

# Concentration inequalities for Stochastic Differential Equations with additive fractional noise

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# Fractional SDE

Let  $Y := (Y_t)_{t \geq 0}$  be an  $\mathbb{R}^d$ -valued process such that

$$Y_t = x + \int_0^t b(Y_s)ds + \sigma B_t. \quad (1.1)$$

where  $x \in \mathbb{R}^d$ ,  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $\sigma \in \mathcal{M}_d(\mathbb{R})$  and  $B$  is a  $d$ -dimensional fractional Brownian motion with Hurst parameter  $H \in (0, 1)$ .

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**Aim** : For all function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  Lipschitz, we wish to control :

- \*  $\mathbb{P} \left( \frac{1}{n} \sum_{k=1}^n (f(Y_{k\Delta}) - E[f(Y_{k\Delta})]) > r \right)$  for a fixed  $\Delta > 0$ ,
- \*  $\mathbb{P} \left( \frac{1}{T} \int_0^T (f(Y_t) - \mathbb{E}[f(Y_t)]) dt > r \right)$

with respect to  $n$  and  $T$ .

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- 4 Sketch of proof

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## Definition

Let  $H \in (0, 1)$ . The *fractional Brownian motion* (fBm) with Hurst parameter  $H$ , denoted by  $(B_t)_{t \geq 0}$ , is a centered Gaussian process with covariance function given by :

$$\mathbb{E}[B_t B_s] = \frac{1}{2} [t^{2H} + s^{2H} - |t - s|^{2H}] \quad \text{for all } t, s \geq 0.$$

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- Stationary increments :

$$\mathbb{E}[(B_t - B_s)^2] = |t - s|^{2H}.$$

- Self-similarity :

$$\mathcal{L}((B_{ct})_{t \geq 0}) = \mathcal{L}(c^H(B_t)_{t \geq 0}) \quad \text{for all } c > 0.$$

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## Remark

The fBm is neither a semimartingale nor a Markov process except for  $H = 1/2$ . In that case,  $B$  is the standard Brownian motion and has independent increments.

## Memory :

$$\begin{aligned}\rho(k) &:= \mathbb{E}[(B_{k+1} - B_k)(B_1 - B_0)] \\ &= \frac{1}{2} \left[ (k+1)^{2H} - 2k^{2H} + (k-1)^{2H} \right] \underset{k \rightarrow +\infty}{\sim} H(2H-1)k^{2H-2}.\end{aligned}$$

- If  $H < 1/2$ ,  $\sum_{k=1}^{+\infty} \rho(k) < +\infty \longrightarrow \text{Short memory}$
- If  $H > 1/2$ ,  $\sum_{k=1}^{+\infty} \rho(k) = +\infty \longrightarrow \text{Long memory}$

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Proposition (Regularity)

Let  $H \in (0, 1)$ . The sample path of fBm with Hurst parameter  $H$  are a.s. locally  $\beta$ -Hölder for all  $\beta < H$ . Namely, for all  $T > 0$  and for all  $0 < \beta < H$ ,

$$\|B\|_{\beta, [0, T]} := \sup_{0 \leq s < t \leq T} \frac{|B_t - B_s|}{|t - s|^\beta} < +\infty \quad \text{a.s.}$$

## Proposition (Volterra representation)

Let  $(W_t)_{t \geq 0}$  be a standard Brownian motion, then the process  $B$  defined by :

$$B_t = \int_0^t K_H(t, s) dW_s, \quad t \geq 0, \tag{2.1}$$

where  $K_H$  is the deterministic kernel given by

$$K_H(t, s) = c_H \left[ \frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}} (t-s)^{H-\frac{1}{2}} - \left( H - \frac{1}{2} \right) \int_s^t \frac{u^{H-\frac{3}{2}}}{s^{H-\frac{1}{2}}} (u-s)^{H-\frac{1}{2}} du \right] \tag{2.2}$$

is a fBm.

