Boundary continuity of solutions to a basic problem in the calculus of variations

Pierre Bousquet

April 9, 2008

1 Introduction

We study the following problem (P) in the multiple integral calculus of variations:

$$\min_u \int_\Omega F(\nabla u(x))\,dx \quad \text{subject to } u \in W^{1,1}(\Omega),\ \operatorname{tr} u_{|\Gamma} = \phi$$

where Ω is a bounded Lipschitz open set in $\mathbb{R}^n$ and $\operatorname{tr} u_{|\Gamma}$ signifies the trace of u on Γ, the boundary of Ω. Throughout the article, we assume that the function $F : \mathbb{R}^n \to \mathbb{R}$ is convex: for any $\theta \in (0,1)$, for any $p, q \in \mathbb{R}^n$,

$$F(\theta p + (1-\theta)q) \leq \theta F(p) + (1-\theta)F(q).$$

Under a coercivity assumption on F, the direct method in the calculus of variations yields the existence of a solution to (P). For instance if F is superlinear, that is, $\lim\limits_{|p|\to\infty} F(p)/|p| = +\infty$, then there exists a minimimum in $W^{1,1}_\phi(\Omega)$, the set of those functions u in $W^{1,1}(\Omega)$ such that $\operatorname{tr} u_{|\Gamma} = \phi$.

In this article, we address the question of the continuity of a minimum on the closure of Ω, $\operatorname{cl}\Omega$. An obvious necessary condition is the continuity of ϕ on Γ. It is an open problem to know whether it is also a sufficient condition when one assumes the convexity of the domain Ω.

The problem of the continuity of the minima of (P) in Ω or in the closure of Ω has been solved under a great number of hypotheses. Most of them require a growth assumption from above for F. This is the case of the works based on a Cacciopoli type inequality and the classes of De Giorgi (see [9] Theorem 7.8, [12], Chapter 5, Theorem 4.1). A growth hypothesis for F is also essential to build most of the barriers used in the theory of elliptic partial differential equations. Barriers have proved to be a useful tool to prove the continuity of minimizers near the boundary (see [8], Chapter 14, part 5). However, Giaquinta [7] has found a Lagrangian F of class C^2 satisfying

$$c_1|\xi|^2 \leq \langle \nabla^2 F(p)\xi, \xi\rangle \leq c_2(1+|p|^2)|\xi|^2$$

for some constants $c_1, c_2 > 0$, such that the minimum is singular along a line. This emphasizes the fact that the growth hypothesis on F must be rather restrictive to get continuity on $\mathrm{cl}\,\Omega$ (Marcellini [13] has provided sharpen hypotheses which guarantee this continuity).

When no growth assumption from above is available on the Lagrangian F, the continuity of a minimum u on Ω or on $\mathrm{cl}\,\Omega$ should depend on some properties of the boundary function ϕ defining the Dirichlet condition and/or on the geometrical or regularity properties of Γ. This is indeed the core of the Hilbert-Haar theory where a classical hypothesis for ϕ is the bounded slope condition. We say that ϕ satisfies the bounded slope condition of constant $Q > 0$ if for any $x \in \Gamma$, there exist $\zeta_x^-, \zeta_x^+ \in \mathbb{R}^n$, $|\zeta_x^-|, |\zeta_x^+| \leq Q$ such that

$$\phi(x) + \langle \zeta_x^-, y - x\rangle \leq \phi(y) \leq \phi(x) + \langle \zeta_x^-, y - x\rangle \quad \forall y \in \Gamma.$$

Under this assumption on ϕ, the Hilbert-Haar's theorem (see [9], chapter 1 and also [15]) asserts that there exists a minimum to problem (P) on the set $W^{1,1}_\phi(\Omega)$ which is globally Lipschitz on Ω. No regularity assumption on F is required here. If F is C^2 and $\nabla^2 F > 0$ (in particular F is strictly convex and the minimum is unique), then the De Giorgi's theorem on the regularity of solutions to uniformly elliptic linear differential equations with bounded measurable coefficients implies that this minimum is locally $C^{1,\alpha}$. The continuity of u up to the boundary is trivially implied by the fact that u is *globally* Lipschitz on Ω.

However, this bounded slope condition is rather restrictive. First, when ϕ is not affine, it implies that Ω is convex (see [10]). Moreover, it implies that ϕ is affine on each affine subset of Γ. For instance, if Ω is a square in $\mathbb{R}^2$, the map ϕ satisfies the bounded slope condition if and only if ϕ is affine on each side of the square.

Recently, Clarke has introduced a new condition: the lower bounded slope condition. We say that ϕ satisfies the lower bounded slope condition of constant $Q > 0$ if for any $x \in \Gamma$, there exist $\zeta_x^- \in \mathbb{R}^n$, $|\zeta_x^-| \leq Q$ such that

$$\phi(x) + \langle \zeta_x^-, y - x\rangle \leq \phi(y) \quad \forall y \in \Gamma.$$

This condition is satisfied if and only if ϕ is the restriction to Γ of a convex function defined on $\mathbb{R}^n$. In particular, it implies that ϕ is Lipschitz continuous. Further caracterizations and properties are provided in [1], where an example shows that the mere lower bounded slope condition does not imply the global Lipschitz continuity of a minimum. Yet, under this assumption and when F is strictly convex and Ω convex, Clarke has proved that any minimum of the problem (P) on $W^{1,1}_{\phi}(\Omega)$ is locally Lipschitz in Ω. Moreover, the continuity on $\mathrm{cl}\,\Omega$ of the minimum was proved when (see [5],[1])

- Ω is $C^{1,1}$ and F is coercive of order $r > (n+1)/2$ (i.e $|F(p)| \geq c|p|^r + d$ for some $c > 0, d \in \mathbb{R}$),
- Ω is a polyhedron,
- Ω is strictly convex.

However, nothing was known for convex sets for which Γ is the union of affine faces and extremal points. For instance, the case when Ω is the intersection of a ball and an half plane was open.

When F is assumed to be merely convex (so that several distinct minima may exist), Mariconda and Treu [16] have generalized [5] to prove the inner regularity of the minima, under a generalized lower bounded slope condition. The question of the continuity up to the boundary remained open.

In this paper, we establish the continuity of a solution of (P) when F is convex and superlinear, Ω is convex and when ϕ is continuous. We also consider the case of convex Lagrangians which are not strictly convex nor superlinear.

We now state our results specifically.

We assume throughout the article that there exists $\bar{u} \in W^{1,1}(\Omega)$, $\mathrm{tr}\,\bar{u} = \phi$ such that $\displaystyle\int_\Omega F(D\bar{u}(x))\,dx < \infty$.

Theorem 1 *Assume that $F : \mathbb{R}^n \to \mathbb{R}$ is convex and superlinear, that $\phi : \Gamma \to \mathbb{R}$ is Lispchitz continuous and let $\gamma \in \Gamma$ such that there exists a supporting hyperplane to Ω at γ. Then any minimum u of (P) on $W^{1,1}_{\phi}(\Omega)$ satisfies*

$$\lim_{x \to \gamma, x \in A} u(x) \quad \textit{exists and is equal to} \quad \phi(\gamma),$$

where A is the set of Lebesgue points of u.

Remark 1 *In Theorem 1, we do not assume that Ω is convex. The existence of a supporting hyperplane means that there exists $\nu \in \mathbb{R}^n \setminus \{0\}$ such that*

$$\Omega \subset \{x \in \mathbb{R}^n : \langle \nu, x - \gamma \rangle \leq 0\}.$$

Corollary 1 *Assume that Ω is convex, that $F : \mathbb{R}^n \to \mathbb{R}$ is convex and superlinear and that ϕ satisfies the lower bounded slope condition. Then any minimum u of (P) on $W^{1,1}_\phi(\Omega)$ is continuous on the closure of Ω.*

In the following theorem, we consider the case when ϕ is merely continuous.

Theorem 2 *Assume that Ω is convex, that $F : \mathbb{R}^n \to \mathbb{R}$ is superlinear and convex and that $\phi : \Gamma \to \mathbb{R}$ is continuous. Then there exists a solution of (P) in $W^{1,1}_\phi(\Omega)$ which is continuous on the closure of Ω.*

As an obvious consequence of Theorem 2, we get

Corollary 2 *Assume that Ω is convex, that $F : \mathbb{R}^n \to \mathbb{R}$ is superlinear and strictly convex and that $\phi : \Gamma \to \mathbb{R}$ is continuous. Then there exists a unique solution of (P) in $W^{1,1}_\phi(\Omega)$. This solution is continuous on the closure of Ω.*

However, we have the following

Open Problem 1 *Under the assumptions of Theorem 2, is it true that any solution of (P) is continuous on the closure of Ω ?*

In Theorem 3, we consider convex Lagrangians which are not necessarily superlinear. This is for instance the case of $F(\xi) := \sqrt{1 + |\xi|^2}$. Before stating the theorem, we introduce some definitions.

