ON A DEGENERATE PROBLEM IN THE CALCULUS OF VARIATIONS

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Abstract. We establish the uniqueness of the solutions for a degenerate scalar problem in the multiple integrals calculus of variations. The proof requires as a preliminary step the study of the regularity properties of the solutions and of their level sets. We exploit the uniqueness and the regularity results to explore some of their qualitative properties. In particular, we emphasize the link between the supports of the solutions and the Cheeger problem.

1. Introduction

Two problems in Optimal Design. We study a problem in the multiple integrals calculus of variations which is both singular and degenerate. This problem arises as the relaxation of the non convex functional introduced by Kohn and Strang in [20, 21, 22]:

$$\mathcal{I}_0 : u \mapsto \int_\Omega \Phi(\nabla u) \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $N \geq 2$, and

$$\Phi(y) := \begin{cases} 0 & \text{if } y = 0, \\ \frac{1}{2}(|y|^2 + 1) & \text{if } |y| > 0. \end{cases}$$

The admissible functions u belong to the Sobolev space $H^1(\Omega)$ and must agree with a given function $\psi : \mathbb{R}^N \to \mathbb{R}$ on the boundary $\partial\Omega$ of Ω.

The functional $\mathcal{I}_0$ is not lower semicontinuous under weak convergence in $H^1(\Omega)$. Therefore, one cannot rely on the direct method in the calculus of variations to find a minimum as the limit of a minimizing sequence. Actually, it may happen that there is no minimum.

However, the infimum of $\mathcal{I}_0$ is equal to the infimum of the relaxed functional (see [20, Theorem 1.1]):

$$I_0 : u \mapsto \int_\Omega \varphi(\nabla u) \tag{1.2}$$

where φ is the convexification of Φ, namely the largest convex function less than or equal to Φ:

$$\varphi(y) = \begin{cases} |y| & \text{if } |y| < 1, \\ \frac{1}{2}(|y|^2 + 1) & \text{if } |y| \geq 1. \end{cases} \tag{1.3}$$

In contrast to the original problem, the relaxed problem is lower semicontinuous under weak H^1 convergence. It follows that it has at least one minimizer, and for every such minimizer u, $I_0(u) = \inf \mathcal{I}_0$. Moreover, the minimizers of I_0 are exactly the weak limits of minimizing sequences of $\mathcal{I}_0$.

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In order to prove the *non existence* of a minimizer for $\mathcal{I}_0$, a possible strategy is to establish the *uniqueness* of the minimizer for I_0, see [24]. Indeed, assume that I_0 has a unique minimum u such that $|\nabla u| < 1$ on a positive measure set. Then $I_0(u) < \mathcal{I}_0(u)$ (here, we use the fact that $\varphi(y) < \Phi(y)$ for every $0 < |y| < 1$) and for every admissible $v \neq u$, $I_0(u) < I_0(v) \leq \mathcal{I}_0(v)$. Since $\inf \mathcal{I}_0 = \inf I_0 = I_0(u)$, this proves that $\mathcal{I}_0$ has no minimum.

For some specific choice of Ω and ψ, this situation may arise, as illustrated in [22, Example 7.4] where Ω is the unit square and ψ a polynomial function of degree 2. However, generally speaking, this is a delicate matter to prove the uniqueness of minimizers for I_0, since I_0 is *not strictly convex*. This is one of the main contributions of this paper to establish this uniqueness property under a mild condition on Ω, see Theorem 1.1 below.

More recently, a related problem was considered in [2] for the same functional I_0, when the boundary condition ψ is equal to 0 everywhere and when Ω is an open bounded subset of $\mathbb{R}^2$ that we assume to be simply connected[1]. More precisely in the context of shape optimization of thin elastic structures, Bouchitté and al. proved in [7] that the section of an optimal torsion rod can be obtained from the following parametrized problem

$$m(s) := \inf \left\{ I_0(u) \quad , \quad \int_\Omega u = s \ , \ u \in H^1_0(\Omega) \right\}. \tag{1.4}$$

Here, s is a positive parameter which gives the intensity of the applied torsion load. If the infimum in $m(s)$ is attained at some $u \in H^1_0(\Omega)$, then the subset $[|\nabla u| > 1]$ corresponds to the optimal subregion where the material should be placed. The *plateau* of u, namely the set $[\nabla u = 0]$, represents the void subregion.

By [2, Proposition 3.8], m is differentiable on $(0, \infty)$. The constraint $\int_\Omega u = s$ can be included in the cost functional through a Lagrange multiplier. More specifically, for every $\lambda \geq 0$, let us define

$$I_\lambda : u \mapsto \int_\Omega \varphi(\nabla u) - \lambda \int_\Omega u. \tag{1.5}$$

Then for every $s > 0$, an admissible u is a solution of $m(s)$ if and only if u minimizes $I_{m'(s)}$ on $H^1_0(\Omega)$, see [2, (3.5) and Proposition 3.3].

A major issue, still beyond the scope of this paper, is the existence or the non existence of a *special solution* for $m(s)$; that is, a solution u with the following property:

$$|\nabla u| \in \{0\} \ \cup \ (1, +\infty) \text{ a.e. in } \Omega.$$

Observe that u is a special solution for I_0 if and only if it is a minimizer of $\mathcal{I}_0$. For every $\lambda \geq 0$, one can construct non spherical domains Ω for which there exist special solutions, see [2, Section 6]. When such a solution exists, an optimal design contains no *homogenized* region corresponding to a fine mixture of material and void. It was proved in [2, Corollary 3.5] that if there exists a special solution, then this is the unique solution of $m(s)$. As a consequence of Theorem 1.1 below, we obtain a deeper result which does not require the existence of a special solution: for every $s > 0$, $m(s)$ has a unique solution.

In order to embrace the two situations arising in [20, 21, 22] and [2, 7], we consider henceforth any Lipschitz function $\psi : \mathbb{R}^N \to \mathbb{R}$ for the boundary condition and any open bounded subset $\Omega \subset \mathbb{R}^N$, $N \geq 2$. Given $\lambda \geq 0$, we study the problem:

$$(P_\lambda) \qquad \text{To minimize } u \mapsto I_\lambda(u) \text{ on } H^1_\psi(\Omega),$$

[1] In fact, this assumption that we introduce here for simplicity, was not required in [2].

where $H^1_\psi(\Omega) := \psi + H^1_0(\Omega)$.[2]

Main results. Our first main result is the uniqueness of solutions for (P_λ) under fairly general assumptions on Ω.

Theorem 1.1. *Assume that Ω is Lipschitz and that $\partial\Omega$ is connected. Then for every $\lambda \in \mathbb{R}$, the solution of (P_λ) is unique.*

The proof of Theorem 1.1 heavily relies on the regularity properties of the solutions of (P_λ). There are serious obstacles to establish them: the function φ is singular at the origin and is not twice differentiable on the unit sphere. Moreover, the Hessian $\nabla^2\varphi(\xi)$ of φ at ξ has a non trivial kernel for every ξ in the unit ball. Hence, (P_λ) is both singular and degenerate. In spite of these facts, recent results for this class of integrands can be used to obtain the following regularity properties:

Proposition 1.2. *Let $\lambda \geq 0$ and u a solution of (P_λ). Then u is bounded on Ω and locally Lipschitz continuous. Moreover, there exists an open set $U \subset \Omega$ such that u is smooth on U. In fact, $|\nabla u| > 1$ on U and $|\nabla u| \leq 1$ a.e. on $\Omega \setminus U$.*

When Ω is assumed to be Lipschitz, then the solutions of (P_λ) are Hölder continuous on $\overline{\Omega}$, see Section 2. On the complement of U, the regularity of u itself is still open. However, generically, the super-level sets $[u \geq t] = \{x \in \Omega : u(x) \geq t\}$, with $t \in \mathbb{R}^+$, satisfy a variational problem on $\Omega \setminus \overline{U}$. It then follows that for a.e. $t \in \mathbb{R}^+$, the level sets $\partial[u \geq t]$ are C^1 hypersurfaces, up to a small singular term:

Proposition 1.3. *Let $\lambda \geq 0$, u a solution of (P_λ) and U as in Proposition 1.2. Then for a.e. $t \in \mathbb{R}$, there exists an open set W in Ω such that $W \cap (\partial^e[u \geq t])$ is a C^1 hypersurface and $\mathcal{H}^s(\Omega \setminus (W \cup \partial U)) = 0$ for every $s > N - 8$.*

Here, $\partial^e[u \geq t]$ is the *essential boundary* of the set $[u \geq t]$, namely the set of those $x \in \mathbb{R}^N$ such that for every $\rho > 0$,

$$0 < |[u \geq t] \cap B_\rho(x)| < |B_\rho(x)|. \tag{1.6}$$

Both Propositions 1.2 and 1.3 are essential steps in the proof of Theorem 1.1. However, when $N = 2$, almost every level set of a Lipschitz function is a Lipschitz curve, see e.g. [1, Theorem 2.5]. It is then possible to rely on this property instead of Proposition 1.3 in order to prove Theorem 1.1.

In the case when Ω is the ball of radius $R > 0$ and $\psi \equiv 0$, the solution has the following explicit expression, see Remark 4.2 below:

$$u(x) = -\frac{\lambda}{2N}\Big(|x|^2 - \frac{N^2}{\lambda^2}\Big)_+ + \frac{\lambda}{2N}\Big(R^2 - \frac{N^2}{\lambda^2}\Big)_+. \tag{1.7}$$

In particular, the solution is Lipschitz continuous, but not even C^1 (except when it is the trivial solution).

The regularity and the uniqueness of the solutions for (P_λ) have important consequences for the study of their qualitative properties. Given Ω and ψ as in Theorem 1.1, for every $\lambda \geq 0$, we denote by u_λ the unique solution of (P_λ) on $H^1_\psi(\Omega)$.

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[2] To be more specific, $H^1_0(\Omega)$ is the set of those $u \in H^1(\Omega)$ such that the extension of u by 0 on $\mathbb{R}^N$ belongs to $H^1(\mathbb{R}^N)$.

Proposition 1.4. *The map* $\lambda \in [0,+\infty) \mapsto u_\lambda \in C^0(\overline{\Omega})$ *is continuous and nondecreasing*[3].

Assume now that $\psi \equiv 0$ and Ω is any bounded open Lipschitz set in $\mathbb{R}^N$. When $\lambda = 0$, the solution is the constant function equal to 0. It turns out that for small values of λ, 0 is still the unique solution of (P_λ). An interesting fact is that the critical value of λ for which 0 is not the solution any more is exactly the Cheeger constant h_Ω of Ω:

$$h_\Omega = \inf_{D\subset\overline{\Omega}} \frac{\text{Per } D}{|D|}. \tag{1.8}$$

Here, Per refers to the perimeter in $\mathbb{R}^N$ (or equivalently in $\overline{\Omega}$ since all the sets that we consider are contained in $\overline{\Omega}$). The precise link of h_Ω with (P_λ) is given in the following statement, which corresponds to [2, (4.5)]:

Proposition 1.5. *Let* $\lambda > 0$

- *If* $\lambda > h_\Omega$, *then* 0 *is not a solution of* (P_λ).
- *If* $\lambda \leq h_\Omega$, *then* 0 *is the unique solution of* (P_λ).

A Cheeger set for Ω is a subset of Ω for which the infimum in (1.8) is attained. Together with the Cheeger constant h_Ω, the Cheeger sets play a natural role in the framework of (P_λ):

Theorem 1.6. *The set*

$$\Omega_0 := \cap_{\lambda > h_\Omega} [u_\lambda > 0]$$

is a solution of the Cheeger problem for Ω.

On some proofs of uniqueness for some multiple integrals variational problems. There are many uniqueness results for variational problems with a lack of strict convexity. We just quote four of them, which have been important sources of inspiration to us. First, in the seminal paper [24, Theorem 3] (see also [25]), Marcellini considers the problem

$$\text{Minimize } v \mapsto \int_\Omega g(|\nabla v(x)|)\, dx$$

with $g : [0,+\infty) \to [0,+\infty)$ an increasing convex function. The bounded open set Ω is assumed to be convex and C^1. If there exists a solution $u \in C^1(\overline{\Omega})$ such that ∇u does not vanish on $\overline{\Omega}$, then u is proved to be the unique solution on the class of Lipschitz functions agreeing with u on $\partial\Omega$. The proof of this result is based on two observations:

- **Step 1** Each level set of a solution u intersects the boundary of the domain $\partial\Omega$.
- **Step 2** If v is another solution, then v is constant on the level sets of u.

Since u and v agree on $\partial\Omega$, it follows from the two above steps that u agrees with v on each level set of u, which finally proves that $u = v$ on $\overline{\Omega}$. The first step is based on the fact that g is (strictly) increasing. The second step uses in a crucial way that the integrand only depends on the *norm* of the gradient of u. Both steps exploit the C^1 regularity of the level sets of u. For a general variational problem which is not stricly convex, such a regularity assumption is difficult to establish (see however [30]). In the two dimensional setting, more precisely when Ω is a bounded open set with connected boundary in $\mathbb{R}^2$, the uniqueness result remains true *without* the two assumptions: u is C^1 and ∇u does not vanish on Ω, see [23].

The above strategy has been exploited in many different contexts. In [31], the authors consider the case when $g(t) = t$ and prove the uniqueness of the solutions in the class $\{u \in BV(\Omega) \cap C^0(\overline{\Omega}) :$

[3]In the sense that if $\lambda_1 \leq \lambda_2$, then $u_{\lambda_1}(x) \leq u_{\lambda_2}(x)$ for every $x \in \Omega$.

$u|_{\partial\Omega} = \psi|_{\partial\Omega}\}$ under additional geometric conditions on Ω. The fact that the (regular components of the) level sets of a solution intersect the boundary of Ω (which corresponds to Step 1 in Marcellini's proof) arises as an essential argument in the proof of [31, Lemma 3.4]. This is also a key tool in [19], see [19, Lemma 4.2], where the authors consider the more general problem

$$\text{To minimize } v \mapsto \int_\Omega g(x, \nabla v(x))\, dx$$

with an integrand g such that $\mapsto g(x, \cdot)$ is a norm for every x. In this extended framework, the uniqueness is established under the same assumptions on Ω as in Theorem 1.1, namely that Ω is Lipschitz and has a connected boundary.