### Remark

$$\mathbb{E}[B_t B_s] = \int_0^{t \wedge s} K_H(t, u) K_H(s, u) du = \frac{1}{2} [t^{2H} + s^{2H} - |t-s|^{2H}].$$

## Back to the fractional SDE

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where  $x \in \mathbb{R}^d$ ,  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $\sigma \in \mathcal{M}_d(\mathbb{R})$  and  $W$  is a  $d$ -dimensional standard Brownian motion.

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Let  $(E, d)$  be a metric space. Given  $p \geq 1$  and two probability measures  $\mu$  and  $\nu$  on  $E$ , the Wasserstein distance is defined by

$$\mathcal{W}_p(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \left( \int_{E \times E} d(x, y)^p d\pi(x, y) \right)^{1/p},$$

where  $\Pi(\mu, \nu) := \{\pi \in \mathcal{M}_1(E \times E) \text{ telles que } \pi(., E) = \mu(.) \text{ et } \pi(E, .) = \nu(.)\}$ .

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The entropy of  $\nu$  with respect to  $\mu$  is defined by

$$\mathbf{H}(\nu|\mu) = \begin{cases} \int \log \left( \frac{d\nu}{d\mu} \right) d\nu, & \text{if } \nu \ll \mu, \\ +\infty & \text{otherwise.} \end{cases}$$

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Then, we say that  $\mu$  satisfies an  $L^p$ -transportation inequality with constant  $C \geq 0$  (denoted by  $\mu \in T_p(C)$ ) if for any probability measure  $\nu$ ,

$$\mathcal{W}_p(\mu, \nu) \leq \sqrt{2C\mathbf{H}(\nu|\mu)}.$$

### Theorem (Bobkov and Götze '99)

Let  $(E, d)$  be a metric space and  $\mu$  a probability measure on  $E$ . Then,  $\mu \in T_1(C)$  if and only if for any  $\mu$ -integrable Lipschitz function  $F : (E, d) \rightarrow \mathbb{R}$  we have for all  $\lambda \in \mathbb{R}$ ,

$$\mathbb{E} [e^{\lambda(F(X) - \mathbb{E}[F(X)])}] \leq \exp\left(\frac{\lambda^2}{2} C \|F\|_{\text{Lip}}^2\right)$$

with  $\mathcal{L}(X) = \mu$ . In that case,

$$\mathbb{P}(F(X) - \mathbb{E}[F(X)] > r) \leq \exp\left(-\frac{r^2}{2C\|F\|_{\text{Lip}}^2}\right), \quad \forall r > 0.$$

## Results for our fractional SDE : $\mu = \mathcal{L}((Y_t)_{t \in [0, T]})$

- For  $H > 1/2$ , Saussereau shows that  $\mu \in T_2(C_T)$  for two metrics on  $\mathcal{C}([0, T], \mathbb{R}^d)$  :

$$d_2(\gamma_1, \gamma_2) = \left( \int_0^T |\gamma_1(t) - \gamma_2(t)|^2 dt \right)^{1/2} \quad \text{and} \quad d_\infty(\gamma_1, \gamma_2) = \sup_{t \in [0, T]} |\gamma_1(t) - \gamma_2(t)|.$$

**Proof:** build a coupling of  $(\nu, \mu)$  (for  $\nu \ll \mu$ ) which is based on Girsanov applied to the underlying Brownian motion.

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- For  $H \in (0, 1)$  (and even for general Gaussian process), Riedel shows that  $\mu \in T_2(C_T)$  for the metric  $d_\infty$ .

**Proof:** show that  $B$  satisfies a transportation inequality and then show that  $B(\omega) \mapsto Y(\omega)$  is Lipschitz.