We denote the epigraph of F by

$$\operatorname{epi} F := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : F(x) \leq t\}.$$

A face of the epigraph of F is a set $\Sigma \subset \mathbb{R}^n \times \mathbb{R}$ such that there exist $x, \zeta \in \mathbb{R}^n$ which satisfy

$$\Sigma := \{(x', F(x')) \in \mathbb{R}^n \times \mathbb{R} : F(x') = F(x) + \langle \zeta, x' - x \rangle\}.$$

The projection of Σ on $\mathbb{R}^n$ is

$$\{x' \in \mathbb{R}^n : F(x') = F(x) + \langle \zeta, x' - x \rangle\}.$$

Then ζ belongs to the convex subdifferential of F at x', for any x' in the projection of Σ on $\mathbb{R}^n$. In Theorem 3, we assume that the projections of the faces of epi F have diameters which are unifomrly bounded. Let us formulate it explicitly : there exists $D > 0$ such that for any $x, x' \in \mathbb{R}^n$, if there exists $\zeta \in \mathbb{R}^n$ satisfying

$$F(x') = F(x) + \langle \zeta, x' - x \rangle,$$

then $|x - x'| \leq D$.

Moreover, this assumption is automatically satisfied when F is strictly convex.

Theorem 3 *Assume that ϕ satisfies a lower bounded slope condition, that the projections of the faces of epi F have diameters which are uniformly bounded and that Ω is convex. Then any solution w is locally Lipschitz continuous on Ω. For any $\gamma \in \Gamma$,*

$$\liminf_{x \in \Omega, x \to \gamma} w(x) = \phi(\gamma). \tag{1}$$

Moreover, w is continuous at $\gamma \in \Gamma$ when one of the following assumptions is satisfied:

i) γ is an extreme point of Γ,

ii) γ belongs to an $n-1$ dimensional face of Γ,

iii) there exists $\zeta_\gamma \in \mathbb{R}^n$ such that

$$\phi(\gamma') + \langle \zeta_\gamma, \gamma' - \gamma \rangle \geq \phi(\gamma) \quad \forall \gamma' \in \Gamma.$$

Remark 2 *i) The first part of Theorem 3 is [5], Theorem 1.2, in case when F is strictly convex and [16], Theorem 4.15 in the general case, except that (1) was only an inequality there: $\liminf_{x \in \Omega, x \to \gamma} w(x) \geq \phi(\gamma)$.*

ii) For the meaning of extreme point and face, we refer the reader to Definition 2.

The third case in Theorem 3 suggests the following

Definition 1 *We say that ϕ satisfies the weak bounded slope condition if for any $\gamma \in \Gamma$, there exists $\zeta_\gamma^-, \zeta_\gamma^+ \in \mathbb{R}^n$ such that*

$$\phi(\gamma) + \langle \zeta_\gamma^-, \gamma' - \gamma \rangle \leq \phi(\gamma') \leq \phi(\gamma) + \langle \zeta_\gamma^+, \gamma' - \gamma \rangle \quad \forall \gamma' \in \Gamma.$$

Remark 3 *i) Using the tools in [2], Appendix A1, it may be seen that in general, the weak bounded slope condition is not equivalent to the classical bounded slope condition.*

ii) The weak bounded slope condition implies the convexity of Ω except when ϕ is affine. This can be seen as for the bounded slope condition (see [10]).

Then we have

Corollary 3 *Assume that F is convex and that the projections on $\mathbb{R}^n$ of the faces of epi$F$ are uniformly bounded. If ϕ satisfies a weak bounded slope condition, then any minimum w is continuous at any point of the boundary in the following sense:*

$$\lim_{x \to \gamma, x \in A} w(x) \quad \textit{exists and is equal to} \quad \phi(\gamma),$$

where A is the set of Lebesgue points of w. Moreover, there exists a minimum which is continuous on the closure of Ω.

We end this introduction with the following

Open Problem 2 *Assume that F is convex and that the projections on $\mathbb{R}^n$ of the faces of the epigraph of F are uniformly bounded. Assume that ϕ satisfies the lower bounded slope condition and that Ω is convex. Is any solution continuous on clΩ ?*

The problem is even open when F is strictly convex (but not superlinear).

In section 2, we prove Theorem 1. Theorems 2 and 3 are proved in section 3 and 4 respectively. In the last section, we generalize these results to more general lagrangians.

2 Proof of Theorem 1

In this section, we consider a convex superlinear Lagrangian $F : \mathbb{R}^n \to \mathbb{R}$ and a Lispchitz continuous map $\phi : \Gamma \to \mathbb{R}$. Then, the problem (P) of minimizing

$$u \in W^{1,1}_\phi(\Omega) \mapsto \int_\Omega F(\nabla u(x))\, dx$$

has a solution. This solution is non necessarily unique, since the Lagrangian is not assumed to be strictly convex. The following observation on the minima of (P) is due to Mariconda and Treu (see [16], Proposition 4.2):

Lemma 1 *There exists a (unique) solution $u \in W^{1,1}_\phi(\Omega)$ of (P) which satisfies $u(x) \geq v(x)$ a.e. $x \in \Omega$, for any other solution v. We call u the maximal minimum of (P) on $W^{1,1}_\phi(\Omega)$.*

Remark 4 *i) When F is strictly convex, the maximal minimum is the unique minimum of (P).*

ii) Analogously, we could define the minimal minimum.

Then, we can state the following comparison principle (see [16], Theorem 2.12):

Lemma 2 *Let u be the maximal minimum of (P) in $W^{1,1}_\phi(\Omega)$. Let $v \in W^{1,1}_\psi(\Omega)$ be a minimum of (P) with respect to another boundary condition $\psi \in L^1(\Gamma)$. Then we have*

$$\phi(\gamma) \geq \psi(\gamma) \text{ a.e. } \gamma \in \Gamma \Longrightarrow u(x) \geq v(x) \text{ a.e. } x \in \Omega.$$

Lemma 2 has the following consequence:

Lemma 3 *Let u be a maximal minimum of (P) with respect to ϕ. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $\psi_{|\Gamma} \leq \phi$. Then*

$$\psi(x) \leq u(x) \quad \text{a.e.} \quad x \in \Omega.$$

Proof: Let $x \in \Omega$ be a Lebesgue point of u and ζ in the convex subdifferential of ψ at x :

$$\psi(y) \geq \psi(x) + \langle \zeta, y - x \rangle \quad \forall y \in \mathbb{R}^n.$$

Consider the affine map $a_x : y \mapsto \psi(x) + \langle \zeta, y - x \rangle$. Then a_x is a minimum of (P) in $W^{1,1}_{a_{x|\Gamma}}(\Omega)$ and

$$a_{x|\Gamma} \leq \psi_{|\Gamma} \leq \phi.$$

Lemma 2 then implies

$$u(y) \geq a_x(y) \quad \text{a.e.} \quad y \in \Omega.$$

In particular, this is true for $y = x$ since x is a Lebesgue point of u. Since $a_x(x) = \psi(x)$, we have

$$u(x) \geq \psi(x),$$

which completes the proof of Lemma 3

□

Remark 5 *Lemma 3 has a natural counterpart where maximal minimum and convex are replaced by minimal minimum and concave.*

We follow [18], section 18, for the following definition and the basic properties of faces.

Definition 2 *A face of* $cl\Omega$ *is a convex subset* Σ *of* $cl\Omega$ *such that every closed line segment in* $cl\Omega$ *with a relative interior point in* Σ *has both endpoints in* Σ *(relative means: with respect to the affine hull topology of* Σ*). The empty set and* $cl\Omega$ *itself are faces of* $cl\Omega$*. The dimension of a face is the dimension of its affine hull. The zero-dimensional faces of* $cl\Omega$ *are called the extreme points of* $cl\Omega$*.*

Thus a point $\gamma \in \mathrm{cl}\,\Omega$ is an extreme point of $\mathrm{cl}\,\Omega$ if there is no way to express x as a convex combination $(1-\lambda)y + \lambda z$ such that $y \in \mathrm{cl}\,\Omega, z \in \mathrm{cl}\,\Omega$ and $0 < \lambda < 1$, except by taking $y = z = x$.

Since a face which is not $\mathrm{cl}\,\Omega$ itself is contained in Γ, we also say a *face* or an *extreme point of* Γ.

We now establish some notation used in [5] and which will be useful in the proof of Lemma 4. For any $x \in \mathrm{cl}\,\Omega$, $y \in \Omega, x \neq y$, we denote by $\pi_\Gamma(x|y)$ the projection of x onto Γ in the direction of y; that is, the unique point $z \in \Gamma$ of the form $x + t(y-x), t > 0$. Then $d_\Gamma(x|y)$, the distance from x to Γ in the direction of y, is given by $d_\Gamma(x|y) := |x - \pi_\Gamma(x|y)|$. We also consider the function $j_x(y) := |x-y|/d_\Gamma(x|y)$, which can be written as

$$j_x(y) := \inf\{\lambda > 0 : \frac{y-x}{\lambda} \in \Omega - x\}. \tag{2}$$

It is well known that for any $x \in \Omega$, j_x is convex on $\mathbb{R}^n$. Remark that even if $\pi_\Gamma(x|y)$ is not defined when $x = y$, we can extend j_x by continuity to $\{x\}$: $j_x(x) = 0$.