The common feature of all the papers quoted above is that the integrand does *not* depend on the variable u. Adding a lower order term of the form $\lambda \int_\Omega u$ as in our problem (P_λ) involves important consequences for the level sets of the solutions, as illustrated by the example of the explicit solution on the ball (1.7). Indeed, generically, level sets do not intersect the boundary of Ω any more. It follows that one cannot directly use the fact that two solutions agree on $\partial\Omega$ to deduce therefrom that they agree on Ω.

Moreover, the regularity of the level sets required by the proof in [24] cannot be established in the framework of (P_λ). Indeed, the explicit example on the ball shows that one cannot expect better than Lipschitz regularity for the solutions. In addition, the assumption that ∇u does not vanish on Ω is far from being satisfied: we can even prove that a solution u is necessarily constant on a positive measure set, see Lemma 2.1 below.

In [31, 19], the general regularity theory for area minimizing sets is exploited in an essential way to get the C^1 regularity of the level sets. In our case, the function $y \mapsto \varphi(y)$ behaves differently depending on whether y is small or outside the unit ball. As a consequence, the super-level sets of u do not minimize a simple variational problem on the whole Ω. We thus have to use two different strategies to establish the regularity of the level sets of u, first on $[|\nabla u| > 1]$ and then on its complement.

Plan of the paper. In the next section, we present the proof of the regularity results Proposition 1.2 and 1.3. For the latter, we need to introduce a minimization problem for the super-level sets of solutions, but only on an open subset of Ω where the gradient is small. In section 3, we establish the uniqueness result Theorem 1.1. The qualitative properties Propositions 1.4 and 1.5 are proved in section 4 as well as a more precise version of Theorem 1.6, see Theorem 4.8. Finally, for the convenience of the reader, we have presented in an appendix the proof of the Lipschitz continuity of the solutions which readily follows from the arguments used in [13, Theorem 2.7].

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2. Regularity and Euler equation

2.1. **Lipschitz regularity.** We first enumerate some continuity properties satisfied by the solutions of (P_λ), $\lambda \geq 0$. These properties readily follow from the classical regularity theory for Lagrangians with quadratic growth: By [18, Theorem 7.5, Theorem 7.6], every solution u is locally Hölder continuous in Ω. If one further assumes that Ω is Lipschitz, then [18, Theorem 7.8] implies that u is globally Hölder continuous (remember that the boundary condition is given by the restriction to $\partial\Omega$ of a *Lipschitz* function $\psi : \mathbb{R}^N \to \mathbb{R}$).

Regarding the L^∞ estimate on u, we have the following result which holds true on every bounded open set Ω:

Lemma 2.1. *There exists $C > 0$ which depends only on N such that for every $x \in \Omega$,*

$$\min_{\partial\Omega} \psi \leq u(x) \leq \max_{\partial\Omega} \psi + C\lambda^N \|(u - \max_{\partial\Omega} \psi)_+\|_{L^1(\Omega)}.$$

Moreover, if $\sup_\Omega u > \max_{\partial\Omega} \psi$, then

$$|\{x \in \Omega : u(x) = \sup_\Omega u\}| \geq \frac{1}{C\lambda^N}.$$

Remark 2.2. That $\sup_\Omega u > \max_{\partial\Omega} \psi$ holds true is closely related to the value of λ. Actually, when Ω is Lipschitz and has a connected boundary, we can prove that there exists $\lambda_* = \lambda_*(\psi, \Omega) \in [0, +\infty)$ such that for every $\lambda \leq \lambda_*$, $\sup_\Omega u = \max_{\partial\Omega} \psi$ while for every $\lambda > \lambda_*$, $\sup_\Omega u > \max_{\partial\Omega} \psi$, see Lemma 4.7 below.

Proof of Lemma 2.1. We denote by $a := \min_{\partial\Omega} \psi$. We first prove that $u \geq a$ on Ω. Let $v := \max(u, a)$. Then v is admissible for (P_λ). From the minimality of u,

$$\int_\Omega \varphi(\nabla u) - \lambda u \leq \int_\Omega \varphi(\nabla v) - \lambda v$$

which implies

$$\int_{[u<a]} \varphi(\nabla u) \leq \lambda \int_{[u<a]} u - a.$$

Since $u - a < 0$ on the set $[u < a]$ while $\varphi(\nabla u) \geq 0$, it follows that $|[u < a]| = 0$; that is, $u \geq a$ on Ω.

We now prove that u is bounded from above. Let $b := \max_{\partial\Omega} \psi$. For every $t \geq b$, let $w := \min(u, t)$. As above, one has

$$\int_{[u>t]} \varphi(\nabla u) \leq \lambda \int_{[u>t]} u - t.$$

Since $\varphi(y) \geq |y|$ for every $y \in \mathbb{R}^N$, this gives

$$\int_{[u>t]} |\nabla u| \leq \lambda \int_{[u>t]} u - t.$$

By the Sobolev inequality in the left hand side and the Hölder inequality in the right hand side, this implies

$$\|(u - t)_+\|_{L^{N'}(\Omega)} \leq C\lambda \|(u - t)_+\|_{L^{N'}(\Omega)} |[u > t]|^{\frac{1}{N}}, \tag{2.1}$$

where C depends only on N. If $\|(u - t)_+\|_{L^{N'}(\Omega)} > 0$, then $|[u > t]| \geq 1/(C\lambda)^N$. Integrating the latter inequality on an interval $[b, T]$ for some $T > b$ yields

$$\frac{T - b}{(C\lambda)^N} \leq \int_{[u\geq b]} \min(u, T) - b \leq \|(u - b)_+\|_{L^1(\Omega)}.$$

This implies that the inequality $|[u > t]| \geq 1/(C\lambda)^N$ can only hold true for $t \leq T_0 := b + (C\lambda)^N \|(u - b)_+\|_{L^1(\Omega)}$. Consequently, $\|(u - t)_+\|_{L^{N'}(\Omega)} = 0$ for every $t > T_0$ and thus u is bounded from above by T_0 on Ω.

Finally, if $\sup_\Omega u > \max_{\partial\Omega} \psi$, there exists a sequence $(t_i)_{i\in\mathbb{N}} \subset (b, \sup_\Omega u)$ converging to $\sup_\Omega u$. Applying (2.1) to t_i gives $|[u > t_i]| \geq 1/(C\lambda)^N$ so that in the limit $|[u = \sup_\Omega u]| \geq 1/(C\lambda)^N$. □

By [13, Theorem 2.7], u is locally Lipschitz continuous in Ω. Actually, this theorem applies to integrands of the form $f(x, \nabla u)$; however, the proof can be easily generalized to integrands of the form $f(\nabla u) - \lambda u$, see Theorem 5.1 in the appendix for a proof in the specific case that we consider in this paper ; for a more general result in this direction, see also [8, Theorem 2.1].

2.2. **The Euler equation.** Let $\lambda \geq 0$ and u a solution of (P_λ). One can establish the Euler equation for u exactly as in the proof of [2, Lemma 3.2] where this is done for the problem stated in (1.4).

Lemma 2.3. *There exists $\sigma \in (L^2 \cap L^\infty_{loc})(\Omega; \mathbb{R}^N)$ such that div $\sigma = -\lambda$ and*

$$\sigma \in \partial\varphi(\nabla u) \quad a.e. \tag{2.2}$$

Proof. We introduce the convex function

$$K_\varphi : g \in L^2(\Omega; \mathbb{R}^N) \mapsto \int_\Omega \varphi(g(x) + \nabla\psi(x))\, dx,$$

and the two continuous linear maps

$$J : v \in H^1_0(\Omega) \mapsto -\lambda \int_\Omega v(x)\, dx \quad , \quad A : v \in H^1_0(\Omega) \mapsto \nabla v \in L^2(\Omega; \mathbb{R}^N).$$

Since $u \in H^1_\psi(\Omega)$ is a minimum of (P_λ),

$$0 \in \partial\left(K_\varphi \circ A + J\right)(u - \psi).$$

where $\partial(\cdots)$ is the convex subgradient. Now, $\partial K_\varphi(g)$ is the set of those $\zeta \in L^2(\Omega; \mathbb{R}^N)$ such that

$$\zeta(x) \in \partial\varphi(g(x) + \nabla\psi(x)) \text{ a.e. } x \in \Omega. \tag{2.3}$$

Since A is linear and continuous, $\partial(K_\varphi \circ A)(u-\psi) = A^*[\partial K_\varphi(A(u-\psi))]$. Finally, J being linear and continuous, one deduces that there exists $\sigma \in L^2(\Omega; \mathbb{R}^N)$ such that $\sigma \in \partial\varphi(\nabla u)$ a.e. on Ω and

$$\lambda \int_\Omega v = \int_\Omega \langle\sigma, \nabla v\rangle \quad \forall v \in H^1_0(\Omega). \tag{2.4}$$

Equivalently, div $\sigma = -\lambda$. Finally, since $\nabla u \in L^\infty_{loc}(\Omega)$, it follows that $\sigma \in L^\infty_{loc}(\Omega; \mathbb{R}^N)$. The proof is complete. □

One can use the Euler equation together with a regularity result due to Colombo and Figalli [11] to prove that any solution is smooth on the set where the norm of its gradient is larger than 1. More precisely,

Lemma 2.4. *There exists an open subset $U \subset \Omega$ such that u is smooth on U, $|\nabla u(x)| > 1$ for every $x \in U$ and $|\nabla u(x)| \leq 1$ for a.e. $x \in \Omega \setminus U$. Moreover, $|\nabla u|$ is continuous on $\overline{U} \cap \Omega$ and $|\nabla u| = 1$ on $\partial U \cap \Omega$.*

Proof. By [11, Theorem 1.1], for every continuous function $\mathcal{H} : \mathbb{R}^N \to \mathbb{R}$ such that $\mathcal{H} = 0$ on $B_1(0)$, the function $\mathcal{H}(\nabla u)$ has a continuous representative on Ω. By applying this result to $\mathcal{H}(y) = (|y| - 1)_+$, one obtains that $(|\nabla u| - 1)_+$ has a continuous representative on Ω and the set $U := [\mathcal{H}(\nabla u) > 0]$ is open.

On the open set U, $|\nabla u| > 1$ a.e. so that the function σ introduced in Lemma 2.3 satisfies $\sigma = \nabla u$ a.e. on U. Hence, u is a (weak and thus a strong) solution of $\Delta u = -\lambda$. In particular, u is smooth on U. Since $|\nabla u| = \mathcal{H}(\nabla u) + 1$ is uniformly continuous on $U \cap \Omega'$ for every $\Omega' \Subset \Omega$, it follows that $|\nabla u|$ extends as a continuous function on $\overline{U} \cap \Omega$ which is equal to 1 on $\partial U \cap \Omega$.

Finally, on $\Omega \setminus U$, $\mathcal{H}(\nabla u) = 0$ and thus $|\nabla u| \leq 1$ a.e. there.

□

The proof of Proposition 1.2 is now complete. In a similar way to [2, Proposition 3.1], we have:

Remark 2.5. On the open set $V := \Omega \setminus \overline{U}$, $|\sigma(x)| \leq 1$ a.e. and since φ is differentiable on $\mathbb{R}^N \setminus \{0\}$, $\sigma(x) = \nabla\varphi(\nabla u(x))$ a.e. on $[\nabla u \neq 0]$. In particular,

$$\langle \sigma(x), \frac{\nabla u(x)}{|\nabla u(x)|} \rangle = 1 \text{ a.e. } x \in [|\nabla u| \neq 0] \cap V. \tag{2.5}$$

In view of (2.4), one also has

$$\lambda \int_\Omega v \leq \int_{[\nabla u = 0]} |\nabla v| + \int_{[\nabla u \neq 0]} \langle \nabla\varphi(\nabla u), \nabla v \rangle \quad , \quad \forall v \in H_0^1(\Omega). \tag{2.6}$$

2.3. **Minimizing properties of the super-level sets.** Using the Euler equation, one can prove that in $V = \Omega \setminus \overline{U}$, the super-level sets of a solution u have constant mean curvature, in a generalized sense. We first recall some basic results on BV functions.

Let $u \in BV(\Omega)$. The distributional gradient of u is a vector valued Radon measure Du such that $|Du|(\Omega) < \infty$, where $|Du|$ is the total variation of Du. Given a Borel set $E \subset \Omega$, we say that E has finite perimeter in Ω if the characteristic function χ_E of E belongs to $BV(\Omega)$. The perimeter $\text{Per}\,(E, \Omega)$ is then defined as the total variation of χ_E on Ω:

$$\text{Per}\,(E, \Omega) = \int_\Omega |D\chi_E| = \sup\left\{ \int_E \text{div}\, g : g \in C_c^1(\Omega; \mathbb{R}^N), |g(x)| \leq 1\,,\, \forall x \in \Omega \right\}.$$

The reduced boundary $\partial^* E$ of E in Ω is the set of those $x \in (\text{supp}\,|D\chi_E|) \cap \Omega$ such that the limit

$$\nu_E(x) := \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} D\chi_E}{\int_{B_\rho(x)} |D\chi_E|}$$

exists in $\mathbb{R}^N$ and satisfies $|\nu_E(x)| = 1$. The reduced boundary $\partial^* E$ is a subset of the essential boudary $\partial^e E$ introduced in (1.6), see [17, Chapter 3]. By the Besicovitch derivation theorem (see e.g. [5, Theorem 2.22]), $|D\chi_E|$ is concentrated on $\partial^* E$ and $D\chi_E = \nu_E |D\chi_E|$. Moreover, $\partial^* E$ is a countably $(N-1)$ rectifiable set and for every Borel set $B \subset \Omega$,

$$\int_B |D\chi_E| = \mathcal{H}^{N-1}(B \cap \partial^* E), \tag{2.7}$$

see e.g. [5, Theorem 3.59] or [17, Theorem 4.4].

For a.e. $s \in \mathbb{R}$, the super-level set $E_s := [u \geq s]$ has finite perimeter in Ω and the *coarea formula* (see e.g. [5, Theorem 3.40]) asserts that for every Borel set $B \subset \Omega$,

$$\int_B |Du| = \int_{\mathbb{R}} ds \int_B |D\chi_{E_s}|. \tag{2.8}$$

In view of (2.7), this equality can be formulated as follows:

$$\int_B |Du| = \int_{\mathbb{R}} \mathcal{H}^{N-1}(B \cap \partial^* E_s)\, ds. \tag{2.9}$$

In terms of the characteristic function of B, this gives:

$$\int_\Omega \chi_B \, d|Du| = \int_{\mathbb{R}} ds \int_{\partial^* E_s} \chi_B \, d\mathcal{H}^{N-1}.$$

By linearity and the monotone convergence theorem, this implies that for every nonnegative Borel function $f : \Omega \to \mathbb{R}^+$,

$$\int_\Omega f \, d|Du| = \int_{\mathbb{R}} ds \int_{\partial^* E_s} f \, d\mathcal{H}^{N-1}. \tag{2.10}$$

This identity remains true for every bounded measurable map f compactly supported in Ω. The following lemma asserts that the gradient of a Sobolev function $u \in W^{1,1}(\Omega)$ is orthogonal (in a generalized sense) to the level sets of u.