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### Theorem (Saussereau '12)

Let  $H > 1/2$ . There exists  $C > 0$  such that for all Lipschitz function  $f : (\mathbb{R}^d, |\cdot|) \rightarrow (\mathbb{R}, |\cdot|)$  and for all  $r \geq 0$ ,

$$\mathbb{P} \left( \frac{1}{T} \int_0^T (f(Y_t) - \mathbb{E}[f(Y_t)]) dt > r \right) \leq \exp \left( - \frac{r^2 T^{2-2H}}{4C \|f\|_{\text{Lip}}^2} \right).$$

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**Hypothesis :** We assume that there exist  $\alpha, L > 0$  such that: for all  $x, y \in \mathbb{R}^d$ ,

$$\langle b(x) - b(y), x - y \rangle \leq -\alpha|x - y|^2 \quad \text{et} \quad |b(x) - b(y)| \leq L|x - y|.$$

### Theorem

Let  $H \in (0, 1)$  and  $\Delta > 0$ . Let  $n \in \mathbb{N}^*$  and  $T \geq 1$ . Then,

- (i) there exists  $C_{H,\Delta} > 0$  such that for all Lipschitz function  $f : (\mathbb{R}^d, |\cdot|) \rightarrow (\mathbb{R}, |\cdot|)$  and for all  $r \geq 0$ ,

$$\mathbb{P}\left(\frac{1}{n} \sum_{k=1}^n (f(Y_{k\Delta}) - E[f(Y_{k\Delta})]) > r\right) \leq \exp\left(-\frac{r^2 n^{2-(2H\vee 1)}}{4C_{H,\Delta} \|f\|_{\text{Lip}}^2}\right).$$

- (ii) there exists  $\tilde{C}_H > 0$  such that for all Lipschitz function  $f : (\mathbb{R}^d, |\cdot|) \rightarrow (\mathbb{R}, |\cdot|)$  and for all  $r \geq 0$ ,

$$\mathbb{P}\left(\frac{1}{T} \int_0^T (f(Y_t) - \mathbb{E}[f(Y_t)]) dt > r\right) \leq \exp\left(-\frac{r^2 T^{2-(2H\vee 1)}}{4\tilde{C}_H \|f\|_{\text{Lip}}^2}\right).$$

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We set

$$F_Y := \frac{1}{n} \sum_{k=1}^n f(Y_{k\Delta}).$$

Assume  $\Delta = 1$  for the sake of simplicity. We denote by  $(\mathcal{F}_t)_{t \geq 0}$  the natural filtration associated to the underlying standard Brownian motion  $W$  (see the Volterra representation). We set  $M_k = \mathbb{E}[F_Y | \mathcal{F}_k]$ .

Then

$$F_Y - \mathbb{E}[F_Y] = M_n = \sum_{k=1}^n M_k - M_{k-1}.$$

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We show the existence of a deterministic sequence  $(u_k)$  such that

$$\mathbb{E} [ e^{\lambda(M_k - M_{k-1})} | \mathcal{F}_{k-1} ] \leq e^{\lambda^2 u_k}$$

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$$\implies \mathbb{E} [ e^{\lambda M_n} ] \leq e^{\lambda^2 \sum_{k=1}^n u_k}$$

### Lemma

Let  $X$  be a centered real valued random variable such that for all  $p \geq 2$ , there exist  $C, \zeta > 0$  such that

$$\mathbb{E}[|X|^p] \leq C\zeta^{p/2} p\Gamma\left(\frac{p}{2}\right).$$

Then

$$\mathbb{E}[e^{\lambda X}] \leq e^{2C'\zeta\lambda^2}$$

with  $C' = 1 \vee C$ .

**Conclusion :** Since  $\mathbb{E}[M_k - M_{k-1} | \mathcal{F}_{k-1}] = 0$ , we are thus reduced to estimate

$$\mathbb{E}[|M_k - M_{k-1}|^p | \mathcal{F}_{k-1}] , \forall p \geq 2.$$

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We can see  $Y_t$  as a functional  $\Phi : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d) \rightarrow \mathbb{R}^d$  depending on the time, the initial condition  $x$  and the Brownian motion :