Lemma 4 *Let* Ω_0 *be a bounded open set in* $\mathbb{R}^n$, $\phi_0 : \partial\Omega_0 \to \mathbb{R}$ *satisfy the lower bounded slope condition and* u_0 *be the maximal minimum of the problem* (P_0) *:*

$$\text{Minimize} \quad u \in W^{1,1}_{\phi_0}(\Omega_0) \mapsto \int_{\Omega_0} F(\nabla u_0(x))\, dx.$$

Let $\gamma \in \partial\Omega_0$ *belong to an* $n-1$ *dimensional face. Then* u_0 *is continuous at* γ*.*

Remark 6 *i) In particular, if* Ω_0 *is a convex polyhedron, then* u_0 *is continuous on* $cl\Omega$*.*

ii) In this lemma, we do not use the fact that F *is superlinear.*

iii) If F *were assumed to be strictly convex, then Lemma 4 would be a consequence of the proof of [5], Theorem 2.2.*

Proof of Lemma 4:

The proof of [14], Theorem 4.15 (which generalizes the proof of [5], Theorem 1.2 to the *non strictly* convex setting) implies that there exists $Q > 0$ such that for any Lebesgue points $x, y \in \Omega_0$ of u_0,

$$u_0(x) \leq u_0(y) + Qj_x(y) = u_0(y) + Q|x-y|/d_{\Gamma_0}(x|y), \tag{3}$$

where $\Gamma_0 := \partial\Omega_0$. In particular, u_0 is locally Lipschitz in Ω_0. Let γ belong to an $n-1$ dimensional face of Ω_0, say Σ. We first prove that

$$\limsup_{x\to\gamma, x\in A} u_0(x) \leq \phi_0(\gamma). \tag{4}$$

Assume first that γ belongs to the relative interior of Σ, (i.e. with respect to the affine hull topology of Σ. Let (x_n) be a sequence of Lebesgue points in Ω_0 converging to γ such that

$$\limsup_{n\to+\infty} u_0(x_n) = \limsup_{x\to\gamma, x\in A} u_0(x).$$

Fix a Lebesgue point $y \in \Omega_0$ of u. Then (3) implies

$$u_0(x_n) \leq u_0(y) + Q\frac{|x_n - y|}{|x_n - z_n|} \ \ , \ n \geq 1,$$

with $z_n := \pi_{\Gamma_0}(x_n|y)$. The sequence (z_n) converges to the unique point $z \in \Gamma_0$ of the form $\gamma + t(y-\gamma)$, $t > 0$. In particular, z does not belong to Σ. This implies that

$$\limsup_{n\to+\infty} u_0(x_n) \leq u_0(y) + Q\frac{|\gamma - y|}{|\gamma - z|} \leq u_0(y) + Q\frac{|\gamma - y|}{\mathrm{d}_{\partial\Sigma}(\gamma)} \ , \text{ a.e } y \in \Omega_0, \tag{5}$$

where $\mathrm{d}_{\partial\Sigma}(\gamma)$ denotes the distance of γ to the relative boundary of Σ. Since ϕ_0 is the trace of u_0, for almost every γ' in the relative interior of Σ, $\gamma' \neq \gamma$, we have

$$\lim_{r\to 0} \frac{1}{|B(\gamma', r)\cap\Omega_0|}\int_{B(\gamma', r)\cap\Omega_0} u_0(y)\, dy = \phi_0(\gamma'). \tag{6}$$

Hence, (5) implies that for a.e. $\gamma' \in \Sigma$,

$$\limsup_{n\to+\infty} u_0(x_n) \leq \phi_0(\gamma') + \frac{Q}{\mathrm{d}_{\partial\Sigma}(\gamma)}|\gamma - \gamma'|.$$

Since ϕ_0 is continuous, this yields (letting $\gamma' \to \gamma$)

$$\limsup_{x\to\gamma, x\in A} u_0(x) = \limsup_{n\to+\infty} u_0(x_n) \leq \phi(\gamma), \tag{7}$$

which completes the proof of (4) when γ belongs to the relative interior of Σ. When γ belongs to the relative boundary of Σ, let γ' be in the interior of Σ. We still have

$$\limsup_{x\to\gamma, x\in A} u_0(x) \le u_0(y) + Q\frac{|\gamma - y|}{|\gamma - z|} \quad \text{a.e.} \quad y \in \Omega,$$

where z is the unique point in Γ_0 of the form $\gamma + t(y-\gamma)$, $t > 0$. By the case above, we know that ϕ_0 is continuous at γ'. Hence, letting $y \to \gamma'$, we get

$$\limsup_{x\to\gamma, x\in A} u_0(x) \le \phi_0(\gamma') + Q\frac{|\gamma - \gamma'|}{|\gamma - z'|} \tag{8}$$

where z' is the unique point in $\partial\Sigma$ (the relative boundary of Σ) of the form $\gamma + t(\gamma' - \gamma)$, $t > 0$. In particular, inequality (8) is true for any $\gamma' \in (\gamma, z')$. Since ϕ_0 is continuous, this yields

$$\limsup_{x\to\gamma, x\in A} u_0(x) \le \phi(\gamma),$$

that is (4).

We now prove that $\liminf_{x\to\gamma, x\in A} u_0(x) \ge \phi_0(\gamma)$. Observe that

$$\phi_0(\gamma') \ge \phi_0(\gamma) + \langle \zeta_\gamma, \gamma' - \gamma \rangle \quad , \quad \gamma' \in \Gamma_0$$

for some $\zeta \in \mathbb{R}^n$ (this follows from the fact that ϕ satisfies the lower bounded slope condition). On the other hand, since affine maps are minimizers of the problem, we have (using Lemma 2)

$$u_0(x) \ge \phi_0(\gamma) + \langle \zeta_\gamma, x - \gamma \rangle \quad \text{a.e.} \ x \in \Omega.$$

Hence, $\liminf_{x\to\gamma, x\in A} u_0(x) \ge \phi_0(\gamma)$, which completes the proof of Lemma 4.

□

We now prove Theorem 1. Fix $\gamma \in \Gamma$ and denote by A the set of Lebesgue points in Ω of a minimum u. We claim that

Lemma 5

$$\limsup_{x\to\gamma, x\in A} u(x) \le \phi(\gamma). \tag{9}$$

We could prove similarly that

$$\liminf_{x\to\gamma, x\in A} u(x) \ge \phi(\gamma). \tag{10}$$

Then (9) and (10) would imply that $\lim_{x\to\gamma, x\in A} u(x) = \phi(\gamma)$. To prove (9), we consider an auxiliary problem. There exists a cube $\Omega_0 \subset \mathbb{R}^n$ such that

i) $\Omega_0 \supset \Omega$,

ii) $\gamma \in \Gamma_0$, where $\Gamma_0 := \partial\Omega_0$.

(the existence of Ω_0 is an easy consequence of the existence of a supporting hyperplane to Ω at γ). Denote by Q a Lipschitz rank of ϕ and define

$$\phi_0(x) := \phi(\gamma) + Q|x-\gamma| \ , \ x \in \mathbb{R}^n.$$

Then ϕ_0 is a convex map which satisfies

$$\forall \gamma' \in \Gamma, \quad \phi_0(\gamma') \geq \phi(\gamma') \quad , \quad \phi_0(\gamma) = \phi(\gamma). \tag{11}$$

Consider the maximal minimum u_0 of the problem (P_0):

$$\text{Minimize} \ \ u \in W^{1,1}_{\phi_0}(\Omega_0) \mapsto \int_{\Omega_0} F(\nabla u_0(x))\, dx.$$

Since ϕ_0 is convex, its restriction to Γ satisfies the lower bounded slope condition (see for instance [1]). By Lemma 4, we know that u_0 is continuous on $\mathrm{cl}\,\Omega$ and locally Lipschitz continuous in Ω. Moreover, Lemma 3 implies that $u_0 \geq \phi_0$ on Ω_0. In particular,

$$u_{0|\Gamma} \geq \phi_{0|\Gamma} = \phi.$$

It is easy to see (by contradiction) that $u_{0|\Omega}$ is still a maximal minimum for (P) in $W^{1,1}_{u_{0|\Gamma}}(\Omega)$. Hence, Lemma 2 implies that

$$u_0(x) \geq u(x) \ \ \text{a.e.} \ \ x \in \Omega.$$

Finally, we get

$$\phi(\gamma) = \phi_0(\gamma) = \lim_{x\to\gamma} u_0(x) \geq \limsup_{x\to\gamma, x\in A} u(x),$$

which proves (9). This completes the proof of Theorem 1.

□

To prove Corollary 1, use [16], Theorem 4.15 to get the continuity inside the domain and Theorem 1 (the Lipschitz continuity is a consequence of the lower bounded slope condition) to get the continuity up to the boundary.

3 Proof of Theorem 2

In this section, F is convex and superlinear, and Ω is bounded and convex. We begin with the following

Lemma 6 *If w_1 and w_2 are two minima of I on $W^{1,1}_\phi(\Omega)$, then for almost every $x \in \Omega$, $\nabla w_1(x)$ and $\nabla w_2(x)$ belong to projection of a same face of epi$F$.*

Proof: Since w_1 and w_2 are minima, we have

$$I(\frac{w_1 + w_2}{2}) \geq \frac{1}{2} I(w_1) + \frac{1}{2} I(w_2). \tag{12}$$

Since F is convex,

$$F(\frac{\nabla w_1(x) + \nabla w_2(x)}{2}) \leq \frac{1}{2} F(\nabla w_1(x)) + \frac{1}{2} F(\nabla w_1(x)) \text{ a.e. } x \in \Omega \tag{13}$$

with equality if and only if $\nabla w_1(x)$ and $\nabla w_2(x)$ belong to the projection of a same face. By integration over Ω, (13) yields

$$I(\frac{w_1 + w_2}{2}) \leq \frac{1}{2} I(w_1) + \frac{1}{2} I(w_2). \tag{14}$$

Inequality (12) implies that (14) is an equality. Hence, for almost every $x \in \Omega$, $\nabla w_1(x)$ and $\nabla w_2(x)$ belong to the projection of a same face.

