Exam (3h30)

Your lecture notes and the manuscript are authorized.

In whole the exercises, one will consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ where $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions. Whole the processes are considered with respect to this filtration.

Exercise 1. (About the supremum of local martingales)

In this exercise, one considers a continuous local martingale $(M_t)_{t\geq 0}$ such that $M_0 = 0$. One sets

$$M_t^* = \sup_{s\in[0,t]} |M_s|, \quad S_t = \sup_{s\in[0,t]} M_s, \quad s_t = \inf_{s\in[0,t]} M_s.$$

A. Preliminaries.

1. Remark that (S_t) is a semimartingale. Show that $\lim_{t\to+\infty} S_t = +\infty$ *a.s.* if $[M]_\infty = +\infty$ *a.s.*
2. Let $0 \leq a < b < +\infty$ such that $S_t > M_t$ for all $t \in]a, b[$. What can we say about the function $t \mapsto S_t$ on this interval ? Deduce that

$$\int_0^{+\infty} 1_{S_t > M_t} dS_t = 0 \quad a.s. \tag{1}$$

   Indication : One will use that every open set of $\mathbb{R}_+$ is a countable union of open intervals.
3. Show that *a.s.*, $[M, S]_t = 0$ and $[M, [M]]_t = 0$ for all $t \geq 0$.

The aim of the sequel of the exercise is to deduce a series of consequences of (1).

B.
1. Let $F : \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ be a $\mathcal{C}^2$-function such that,

$$\forall\,(x,y,z) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+, \quad \begin{cases} \frac{\partial F}{\partial y}(x,y,z) = 0 & \text{if } x = y \\ \left(\frac{\partial F}{\partial z} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right)(x,y,z) = 0. & \end{cases}$$

   Set $Z_t = F(M_t, S_t, [M]_t)$. Show that $(Z_t)_{t\geq 0}$ is a local martingale.
2. Deduce that $((S_t - M_t)^2 - [M]_t)_{t\geq 0}$ is a local martingale and that $(N_t) = (S_t^2 - 2S_tM_t)$ is also a local martingale.
3. Using the previous question and Cauchy-Schwarz inequality, (re)-prove the inequality

$$\mathbb{E}[S_t^2] \leq 4\mathbb{E}[[M]_t].$$

   Indication : One can begin by the case where M is bounded.

4. Let β be a positive number. Show that (Y_t) defined by

$$Y_t = (\text{ch}\,(\beta(S_t - M_t)))\exp\left(-\frac{\beta^2}{2}[M]_t\right)$$

is a local martingale.

5. Suppose in this question that $M = B$ where B is a Brownian Motion with $B_0 = 0$. For $a > 0$, set $T_a := \inf\{t, S_t - B_t = a\}$. Show that $T_a < +\infty$ *a.s.* Show that

$$\mathbb{E}\left[\exp\left(-\frac{\beta^2}{2}T_a\right)\right] = \frac{1}{\text{ch}(\beta a)}.$$

C. We suppose in this part that $(M_t)_{t\geq 0}$ is a martingale bounded in L^2. We consider a non-decreasing continuous process $(A_t)_{t\geq 0}$ such that $A_0 \geq a_0 > 0$.

1. Show that

$$(M_t - S_t)^2 = 2\int_0^t (M_s - S_s)dM_s + [M]_t.$$

2. Deduce that

$$a.s., \quad \forall t \geq 0, \quad A_t^{-1}(S_t - M_t)^2 \leq 2\int_0^t A_s^{-1}(M_s - S_s)dM_s + \int_0^t A_s^{-1}d[M]_s.$$

3. Deduce that for all $t \geq 0$,

$$\mathbb{E}[A_t^{-1}(S_t - M_t)^2] \leq \mathbb{E}\left[\int_0^t A_s^{-1}d[M]_s\right].$$

Indication : One can introduce a sequence of localizing stopping times.

4. Show that

$$\mathbb{E}[A_\infty^{-1}(S_\infty - M_\infty)^2] \leq \mathbb{E}\left[\int_0^{+\infty} A_s^{-1}d[M]_s\right].$$

5. Show that $M_t^* \leq |S_t - s_t|$. Deduce that

$$(M_t^*)^2 \leq 2[(S_t - M_t)^2 + (M_t - s_t)^2].$$

6. Deduce that

$$\mathbb{E}[A_\infty^{-1}(M_\infty^*)^2] \leq 4\mathbb{E}\left[\int_0^{+\infty} A_s^{-1}d[M]_s\right].$$

7. Deduce that

$$\mathbb{E}[[M]_\infty^{-\frac{1}{2}}(M_\infty^*)^2] \leq 4\mathbb{E}\left[\int_0^{+\infty} [M]_s^{-\frac{1}{2}}d[M]_s\right]. \quad (2)$$

Indication : One will be careful of the fact that $[M]_0 = 0$.