$$\forall t \geq 0, \quad Y_t := \Phi_t(x, (W_s)_{s \in [0, t]}). \quad (5.1)$$

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Let  $k \geq 1$ ,

$$\begin{aligned} & |M_k - M_{k-1}| \\ &= |\mathbb{E}[F_Y | \mathcal{F}_k] - \mathbb{E}[F_Y | \mathcal{F}_{k-1}]| = \left| \frac{1}{n} \sum_{t=k}^n \mathbb{E}[f(Y_t) | \mathcal{F}_k] - \mathbb{E}[f(Y_t) | \mathcal{F}_{k-1}] \right| \\ &\leq \frac{\|f\|_{\text{Lip}}}{n} \int_{\Omega} \sum_{t=k}^n |\Phi_t(x, W_{[0, k]} \sqcup \tilde{w}_{[k, t]}) - \Phi_t(x, W_{[0, k-1]} \sqcup \tilde{w}_{[k-1, t]})| \mathbb{P}_W(d\tilde{w}) \\ &\leq \frac{\|f\|_{\text{Lip}}}{n} \int_{\Omega} \sum_{u=1}^{n-k+1} |X_u - \tilde{X}_u| \mathbb{P}_W(d\tilde{w}) \end{aligned}$$

Using the SDE, we get for all  $u \geq 1$ ,

$$\begin{aligned} X_u - \tilde{X}_u &= \int_0^u b(X_s) - b(\tilde{X}_s) ds + \sigma \int_{k-1}^k K_H(u+k-1, s) d(W - \tilde{w})_s \\ &= \int_0^u b(X_s) - b(\tilde{X}_s) ds + \sigma \int_0^1 K_H(u+k-1, s+k-1) d(W^{(k)} - \tilde{w}^{(k)})_s \end{aligned}$$

where we have set  $(W_s^{(k)})_{s \geq 0} := (W_{s+k-1} - W_{k-1})_{s \geq 0}$  and  
 $(\tilde{w}_s^{(k)})_{s \geq 0} := (\tilde{w}_{s+k-1} - \tilde{w}_{k-1})_{s \geq 0}$ .

Key role of Hypothesis  $\langle x - y, b(x) - b(y) \rangle \leq -\alpha|x - y|^2$

For the sake of simplicity, we set

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For  $u > 1$ ,

$$\begin{aligned} \frac{d}{du} |X_u - \tilde{X}_u|^2 &= 2\langle X_u - \tilde{X}_u, \frac{d}{du}(X_u - \tilde{X}_u) \rangle \\ &= 2\langle X_u - \tilde{X}_u, b(X_u) - b(\tilde{X}_u) \rangle + 2\langle X_u - \tilde{X}_u, \partial_u \tilde{B}_u^{(k)} \rangle \\ &\leq -\alpha |X_u - \tilde{X}_u|^2 + \frac{1}{\alpha} |\partial_u \tilde{B}_u^{(k)}|^2 \end{aligned}$$

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**Gronwall's lemma :** for all  $u_0 > 1$ ,

$$|X_u - \tilde{X}_u|^2 \leq e^{-\alpha(u-u_0)} |X_{u_0} - \tilde{X}_{u_0}|^2 + \frac{1}{\alpha} \int_{u_0}^u e^{-\alpha(u-v)} |\partial_v \tilde{B}_v^{(k)}|^2 dv.$$

## Proposition

For all  $u \geq 1$  and  $k \in \mathbb{N}^*$ ,

$$\begin{aligned} & |X_u - \tilde{X}_u| \\ & \leq \Psi_{u,k} \left( \|W_v^{(k)} - \tilde{w}_v^{(k)}\|_{\infty,[0,1]} + \sup_{v \in [0,2]} |G_v^{(k)}| \right. \\ & \quad \left. + \sup_{v \in [0,1/2]} \left| \int_0^1 s^{\frac{1}{2}-H} (1-vs)^{H-\frac{3}{2}} d(W^{(k)} - \tilde{w}^{(k)})_s \right| \right) \end{aligned}$$

where  $G^{(k)}$  is given by

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## Technical difficulties :

- Get a “good” sequence  $\Psi_{u,k}$  by using sharp upper-bounds with the following tools : Gronwall’s lemma + precise estimates on the kernel  $K_H$ .
- Get the Sub-Gaussianity of the supremum of  $G^{(k)}$  **uniformly in  $k$**  (actually, we show this for Hölder seminorm).

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Thank you !