□

We now prove Theorem 2. Assume first that ϕ is Lipschitz continuous and denote by w the maximal minimum of (P) in $W^{1,1}_\phi(\Omega)$.

Let us extend w by ϕ out of Ω. The resulting function is denoted by $\bar{w}$. By Theorem 1, for any $\gamma \in \Gamma$, $\bar{w}$ satisfies

$$\lim_{y \to \gamma,\, y \in A \cup (\mathbb{R}^n \setminus \Omega)} \bar{w}(y) = \phi(\gamma), \tag{15}$$

where A is the set of Lebesgue points of w in Ω.

Let ρ_ϵ be a smooth kernel and consider

$$\bar{w}_\epsilon := \bar{w} * \rho_\epsilon. \tag{16}$$

We may assume , without loss of generality, that $0 \in \Omega$.

We define an increasing family $\{\Omega_\epsilon\}_{\epsilon \to 0}$ of strictly convex subsets of Ω such that

$$\max_{y \in \partial \Omega_\epsilon} \text{dist}\,(y, \partial\Omega) \leq \epsilon \tag{17}$$

as follows

$$\Omega_\epsilon := \{x \in \mathbb{R}^n : j_0(x) + \epsilon' |x|^2 < 1\},$$

with $\epsilon' := \epsilon / \max_{y \in \Omega}(1+|y|)^3$ (see (2) for the definition of j_0).

We say that a function $f\Omega \to \mathbb{R}$ is uniformly convex if there exists $\mu > 0$ such that for any $x \in \Omega$, there exists $\zeta \in \partial f(x)$ which satisfies

$$f(y) \geq f(x) + \langle \zeta, y - x\rangle + \mu |y - x|^2 \quad , \; y \in \Omega.$$

The map $x \mapsto j_0(x) + \epsilon' |x|^2$ is uniformly convex, as the sum of a convex and a uniformly convex map Hence, Ω_ϵ is uniformly convex; that is, there exists $\mu > 0$ such that $\forall x \in \partial\Omega_\epsilon$, there exists $n_x \in \mathbb{R}^n$, $|n_x| = 1$ which satisfies

$$\langle n_x, y - x\rangle \geq \mu |y - x|^2 \quad y \in \partial\Omega_\epsilon.$$

Let v_ϵ be the maximal minimum of $v \mapsto \displaystyle\int_{\Omega_\epsilon} F(\nabla v)$ on $W^{1,1}_{\bar{w}_\epsilon|\partial\Omega_\epsilon}(\Omega_\epsilon)$. We claim that

Lemma 7 *The map v_ϵ is continuous on $cl\Omega_\epsilon$.*

Proof: Since $\bar{w}_\epsilon$ is smooth, its restriction to the boundary of the uniformly convex set Ω_ϵ satisfies the bounded slope condition (see [17]). Hence, there exists a minimum $\tilde{v}$ of $v \mapsto \displaystyle\int_\Omega F(\nabla v)$ on $W^{1,1}_{\bar{w}_\epsilon|\partial\Omega_\epsilon}(\Omega_\epsilon)$ which is Lipschitz continuous on $\mathrm{cl}\,\Omega_\epsilon$. By Lemma 6, for almost every $x \in \Omega_\epsilon$, $\nabla\bar{v}(x)$ and ∇v_ϵ belong to the projection of the same face of $\mathrm{epi}\,F$.

Let $M > 0$ be such that $|\nabla\bar{v}(x)| \leq M$ for almost every $x \in \Omega$. We claim that there exists $K > 0$ such that for any $p, q \in \mathbb{R}^n$, $|p| \leq M$, if there exists $\zeta \in \mathbb{R}^n$ satisfying

$$F(q) - F(p) = \langle \zeta, p - q\rangle, \tag{18}$$

then $|q| \leq K$ (observe that (18) means that p and q belong to the projection of the same face of $\mathrm{epi}\,F$ and that$\zeta \in \partial F(p)$). Indeed, assume by contradiction that there exists $p_i, q_i \in \mathbb{R}^n$, $|p_i| \leq K$, $|q_i| \geq i$ and $\zeta_i \in \partial F(p_i)$ such that $F(q_i) - F(p_i) = \langle \zeta_i, q_i - p_i\rangle$. Then we get

$$\frac{F(q_i)}{|q_i|} = \frac{F(p_i)}{|q_i|} + \langle \zeta_i, \frac{q_i - p_i}{|q_i|}\rangle.$$

Since F is Lipschitz on $B(0, M)$, the sequence (ζ_i) is bounded. Hence, we get a contradiction when $i \to +\infty$.

This implies (with M a Lipschitz rank for $\nabla\bar{v}$) that there exists $K > 0$ such that v_ϵ is Lipschitz of rank K on $\mathrm{cl}\,\Omega$.

□

Using (15), it is easy to prove (by contradiction) that for any $\eta > 0$, there exists $\delta := \delta(\eta) > 0$ such that for any $y \in A \cup (\mathbb{R}^n \setminus \Omega), x \in \Gamma$, we have

$$||x - y|| \leq \delta \Longrightarrow |\phi(x) - \bar{w}(y)| < \eta.$$

Fix $\eta > 0$. By the Coarea's formula (see [6]), for almost every $\epsilon > 0$, almost every $y \in \partial\Omega_\epsilon$ is a Lebesgue point of w. For any $\epsilon < \delta(\eta)/2$ satisfying this property and any Lebesgue point $y \in \partial\Omega_\epsilon$, there exists $x \in \Gamma$ such that $||y - x|| \leq \epsilon$ (see (17)). Hence, $|w(y) - \phi(x)| \leq \eta$. Moreover,

$$|\bar{w}_\epsilon(y) - \phi(x)| \leq \int_{B_\epsilon(0)} \rho_\epsilon(z)|\bar{w}(y - z) - \phi(x)|\, dz \leq \eta$$

since for any $z \in B_\epsilon(0)$, $||y - z - x|| \leq 2\epsilon \leq \delta(\eta)$ and for almost every $z \in B_\epsilon(0)$, $y - z$ is a Lebesgue point of $\bar{w}$.

We have thus proved that for almost every $y \in \partial\Omega_\epsilon$, $|\bar{w}_\epsilon(y) - w(y)| < 2\eta$. Since $(w + 2\eta)_{|\Omega_\epsilon}$ is a maximal minimum in $W^{1,1}_{(w+2\eta)_{|\partial\Omega_\epsilon}}(\Omega)$ (this can be easily seen by contradiction), we have (using Lemma 2)

$$v_\epsilon(x) \leq w(x) + 2\eta \quad \text{a.e.} \quad x \in \Omega_\epsilon.$$

Since v_ϵ is a maximal minimum in $W^{1,1}_{\bar{w}_{\epsilon|\partial\Omega_\epsilon}}(\Omega)$, we also have

$$v_\epsilon(x) \geq w(x) - 2\eta \quad \text{a.e.} \quad x \in \Omega_\epsilon.$$

Finally,

$$||v_\epsilon - w||_{L^\infty(\Omega_\epsilon)} \leq 2\eta.$$

This proves that for any compact subset $K \subset \Omega$, (v_ϵ) converges to w uniformly on K. By Lemma 7, v_ϵ is continuous. Hence, w is continuous on Ω and $A = \Omega$. In view of (15), w is then continuous on the closure of Ω. This completes the proof of Theorem 2 in case when ϕ is Lipschitz continuous.

When ϕ is merely continuous, consider the maximal minimum w of (P) in $W^{1,1}_\phi(\Omega)$. Let (ϕ_i) be a sequence of Lipschitz maps converging uniformly to ϕ on Γ. In light of the proof above, for each i, the maximal minimum w_i of I on $W^{1,1}_{\phi_i}(\Omega)$ is continuous on $\mathrm{cl}\,\Omega$. Since w and w_i are two maximal minima, we have (as above)

$$||w_i - w||_{L^\infty(\Omega)} \leq ||\phi_i - \phi||_{L^\infty(\Gamma)}$$

which implies that w_i converges uniformly to w on $\mathrm{cl}\,\Omega$. Hence, w is continuous on $\mathrm{cl}\,\Omega$. This completes the proof of Theorem 2.

□

4 Non superlinear Lagrangians

In this section, we assume that the projections of the faces of $\operatorname{epi} F$ have diameters which are uniformly bounded (and that F is convex). We do not assume that F is superlinear. The map ϕ satisfies the lower bounded slope condition and the set Ω is convex.

We have the following counterpart of Lemma 1, which is also due to Mariconda and Treu (see [16], Proposition 4.2):

Lemma 8 *Assume that there exists a solution to (P) in $W^{1,1}_\phi(\Omega)$. Then, there exists a (unique) solution $u \in W^{1,1}_\phi(\Omega)$ to the problem (P) which satisfies $u(x) \geq v(x)$ a.e. $x \in \Omega$, for any other solution v. We call u the maximal minimum of (P) on $W^{1,1}_\phi(\Omega)$.*

Assume that there exists a solution of (P) in $W^{1,1}_\phi(\Omega)$. By Lemma 8 (and its counterpart for the minimum of the minima), there exist $w_-, w_+ \in W^{1,1}_\phi(\Omega)$ the minimum and the maximum of the minima respectively.