Lemma 2.6. *Let $u \in W^{1,1}(\Omega)$. Then for a.e. $s \in \mathbb{R}$ and $\mathcal{H}^{N-1}$ a.e. $x \in \partial^* E_s$, $\nabla u(x) \neq 0$. Moreover, one has*

$$\nu_{E_s} = \frac{D\chi_{E_s}}{|D\chi_{E_s}|} = \frac{\nabla u}{|\nabla u|} \qquad |D\chi_{E_s}| \ a.e. \tag{2.11}$$

Proof. Since $u \in W^{1,1}(\Omega)$, the measure Du coincides with the L^1 function ∇u. Let S_* be a negligeable Borel set such that every point in $\mathbb{R}^N \setminus S_*$ is a Lebesgue point of ∇u and let $S = S_* \cup [\nabla u = 0]$. By (2.9) with $B = S$, one gets

$$\int_{\mathbb{R}} \mathcal{H}^{N-1}(\partial^* E_s \cap S) \, ds = \int_S |\nabla u(x)| \, dx = 0.$$

Hence, for a.e. $s \in \mathbb{R}$, $\nabla u(x) \neq 0$ for $\mathcal{H}^{N-1}$ a.e. $x \in \partial^* E_s$.

For every $a < b$, define $u_{a,b} := (\min(u,b) - a)_+$. For every $\kappa \in C_c^\infty(\Omega; \mathbb{R}^N)$,

$$\int_{[a \le u \le b]} \langle \nabla u, \kappa \rangle = \int_\Omega \langle \nabla u_{a,b}, \kappa \rangle = -\int_\Omega u_{a,b} \operatorname{div} \kappa. \tag{2.12}$$

On the set $[\nabla u_{a,b} \neq 0]$, we define

$$f(x) := \langle \frac{\nabla u_{a,b}(x)}{|\nabla u_{a,b}(x)|}, \kappa \rangle = \langle \frac{\nabla u(x)}{|\nabla u(x)|}, \kappa \rangle.$$

We then extend f by 0 on Ω. Then $\langle \nabla u_{a,b}, \kappa \rangle = f |\nabla u_{a,b}|$ on Ω. Hence, by (2.10),

$$\int_{[a \le u \le b]} \langle \nabla u, \kappa \rangle = \int_{\mathbb{R}} ds \int_{\partial^* [u_{a,b} \ge s]} f \, d\mathcal{H}^{N-1}.$$

For $s \le 0$, $[u_{a,b} \ge s] = \Omega$ while for $s > b - a$, $[u_{a,b} \ge s] = \emptyset$. Moreover, for $s \in (0, b-a)$, $[u_{a,b} \ge s] = [u \ge s + a]$. Moreover, for a.e. $s \in (0, b-a)$, $\nabla u_{a,b}(x) \neq 0$ for $\mathcal{H}^{N-1}$ a.e. $x \in \partial^*[u_{a,b} \ge s]$ and thus $f(x) = \langle \frac{\nabla u(x)}{|\nabla u(x)|}, \kappa \rangle$. It follows that

$$\int_{[a \le u \le b]} \langle \nabla u, \kappa \rangle = \int_0^{b-a} ds \int_{\partial^*[u_{a,b} \ge s]} \langle \frac{\nabla u}{|\nabla u|}, \kappa \rangle \, d\mathcal{H}^{N-1} = \int_a^b ds \int_{\partial^* E_s} \langle \frac{\nabla u}{|\nabla u|}, \kappa \rangle \, d\mathcal{H}^{N-1}.$$

Inserting this identity in (2.12) and using Fubini theorem in the right hand side, one gets

$$\int_a^b ds \int_{\partial^* E_s} \langle \frac{\nabla u}{|\nabla u|}, \kappa \rangle \, d\mathcal{H}^{N-1} = -\int_a^b ds \int_{E_s} \operatorname{div} \kappa.$$

Dividing by $b-a$ and letting $b-a \to 0$, this gives for a.e. $s \in \mathbb{R}$,

$$\int_{\partial^* E_s} \langle \frac{\nabla u}{|\nabla u|}, \kappa \rangle \, d\mathcal{H}^{N-1} = -\int_{E_s} \text{div } \kappa.$$

Using (2.7), one has

$$\int_{\partial^* E_s} \langle \kappa, \frac{D\chi_{E_s}}{|D\chi_{E_s}|} \rangle \, d\mathcal{H}^{N-1} = \int_\Omega \langle \kappa, \frac{D\chi_{E_s}}{|D\chi_{E_s}|} \rangle \, d|D\chi_{E_s}| = -\int_\Omega \chi_{E_s} \text{div } \kappa = -\int_{E_s} \text{div } \kappa.$$

Hence,

$$\int_{\partial^* E_s} \langle \kappa, \frac{D\chi_{E_s}}{|D\chi_{E_s}|} \rangle \, d\mathcal{H}^{N-1} = \int_{\partial^* E_s} \langle \frac{\nabla u}{|\nabla u|}, \kappa \rangle \, d\mathcal{H}^{N-1}.$$

Since κ is arbitrary, this implies (2.11). The proof is complete. □

We now present the main result of this section : the super-level sets of a solution u satisfy a minimization problem on the subset of Ω where the gradient of u is lower than 1:

Proposition 2.7. *Given $\lambda \geq 0$, let u be a solution of (P_λ). Let $V := \Omega \setminus \overline{U}$ where U is the open set $[|\nabla u| > 1]$ introduced in Lemma 2.4. For a.e. $s \in \mathbb{R}$, for every set $F \subset \Omega$ with finite perimeter in Ω such that $F \Delta E_s \Subset V$,*

$$Per\ (E_s, V) - \lambda |E_s \cap V| \leq Per\ (F, V) - \lambda |F \cap V|. \tag{2.13}$$

Here, $F\Delta E_s$ is the set $(F \setminus E_s) \cup (E_s \setminus F)$. The argument below is inspired from [3, Proposition 4] which in turn is based on [6, Proposition 2.7]. For the convenience of the reader, we present a self-contained proof.

Proof. We divide the proof into three steps.
Step 1. *In the first step, we introduce an approximation of the map σ introduced in the Euler equation, see Lemma 2.3.* We extend σ by 0 outside Ω and we define $\sigma_j := \sigma * \rho_j$, where $(\rho_j)_{j\geq 1} \subset C_c^\infty(B_{1/j})$ is a standard mollifier. Then σ_j converges to σ a.e. on Ω. Moreover, for every compact $K \Subset \Omega$, for every $j \geq 1/\text{dist }(K, \partial\Omega)$,

$$\text{div } \sigma_j = (\text{div } \sigma) * \rho_j = -\lambda * \rho_j = -\lambda \quad \text{a.e. on } K.$$

We claim that there exists a subsequence, still denoted by $(\sigma_j)_{j\geq 1}$, such that for a.e. $s > 0$, for every $K \Subset \Omega$,

$$\lim_{j\to+\infty} \int_{K\cap \partial^* E_s} |\sigma_j - \sigma| = 0.$$

Indeed, let $K \Subset \Omega$. By the coarea formula (2.10), for every $j \geq 1$,

$$\int_{\mathbb{R}} ds \int_{K\cap\partial^* E_s} |\sigma_j - \sigma| \, d\mathcal{H}^{N-1} = \int_K |\nabla u||\sigma_j - \sigma|. \tag{2.14}$$

Let $j_0 \geq 1$ be such that $K + B_{1/j_0} \Subset \Omega$. For every $j \geq j_0$, the integrand in the right hand side is bounded from above by

$$\|\nabla u\|_{L^\infty(K)}|(\|\sigma_j\|_{L^\infty(K)} + \|\sigma\|_{L^\infty(K)}) \leq 2\|\nabla u\|_{L^\infty(K)}\|\sigma\|_{L^\infty(K+B_{1/j_0})}.$$

Using that (σ_j) converges a.e. to σ on Ω, the dominated convergence theorem implies that

$$\lim_{j\to+\infty} \int_K |\nabla u||\sigma_j - \sigma| = 0.$$

In view of (2.14), there exists a subsequence (we do not relabel) such that for a.e. $s \in \mathbb{R}$,

$$\lim_{j\to+\infty} \int_{K\cap\partial^* E_s} |\sigma_j - \sigma| \, d\mathcal{H}^{N-1} = 0. \tag{2.15}$$

Let $(K_n)_{n\geq 1}$ be a sequence of compact subsets of Ω such that $\cup_{n\geq 1}\text{int } K_n = \Omega$. By applying the above reasoning to each K_n, one can extract, through a diagonal process, a subsequence, still denoted by $(\sigma_j)_{j\geq 1}$, such that for a.e. $s \in \mathbb{R}$, (2.15) holds true for *every* K_n, and thus for every compact subset $K \Subset \Omega$.

Step 2. *In the second step, we prove that for every $F \subset \Omega$ as in the statement of the proposition, for a.e. $s \in \mathbb{R}$ and for every $\theta \in C_c^\infty(V)$ such that $\theta \equiv 1$ on $F\Delta E_s$ and $0 \leq \theta \leq 1$, we have*

$$\lambda \int_{\mathbb{R}^N} (\chi_F - \chi_{E_s})\theta \leq \text{Per } (F, V) - \int_{\partial^* E_s} \theta d\mathcal{H}^{N-1}. \tag{2.16}$$

By the coarea formula and (2.5), for a.e. $s \in \mathbb{R}$, for $\mathcal{H}^{N-1}$ a.e. $x \in V \cap \partial^* E_s$, $\nabla u(x) \neq 0$ and $\langle \sigma(x), \nabla u(x)/|\nabla u(x)|\rangle = 1$. We fix any s for which this property as well as (2.11) and (2.15) hold true. Observe that a.e. s satisfies these conditions.

Let $\theta \in C_c^\infty(V)$ such that $\theta \equiv 1$ on $F\Delta E_s$ and $0 \leq \theta \leq 1$ on V.

Since $\nabla\theta = 0$ on $E_s \Delta F$,

$$-\int_{\mathbb{R}^N} (\chi_F - \chi_{E_s})\theta \text{div } \sigma_j = -\int_{\mathbb{R}^N} (\chi_F - \chi_{E_s}) \text{div } (\theta\sigma_j). \tag{2.17}$$

Remember that $\sigma_j = \sigma * \rho_j$ and $|\sigma| \leq 1$ a.e. on V. By construction, $\theta\sigma_j \in C_c^\infty(V)$ and for every $j > \frac{1}{\text{dist (supp } \theta, \partial V)}$, $|\theta\sigma_j| \leq 1$. Hence,

$$\left| \int_{\mathbb{R}^N} \chi_F \text{div } (\theta\sigma_j) \right| \leq |D\chi_F|(V) = \text{Per } (F, V). \tag{2.18}$$

Using (2.11),

$$\begin{aligned} \int_{\mathbb{R}^N} \chi_{E_s} \text{div } (\theta\sigma_j) &= -\int_{\mathbb{R}^N} \theta\langle \sigma_j, \frac{D\chi_{E_s}}{|D\chi_{E_s}|}\rangle d|D\chi_{E_s}| \\ &= -\int_{\mathbb{R}^N} \theta\langle \sigma_j, \frac{\nabla u}{|\nabla u|}\rangle d|D\chi_{E_s}| \\ &= -\int_{\partial^* E_s} \theta\langle \sigma_j, \frac{\nabla u}{|\nabla u|}\rangle d\mathcal{H}^{N-1}. \end{aligned} \tag{2.19}$$

The last line relies on (2.7). Now,

$$\left| \int_{\partial^* E_s} \theta\langle \sigma_j, \frac{\nabla u}{|\nabla u|}\rangle d\mathcal{H}^{N-1} - \int_{\partial^* E_s} \theta\langle \sigma, \frac{\nabla u}{|\nabla u|}\rangle d\mathcal{H}^{N-1} \right| \leq \int_{\text{supp } \theta\cap\partial^* E_s} |\sigma_j - \sigma| \, d\mathcal{H}^{N-1}.$$

In view of (2.15), the right hand side goes to 0 when $j \to +\infty$. Hence,

$$\lim_{j\to+\infty} \int_{\partial^* E_s} \theta\langle \sigma_j, \frac{\nabla u}{|\nabla u|}\rangle d\mathcal{H}^{N-1} = \int_{\partial^* E_s} \theta\langle \sigma, \frac{\nabla u}{|\nabla u|}\rangle d\mathcal{H}^{N-1}. \tag{2.20}$$

But for $\mathcal{H}^{N-1}$ a.e. $x \in V \cap \partial^* E_s$, $\langle \sigma(x), \nabla u(x)/|\nabla u(x)|\rangle = 1$. It thus follows from (2.19) and (2.20) that

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \chi_{E_s} \text{div } (\theta\sigma_j) = -\int_{\partial^* E_s} \theta d\mathcal{H}^{N-1}. \tag{2.21}$$

In the left hand side of (2.17), we use the fact that $F \Delta E_s$ is compactly contained in Ω, so that for every j sufficiently large, $\operatorname{div} \sigma_j = -\lambda$ on $F \Delta E_s$. Our claim (2.16) is now a consequence of (2.17), (2.18) and (2.21).
Step 3. *Completion of the proof.*

We replace the function θ introduced in the previous step by a sequence $(\theta_k)_{k\geq}$ such that each θ_k satisfies the same assumptions as θ, and $\theta_k \to 1$ a.e. on V. By letting $k \to +\infty$, we thus obtain

$$\lambda \int_{\mathbb{R}^N} (\chi_F - \chi_{E_s}) \leq \text{Per } (F, V) - \mathcal{H}^{N-1}(V \cap \partial^* E_s) = \text{Per } (F, V) - \text{Per } (E_s, V). \tag{2.22}$$

Since $F \setminus V = E_s \setminus V$,

$$\lambda(|F \cap V| - |E_s \cap V|) = \lambda(|F| - |E_s|) = \lambda \int_{\mathbb{R}^N} (\chi_F - \chi_{E_s}).$$

Together with (2.22), this implies that

$$\text{Per } (E_s, V) - \lambda |E_s \cap V| \leq \text{Per } (F, V) - \lambda |F \cap V|.$$

The proof is complete. □

2.4. **Regularity of the level sets.** This section is devoted to the proof of Proposition 1.3 which states that the level sets of a solution are C^1 up to a small singular set.