8. With the help of the Itô formula, show that

$$[M]_\infty^{\frac{1}{2}} = \frac{1}{2}\int_0^{+\infty} [M]_s^{-\frac{1}{2}}d[M]_s \quad a.s.$$

9. With the help of the previous questions, show that

$$\mathbb{E}[M^*_\infty] \leq 2\sqrt{2}\mathbb{E}[\sqrt{[M]_\infty}]. \tag{3}$$

Indication : One will begin by using the Cauchy-Schwarz inequality to go back to inequality (2).

10. (3) has been obtained under the assumption "M bounded in L^2". Show that this result is still true for any local martingale.

11. Deduce that if $\mathbb{E}[\sqrt{[M]_\infty}] < +\infty$, alors $(M_t)_{t\geq 0}$ is a martingale which converges *a.s.* and in L^1 to a random variable M_∞.

Exercise 2. (Poisson Equation) Let $(X_t)_{t\geq 0}$ be the solution to the equation

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

where (B_t) is a real brownian motion and $\mathcal{L}$ is its infinitesimal generator. The functions $b : \mathbb{R} \mapsto \mathbb{R}$ and $\sigma : \mathbb{R} \mapsto \mathbb{R}$ are Lipschitz continuous. One also assumes that

$$\exists \sigma_0 > 0, \forall x \in \mathbb{R}, \quad \sigma(x) \geq \sigma_0.$$

One is interested by the probabilistic interpretation of the solutions of the equation

$$\begin{cases} \mathcal{L}u(x) - \lambda(x)u(x) = -f & \text{si } x \in I \\ u(x) = 0 & \text{si } x \in \partial I \end{cases}$$

where $I =]0, a[$ is an open and bounded interval of $\mathbb{R}$, λ is a positive continuous function on $\mathbb{R}$ and f is a function (a minima) continuous on $[0, a]$. One will suppose that there exists a solution u to this PDE which is $\mathcal{C}^2$ on I and continuous on $\bar{I}$. Finally, we set

$$\tau := \inf\{t \geq 0, X_t \in \partial I\}.$$

One will assume that

$$\sup_{x\in I} \mathbb{E}_x[\tau] < +\infty. \tag{4}$$

In whole the exercise, $X_0 = x \in I$.

A. We set $Z_t = e^{-\int_0^t \lambda(X_s)ds}u(X_t)$ and $\tau^N = \inf\{t \geq 0, d(X_t, \partial I) \leq \frac{1}{N}\}$.

1. Show that

$$Z_t^{\tau^N} = u(x) + \int_0^{t\wedge\tau^N} H_s ds + M_{t\wedge\tau_N}$$

where H is a locally bounded process and M is a local martingale. One will provide an explicit expression of H_t and M_t.

2. Show that

$$\mathbb{E}_x[Z_{\tau^N}] = u(x) + \mathbb{E}_x\left[\int_0^{\tau^N} H_s ds\right].$$

3. Deduce that

$$u(x) = \mathbb{E}_x\left[\int_0^\tau e^{-\int_0^s \lambda(X_u)du} f(X_u)du\right].$$

B. One is now interested by another representation of the solution based on the Girsanov theorem. One denotes by (Y_t) a process which is a solution to

$$dY_t = \sigma(Y_t)dB_t$$

where σ and B are defined as previously. One sets $\rho = b\sigma^{-1}$. One will suppose in this part that $\mathbb{E}\left[\exp\left(\frac{\|\rho\|_\infty^2}{2}\tau\right)\right] < +\infty$ where $\|\rho\|_\infty = \sup_{x\in I}|\rho(x)|$.

1. Without calculus, show that $(M_t)_{t\geq 0}$ defined by

$$M_t = \exp\left(\int_0^t \rho(Y_s)dB_s - \frac{1}{2}\int_0^t \rho(Y_s)^2 ds\right), \quad t \geq 0$$

is a local martingale.

2. With the help of a well-chosen criterion, show that M^τ is uniformly integrable.

3. One then defines $\mathbb{Q} = M_\tau \mathbb{P}$. Show that $(N_t) = (Y_t - \int_0^t b(Y_s)ds)_{t\geq 0}$ is a local martingale under $\mathbb{Q}$.

4. Show that W defined by

$$W_t = \int_0^t \frac{1}{\sigma(Y_s)} dN_s$$

is a brownian motion under $\mathbb{Q}$.

5. Remark that

$$dY_t = b(Y_t)dt + \sigma(Y_t)dW_t.$$

6. Deduce from the previous part that

$$u(x) = \mathbb{E}\left[\int_0^\tau e^{-\int_0^s \lambda(Y_u)du} f(Y_s)M_s ds\right].$$