When ϕ satisfies the lower bounded slope condition, it is known that (see [5] for the case when F is strictly convex and [16] for the generalization when F is not necessarily strictly convex) that

i) any minimum w is bounded. Actually,

$$||w||_{L^\infty(\Omega)} \leq ||\phi||_{L^\infty(\Gamma)} + R_0 \operatorname{diam} \Omega$$

where R_0 is the radius of any ball containing the projection of the face of the epigraph of F which contains $(0, F(0))$.

ii) Each minimum w is locally Lipschitz on Ω.

iii) The minimum and the maximum of the minima satisfy

$$\text{there exists } K > 0 \text{ such that } w_\pm(x) \leq w_\pm(y) + K j_x(y) \ , \ \forall x, y \in \Omega. \tag{19}$$

Moreover, K depends on ϕ and Ω but not on F.

To prove that any minimum w is continuous at a point $\gamma \in \Gamma$, it is enough to prove that

$$\liminf_{x \to \gamma} w_-(x) \geq \phi(\gamma) \tag{20}$$

$$\limsup_{x \to \gamma} w_+(x) \leq \phi(\gamma). \tag{21}$$

Indeed, if (20) and (21) are satisfied, then for any minimum w,

$$\phi(\gamma) \leq \liminf_{x\to\gamma} w_-(x) \leq \liminf_{x\to\gamma} w(x) \leq \limsup_{x\to\gamma} w(x) \leq \limsup_{x\to\gamma} w_+(x) \leq \phi(\gamma).$$

This will prove the continuity of w at γ. Property (20) is true for any $\gamma \in \Gamma$. Indeed, let $a_\gamma(x) := \phi(\gamma) + \langle \zeta_\gamma, x - \gamma\rangle$ such that $\phi \geq a_\gamma$ on Γ (the existence of a_γ follows from the lower bounded slope condition). Then, a_γ is a minimum of I on $W^{1,1}_{a_{\gamma|\Gamma}}(\Omega)$. By Lemma 8, there exists a minimal minimizer a_γ^- of I on $W^{1,1}_{a_{\gamma|\Gamma}}(\Omega)$. By Lemma 6, we know that $\nabla a_\gamma(x)$ and $\nabla a_\gamma^-(x)$ belong to the projection of the same face of $\operatorname{epi} F$, for almost every $x \in \Omega$. Since $\nabla a_\gamma^-(x) = \zeta$ and the faces of the epigraph are bounded, the map a_γ^- is Lispchitz continuous, and in particular continuous on $\operatorname{cl}\Omega$. By Lemma 2 (more specifically, its counterpart for minimal minimum), we have $w_- \geq a_\gamma^-$ almost everywhere on Ω. This implies (20).

We now prove Theorem 3 iii). Assume that there exists $\zeta_\gamma^+ \in \mathbb{R}^n$ such that

$$\phi(\gamma') \leq \phi(\gamma) + \langle \zeta_\gamma^+, \gamma' - \gamma\rangle \quad \forall \gamma' \in \Gamma.$$

Then (21) holds exactly for the same reasons as (20). This proves Theorem 3 iii).

Lemma 4 and Remark 6 imply that (21) also holds when γ belongs to an $n-1$ dimensional face. This proves Theorem 3 ii). It remains to consider the case when γ is extremal:

Lemma 9 *If $\gamma \in \Gamma$ is an extreme point, then (21) holds.*

Proof: For any $x \in \mathbb{R}^N$, define

$$S(x) := \{(T, \mu_1, ...\mu_m, x_1, ..., x_m) : x = \sum_{i=1}^m \mu_i x_i \ , \ T \geq 0, \mu_i \geq 0,$$

$$\sum_{i=1}^m \mu_i = 1, x_i \in \Gamma, m > 0\},$$

and for any $s > ||\phi||_{L^\infty(\Gamma)}$,

$$\phi^s(x) := \sup_{S(x)} \{s + T \sum_{i=1}^m \mu_i(\phi(x_i) - s)\}.$$

The map ϕ^s has been introduced in [10]. It is easy to see that ϕ^s is a concave function on $\mathbb{R}^n$ and that $\phi^s \geq \phi$ on Γ. Let $a_\gamma^s(x) := \phi^s(\gamma) + \langle \zeta, x - \gamma\rangle$, where

ζ is in the concave subdifferential of ϕ^s at γ. Let a_γ^{s+} be the maximum of the minima of I on $W^{1,1}_{a^s_{\gamma|\Gamma}}(\Omega)$. Then, as in the proof of (20), we may see that a_γ^{s+} is continuous on $\mathrm{cl}\,\Omega$ and $a_\gamma^{s+} \geq w_+$ almost everywhere on Ω.

Moreover, $\phi^s(\gamma)$ converges to $\phi(\gamma)$ when γ is an extreme point of Γ (see [10], Proposition 3.5). Then for any $s > ||\phi||_{L^\infty(\Gamma)}$,

$$\limsup_{x\in\Omega, x\to\gamma} w_+(x) \leq \limsup_{x\in\Omega, x\to\gamma} a_\gamma^{s+}(x) = \phi^s(\gamma).$$

Now, let $s \to \infty$. We get

$$\limsup_{x\in\Omega, x\to\gamma} w_+(x) \leq \phi(\gamma).$$

This completes the proof of Lemma 9.

□

To complete the proof of Theorem 3, it remains to prove (1).

In the following, we assume (without loss of generality) that $0 \in \Omega$.

Assume that ϕ satisfies the lower bounded slope condition. We may extend it as a convex function on $\mathbb{R}^n$, still denoted by ϕ. Let w_+ be the maximum of the minima in $W^{1,1}_\phi(\Omega)$. Lemma 3 then shows that $\phi \leq w_+$ on Ω. In particular, for any $\gamma \in \Gamma$,

$$\liminf_{x\in\Omega, x\to\gamma} w_+(x) \geq \phi(\gamma). \tag{22}$$

Actually, we prove below that this inequality is an equality. We need first the following

Lemma 10 *Let $\gamma \in \Gamma$ and (x_k) be a sequence in Ω such that x_k converges to γ. Then, there exists a subsequence of $\{x_k\}$ (we do not relabel) and $y_k \in [\gamma, 0]$ such that $y_k \to \gamma$ and*

$$j_{x_k}(y_k) \to 0.$$

Proof: The function $j_x(y)$ has been defined just before Lemma 4. We may assume that $\forall k \geq 1$, $x_k \notin [0,\gamma]$ (otherwise, we define $y_k := x_k$). For any k, we define y_k as any point in $[0,\gamma]$ such that j_{x_k} attains its minimum on $[0,\gamma]$ at y_k. Without relabeling, we may assume that y_k converges to some point $y \in [0,\gamma]$. We claim that $y = \gamma$. Assume by contradiction that $y \neq \gamma$. Then, $\pi_\Gamma(x_k|y_k)$ converges to $\pi_\Gamma(\gamma|y)$. This implies that $j_{x_k}(y_k) = \dfrac{|x_k - y_k|}{|x_k - \pi_\Gamma(x_k|y_k)|}$ converges to $j_\gamma(y)$. Moreover, for any $y' \in \Omega$, $j_{x_k}(y')$ converges to $j_\gamma(y')$.

Since $j_{x_k}(y_k) \leq j_{x_k}(y')$ for any $y' \in (0, \gamma)$, we have

$$j_\gamma(y) \leq j_\gamma(y'),$$

which cannot hold for any $y' \in (y, \gamma)$. (Here, we use the fact that $\pi_\Gamma(\gamma|0) = \pi_\Gamma(\gamma|y) = \pi_\Gamma(\gamma|y')$). Hence, $y = \gamma$ and (the whole sequence) (y_k) converges to γ.

Finally, we show that $j_{x_k}(y_k) \to 0$. The sequence j_{x_k} converges pointwisely on $\{t\gamma : t \in (0,1)\}$ to the continuous function j_γ which satisfies

$$j_\gamma(t\gamma) = \frac{(1-t)|\gamma|}{\mathrm{d}_\Gamma(\gamma|0)}.$$

Define $a := |\gamma|/\mathrm{d}_\Gamma(\gamma|0)$. Let $\epsilon > 0$. Restricted to the compact interval $I := \{t\gamma : t \in [1 - \frac{\epsilon}{2a}, 1 - \frac{\epsilon}{4a}]\}$, the family of convex nonnegative uniformly bounded functions (j_{x_k}) converges uniformly to j_γ. Then, there exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$, $j_{x_k} \leq j_\gamma + \epsilon/2 \leq \epsilon$ on I. This implies that

$$j_{x_k}(y_k) = \min_{[0,\gamma]} j_{x_k} \leq \min_I j_{x_k} \leq \epsilon$$

for any $k \geq k_0$. Hence, $j_{x_k}(y_k)$ converges to 0. This completes the proof of Lemma 10.