Proposition 2.8. *Given $\lambda \geq 0$, let u be a solution of (P_λ) and U be the open set in Ω defined by $[|\nabla u| > 1]$. Then for a.e. $t \in \mathbb{R}$, there exists an open set W in Ω such that $W \cap \partial^e [u \geq t])$ is a C^1 hypersurface and $\mathcal{H}^s(\Omega \setminus (W \cup \partial U)) = 0$ for every $s > N - 8$.*

Remember that $\partial^e[u \geq t]$ is the essential boundary of the set $[u \geq t]$, see (1.6).

Proof of Proposition 2.8. On the open set U, u is smooth and ∇u does not vanish. Hence each level set is a smooth hypersurface. Let $t \in \mathbb{R}$ such that the super-level set $E_t := [u \geq t]$ satisfies the conclusion of Proposition 2.7. Then by [26, Theorems 5.1 and 5.2], there exists an open set W_0 in $V = \Omega \setminus \overline{U}$ such that $\partial^e E_t \cap W_0$ is an $N-1$ dimensional manifold of class $C^{1,\alpha}$ for some $0 < \alpha < 1$ and $\mathcal{H}^s(V \setminus W_0) = 0$ for every $s > N - 8$. The set $W = U \cup W_0$ satisfies the required properties. □

Remark 2.9. The proof of [26, Theorems 5.1 and 5.2] shows that for every $x \in W_0 \cap \partial^e E_t$, there exists $\delta > 0$ such that $B_\delta(x) \cap \partial^e E_t = B_\delta(x) \cap \partial^* E_t$. In particular, the vector $\nu_{E_t}(y)$ is defined for every $y \in B_\delta(x) \cap \partial^e E_t$ and it is proved to be (Hölder) continuous on this set. If we choose t as in Lemma 2.6: $\nabla u(x) \neq 0$ for $\mathcal{H}^{N-1}$ a.e. $x \in \partial^* E_t$ and (2.11) holds true, we deduce that $\frac{\nabla u}{|\nabla u|}$ is equal to the continuous function ν_{E_t} $\mathcal{H}^{N-1}$ a.e. on $W_0 \cap \partial^e E_t$. Hence, the smooth hypersurface $W_0 \cap \partial^e E_t$ has a well defined *orientation* which is given by (the continuous extension of) $\frac{\nabla u}{|\nabla u|}\,|_{W_0 \cap \partial^e E_t}$.

3. Uniqueness

In this section, we present the proof of Theorem 1.1, namely the uniqueness of the solutions of (P_λ). We first explain the strategy that we follow:

Let $\lambda \geq 0$ and u, v be two solutions of (P_λ).

Step 1 The two open sets $[|\nabla u| > 1]$ and $[|\nabla v| > 1]$ coincide : this is an easy consequence of the fact that φ is strictly convex outside the unit ball. We denote by U this open set. Still by strict convexity, $\nabla u = \nabla v$ on U. This step corresponds to Lemma 3.1.

Step 2 Every connected component of U intersects the boundary $\partial\Omega$. This is Lemma 3.3.

It follows from the two above steps that $u = v$ on U.

Step 3 The fact that φ only depends on the norm of the gradient of u implies that ∇u and ∇v are colinear a.e. (Lemma 3.1 again). By using the regularity of the level sets of u, one can deduce that v is constant on the level sets of u.

Step 4 For a.e. $t \in \mathbb{R}$, every connected component of $u^{-1}(t)$ with positive measure intersect $U \cup \partial\Omega$. This step involves a geometrical result, see Lemma 3.8, which is a weak generalization of the fact that a compact hypersurface without boundary is the boundary of a bounded open set.

From Step 3 and Step 4 together with the fact that $u = v$ on U, we finally deduce that $u = v$ on almost every level set of u, which is enough to conclude that $u = v$ on Ω.

3.1. **A comparison principle.**

Lemma 3.1. *Let $\lambda \geq 0$ and let u, v be two solutions of (P_λ). Then for a.e. $x \in \Omega$, either*

$$\max(|\nabla u(x)|, |\nabla v(x)|) \leq 1 \quad \text{and} \quad \nabla u(x), \nabla v(x) \text{ are colinear}$$

or

$$\nabla u(x) = \nabla v(x).$$

Proof. Since u is a solution of (P_λ),

$$I_\lambda(u) \leq I_\lambda\left(\frac{u+v}{2}\right).$$

By convexity of φ,

$$\begin{aligned} I_\lambda\left(\frac{u+v}{2}\right) &= \int_\Omega \varphi\left(\frac{\nabla u + \nabla v}{2}\right) - \lambda \frac{u+v}{2} \\ &\leq \frac{1}{2}\int_\Omega (\varphi(\nabla u) - \lambda u) + \frac{1}{2}\int_\Omega (\varphi(\nabla v) - \lambda v) \qquad (3.1) \\ &= \frac{1}{2} I_\lambda(u) + \frac{1}{2} I_\lambda(v). \end{aligned}$$

Since v is another solution,

$$I_\lambda(u) = \frac{1}{2} I_\lambda(u) + \frac{1}{2} I_\lambda(v).$$

This implies that a.e. on Ω,

$$\varphi\left(\frac{\nabla u + \nabla v}{2}\right) - \lambda \frac{u+v}{2} = \frac{1}{2}(\varphi(\nabla u) + \varphi(\nabla v)) - \frac{\lambda}{2}(u+v),$$

or equivalently,

$$\varphi\left(\frac{\nabla u + \nabla v}{2}\right) = \frac{1}{2}(\varphi(\nabla u) + \varphi(\nabla v)).$$

Hence for a.e. $x \in \Omega$, φ is affine on the segment $[\nabla u(x), \nabla v(x)]$. In view of the definition of φ, this implies the desired conclusion.

□

Lemma 3.2. *Assume that the boundary condition ψ satisfies $\psi \equiv 0$. Given $\lambda \geq 0$, let u be a solution of (P_λ). Then for every $\mu > 0$, the super-level set $E_\mu := [u \geq \mu]$ has finite perimeter and*

$$Per\ (E_\mu, \Omega) \leq \lambda |E_\mu|.$$

Proof. For every $0 < \mu < \nu$, let

$$u_{\mu,\nu} = \max(\min(u,\mu), u - \nu + \mu) = \begin{cases} u & \text{if } u \leq \mu, \\ \mu & \text{if } \mu \leq u \leq \nu, \\ u - \nu + \mu & \text{if } \nu \leq u. \end{cases}$$

Then $u_{\mu,\nu} \in H^1_0(\Omega)$. Hence $I_\lambda(u) \leq I_\lambda(u_{\mu,\nu})$ and thus

$$\int_{E_\mu \setminus E_\nu} \varphi(\nabla u) \leq \lambda \int_\Omega (u - u_{\mu,\nu}).$$

Since $\varphi(\nabla u) \geq |\nabla u|$, this implies

$$\int_{E_\mu \setminus E_\nu} |\nabla u| \leq \lambda \int_\Omega (u - u_{\mu,\nu}).$$

By the coarea formula, we get

$$\int_\mu^\nu \text{Per } (E_t, \Omega)\, dt \leq \lambda \int_\Omega (u - u_{\mu,\nu}).$$

By definition of $u_{\mu,\nu}$, it follows that

$$\begin{aligned} \int_\mu^\nu \text{Per } (E_t, \Omega)\, dt &\leq \lambda \int_{E_\mu \setminus E_\nu} (u - \mu) + \lambda \int_{E_\nu} (\nu - \mu) \\ &\leq \lambda(\nu - \mu)|E_\mu|. \end{aligned}$$

Assume first that μ is a Lebesgue point of the map $t \mapsto \text{Per } (E_t, \Omega)$. Then, dividing the above inequality by $\nu - \mu$ and letting $\nu \to \mu$ yield the desired result.

Now, for every $\mu > 0$, there exists an increasing sequence of such Lebesgue points μ_i converging to μ. Hence, $E_\mu = \cap_i E_{\mu_i}$ and thus, $\lim_{i \to +\infty} |E_{\mu_i}| = |E_\mu|$. Moreover, by semicontinuity of the total variation on $BV(\Omega)$,

$$\text{Per } (E_\mu, \Omega) \leq \liminf_{i \to +\infty} \text{Per } (E_{\mu_i}, \Omega).$$

By the previous case, for every $i \geq 0$, $\text{Per } (E_{\mu_i}, \Omega) \leq \lambda |E_{\mu_i}|$. This implies

$$\text{Per } (E_\mu, \Omega) \leq \lambda |E_\mu|.$$

The proof is complete.

□

3.2. **Uniqueness on the set U.** A crucial step in the proof of Theorem 1.1 is the uniqueness of the solution on the open set where the norms of the gradients are larger than 1.

Lemma 3.3. *Given $\lambda \geq 0$, let $u \in H^1_0(\Omega)$ be a solution of (P_λ). Then each connected component U_0 of the open set $U = [|\nabla u| > 1]$ satisfies*

$$\partial U_0 \cap \partial \Omega \neq \emptyset.$$

The proof below is inspired from the one of [2, Proposition 7.3].

Proof. Assume by contradiction that $\overline{U_0} \subset \Omega$. Since U is open and $|\nabla u|$ can be extended as a uniformly continuous function on $\overline{U_0}$, $|\nabla u| \equiv 1$ on ∂U_0, see Lemma 2.4. But on the set U_0, where u is smooth,

$$\Delta|\nabla u|^2 = 2\sum_{i,j}(\partial^2_{ij}u)^2 + 2\sum_j \partial_j u \partial_j \Delta u = 2\sum_{i,j}(\partial^2_{ij}u)^2 \geq 0. \tag{3.2}$$

Here, we have used that $\Delta u = -\lambda$ on U so that $\partial_j \Delta u = 0$ for every j. This proves that $|\nabla u|^2$ is subharmonic on U_0, and thus $|\nabla u|^2$ attains its maximum on ∂U_0. But this contradicts the facts that $|\nabla u| \equiv 1$ on ∂U_0 and $|\nabla u| > 1$ on U_0. Hence, $\partial U_0 \cap \partial\Omega \neq \emptyset$, as desired. □

Remark 3.4. It also follows from (3.2) that the interior of the set $[|\nabla u| = 1]$ is empty, when $\lambda > 0$. Indeed, assume by contradiction that there exists an open connected set $W \subset \Omega$ where $|\nabla u| = 1$ a.e. Then the Euler equation on W is $\int_\Omega \langle \nabla u, \nabla\theta\rangle - \lambda\theta = 0$, for every $\theta \in C^\infty_c(W)$. Hence, $\Delta u = -\lambda$ on W (in the sense of distributions and thus in the classical sense). Since $|\nabla u| = 1$ on W, $\Delta|\nabla u|^2 = 0$. From (3.2), we deduce that $|\nabla^2 u| = 0$, which implies that u is affine, and thus $\Delta u = 0$, a contradiction with $\lambda > 0$. This proves the remark.

In the proof of Theorem 1.1, we will also need the following variant of Lemma 3.3.

Lemma 3.5. *Given $\lambda \geq 0$, let u be a solution of (P_λ) and $U := [|\nabla u| > 1]$. For every $r > 0$, let $H_r := \{x \in \mathbb{R}^N : \text{dist}\ (x, \mathbb{R}^N \setminus \Omega) \leq r\}$. If $\partial\Omega$ is connected, then the set $\overline{U} \cup H_r$ is connected.*

Proof. We first prove that the set $K_r := \{x \in \overline{\Omega} : \text{dist}\ (x, \partial\Omega) \leq r\}$ is connected. Consider a continuous function $\chi : K_r \to \mathbb{R}$ with values into $\{0, 1\}$. Since $K_0 = \partial\Omega$ is connected, χ is constant on K_0, e.g. $\chi \equiv 0$ on K_0. Let $r_* := \inf\{s \in [0, r] : \chi \not\equiv 0 \text{ on } K_s\}$. Then by continuity of χ, $\chi^{-1}(0)$ is an open set of K_r containing K_0 and thus $r_* > 0$. Moreover, for every $0 < r' < r_*$, $\chi \equiv 0$ on $K_{r'}$ so that $\chi \equiv 0$ on K_{r_*}. Assume by contradiction that $r_* < r$. Then there exists $r_i \downarrow r_*$ and $x_i \in K_{r_i}$ such that $\chi(x_i) = 1$. Up to a subsequence, $(x_i)_i$ converges to some x in K_{r_*}, which implies $\chi(x) = 1$, a contradiction. This proves that $r_* = r$; that is, $\chi \equiv 0$ on K_r. Hence, K_r is connected. We can prove in a similar way that $\tilde{K}_s := \{x \in \mathbb{R}^N \setminus \Omega : \text{dist}\ (x, \partial\Omega) \leq s\}$ is connected for every $s \geq 0$. Since every $\tilde{K}_s$ contains $\partial\Omega$, it follows that $\mathbb{R}^N \setminus \Omega = \cup_{s\geq 0}\tilde{K}_s$ is connected and so is $H_r = K_r \cup (\mathbb{R}^N \setminus \Omega)$.

For every connected component U_0 of U, Lemma 3.3 implies that $U_0 \cap H_r \neq \emptyset$. Hence, $U_0 \cup H_r$ is connected. It follows that the set $U \cup H_r$ is connected and the same is true for its closure

$$\overline{U \cup H_r} = \overline{U} \cup \overline{H_r} = \overline{U} \cup H_r.$$

The lemma is proved. □

We now establish the main result of this section: two solutions agree on the set where their gradients do not belong to the unit ball.

Lemma 3.6. *Given $\lambda \geq 0$, let $u, v \in H^1_0(\Omega)$ be two solutions of (P_λ). Then*

- *the two open sets $[|\nabla u| > 1]$ and $[|\nabla v| > 1]$ coincide. We denote by U the corresponding set.*
- *$u = v$ on U.*

Proof. The first assertion follows from Lemma 3.1. Still by Lemma 3.1, $\nabla(u - v) = 0$ on U and thus $u - v$ is constant on each connected component of U. Since $u - v = 0$ on $\partial\Omega$, the second assertion is a consequence of Lemma 3.3. □

Another important tool in the proof of Theorem 1.1 is the fact that a level set of a solution cannot be contained in $\Omega \setminus U$ except if its interior is not empty. Here is a first result in that direction.

