 - Z_1| \leq M_K) = 1. \quad (3.5)$$

Étape 1

On veut coupler à l'instant $\tau_{j-1} + 1 \implies$ on applique le lemme 1 avec
 $(x, x', y, y') := \left(X_{\tau_{j-1}}^1, X_{\tau_{j-1}}^2, \sum_{k=1}^{+\infty} a_k \xi_{\tau_{j-1}+1-k}^1, \sum_{k=1}^{+\infty} a_k \xi_{\tau_{j-1}+1-k}^2 \right)$ et on pose

$$(\xi_{\tau_{j-1}+1}^1, \xi_{\tau_{j-1}+1}^2) = (\mathbb{1}_{\Omega_{K,\alpha,\tau_{j-1}}} Z_1 + \mathbb{1}_{\Omega_{K,\alpha,\tau_{j-1}}^c} \xi, \quad \mathbb{1}_{\Omega_{K,\alpha,\tau_{j-1}}} Z_2 + \mathbb{1}_{\Omega_{K,\alpha,\tau_{j-1}}^c} \xi)$$

où $\xi \sim \mathcal{N}(0, 1)$ indépendante de (Z_1, Z_2) .

- $\mathbb{P}(\text{succès de l'étape 1} | \Omega_{K,\alpha,\tau_{j-1}}) \geq \delta_K > 0$
- $|g_{\tau_{j-1}}| = |\xi_{\tau_{j-1}+1}^1 - \xi_{\tau_{j-1}+1}^2| \leq M_K \quad p.s$

On pose $\mathcal{A}_{j,\ell} := \{\text{échec étape 2 de la tentative } j \text{ après } \ell \text{ essais exactement}\}$

$$\begin{aligned} & \mathbb{E}[|\Delta\tau_j|^p \mathbb{1}_{\{\Delta\tau_j < +\infty\}} \mid \{\tau_{j-1} < +\infty\}] \\ &= \sum_{\ell=1}^{+\infty} \mathbb{E}[\mathbb{1}_{\mathcal{A}_{j,\ell}} |\Delta\tau_j|^p \mathbb{1}_{\{\Delta\tau_j < +\infty\}} \mid \{\tau_{j-1} < +\infty\}]. \end{aligned}$$

Sur l'événement $\mathcal{A}_{j,\ell}$,

$$\Delta\tau_j = c_2 2^{\ell+1} + \Delta t_3^{(j)} \leq C \varsigma^j 2^{(\theta \vee 1)\ell}.$$

De plus, d'après le lemme de couplage de l'étape 2,

$$\mathbb{P}(\mathcal{A}_{j,\ell} \mid \{\tau_{j-1} < +\infty\}) = \mathbb{P}(\mathcal{B}_{j,\ell}^c \mid \mathcal{B}_{j,\ell-1}) \leq 2^{-\tilde{\alpha}\ell}$$

donc

$$\mathbb{E}[|\Delta\tau_j|^p \mathbb{1}_{\{\Delta\tau_j < +\infty\}} \mid \{\tau_{j-1} < +\infty\}] \leq C \varsigma^{jp} \iff p \in \left(0, \frac{\tilde{\alpha}}{\theta \vee 1}\right).$$