□

Lemma 11 *Let $\gamma \in \Gamma$ be such that $\lim_{t\to 1^-} w_+(t\gamma) = \phi(\gamma)$. Then*

$$\lim_{x\in\Omega, x\to\gamma} w_+(x) = \phi(\gamma).$$

Proof: We prove Lemma 11 by contradiction. In light of (22), this means that there exists $\epsilon > 0$ and a sequence of points x_k in Ω such that x_k converges to γ and

$$\lim_{k\to\infty} w_+(x_k) = \phi(\gamma) + \epsilon.$$

Then, up to a subsequence (we do not relabel) and using Lemma 10, there exists $y_k \in [\gamma, 0]$ such that (y_k) converges to γ, and

$$\frac{|x_k - y_k|}{d_\Gamma(x_k|y_k)} \to 0.$$

Since w_+ satisfies property (19), we have

$$w_+(x_k) \leq w_+(y_k) + K\frac{|x_k - y_k|}{d_\Gamma(x_k|y_k)}$$

which implies (using the fact that w_+ is continuous on $[\gamma, 0]$)),

$$w_+(\gamma) + \epsilon \le \limsup_{k\to\infty}(w_+(y_k) + K\frac{|x_k - y_k|}{d_\Gamma(x_k|y_k)}) = w_+(\gamma).$$

This contradiction completes the proof of Lemma 11.

□

Corollary 4 *For almost every* $\gamma \in \Gamma$, $\lim_{x\in\Omega, x\to\gamma} w_+(x) = \phi(\gamma)$.

Proof : Let $\epsilon > 0$ be such that $B(0,\epsilon) \subset \Omega$. Denote by $\tilde{w}_+$ the function which is equal to w_+ on Ω and to ϕ on $\mathbb{R}^N \setminus \Omega$. For almost every $\gamma \in \Gamma$, the restriction of $\tilde{w}_+$ to $I := \{t\gamma, t > \epsilon/|\gamma|\}$ belongs to $W^{1,1}(I)$, hence is continuous on I. Then, Lemma 11 implies that $\lim_{x\in\Omega, x\to\gamma} w_+(x) = \phi(\gamma)$.
□

Remark 7 *Actually, one may improve Corollary 4 using exercise 3.15 in [19] where it is shown that for* $B_{1,r}$ *quasi-every* $x \in \mathbb{R}^n$, $\tilde{w}_+ \in W^{1,r}(\mathbb{R}^n)$, $r \ge 1$, *is continuous on almost every ray* λ_x *whose endpoint is* x *(here,* $B_{1,r}$ *refers to the Bessel capacity). Hence,* $\lim_{x\in\Omega, x\to\gamma} w_+(x) = \phi(\gamma)$ *for* $B_{1,r}$ *quasi-every* $\gamma \in \Gamma$.

Lemma 12 *For any* $\gamma \in \Gamma$, *we have*

$$\liminf_{x\in\Omega, x\to\gamma} w_+(x) = \phi(\gamma).$$

Proof: Fix $\gamma \in \Gamma$. By Corollary 4, there exists a sequence $(\gamma_i) \subset \Gamma$ such that γ_i converges to γ and $\lim_{x\in\Omega, x\to\gamma_i} w_+(x) = \phi(\gamma_i)$. For each i, there exists $x_i \in \Omega$ such that $|w_+(x_i) - \phi(\gamma_i)| \le 1/i$. Since ϕ is continuous, this implies that $w_+(x_i)$ converges to $\phi(\gamma)$. The lemma is proven.

□

Since (1) is satisfied for w_+, it is automatically satisfied by any minimum w. This completes the proof of Theorem 3.

□

Proof of Corollary 3 We now assume that ϕ satisfies a weak bounded slope condition. Then (20) and (21) hold (this can be seen exactly as for the proof of (20) in the proof of Theorem 3). Hence, any minimizer is continuous at the boundary. The existence of a continuous minimum on $\operatorname{cl}\Omega$ can be proved as in the proof of Theorem 2 (Lemma 7 remains true for Lagrangians which are not necessarily superlinear but such that the faces of their epigraphs have projections which are uniformly bounded. The proof is easier and we omit it).

□

5 More general Lagrangians

We now consider the following problem

$$\min_u \int_\Omega F(Du(x)) + G(x, u(x))\,dx \text{ subject to } u \in W^{1,1}(\Omega),\ \operatorname{tr} u = \phi.$$

We still assume that Ω is convex. We now require that F be uniformly elliptic, and that G be locally Lipschitz in u. More precisely:

(HF) For some $\mu > 0, F$ satisfies, for all $\theta \in (0,1)$ and $p, q \in \mathbb{R}^n$:

$$\theta F(p) + (1-\theta)F(q) \geq F(\theta p + (1-\theta)q) - (\mu/2)\theta(1-\theta)|p-q|^2.$$

(HG) $G(x,u)$ is measurable in x and differentiable in u and for every bounded interval U in $\mathbb{R}$, there is a constant L such that for almost all $x \in \Omega$,

$$|G(x,u) - G(x,u')| \leq L|u-u'|\ \forall u, u' \in U.$$

We also postulate as part of (HG) that for some bounded function b, the integral $\int_\Omega G(x, b(x))\,dx$ is well-defined and finite. It follows that the same is true for all bounded measurable functions w. In the presence of (HF) and (HG), it follows that

$$I(w) := \int_\Omega F(Dw(x)) + G(x, w(x))\,dx$$

is well-defined for all $w \in W^{1,1}(\Omega)$ for which w is bounded. We say that u is a solution relative to $L^\infty(\Omega)$ if u is itself bounded, and if we have $I(u) \leq I(w)$ for all bounded $w \in W^{1,2}(\Omega), \operatorname{tr} w = \phi$.

We then have (see [3])

Theorem 4 *Under the hypotheses (HF) and (HG), when Ω is bounded and convex, if ϕ satisfies the lower bounded slope condition, then any solution w relative to $L^\infty(\Omega)$ satisfies*

$$w(x) \leq w(y) + K\frac{|x-y|}{|x - \pi_\Gamma(x|y)|} \quad \forall x, y \in \Omega, \tag{23}$$

for some $K > 0$. In particular, w is locally Lipschitz in Ω.
is locally Lipschitz in Ω.

As in Theorem 1, we address the question whether u is continuous on $\operatorname{cl}\Omega$. The following theorem significantly improves [3], Theorems 4 and 5.

Theorem 5 *Under the hypotheses* (HF) *and* (HG) *and when* Ω *is a bounded open convex set, if* ϕ *satisfies a lower bounded slope condition and* w *is a solution relative to* $L^\infty(\Omega)$, *then for any* $\gamma \in \Gamma$,

$$\liminf_{x\in\Omega, x\to\gamma} w(x) = \phi(\gamma). \tag{24}$$

Moreover, w *is continuous at* $\gamma \in \Gamma$ *when one of the following assumptions is satisfied:*

i) γ *is an extreme point of* Γ,

ii) there exists $1 \le k \le n-1$ *such that* γ *belongs to an* $n-k$ *dimensional face of* Γ *and*

  a) $k = 1$,

  b) F *is coercive of order* r *with* $r \ge k$,

  c) Ω *is locally* $C^{1,\alpha}$ *near* γ *for some* $0 \le \alpha \le 1$ *and* F *is coercive of order* r *with* $r \ge (k+\alpha)/(1+\alpha)$.

Proof of Theorem 5 We begin with the following

Lemma 13 *Let* $\psi : \mathbb{R}^n \to \mathbb{R}$ *be a convex function such that* $\psi_{|\Gamma} \le \phi$. *Then there exists* $T > 0$ *such that for any* $\gamma \in \Gamma$, *there exists* $\zeta_\gamma \in \mathbb{R}^n$, $\nu_\gamma \in \mathbb{R}^n$, $|\nu_\gamma| = 1$ *such that*

$$w(x) \ge \psi(\gamma) + \langle \zeta_\gamma, x-\gamma\rangle - T(1 - e^{\langle \nu_\gamma, x-\gamma\rangle}) \quad \forall x \in \Omega.$$

Remark 8 *In Lemma 13 , we do not use the fact that* ϕ *satisfies the lower bounded slope condition. Moreover, the analogue for concave functions holds true: if* ψ *is concave and* $\psi_{|\Gamma} \ge \phi$, *then*

$$w(x) \le \psi(\gamma) + \langle \zeta_\gamma, x-\gamma\rangle + T(1 - e^{\langle \nu_\gamma, x-\gamma\rangle}) \;\; \forall x \in \Omega$$

for some $\zeta_\gamma \in \mathbb{R}^n$, $\nu_\gamma \in \mathbb{R}^n$, $|\nu_\gamma| = 1$ *and* $T > 0$.

For a proof of Lemma 13, see [3], Theorem 2.

Lemma 13 then implies that for any $\gamma \in \Gamma$, we have

$$\liminf_{x\in\Omega, x\to\gamma} w(x) \ge \phi(\gamma). \tag{25}$$

The fact that equality holds in (25) can be proven using Lemma 10, 11 and 12 exactly as in the proof of Theorem 3.