Lemma 3.7. *Given $\lambda \geq 0$, let u be a solution of (P_λ). Let G be an open subset of $\Omega \setminus U$, where U is the open set defined by $U = [|\nabla u| > 1]$. Assume that u is continuous on $\overline{G}$* [4] *and that u is constant on ∂G. Then u is constant on G.*

Proof. By assumption, there exists $c \in \mathbb{R}$ such that $u|_G - c$ belongs to $H^1_0(G)$. Moreover, it minimizes the functional

$$v \mapsto \int_G \varphi(\nabla v) - \lambda v. \tag{3.3}$$

By Lemma 2.1 applied to this functional on $H^1_0(G)$, $u \geq c$ on G. We now prove that $u \leq c$ on G. Assume by contradiction that $\max_{\overline{G}} u > c$. Then

$$A := \{x \in G : u(x) = \max_{\overline{G}} u\}$$

is a compact subset contained in G. By Lemma 3.2 with $\mu := \max_{\overline{G}} u$,

$$\text{Per } A = \text{Per } (A, G) \leq \lambda |A|.$$

Since $A \Subset G$, $sA \Subset G$ for every s sufficiently close to 1. Fix such an $s > 1$. Then

$$\text{Per } sA = s^{N-1} \text{Per } A \leq s^{N-1} \lambda |A| < \lambda |sA|. \tag{3.4}$$

We claim that there exists $v \in H^1_0(G)$ such that

$$\int_G |\nabla v| < \lambda \int_G v. \tag{3.5}$$

Indeed, let $(\rho_k)_{k \geq 1} \subset C^\infty_c(B_{\frac{1}{k}})$ be a regularization kernel. Then for $k > \frac{1}{\text{dist}\,(sA, \partial G)}$, the map $v_k := 1_{sA} * \rho_k$ belongs to $H^1_0(G)$ and converges in $L^1(G)$ to 1_{sA}. Moreover, by [17, Remark 1.16],

$$\lim_{k \to +\infty} \int_G |\nabla v_k| = \lim_{k \to +\infty} \int_{\mathbb{R}^N} |\nabla v_k| = \text{Per } sA.$$

In view of (3.4), there exists $k \in \mathbb{N}$ such that

$$\int_G |\nabla v_k| < \lambda \int_G v_k.$$

This proves the claim (3.5) with $v := v_k$.

However, by (2.6) applied to the minimization problem (3.3),

$$\lambda \int_G v \leq \int_{[\nabla u = 0] \cap G} |\nabla v| + \int_{[\nabla u \neq 0] \cap G} \langle \nabla \varphi(\nabla u), \nabla v \rangle.$$

Using the fact that $G \subset \Omega \setminus U$, this yields

$$\lambda \int_G v \leq \int_{[\nabla u = 0] \cap G} |\nabla v| + \int_{[|\nabla u| \leq 1] \cap G} |\nabla v| \leq \int_G |\nabla v|.$$

This contradicts (3.5). Hence $u \equiv c$ on G. □

[4] This is the case when $\overline{G} \subset \Omega$ or when Ω is Lipschitz.

3.3. **A geometrical result.** The following lemma will be crucial to prove that generically, the level sets of u which intersect $V = \Omega \setminus \overline{U}$ are not contained in V.

Lemma 3.8. *Let V be an open bounded subset of $\mathbb{R}^N$ such that $\mathbb{R}^N \setminus V$ is connected. Let R be a C^1 orientable hypersurface compactly contained in V. If $\mathcal{H}^{N-1}(R) < \infty$ and $\mathcal{H}^{N-2}(\overline{R} \setminus R) = 0$, then there exists a non empty open set $E \subset V$ such that $\partial E \subset \overline{R}$.*

Remark 3.9. If one further assumes that R is connected, then one can prove a deeper result: there exists a set $F \Subset V$ with finite perimeter such that $\partial F = \overline{R}$, see [19, Lemma 4.2] which inspired us for the first paragraph of the proof below.

Proof. Since R is an orientable C^1 hypersurface of $\mathbb{R}^N$ with $\mathcal{H}^{N-1}(R) < \infty$, it defines an integer multiplicity rectifiable current that we denote by $[R]$. We first claim that the boundary (in the sense of currents) of $[R]$ is trivial: $\partial[R] = 0$. Indeed, $\partial[R]$ is a flat chain as the boundary of an integer multiplicity rectifiable current, see e.g. [27, Section 4.3]. Since $\partial[R]$ is supported in $\overline{R} \setminus R$ and $\mathcal{H}^{N-2}(\overline{R} \setminus R) = 0$, this implies that $\partial[R] = 0$, see e.g. [27, Theorem 4.7]. It follows that there exists an N dimensional integer multiplicity rectifiable current H with finite mass such that $\partial H = [R]$, see e.g. [14, Sections I.2.3 and I.2.4]. Actually, one can take for H the cone over $[R]$, namely the push-forward by the map $h(t, x) = tx$ of the product of the two currents $[0, 1] \times R$: $H = h_\sharp([0, 1] \times R)$.

We now prove that $\mathbb{R}^N \setminus \overline{R}$ is *not* connected. Assume by contradiction that this is not the case. Since the support of $[R]$ is contained in $\overline{R}$, H has no boundary (in the sense of currents) in the open set $\mathbb{R}^N \setminus \overline{R}$. It follows from the Constancy theorem, see e.g. [14, Theorem I.2.3.4], that there exists $r \in \mathbb{R}$ such that for every smooth N form ω compactly supported in $\mathbb{R}^N \setminus \overline{R}$,

$$H(\omega) = r \int_{\mathbb{R}^N \setminus \overline{R}} \omega. \tag{3.6}$$

Since H is an N dimensional integer rectifiable current and $\mathcal{H}^N(\overline{R}) = 0$, it follows that (3.6) remains true for every smooth N form with compact support. Hence, H is constant as a current on $\mathbb{R}^N$ and thus $\partial H = [R] = 0$, a contradiction. This proves that $\mathbb{R}^N \setminus \overline{R}$ is not connected.

Since $\mathbb{R}^N \setminus \overline{R}$ is open, each connected component of $\mathbb{R}^N \setminus \overline{R}$ is open. By assumption, $\overline{R} \subset V$ and thus $\mathbb{R}^N \setminus V \subset \mathbb{R}^N \setminus \overline{R}$. Since $\mathbb{R}^N \setminus \overline{R}$ is not connected, there exists at least one connected component E of $\mathbb{R}^N \setminus \overline{R}$ which does not intersect the open connected component of $\mathbb{R}^N \setminus \overline{R}$ containing the connected set $\mathbb{R}^N \setminus V$. This implies that $E \subset V$. Moreover, $\partial E \subset \overline{R}$. This completes the proof. □

3.4. **Proof of Theorem 1.1.** We can now complete the proof of Theorem 1.1. We assume throughout this section that Ω is Lipschitz and has a connected boundary.

Proof. Let u and v be two solutions of (P_λ). By Lemma 3.1, $\nabla u(x)$ and $\nabla v(x)$ are colinear for a.e. $x \in \Omega$.

Let S be the set of those $x \in \Omega$ such that

(1) either u or v is not differentiable at x,
(2) or $\nabla u(x) = 0$,
(3) or $\nabla u(x)$ and $\nabla v(x)$ are not colinear.

Since $\int_S |\nabla u| = 0$, the coarea formula yields the existence of a negligeable set $N_0 \subset \mathbb{R}$ such that for every $t \in \mathbb{R} \setminus N_0$, $[u \geq t]$ is a set of finite perimeter with $\mathcal{H}^{N-1}(u^{-1}(t) \cap S) = 0$. In view of Proposition 2.8 and Remark 2.9, there exists a negligeable set $N_1 \subset \mathbb{R}$ with the following property:

for every $t \in \mathbb{R} \setminus N_1$, there exists an open set W_t such that $\partial^e[u \geq t] \cap W_t$ is an orientable C^1 hypersurface and $\mathcal{H}^s(\Omega \setminus (W_t \cup \partial U)) = 0$ for every $s > N - 8$ (as usual, U is the open subset of Ω defined by $[|\nabla u| > 1]$).

Let $N := N_0 \cup N_1$ and $t \in (0, \infty) \setminus N$. We define $E_t := [u \geq t]$. Let R be a connected component of $\partial^e E_t \cap W_t$. Since u is constant on the C^1 hypersurface R and $\nabla v(x)$ is colinear to $\nabla u(x)$ $\mathcal{H}^{N-1}$ a.e. $x \in R$, it follows that for such $x \in R$, $\nabla v(x)$ is orthogonal to R at x. This implies that the tangential gradient of v on the connected manifold R vanishes. Thus v is constant on R and, by continuity of v, on $\overline{R}$ as well.

We claim that

$$\overline{R} \cap (\partial\Omega \cup \overline{U}) \neq \emptyset. \tag{3.7}$$

Indeed, assume by contradiction that $\overline{R} \cap (\partial\Omega \cup \overline{U}) = \emptyset$. Then there exists $r > 0$ such that $\overline{R} \cap H_r = \emptyset$ where $H_r := \{x \in \mathbb{R}^N : \text{dist}\ (x, \mathbb{R}^N \setminus \Omega) \leq r\}$. We set $V_r := \Omega \setminus (\overline{U} \cup H_r) = \mathbb{R}^N \setminus (\overline{U} \cup H_r)$. Then by Lemma 3.5, $\mathbb{R}^N \setminus V_r = \overline{U} \cup H_r$ is connected.

The essential boundary $\partial^e E_t$ is a closed subset of $\mathbb{R}^N$, as the complement of the open set $E_t^0 \cup E_t^1$ where

$$E_t^0 = \{x \in \mathbb{R}^N : \exists \rho > 0 \text{ such that } |E_t \cap B_\rho(x)| = 0\},$$
$$E_t^1 = \{x \in \mathbb{R}^N : \exists \rho > 0 \text{ such that } |E_t \cap B_\rho(x)| = |B_\rho(x)|\}.$$

(The fact that E_t^0 and E_t^1 are open is proved in [17, Proposition 3.1]). Since $R \subset \partial^e E_t$, we also have $\overline{R} \subset \partial^e E_t$. Since the closure of a connected component of the C^1 hypersurface $\partial^e E_t \cap W_t$ cannot intersect another connected component of this hypersurface, this implies that $(\overline{R} \setminus R) \cap W_t = \emptyset$ and thus

$$\overline{R} \setminus R \subset V_r \setminus W_t \subset \Omega \setminus (W_t \cup \partial U).$$

It follows that $\mathcal{H}^{N-2}(\overline{R} \setminus R) = 0$. By Lemma 3.8 applied to V_r and R, there exists a non empty open set $E \subset V_r$ such that $\partial E \subset \overline{R}$. Observe that $u \equiv t$ on $\overline{R} \supset \partial E$. Applying Lemma 3.7 to E yields $u \equiv t$ on E. In particular, $\nabla u = 0$ $\mathcal{H}^N$ a.e. on E and thus $\mathcal{H}^N(E \setminus S) = 0$. Since $E \subset u^{-1}(t)$ and $\mathcal{H}^{N-1}(u^{-1}(t) \cap S) = 0$, one has $\mathcal{H}^{N-1}(E \cap S) = 0$. It follows that $\mathcal{H}^N(E) = 0$, a contradiction. Our claim (3.7) is thus proved.

Since Ω is assumed to be Lipschitz, the solutions u and v are continuous on $\overline{\Omega}$, see section 2.1. Moreover, $u = v$ on $\partial\Omega \cup \overline{U}$ and v and u are constant on $\overline{R}$. It then follows from (3.7) that $v = u = t$ on $\overline{R}$. Since this is true for every component of $\partial^e E_t \cap W_t$, one gets $v = t$ on $\partial^e E_t \cap W_t$.

For every $x \in u^{-1}(t) \setminus S$, u is differentiable at x and $\nabla u(x) \neq 0$. This implies that for every $\rho > 0$ sufficiently small,

$$\{y \in B_\rho(x) : \langle \frac{\nabla u(x)}{|\nabla u(x)|}, y - x\rangle \geq \frac{1}{2}|y-x|\} \subset B_\rho(x) \cap E_t \subset \{y \in B_\rho(x) : \langle \frac{\nabla u(x)}{|\nabla u(x)|}, y-x\rangle \geq \frac{-1}{2}|y-x|\},$$

which proves that $x \in \partial^e E_t$. It follows that $u^{-1}(t) \setminus S \subset \partial^e E_t$. Since $v = t$ on $\partial^e E_t \cap (W_t \cup \overline{U})$, one thus gets

$$\mathcal{H}^{N-1}(u^{-1}(t) \cap [u \neq v]) \leq \mathcal{H}^{N-1}\left((u^{-1}(t) \cap S) \cup (\Omega \setminus (W_t \cup \partial U))\right) = 0.$$

By the coarea formula, this implies

$$\int_{[u \neq v]} |\nabla u| = 0.$$

Hence, $\nabla u = 0$ a.e. on the open set $[u \neq v]$. Similarly, $\nabla v = 0$ a.e. on $[u \neq v]$. Since $\nabla(u - v) = 0$ a.e. on the set $[u = v]$, it follows that $\nabla(u - v) = 0$ a.e. on Ω. But $u = v = 0$ on $\partial\Omega$. This implies that $u = v$ on Ω. The proof is complete.

□

Remark 3.10. When ψ is a constant map, the assumption that Ω is Lipschitz is unnecessary. Indeed, this regularity assumption was made to guarantee the continuity of the solutions up to the boundary. But in the case when ψ is constant, the level sets of the solutions do not intersect the boundary (except for the level set corresponding to the value of ψ) and the continuity of the solutions inside Ω is enough for the above argument to remain true.

4. Some qualitative properties of the solutions

In this section, we assume that Ω is Lipschitz. This implies that every solution is continuous on $\overline{\Omega}$, see section 2.1.

4.1. **A comparison principle.**

Lemma 4.1. *Let $\lambda_2 \geq \lambda_1 > 0$, u_1 a solution of (P_{λ_1}) on $u_1 + H_0^1(\Omega)$ and u_2 a solution of (P_{λ_2}) on $u_2 + H_0^1(\Omega)$. We also assume that $u_1|_{\partial\Omega} \leq u_2|_{\partial\Omega}$. If $\lambda_2 > \lambda_1$ or if u_1 is the unique minimum of (P_{λ_1}) on $u_1 + H_0^1(\Omega)$ or if u_2 is the unique minimum of (P_{λ_2}) on $u_2 + H_0^1(\Omega)$, then $u_1 \leq u_2$ on Ω.*

The assumption $u_1|_{\partial\Omega} \leq u_2|_{\partial\Omega}$ means that $(u_1 - u_2)_+ \in H_0^1(\Omega)$.

Proof. Since $I_{\lambda_1}(u_1) \leq I_{\lambda_1}(\min(u_1, u_2))$, we have

$$\int_{[u_1 > u_2]} \varphi(\nabla u_1) - \lambda_1 u_1 \leq \int_{[u_1 > u_2]} \varphi(\nabla u_2) - \lambda_1 u_2. \tag{4.1}$$

Since $I_{\lambda_2}(u_2) \leq I_{\lambda_2}(\max(u_1, u_2))$,

$$\int_{[u_1 > u_2]} \varphi(\nabla u_2) - \lambda_2 u_2 \leq \int_{[u_1 > u_2]} \varphi(\nabla u_1) - \lambda_2 u_1. \tag{4.2}$$

The sum of (4.1) and (4.2) gives

$$0 \leq (\lambda_2 - \lambda_1) \int_{[u_2 < u_1]} u_2 - u_1.$$

We first assume that $\lambda_2 > \lambda_1$. Since $u_2 - u_1$ is nonpositive on the set $[u_2 < u_1]$, it follows that $u_1 \leq u_2$ on Ω, which completes the proof in that case. Otherwise, $\lambda_1 = \lambda_2$ and assume for instance that u_1 is the unique minimum of (P_{λ_1}) on $u_1 + H_0^1(\Omega)$. Then (4.1), (4.2) imply

$$\int_{[u_1 > u_2]} \varphi(\nabla u_1) - \lambda_1 u_1 = \int_{[u_1 > u_2]} \varphi(\nabla u_2) - \lambda_1 u_2.$$

This yields

$$I_{\lambda_1}(u_1) = I_{\lambda_1}(\min(u_1, u_2)).$$

Since u_1 is the unique minimum, it follows that $u_1 = \min(u_1, u_2)$. Hence, $u_1 \leq u_2$ on Ω, which completes the proof in that case as well.

□

As a consequence of Theorem 1.1, the above comparison principle applies when Ω is Lipschitz and has a connected boundary, since under these assumptions, the solutions of (P_λ) are unique. In the particular case when Ω is a ball, we even know the solution explicitly:

Remark 4.2. If Ω is the ball $B_R(x_0)$ of center $x_0 \in \mathbb{R}^N$ and radius $R > 0$ and if $\psi \equiv 0$, then the unique solution of (P_λ) on $B_R(x_0)$ is the function

$$\xi_{\lambda,x_0,R} = \frac{-N}{2\lambda}\left(\left(\frac{\lambda^2}{N^2}|x-x_0|^2 - 1\right)_+ - \left(\frac{\lambda^2}{N^2}R^2 - 1\right)_+\right). \tag{4.3}$$

The above remark can be seen as a consequence of a more general result due to Cellina, see [10, Theorem 1]:

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set in $\mathbb{R}^N$ and $F : \mathbb{R}^N \to \mathbb{R}$ a convex function. We assume that F is superlinear: $\lim_{x\to+\infty} \frac{F(x)}{|x|} = +\infty$. Given $c \in \mathbb{R}$, $x_0 \in \mathbb{R}^N$, let*

$$h_{x_0,c}(x) = \frac{-N}{\lambda} F^*\left(\frac{-\lambda}{N}(x - x_0)\right) + c \quad , \quad x \in \Omega,$$

where $F^*(y) := \sup_{x\in\mathbb{R}^N} (\langle x, y\rangle - F(x))$. Then $h_{x_0,c}$ is the unique minimum of the variational problem:

$$\text{To Minimize } u \mapsto \int_\Omega F(\nabla u) - \lambda u$$

on $h_{x_0,c} + H^1_0(\Omega)$.

Proof. Let ζ be the affine map $x \mapsto \frac{-\lambda}{N}(x - x_0)$. Then $h_{x_0,c} = \frac{-N}{\lambda} F^* \circ \zeta + c$. Since F^* is convex, $h_{x_0,c}$ is concave. Hence, both F^* and $h_{x_0,c}$ are locally Lipschitz continuous and thus differentiable a.e. Moreover, for a.e. $x \in \Omega$, the convex subdifferential $\partial F^*(\zeta(x))$ is reduced to the singleton $\{\nabla F^*(\zeta(x))\}$ and

$$\nabla h_{x_0,c}(x) = \nabla F^*(\zeta(x)). \tag{4.4}$$

By convex duality, this implies that

$$\zeta(x) \in \partial F(\nabla h_{x_0,c}(x)).$$

Let $u \in h_{x_0,c} + H^1_0(\Omega)$. Then by definition of a convex subgradient,

$$\int_\Omega F(\nabla u) - F(\nabla h_{x_0,c}) - \lambda(u - h_{x_0,c}) \geq \int_\Omega \langle \zeta, \nabla u - \nabla h_{x_0,c}\rangle - \lambda(u - h_{x_0,c}). \tag{4.5}$$

Since $u - h_{x_0,c} \in H^1_0(\Omega)$, Stokes formula implies

$$\int_\Omega F(\nabla u) - F(\nabla h_{x_0,c}) - \lambda(u - h_{x_0,c}) \geq \int_\Omega \lambda(u - h_{x_0,c}) - \lambda(u - h_{x_0,c}) = 0.$$

This proves that $h_{x_0,c}$ is a minimum. If u is another minimum, then the left hand side in (4.5) is 0 which implies that

$$F(\nabla h_{x_0,c}(x)) = F(\nabla u(x)) + \langle \zeta(x), \nabla h_{x_0,c}(x) - \nabla u(x)\rangle \quad \text{a.e. } x \in \Omega.$$

Hence, $\zeta(x) \in \partial F(\nabla u(x))$. By convex duality together with (4.4), we get for a.e. $x \in \Omega$,

$$\nabla u(x) \in \partial F^*(\zeta(x)) = \{\nabla h_{x_0,c}(x)\}.$$

Hence $\nabla u(x) = \nabla h_{x_0,c}(x)$ a.e. and thus $h_{x_0,c} = u$. This proves that $h_{x_0,c}$ is the unique solution on $h_{x_0,c} + H^1_0(\Omega)$.

□

Observe that Remark 4.2 follows from Theorem 4.3 since the convex conjugate of $F = \varphi$ is $F^*(y) = \frac{1}{2}(|y|^2 - 1)_+$. We can now use the explicit solution on the ball as a barrier which, together with the comparison principle Lemma 4.1, yields certain bounds on the solutions when Ω is *any* bounded open set. As an illustration, we give an explicit upper bound on the solutions of (P_λ) which does not depend on the L^1 norm of the solutions, in contrast to Lemma 2.1, but only on Ω, λ and the boundary condition ψ:

Lemma 4.4. *Let $\lambda > 0$ and u a solution of (P_λ) on $\psi|_\Omega + H_0^1(\Omega)$. Then*

$$\max_\Omega u \le \frac{N}{2\lambda}\left(\frac{\lambda^2}{N^2}R^2 - 1\right)_+ + \max_{\partial\Omega}\psi,$$

where $R = diam\,\Omega$.

Proof. The proof is based on the following elementary observation. If v is the unique solution of (P_λ) on $v + H_0^1(\Omega_1)$, for some bounded open set $\Omega_1 \subset \mathbb{R}^N$, then for every bounded open subset $\Omega_2 \subset \Omega_1$, $v|_{\Omega_2}$ is the unique solution on $v|_{\Omega_2} + H_0^1(\Omega_2)$. Indeed, if $\tilde{v}$ were another solution on Ω_2, then the extension of $\tilde{v}$ by v on Ω_1 would be another solution on Ω_1, which would contradict the uniqueness of the solution on Ω_1.

Now, let $R = diam\,\Omega$ and $x_0 \in \Omega$. Then $\Omega \subset B_R(x_0)$. The unique solution of (P_λ) on $H_0^1(B_R(x_0))$ is the map $\xi_{\lambda,x_0,R}$ defined in (4.3). Then $v := \xi_{\lambda,x_0,R}|_\Omega + \max_{\partial\Omega}\psi$ is the unique minimum of (P_λ) on $v + H_0^1(\Omega)$. Since $\xi_{\lambda,x_0,R} \ge 0$ on $B_R(x_0)$, it follows that $v \ge \max_{\partial\Omega}\psi \ge \psi$ on $\partial\Omega$. By Lemma 4.1, this implies that on Ω

$$u \le v \le \max_{B_R(x_0)} \xi_{\lambda,x_0,R} + \max_{\partial\Omega}\psi = \frac{N}{2\lambda}\left(\frac{\lambda^2}{N^2}R^2 - 1\right)_+ + \max_{\partial\Omega}\psi,$$

and the lemma follows. □

The next lemma gives a lower bound on the solutions of (P_λ):

Lemma 4.5. *Let $B_R(x_0) \subset B_{R'}(x_0) \subset \Omega$. Let $\lambda \ge N/R$ and u be a solution of (P_λ) on $\psi + H_0^1(\Omega)$. Then*

$$\min_{B_R(x_0)} u \ge \frac{\lambda}{2N}(R'^2 - R^2) + \min_{\partial\Omega}\psi.$$

Proof. Let $\xi_{\lambda,x_0,R'}$ be the unique solution of (P_λ) on $H_0^1(B_{R'}(x_0))$ given by (4.3). Then $\xi_{\lambda,x_0,R'} + \min_{\partial\Omega}\psi$ is the unique solution of (P_λ) on $\min_{\partial\Omega}\psi + H_0^1(B_{R'}(x_0))$. Since $u|_{B_{R'}(x_0)}$ is a solution of (P_λ) on $u|_{B_{R'}(x_0)} + H_0^1(B_{R'}(x_0))$ and inferring from Lemma 2.1 that $u \ge \min_{\partial\Omega}\psi$ on Ω and thus on $\partial B_{R'}(x_0)$, it follows from Lemma 4.1 that on $B_{R'}(x_0)$,

$$u \ge \xi_{\lambda,x_0,R'} + \min_{\partial\Omega}\psi.$$

In particular, if $\lambda \ge N/R$, then on $B_R(x_0)$,

$$u \ge \frac{\lambda}{2N}(R'^2 - R^2) + \min_{\partial\Omega}\psi,$$

which implies the desired result.

□

4.2. **On the family $\{u_\lambda\}_{\lambda>0}$.** In this section, we assume that Ω is Lipschitz and that $\partial\Omega$ is connected. We also fix the boundary condition ψ which is assumed to be Lipschitz continuous. As a consequence of Theorem 1.1, for every $\lambda \geq 0$, there exists a unique solution u_λ of (P_λ) on $H^1_\psi(\Omega) = \psi + H^1_0(\Omega)$. We then consider the map $\lambda \mapsto u_\lambda$. A continuity property is established in the next proposition.

Proposition 4.6. *The function $\lambda \mapsto u_\lambda \in C^0(\overline{\Omega})$ is continuous.*

Proof. Since Ω is Lipschitz, for every $\lambda \geq 0$, u_λ is Hölder continuous on $\overline{\Omega}$. Actually, for every bounded subset $\Lambda \subset [0, +\infty)$, there exist $C > 0$ and $\alpha > 0$ (which depend on Λ, ψ and Ω) such that for every $\lambda \in \Lambda$, for every $x, y \in \overline{\Omega}$,

$$|u_\lambda(x) - u_\lambda(y)| \leq C|x - y|^\alpha. \tag{4.6}$$

Indeed, the functional in (P_λ) satisfies the following growth assumptions:

$$\frac{1}{2}|\nabla u|^2 - A|u| \leq \varphi(\nabla u) - \lambda u \leq \frac{1}{2}|\nabla u|^2 + A|u| + \frac{1}{2},$$

with $A := \sup \Lambda$. The proof of [18, Theorem 7.8] implies that the Hölder norm and the Hölder exponent of u_λ can be estimated[5] only in terms of A, ψ and Ω. In particular, the constants C and α in (4.6) can be estimated independently of $\lambda \in \Lambda$.

Fix $\lambda \geq 0$. Let $\lambda_i \to \lambda$ and u_{λ_i} the solution of (P_{λ_i}). We claim that u_{λ_i} uniformly converges to u_λ. Indeed, since by (4.6) the sequence $(u_{\lambda_i})_{i\geq 1}$ is equicontinuous on $\overline{\Omega}$, Arzela-Ascoli theorem implies that a subsequence (we do not relabel) uniformly converges to a function $v \in C^0(\overline{\Omega})$. Since $(u_{\lambda_i})_{i\geq 1}$ is bounded in $H^1_\psi(\Omega)$, there exists a subsequence (we do not relabel) which weakly converges in $H^1_\psi(\Omega)$, necessarily to v. In particular, $v \in H^1_\psi(\Omega)$. Now, let $w \in H^1_\psi(\Omega)$. Then for every $i \geq 1$,

$$\int_\Omega \varphi(\nabla u_{\lambda_i}) - \lambda_i u_{\lambda_i} \leq \int_\Omega \varphi(\nabla w) - \lambda_i w.$$

Letting $i \to +\infty$ and using the weak lower semicontinuity in the left hand side, one gets

$$\int_\Omega \varphi(\nabla v) - \lambda v \leq \int_\Omega \varphi(\nabla w) - \lambda w.$$

This proves that v is a solution of (P_λ). By uniqueness for the problem (P_λ), this implies that $v = u_\lambda$. Hence, by uniqueness of the limit, the whole original sequence $(u_{\lambda_i})_{i\geq 1}$ uniformly converges to u_λ, which completes the proof of the lemma. □

In the next lemma, we prove the existence of a critical value of λ for which the supremum of u on Ω becomes larger than the supremum of u on $\partial\Omega$.

Lemma 4.7. *There exists $\lambda_* = \lambda_*(\psi, \Omega) \in [0, +\infty)$ such that for every $0 \leq \lambda \leq \lambda_*$, $\sup_\Omega u = \max_{\partial\Omega} \psi$ while for every $\lambda > \lambda_*$, $\sup_\Omega u > \max_{\partial\Omega} \psi$*

Proof. By Proposition 4.6, the function $\lambda \mapsto u_\lambda \in C^0(\overline{\Omega})$ is continuous. This implies that the function $\ell : \lambda \mapsto \max_{\overline{\Omega}} u_\lambda$ is continuous as well. By Lemma 4.1, ℓ is also nondecreasing. From Lemma 4.5, we deduce that for every $B_R(x_0) \Subset \Omega$,

$$\lim_{\lambda\to+\infty} \min_{\overline{B}_R(x_0)} u_\lambda = +\infty.$$

[5] In particular, the Lagrangian $F(u, \nabla u) = 2\varphi(\nabla u) - 2\lambda u$ satisfies [18, Assumption (7.2)] with $\gamma = p = 2$, $s = \sigma = \infty$, $\varepsilon = 2/N$, $L = 1$, $b = 2A$ and $a = 2A + 1$.