The proof of Lemma 9 can be generalized as follows: we do not assert that $w \leq \phi^s$ any more. However, using Remark 8 for fixed $s > 0$ and $\gamma \in \Gamma$, we get

$$w(x) \leq \phi^s(\gamma) + \langle \zeta_\gamma, x - \gamma \rangle + T(1 - e^{\langle \nu_\gamma, x-\gamma \rangle}) \quad \forall x \in \Omega$$

for some $\zeta_\gamma \in \mathbb{R}^n$, $\nu_\gamma \in \mathbb{R}^n$, $|\nu_\gamma| = 1$ and $T > 0$. We now let $x \to \gamma$:

$$\limsup_{x \in \Omega, x \to \gamma} w(x) \leq \phi^s(\gamma).$$

Now, when γ is an extreme point, let $s \to \infty$. We get

$$\limsup_{x \in \Omega, x \to \gamma} w(x) \leq \phi(\gamma).$$

This completes the proof of the analogue of Theorem 5 i).

Lemma 14 *Let $\Sigma \subset \Gamma$ be a face of Γ and denote its dimension by k. Assume that $1 \leq k \leq n-1$. Let γ be a relative interior point in Σ. Consider the $n-k$ dimensional affine plane G perpendicular to Σ at γ. Then $G \cap \Omega \neq \emptyset$.*

Proof: Let ζ be the minimal norm subgradient in $\partial j_0(\gamma)$ ($||\zeta|| > 0$, since $0 \notin \partial j_0(\gamma)$). Then (see [4]) there exists $\delta > 0$ such that $j_0(\gamma - t\zeta) < j_0(\gamma)$ for any $t \in (0, \delta)$. It implies that $\gamma - t\zeta \in \Omega$. Since $j_0 = 1$ on $\Sigma \subset \Gamma$ and $\gamma \in \operatorname{int} \Sigma$, ζ is in the vector space generated by G. Thus

$$\{\gamma - t\zeta : t \in (0, \delta)\} \subset \Omega \cap G.$$

This completes the proof of Lemma 14.

□

Definition 3 *The set Ω is said to be locally $C^{1,\alpha}$ near $\gamma \in \Gamma$ if one can choose coordinates with the origin at γ and a neighborhood U of γ such that*

$$U \cap cl\Omega = \{x \in U : x_1 \geq f(x')\}$$

where $x' = (x_2, ..., x_n)$ and $f \in C^{1,\alpha}$.

Necessarily, $f(0) = 0$. Since Ω is convex, f is convex. Up to a change of coordinates, one may further assume that $f \geq 0$ and $x_1 = \langle x, n \rangle$, where $-n$ is the unit outer normal vector to Ω at γ. Since f is $C^{1,\alpha}$, there exists $d > 0$ such that $f(x') \leq d(x')^{1+\alpha}$.

Lemma 15 *Let $v \in W^{1,p}(\Omega)$ satisfy (23). If $p \geq n$, then v is continuous on $cl\Omega$.*

Proof: The lemma is obvious when $p > n$ by the Morrey-Sobolev embeddings. When $p = n$, it is based on a modification of Lemma 2.12 in [5]. Let $\gamma \in \Gamma$. Since Ω is convex, there exist $\rho, a \in (0,1)$ and a unit vector n such that

$$\mathcal{C} := \{x \in B(\gamma,\rho) \setminus \{\gamma\} : \langle n, \frac{x-\gamma}{|x-\gamma|}\rangle > a\} \subset \Omega.$$

Moreover, n can be chosen such that $-n$ is in the normal cone to Ω at γ. We may assume that $0 \in \mathcal{D}$ where $\mathcal{D} := \{\gamma + tn : t > 0\} \cap B(\gamma, a\rho)$. For any $x \in \mathcal{D}$, we consider the affine hyperplane $\mathcal{H}_x$ perpendicular to $\mathcal{D}$ at x. Then

$$\mathcal{B}(x,R) := \{y \in \mathcal{H}_x : |x-y| < R\} \subset \mathcal{C},$$

for any $R \leq R_x := |x-\gamma|\dfrac{\sqrt{1-a^2}}{a}$.

Let $\alpha : x \in [\gamma, \gamma/2] \to \alpha_x \in (0,\infty)$ be such that $\alpha_x \leq R_x$ and $\alpha_x = o(R_x)$ when $x \to \gamma$. The map α_x will be subject to further restrictions below. Fix $x \in (\gamma, \gamma/2]$. We denote $\mathcal{B}(x,\alpha_x)$ by $\mathcal{B}_{\alpha_x}$. For any $y \in \mathcal{B}_{\alpha_x}$ such that $v_{|[y,\pi_\Gamma(0|y)]} \in W^{1,1}((y, \pi_\Gamma(0|y)))$ (a.e. $y \in \mathcal{B}_{\alpha_x}$ satisfies this condition), we have (using (23))

$$v(x) \leq v(y) + K\frac{|x-y|}{|x-\pi_\Gamma(x|y)|} \leq v(y) + K\frac{\alpha_x}{R_x}$$

$$\leq \phi(\pi_\Gamma(0|y)) + \int_{[y,\pi_\Gamma(0|y)]} |\nabla v| + K\frac{\alpha_x}{R_x}.$$

We now integrate this inequality on $\mathcal{B}_{\alpha_x}$:

$$v(x) \leq \frac{1}{|\mathcal{B}_{\alpha_x}|}\int_{\mathcal{B}_{\alpha_x}} \phi(\pi_\Gamma(0|y))\,dy + \frac{1}{|\mathcal{B}_{\alpha_x}|}\int_{\mathcal{M}_x} |\nabla v| + K\frac{\alpha_x}{R_x}$$

where we have denoted by $\mathcal{M}_x$ the set

$$\mathcal{M}_x := \{ty + (1-t)\pi_\Gamma(0|y) : 0 \leq t \leq 1, y \in \mathcal{B}_{\alpha_x}\}.$$

The first term in the right hand side converges to $\phi(\gamma)$ when $x \to \gamma$ (here we use the fact that $\phi \circ \pi_\Gamma(0|\cdot)$ is continuous near γ). The third term goes to 0 (in light of the assumption on α_x). It remains to show that

$$\frac{1}{|\mathcal{B}_{\alpha_x}|}\int_{\mathcal{M}_x} |\nabla v| = o(1)\ ,\ x \to \gamma. \tag{26}$$

By Hölder's inequality, we have

$$\int_{\mathcal{M}_x} |\nabla v| \leq |\mathcal{M}_x|^{1-1/n} (\int_{\mathcal{M}_x} |\nabla v|^n)^{1/n}. \tag{27}$$

For any $y \in \Omega$ sufficiently close to γ we denote by $\pi_{\mathcal{H}_\gamma}(0|y)$ the unique point of $\mathcal{H}_\gamma$ of the form ty for some $t > 0$. Then, for x sufficiently close to γ, $\pi_{\mathcal{H}_\gamma}(0|y)$ is well defined for any $y \in \mathcal{B}_{\alpha_x}$. Moreover, we have

$$\mathcal{M}_x \subset \{ty + (1-t)\pi_{\mathcal{H}_\gamma}(0|y), y \in \mathcal{B}_{\alpha_x}\}.$$

Using the fact that $\pi_{\mathcal{H}_\gamma}(0|x) = \gamma$ and $0 \in \{\gamma + tn, t > 0\}$ where $-n$ is normal to $\mathcal{H}_\gamma$, we easily get $|\mathcal{M}_x| \leq C'|\mathcal{B}_{\alpha_x}||x - \gamma|$, for some constant $C' > 0$. Thus, (26) is true if and only if

$$\frac{|x-\gamma|^{1-1/n}|\mathcal{B}_{\alpha_x}|^{1-1/n}}{|\mathcal{B}_{\alpha_x}|} ||\nabla v||_{L^n(\mathcal{M}_x)} = o(1) \ , \ x \to \gamma.$$

Taking into account the fact that $|\mathcal{B}_{\alpha_x}| = \beta \alpha_x^{n-1}$ (where β depends only on n), this is equivalent to

$$(\frac{|x-\gamma|}{\alpha_x})^{1-1/n} ||\nabla v||_{L^n(\mathcal{M}_x)} = o(1) \ , \ x \to \gamma.$$

Since $\mathcal{M}_x \subset \mathcal{N}_x := \{ty + (1-t)\pi_\Gamma(0|y) : 0 \leq t \leq 1, y \in \mathcal{B}_{R_x}\}$, we may define

$$\alpha_x = |x - \gamma| ||\nabla v||_{L^n(\mathcal{N}_x)}.$$

(We may assume that $||\nabla v||_{L^n(\mathcal{N}_x)} > 0$ since otherwise, v is constant on a neighborhood of γ and the result is obvious). Then,

$$(\frac{|x-\gamma|}{\alpha_x})^{1-1/n} ||\nabla v||_{L^n(\mathcal{M}_x)} = ||\nabla v||_{L^n(\mathcal{N}_x)}^{1/n} = o(1)$$

and it is easy to check that $\alpha_x = o(R_x)$, $x \to \gamma$. This completes the proof of Lemma 15.