It follows that $\lim_{\lambda\to+\infty}\ell(\lambda)=+\infty$. Finally, when $\lambda=0$, the constant function $v:\equiv\max_{\partial\Omega}\psi$ is a solution of (P_0) and $v\geq u_0$ on $\partial\Omega$. It follows from Lemma 4.1 that $u_0\leq\max_\Omega\psi$ on Ω and thus $\ell(0)=\max_{\partial\Omega}\psi$. Hence $\ell([0,+\infty))=[\max_{\partial\Omega}\psi,+\infty)$. This proves the existence of the value λ_* as in the statement. □

4.3. **On the supports of the solutions.** In this section, we describe some properties of the supports of the solutions and present some connections with the Cheeger subsets of Ω. We assume throughout that Ω is *any* bounded open Lipschitz set but that the boundary condition ψ is constant and equal to 0. This implies by Lemma 2.1 that every solution is nonnegative on $\overline{\Omega}$.

As explained in the introduction, the Cheeger constant of Ω is given by

$$h_\Omega=\inf_{D\subset\overline{\Omega}}\frac{\text{Per } D}{|D|}.$$

Alternatively (see for instance [29]), h_Ω can be defined as the minimum value of a variational problem on $H^1_0(\Omega)$:

$$h_\Omega=\inf\left(\int_\Omega|\nabla u|:u\in H^1_0(\Omega),\int_\Omega u=1\right).$$

This constant arises naturally in the framework of (P_λ), as stated in Proposition 1.5 that we now prove:

Proof of Proposition 1.5. Assume first that 0 is a solution of (P_λ). Then for every $v\in H^1_0(\Omega)$,

$$\lambda\int_\Omega v\leq\int_\Omega\varphi(\nabla v)\leq\int_{[|\nabla v|\leq1]}|\nabla v|+\int_{[|\nabla v|>1]}|\nabla v|^2.$$

By replacing v by sv for some $s>0$ and dividing by s, one gets

$$\lambda\int_\Omega v\leq\int_{[|\nabla v|\leq s^{-1}]}|\nabla v|+s\int_{[|\nabla v|>s^{-1}]}|\nabla v|^2.$$

We now let $s\to0^+$:

$$\lambda\int_\Omega v\leq\int_\Omega|\nabla v|.$$

This implies that $\lambda\leq h_\Omega$. Moreover, if u is another solution of (P_λ), then it follows from Lemma 3.1 and the fact that 0 is a solution that $|\nabla u|\leq1$ a.e. Hence $\Omega=\Omega\setminus[|\nabla u|>1]$ and u is constant on $\partial\Omega$. From Lemma 3.7, we deduce that u is constant on Ω: $u\equiv0$; that is, 0 is the unique solution of (P_λ).

Assume now that $\lambda\leq h_\Omega$. Then for any $u\in H^1_0(\Omega)$,

$$\int_\Omega\varphi(\nabla u)-\lambda\int_\Omega u\geq\int_\Omega|\nabla u|-h_\Omega\int_\Omega u\geq0.$$

This proves that 0 is a solution of (P_λ). □

When λ increases and becomes larger than h_Ω, a non trivial solution suddenly appears inside the Cheeger sets of Ω. Here is a precise result, which implies Theorem 1.6 stated in the introduction:

Theorem 4.8. *For $\lambda>h_\Omega$, let u_λ be a solution of (P_λ). Then the family $([u_\lambda>0])_{\lambda>h_\Omega}$ is nondecreasing. Moreover, the set*

$$\Omega_0:=\cap_{\lambda>h_\Omega}[u_\lambda>0]$$

is a solution of the Cheeger problem for Ω*:*

$$\frac{Per\ \Omega_0}{|\Omega_0|} = h_\Omega.$$

Remark 4.9. The Cheeger problem has a unique solution when the domain is convex, see [4]. Hence, when Ω is convex, Ω_0 is the unique Cheeger set contained in Ω. When Ω is not convex, we conjecture that Ω_0 is the maximal Cheeger set (for the characterization of the maximal Cheeger set, we refer for instance to [9]).

Proof of Theorem 4.8. By Lemma 4.1, if $h_\Omega < \lambda < \lambda'$, then $u_\lambda \leq u_{\lambda'}$. Hence $[u_\lambda > 0] \subset [u_{\lambda'} > 0]$ and the first assertion follows.

By Lemma 3.2, for every $\varepsilon > 0$,

$$\text{Per } [u_\lambda \geq \varepsilon] \leq \lambda |[u_\lambda \geq \varepsilon]|.$$

Here, we also use the fact that $\text{Per } [u_\lambda \geq \varepsilon] = \text{Per } ([u_\lambda \geq \varepsilon], \Omega)$ since $[u \geq \varepsilon] \Subset \Omega$. For every $\lambda > h_\Omega$, the family of characteristic functions $\chi_{[u_\lambda \geq \varepsilon]}$ converges to $\chi_{[u_\lambda > 0]}$ a.e. when $\varepsilon \to 0$ and thus, by the dominated convergence theorem, in $L^1(\Omega)$. The lower semicontinuity of the perimeter then yields

$$\text{Per } [u_\lambda > 0] \leq \lambda |[u_\lambda > 0]|. \tag{4.7}$$

On the other hand, by definition of h_Ω, for every $\lambda > h_\Omega$,

$$\text{Per } [u_\lambda > 0] \geq h_\Omega |[u_\lambda > 0]|. \tag{4.8}$$

When $\lambda \to h_\Omega$, $|[u_\lambda > 0]|$ tends to $|\Omega_0|$. From the inequalities (4.7) and (4.8) above, one thus gets

$$\lim_{\substack{\lambda \to h_\Omega \\ \lambda > h_\Omega}} \text{Per } [u_\lambda > 0] = h_\Omega |\Omega_0|.$$

Since by lower semicontinuity of the perimeters,

$$\liminf_{\lambda \to h_\Omega} \text{Per } [u_\lambda > 0] \geq \text{Per } \Omega_0,$$

this gives

$$\text{Per } \Omega_0 \leq h_\Omega |\Omega_0|. \tag{4.9}$$

For every $\lambda > h_\Omega$, Lemma 1.5 implies that $u_\lambda \not\equiv 0$. Hence, for every $\varepsilon > 0$ sufficiently small, $|[u_\lambda \geq \varepsilon]| > 0$. It then follows from the proof of Lemma 2.1, see (2.1), that $|[u_\lambda \geq \varepsilon]| \geq (C\lambda)^{-N}$ for some constant C which depends only on N. This implies $|[u_\lambda > 0]| \geq (C\lambda)^{-N} > 0$ and finally letting $\lambda \to h_\Omega$,

$$|\Omega_0| \geq (Ch_\Omega)^{-N} > 0.$$

Together with (4.9), this completes the proof of Theorem 4.8. □

Remark 4.10. If Ω is convex and a Cheeger set of itself, then $\Omega = \Omega_0$ and thus for every $\lambda > h_\Omega$, $u_\lambda > 0$ on Ω. On the contrary, if $\frac{\text{Per }\Omega}{|\Omega|} > h_\Omega$, then for $\lambda \in (h_\Omega, \frac{\text{Per }\Omega}{|\Omega|})$, we deduce from the inequality (4.7) that $[u_\lambda > 0] \neq \Omega$; that is, u_λ vanishes on a subset of Ω of positive measure. This emphasizes the fact that the strong maximum principle does not apply for this degenerate (not uniformly elliptic) problem.

For a fixed $\lambda > h_\Omega$, the support of a solution u_λ is related to the geometry of Ω in the following sense:

Proposition 4.11. *Given $\lambda > h_\Omega$, let u_λ be a solution of (P_λ). Then $[u_\lambda > 0]$ contains the set Ω_λ defined as the union of all open convex sets $A \subset \Omega$ which are Cheeger sets of themselves with $h_A < \lambda$.*

Proof. Indeed, let $A \subset \Omega$ be an open convex set which is a Cheeger set of itself. Then by [4], A is the unique Cheeger set contained in A. Hence $A = A_0$, where A_0 is defined in Theorem 4.8, with A instead of Ω. Let $\lambda > h_A$. By Theorem 1.1, there exists a unique solution $u_{A,\lambda}$ to the problem (P_λ) on A with a homogeneous Dirichlet boundary condition. It follows from Theorem 4.8 that $u_{A,\lambda}$ is stricly positive on A. Since $u_\lambda|_{\partial A} \geq 0 = (u_{A,\lambda})|_{\partial A}$, Lemma 4.1 implies that $u_\lambda \geq u_{A,\lambda} > 0$ on A. Finally, $u_\lambda > 0$ on Ω_λ. □

5. Appendix

In this appendix, we detail the proof of the Lipschitz continuity of the solutions of (P_λ). Our strategy is essentially the same as the one introduced in [13, Theorem 2.7] for Lagrangians which depend only on x and ∇u. In our situation, the dependence on u is linear, so that it plays no role and does not involve any additional difficulty in the proof. Moreover, the specific form of the function φ outside the unit ball simplifies the main estimates obtained in [13].

Theorem 5.1. *Let u be a solution of (P_λ). Then the function u is locally Lipschitz on Ω.*

We first introduce a sequence of variational problems approximating (P_λ) and for which the solutions v_j, $j \geq 1$, are known to be smooth. This is the role of Lemma 5.2 below. We then obtain a uniform Lipschitz bound on v_j in Lemma 5.3, which finally implies Theorem 5.1.

Lemma 5.2. *There exists a sequence $(\varphi_j)_{j\geq 1}$ of C^∞ uniformly convex[6] functions which converges uniformly to φ on bounded sets and such that for every $j \geq 1$,*

$$\forall y \in \mathbb{R}^N \setminus B_2(0) \quad , \quad \nabla^2 \varphi_j(y) = (1 + \frac{1}{j})\ Id, \tag{5.1}$$

$$\forall y \in \mathbb{R}^N \quad , \quad \varphi(y) \leq \varphi_j(y) \leq 2(\varphi(y) + 1). \tag{5.2}$$

Proof. Let $\rho \in C_c^\infty(B_1(0))$ such that $\rho \geq 0$, $\int_{\mathbb{R}^N} \rho = 1$ and $\rho(-y) = \rho(y)$ for every $y \in \mathbb{R}^N$. We then define $\rho_j(\cdot) = j^N \rho(j\cdot)$ and

$$\varphi_j(y) = \varphi * \rho_j(y) + \frac{1}{2j}|y|^2 \quad , \quad y \in \mathbb{R}^N.$$

Then φ_j is a smooth uniformly convex function which uniformly converges to φ on bounded sets. By the change of variables $z \mapsto -z$ and the property $\rho_j(-z) = \rho_j(z)$, one has

$$\int_{\mathbb{R}^N} z\rho_j(z)\, dz = 0.$$

[6]By this, we mean that there exists $m_j > 0$ such that $\nabla^2\varphi_j \geq m_j$ Id.

By convexity of φ, Jensen inequality implies $\varphi * \rho_j(y) \geq \varphi(y)$. Moreover, for every $|y| \geq 2$,

$$\begin{aligned}\varphi * \rho_j(y) &= \int_{\mathbb{R}^N} \varphi(y-z)\rho_j(z)\,dz = \int_{\mathbb{R}^N} \frac{1}{2}(|y-z|^2+1)\rho_j(z)\,dz \\ &= \frac{1}{2}(|y|^2+1)\int_{\mathbb{R}^N} \rho_j(z)\,dz + \frac{1}{2}\int_{\mathbb{R}^N} |z|^2\rho_j(z)\,dz + \int_{\mathbb{R}^N} \langle y,z\rangle \rho_j(z)\,dz \\ &= \frac{1}{2}(|y|^2+1) + \frac{1}{2}\int_{\mathbb{R}^N} |z|^2\rho_j(z)\,dz = \varphi(y) + c_j\end{aligned}$$

where $c_j := \frac{1}{2}\int_{\mathbb{R}^N} |z|^2\rho_j(z)\,dz$ belongs to $(0, \frac{1}{2j^2})$. When $|y| \leq 2$, using the fact that $\varphi(y-z) \leq (|y-z|^2+1)/2$ for every $z \in \mathbb{R}^N$, the same calculation leads to $\varphi * \rho_j(y) \leq \varphi(y) + c_j$. Then (5.1) and (5.2) easily follow. The proof is complete.

□

Let B_R be a ball compactly contained in Ω. For every $j \geq 1$, we consider the variational problem (Q_j)

$$\text{To minimize } v \mapsto \int_{B_R} \varphi_j(\nabla v) - \lambda v$$

on the set of those $v \in H^1(B_R)$ which coincide with u on ∂B_R.

By the direct method in the calculus of variations, for every $j \geq 1$, there exists a unique solution v_j to the above problem. By minimality of v_j,

$$\int_{B_R} \varphi_j(\nabla v_j) - \lambda v_j \leq \int_{B_R} \varphi_j(\nabla u) - \lambda u. \tag{5.3}$$

Since for every $j \geq 1$, $y \in \mathbb{R}^N$,

$$\frac{1}{2}|y|^2 \leq \varphi(y) \leq \varphi_j(y) \leq 2(\varphi(y)+1),$$

the sequence $(v_j)_{j\geq 1}$ is bounded in $H^1(B_R)$. Hence, we can extract (we do not relabel) a subsequence which converges weakly in $H^1(B_R)$ and strongly in $L^2(B_R)$ to a certain function $v \in H^1(B_R)$ which agrees with u on ∂B_R. Using that $\varphi \leq \varphi_j$, one has

$$\int_{B_R} \varphi(\nabla v_j) \leq \int_{B_R} \varphi_j(\nabla v_j).$$

By weak lower semicontinuity, this gives

$$\int_{B_R} \varphi(\nabla v) \leq \liminf_{j\to+\infty} \int_{B_R} \varphi_j(\nabla v_j).$$

Together with (5.3) and the L^2 convergence of v_j, this implies

$$\int_{B_R} \varphi(\nabla v) - \lambda \int_{B_R} v \leq \liminf_{j\to+\infty} \int_{B_R} \varphi_j(\nabla u) - \lambda \int_{B_R} u.$$

Since $\varphi_j \leq 2(\varphi+1)$ and φ_j converges pointwisely to φ, one may apply the dominated convergence theorem to get

$$\int_{B_R} \varphi(\nabla v) - \lambda \int_{B_R} v \leq \int_{B_R} \varphi(\nabla u) - \lambda \int_{B_R} u.$$

By minimality of u, the opposite inequality is true. Hence, v is a minimum. By Lemma 3.1, this implies that $|\nabla v - \nabla u| \leq 1$. Hence, Theorem 5.1 follows from the following lemma.

Lemma 5.3. *The function v is locally Lipschitz on B_R.*

Proof. We only need to prove that there exists a constant C_0 such that for every $j \geq 1$,

$$\|\nabla v_j\|_{L^\infty(B_{R/2})} \leq C_0. \tag{5.4}$$

Since φ_j is smooth, uniformly convex and has a bounded Hessian, it follows from the standard elliptic regularity theory that v_j is smooth on B_R and thus satisfies the Euler equation: for every $\theta \in C_c^\infty(B_R)$

$$\int_{B_R} \langle \nabla\varphi_j(\nabla v_j), \nabla\theta\rangle = \lambda \int_{B_R} \theta.$$

Let $1 \leq s \leq N$ and take $\theta = \partial_s\zeta$, with $\zeta \in C_c^\infty(B_R)$, in the above equality. By integration by parts, this gives

$$\int_{B_R} \sum_{i,k} \partial_{ik}\varphi_j \partial_{ks} v_j \partial_i\zeta = 0. \tag{5.5}$$

This equality holds true for every $\zeta \in H^1(B_R)$ with compact support in B_R. Set

$$V_+ = 1 + \sum_{h=1}^N (\partial_h v_j - 2)_+^2 \quad , \quad V_- = 1 + \sum_{h=1}^N (\partial_h v_j + 2)_-^2.$$

Let $\eta \in C_c^\infty(B_R)$ and take $\zeta := \eta^2(\partial_s v_j - 2)_+ V_+^\beta$, where $\beta \geq 0$. This yields

$$\int_{B_R} \eta^2 \sum_{i,k} \partial_{ik}\varphi_j \partial_{ks} v_j V_+^\beta \partial_i(\partial_s v_j - 2)_+ + \beta \int_{B_R} \eta^2 \sum_{i,k} \partial_{ik}\varphi_j \partial_{ks} v_j V_+^{\beta-1}(\partial_s v_j - 2)_+ \partial_i V_+$$
$$= -2\int_{B_R} \eta \sum_{i,k} \partial_{ik}\varphi_j \partial_{ks} v_j (\partial_s v_j - 2)_+ V_+^\beta \partial_i\eta.$$

Since the integrals above vanish when $\partial_s v \leq 2$, one gets

$$\int_{B_R} \eta^2 \sum_{i,k} \partial_{ik}\varphi_j \partial_k(\partial_s v_j - 2)_+ V_+^\beta \partial_i(\partial_s v_j - 2)_+ + \beta \int_{B_R} \eta^2 \sum_{i,k} \partial_{ik}\varphi_j \partial_k(\partial_s v_j - 2)_+ V_+^{\beta-1}(\partial_s v_j - 2)_+ \partial_i V_+$$
$$= -2\int_{B_R} \eta \sum_{i,k} \partial_{ik}\varphi_j \partial_k(\partial_s v_j - 2)_+ (\partial_s v_j - 2)_+ V_+^\beta \partial_i\eta.$$

Using that $\nabla^2\varphi_j(y) = (1 + \frac{1}{j})\,\mathrm{Id}$ when $|y| \geq 2$, this implies

$$\int_{B_R} \eta^2 \sum_i (\partial_i(\partial_s v_j - 2)_+)^2 V_+^\beta + \beta \int_{B_R} \eta^2 \sum_i \partial_i(\partial_s v_j - 2)_+ V_+^{\beta-1}(\partial_s v_j - 2)_+ \partial_i V_+$$
$$= -2\int_{B_R} \eta \sum_i \partial_i(\partial_s v_j - 2)_+ (\partial_s v_j - 2)_+ V_+^\beta \partial_i\eta.$$

Summing over s and differentiating $\partial_i V_+$ in the second term of the left hand side, we get

$$(5.6)\quad \int_{B_R} \eta^2 \sum_{i,s} (\partial_i(\partial_s v_j - 2)_+)^2 V_+^\beta + 2\beta \int_{B_R} \eta^2 V_+^{\beta-1} \sum_i \left(\sum_s A_{is}\right)^2 = -2\int_{B_R} \eta \sum_{i,s} \partial_i(\partial_s v_j - 2)_+ (\partial_s v_j - 2)_+ V_+^\beta \partial_i \eta.$$

where

$$A_{is} := (\partial_s v_j - 2)_+ \partial_i (\partial_s v_j - 2)_+.$$

Since $\sum_{s=1}^N (\partial_s v_j - 2)_+^2 \leq V_+$, and using the Cauchy-Schwarz inequality, we obtain

$$\int_{B_R} \eta^2 V_+^{\beta-1} \sum_i \left(\sum_s A_{is}\right)^2 \leq \int_{B_R} \eta^2 \sum_{i,s} (\partial_i(\partial_s v_j - 2)_+)^2 V_+^\beta.$$

In view of the above inequality, (5.6) implies

$$(5.7)\qquad (1+2\beta)\int_{B_R} \eta^2 V_+^{\beta-1} \sum_i \left(\sum_s A_{is}\right)^2 \leq -2\int_{B_R} \eta \sum_{i,s} \partial_i(\partial_s v_j - 2)_+ (\partial_s v_j - 2)_+ V_+^\beta \partial_i \eta.$$

Writing $A_{is} = \frac{1}{2}\partial_i(\partial_s v_j - 2)_+^2$ and taking into account the definition of V_+, this gives

$$\int_{B_R} \eta^2 |\nabla V_+|^2 V_+^{\beta-1} \leq C \left| \int_{B_R} \eta V_+^\beta \sum_i \partial_i V_+ \partial_i \eta \right| \leq C \int_{B_R} \eta V_+^\beta |\nabla V_+| |\nabla \eta|,$$

where $C = 4/(1+2\beta)$. Writing that $V_+^\beta = V_+^{\frac{\beta-1}{2}} V_+^{\frac{\beta+1}{2}}$, this implies

$$\int_{B_R} \eta^2 |\nabla V_+|^2 V_+^{\beta-1} \leq C^2 \int_{B_R} |\nabla \eta|^2 V_+^{\beta+1}.$$

Inserting now the function $\zeta = \eta^2 (\partial_s v_j + 2)_- V_-^\beta$ in (5.5), a similar calculation leads to

$$\int_{B_R} \eta^2 |\nabla V_-|^2 V_-^{\beta-1} \leq C^2 \int_{B_R} |\nabla \eta|^2 V_-^{\beta+1}.$$

By summing the two last inequalities, one gets

$$\int_{B_R} \eta^2 |\nabla V|^2 V^{\beta-1} \leq 2C^2 \int_{B_R} |\nabla \eta|^2 V^{\beta+1}.$$

where $V := \max(V_+, V_-)$. Equivalently,

$$\int_{B_R} \eta^2 |\nabla V^\gamma|^2 \leq 2C^2\gamma^2 \int_{B_R} |\nabla \eta|^2 V^{2\gamma},$$

with $\gamma = (\beta+1)/2$. Hence,

$$\int_{B_R} |\nabla(\eta V^\gamma)|^2 \leq (4C^2\gamma^2 + 2) \int_{B_R} |\nabla \eta|^2 V^{2\gamma}.$$

By the Sobolev inequality and the arbitrariness of $\beta \geq 0$, we get that for every $\gamma \geq 1/2$,

$$\|V^\gamma \eta\|_{L^{2\chi}(B_R)} \leq c \|V^\gamma |\nabla \eta|\|_{L^2(B_R)},$$

where $\chi = N/(N-2)$ if $N \geq 3$ or any number > 1 if $N = 2$, and c is a constant which depends only on N (and on R when $N = 2$). Considering the sequence of radii $r_i := (1/2 + 1/2^i)R$ for $i \geq 0$, we apply the above inequality to $\gamma = \gamma_i := \chi^i/2$, and choose $\eta \in C_c^1(B_{r_i})$ such that $\eta = 1$ on $B_{r_{i+1}}$, $0 \leq \eta \leq 1$, $|D\eta| \leq c_0 2^i$. This yields

$$\|V\|_{L^{2\gamma_{i+1}}(B_{r_i+1})} \leq (c_1 2^i)^{\frac{1}{\gamma_i}} \|V\|_{L^{2\gamma_i}(B_{r_i})}.$$

Iterating the above formula and letting $i \to +\infty$, one gets

$$\|V\|_{L^\infty(B_{R/2})} \leq c_2 \|V\|_{L^1(B_R)} \leq c_3 (1 + \|\nabla v_j\|^2_{L^2(B_R)}).$$

Since the sequence $(v_j)_{j\geq 1}$ is bounded in $H^1(B_R)$, this implies that $\|V\|_{L^\infty(B_{R/2})}$ can be bounded independently of j. In view of the definition of V, (5.4) follows. The proof of Lemma 5.3 is complete. □

References

[1] G. Alberti, S. Bianchini, G. Crippa, Structure of level sets and Sard-type properties of Lipschitz maps, Ann. Sc. Norm. Super. Pisa Cl. Sci., **12** (2013), 863–902.

[2] J.-J. Alibert, G. Bouchitté, I. Fragalà, I. Lucardesi, A nonstandard free boundary problem arising in the shape optimization of thin torsion rods, Interfaces Free Bound., **15** (2013), 95–119.

[3] F. Alter, V. Caselles, A. Chambolle, A characterization of convex calibrable sets in $\mathbb{R}^N$, Math. Ann., **332** (2005), 329–366.

[4] F. Alter, V. Caselles, Uniqueness of the Cheeger set of a convex body, Nonlinear Anal., **70** (2009), 32–44.

[5] L. Ambrosio, N. Fusco, D. Pallara, Functions of bounded variation and free discontinuity problems, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[6] G. Anzellotti, Pairings between measures and bounded functions and compensated compactness, Ann. Mat. Pura Appl., **135** (1983), 293–318.

[7] G. Bouchitté, I. Fragalà, I. Lucardesi, P. Seppecher, Optimal thin torsion rods and Cheeger sets, SIAM J. Math. Anal. **44** (2012), 483-512.

[8] L. Brasco, Global L^∞ gradient estimates for solutions to a certain degenerate elliptic equation, Nonlinear Anal., **74** (2011), 516–531.

[9] V. Caselles, A. Chambolle, M. Novaga, Some remarks on uniqueness and regularity of Cheeger sets, Rend. Semin. Mat. Univ. Padova, **123** (2010), 191–201.

[10] A. Cellina, Uniqueness and comparison results for functionals depending on ∇u and on u, SIAM J. Optim., **18** (2007), 711–716.

[11] M. Colombo, A. Figalli, Regularity results for very degenerate elliptic equations, J. Math. Pures Appl., **101** (2014), 94–117.

[12] I. Ekeland, R. Témam, Convex analysis and variational problems, Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), 1999.

[13] I. Fonseca, N. Fusco, P. Marcellini, An existence result for a nonconvex variational problem via regularity, ESAIM COCV, **7** (2002), 69–95.

[14] M. Giaquinta, G. Modica, J. Souček, Cartesian currents in the calculus of variations. I, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 1998.

[15] D. Gilbarg, N. Trudinger, Elliptic partial differential equations of second order, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[16] E. Giusti, On the equation of surfaces of prescribed mean curvature. Existence and uniqueness without boundary conditions, Invent. Math., **46** (1978), 111–137.

[17] E. Giusti, Minimal surfaces and functions of bounded variation, Monographs in Mathematics, Birkhäuser Verlag, Basel, 1984.

[18] E. Giusti, Direct methods in the calculus of variations, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[19] R. Jerrard, A. Moradifam, A. Nachman, to appear in Jour. Rein. Angew. Math.

[20] R.V. Kohn, G. Strang, Optimal design and relaxation of variational problems. I, Comm. Pure Appl. Math., **39** (1986), 113–137.
[21] R.V. Kohn, G. Strang, Optimal design and relaxation of variational problems. II, Comm. Pure Appl. Math., **39** (1986), 139–182.
[22] R.V. Kohn, G. Strang, Optimal design and relaxation of variational problems. III, Comm. Pure Appl. Math., **39** (1986), 353–377.
[23] L. Lussardi, E. Mascolo, A uniqueness result for a class of non-strictly convex variational problems, J. Math. Anal. Appl., **446** (2017), 1687–1694.
[24] P. Marcellini, A relation between existence of minima for nonconvex integrals and uniqueness for non strictly convex integrals of the calculus of variations, Mathematical Theories of Optimization, Proceedings, edited by J.P. Cecconi and T. Tolezzi, Lecture Notes in Math., Springer, **979** (1983), 216–231.
[25] P. Marcellini, Some remarks on uniqueness in the calculus of variations, Collège de France seminar, edited by H. Brezis and J.L. Lions, Research Notes in Math., Pitman, **84**, 148–153.
[26] U. Massari, Esistenza e regolarità delle ipersuperfice di curvatura media assegnata in R^n, Arch. Rational Mech. Anal., **55** (1974), 357–382.
[27] F. Morgan, Geometric measure theory, Fourth edition, Elsevier/Academic Press, Amsterdam, 2009.
[28] T. Napier, M. Ramachandran, An introduction to Riemann surfaces, Birkhäuser/Springer, New York, 2011.
[29] E. Parini, An introduction to the Cheeger problem, Surv. Math. Appl., **6** (2011), 9–21.
[30] J.P. Raymond, An anti-plane shear problem, J. Elasticity, **33** (1993), 213-231.
[31] P. Sternberg, G. Williams, W. Ziemer, Existence, uniqueness, and regularity for functions of least gradient, J. reine angew. Math., **430** (1992), 35–60.
[32] P. Sternberg, G. Williams, W. Ziemer, The constrained least gradient problem in $\mathbf{R}^n$, Trans. Amer. Math. Soc., **339** (1993), 403–432.
[33] I. Tamanini, Boundaries of Caccioppoli sets with Hölder-continuous normal vector, J. Reine Angew. Math., **334** (1982), 27–39.
[34] I. Tamanini, Il problema della capillarità su domini non regolari, Rend. Sem. Mat. Univ. Padova, **56** (1976), 169–191.

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