□

Lemma 16 *Let $v \in W^{1,p}(\Omega)$ satisfy (23). If Ω is locally $C^{1,\alpha}$ near $\gamma \in \Gamma$ for some $0 \leq \alpha \leq 1$ and $p \geq (n+\alpha)/(1+\alpha)$, then v is continuous at γ.*

Proof : We indicate here the minor modifications with respect to the proof of Lemma 15. The cone $\mathcal{C}$ now becomes a $C^{1,\alpha}$ paraboloid. More specifically, we define

$$\mathcal{C} := \{x \in B(\gamma, \rho) \setminus \{\gamma\} : |x - \gamma| \cos\theta \geq d(|x-\gamma| \sin\theta)^{1+\alpha}$$

$$\text{where} \quad \cos\theta = \langle \frac{x-\gamma}{|x-\gamma|}, n\rangle\}$$

for some $\rho, d > 0$ such that $\mathcal{C} \subset \Omega$. We now set

$$R_x := (\frac{|x-\gamma|}{d})^{1/(1+\alpha)} \quad \text{and} \quad \alpha_x := |x-\gamma|^{1/(1+\alpha)} ||\nabla v||_{L^p(\mathcal{N}_x)}.$$

Using now Holder's inequality with p instead of n in (27) we get the result. □

We now prove Theorem 5ii).

Corollary 5 *Assume that $\gamma \in \Gamma$ belongs to an $n-k$ dimensional face where $1 \le k \le n-1$. Then w is continuous at γ if one of the three following assumptions holds true:*

i) $k = 1$,

ii) F is coercive of order r with $r \ge k$,

iii) Ω is locally $C^{1,\alpha}$ near γ for some $0 \le \alpha \le 1$ and F is coercive of order r with $r \ge (k+\alpha)/(1+\alpha)$.

Proof: We may assume that γ is not an extreme point. Let Σ be an $n-k$ dimensional face such that $\gamma \in \Sigma$. We denote by $\operatorname{int}\Sigma$, the interior of Σ for the relative topology in the affine hull of Σ. For any $\gamma' \in \operatorname{int}\Sigma$, we consider the affine k dimensional plane $H_{\gamma'}$ perpendicular to Σ at γ'. By Lemma 14, $H_{\gamma'} \cap \Omega$ is not empty. Since $w \in W^{1,r}(\Omega)$ with $r \ge 1$, for almost every γ', the restriction of w to $H_{\gamma'} \cap \Omega$ belongs to $W^{1,r}(H_{\gamma'} \cap \Omega)$ and satisfies (19). Hence, Lemma 15 or Lemma 16 imply that $w_{|H_{\gamma'} \cap \Omega}$ is continuous. (Here, we use the fact that if Ω is locally $C^{1,\alpha}$ near γ', then $\Omega \cap H_{\gamma'}$ is locally $C^{1,\alpha}$ near γ'). Fix such a $\gamma' \in \operatorname{int}\Sigma$, $\gamma' \neq \gamma$. For any $x^\star \in H_{\gamma'} \cap \Omega$, $w_{|[\gamma', x^\star]}$ is continuous. We claim that

$$\lim_{y \in (\gamma', x^\star), y \to \gamma'} \frac{|\gamma - y|}{\mathrm{d}_\Gamma(\gamma|y)} = \frac{|\gamma - \gamma'|}{|\gamma - \bar{z}|} \tag{28}$$

where $\bar{z} \in \Sigma$ is defined by $\bar{z} = \gamma + \bar{t}(\gamma' - \gamma)$ with $\bar{t} = \max\{t > 0 : \gamma + t(\gamma' - \gamma) \in \Sigma\}$.

Indeed, let $(y_i) \subset (\gamma', x^\star)$ converging to γ'. Consider $z_i := \pi_\Gamma(\gamma|y_i)$. There exists $t_i > 0$ such that $z_i = \gamma + t_i(y_i - \gamma)$. Since $\gamma' \neq \gamma$, the sequence (t_i) is bounded, so that, up to a subsequence, converges to some $t \ge 0$. Then,

(z_i) converges to $z = \gamma + t(\gamma' - \gamma)$. We now prove that $t = \bar{t}$. Assume by contradiction that

$$\text{there exists } \ t' > t \ \text{ such that } \ \gamma + t'(\gamma' - \gamma) \in \Sigma. \tag{29}$$

Let $\Omega' := \Omega \cap \mathcal{P}$, $\Gamma' := \Gamma \cap \mathcal{P}$ where $\mathcal{P}$ is the 2 dimensional plane defined by the three points γ, γ' and $x^\star$. Then Ω' is a 2 dimensional convex set and $z_i \in \Gamma \cap \mathcal{P}$. Consider the 1 dimensional face of Γ', $\Sigma' := \Sigma \cap \mathcal{P}$. The assumption (29) implies that z belongs to the relative interior of Σ'. Hence, z_i belongs to the relative interior of Σ' for sufficiently large i. But this implies that $y_i \in [\gamma, z_i] \subset \Gamma$, a contradiction. Then $z = \bar{z}$ and the claim (28) is proven.

Since almost every $\gamma' \in \operatorname{int} \Sigma$ satisfies $w \in W^{1,r}(\Omega \cap H_{\gamma'})$, it follows from Fubini's Theorem and the use of spherical coordinates that there exists $f^\star \in \operatorname{int} \Sigma$ such that almost every $\gamma' \in (f^\star, \gamma)$ satisfies

$$w \in W^{1,r}(\Omega \cap H_{\gamma'}).$$

Hence, there exists a sequence $(\gamma_i) \subset (f^\star, \gamma)$ such that γ_i converges to γ and w is continuous on $[\gamma_i, x_i^\star]$ for some $x_i^\star \in H_{\gamma_i} \cap \Omega$.

Since w satisfies (23), for any $i \geq 1$ and any $y \in (x_i^\star, \gamma_i)$, we have

$$\limsup_{x \in \Omega, x \to \gamma} w(x) \leq w(y) + K \frac{|\gamma - y|}{\mathrm{d}_\Gamma(\gamma | y)}. \tag{30}$$

Then (28) (for $\gamma' = \gamma_i$) implies that for each $i \geq 1$,

$$\limsup_{x \in \Omega, x \to \gamma} w(x) \leq \phi(\gamma_i) + K \frac{|\gamma - \gamma_i|}{|\gamma - \bar{z}|} \tag{31}$$

where $\bar{z} \in \Sigma$ is defined by $\bar{z} = \gamma + \bar{t}(f^\star - \gamma)$ with $\bar{t} = \max\{t > 0 : \gamma + t(f^\star - \gamma) \in \Sigma\}$.

Finally, letting $i \to +\infty$ in (31), we get

$$\limsup_{x \in \Omega, x \to \gamma} w(x) \leq \phi(\gamma).$$

This completes the proof of Corollary 5.

□

Acknowledgement This paper was completed during a visit of the author in the department of mathematics of the university of Padua supported by a G.N.A.M.P.A. research project. The author warmly thanks Carlo Mariconda and Giulia Treu for very interesting comments and improvements on the original paper. He also thanks Francis Clarke for some extremely useful discussions.

References

[1] P. Bousquet, *On the Lower Bounded Slope Condition*, J. Convex Anal. 14, no 1, (2007), 119–136.

[2] P. Bousquet, *Régularité en Calcul des Variations. Espaces de Sobolev fractionnaires*, PhD Université Lyon 1, available on www.pierre.bousquet9.free.fr.

[3] P. Bousquet, F. Clarke, *Local Lipschitz continuity of solutions to a problem in the calculus of variations*, J. Differential Equations, 243(2), (2007), 489–503.

[4] F. Clarke, Yu. S. Ledyaev, R. J. Stern and P. R. Wolenski, N̈onsmooth analysis and control theory", Graduate Texts in Mathematics, 178. Springer-Verlag, New York, 1998.

[5] F. Clarke, *Continuity of solutions to a basic problem in the calculus of variations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 4, no. 3, (2005), 511–530.

[6] L.C. Evans, R.F. Gariepy "Measure theory and fine properties of functions", Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[7] M. Giaquinta, *Growth conditions and regularity, a counterexample*, Manuscripta Math. 59, no. 2, (1987), 245–248.

[8] D. Gilbarg, N. Trudinger, "Elliptic partial differential equations of second order", reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[9] E. Giusti, "Direct methods in the calculus of variations", World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[10] P. Hartman, *On the bounded slope condition*, Pacific J. Math. 18(3) (1966), 495–511.

[11] P. Hartman, *Convex sets and the bounded slope condition*, Pacific J. Math. 25 (1968) 511–522.

[12] O. Ladyzhenskaya, N. Uraltseva, "Linear and quasilinear elliptic equations", translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis Academic Press, New York-London, 1968.

[13] P. Marcellini, *Regularity for some scalar variational problems under general growth conditions*, J. Optim. Theory Appl. 90, no. 1 (1996), 161–181.

[14] C. Mariconda, G. Treu, *Gradient Maximum Principle for minima*, J. Optim. Theory Appl. 112 no. 1, (2002), 167–186.

[15] C. Mariconda, G. Treu, *Existence and Lipschitz regularity for minima*, Proc. Amer. Math. Soc. 130(2) 2002, 395–404.

[16] C. Mariconda, G. Treu, *Local Lipschitz regularity of minima for a scalar problem of the calculus of variations*, Contemp. Math. In press 2008.

[17] M. Miranda, *Un teorema di esistenza e unicità per il problema dell' area minima in n variabili*, Ann. Scuola Norm. Sup. Pisa, (3) 19 (1965) 233–249.

[18] Rockafellar, "Convex Analysis", reprint of the 1970 original. Princeton Landmarks in Mathematics. Princeton Paperbacks. Princeton University Press, Princeton, NJ, 1997.

[19] W. Ziemer, "Weakly differentiable functions. Sobolev spaces and functions of bounded variation", Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989.