Continuity of solutions of a problem in the calculus of variations

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17th December 2010

Abstract

We study the problem of minimizing $\int_\Omega L(x, u(x), Du(x))\, dx$ over the functions $u \in W^{1,p}(\Omega)$ that assume given boundary values ϕ on $\partial\Omega$. We assume that $L(x, u, Du) = F(Du) + G(x, u)$ and that F is convex. We prove that if ϕ is continuous and Ω is convex, then any minimum u is continuous on the closure of Ω. When Ω is not convex, the result holds true if $F(Du) = f(|Du|)$. Moreover, if ϕ is Lipschitz continuous, then u is Hölder continuous.

1 Introduction

We study the regularity of solutions to the following problem (P) in the multiple integral calculus of variations:

$$\min_u \int_\Omega L(x, u(u), Du(x))\, dx \quad \text{subject to} \quad u \in W_0^{1,1}(\Omega) + \phi,$$

where Ω is a bounded domain in $\mathbb{R}^n$, u is scalar-valued and $L : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is convex with respect to the last variable. Under suitable assumptions on L, one may invoke the *direct method* to deduce the existence of a solution u to problem (P). The issue then becomes the regularity of u. In particular, if u is continuous on $\overline{\Omega}$, the boundary conditions are assumed pointwise (rather than in the sense of trace).

Following the pioneering work of De Giorgi ([7]), Giaquinta and Giusti ([11, Theorem 3.1]) proved the Hölder continuity of any bounded solution u when ϕ is Hölder continuous and the Lagrangian L satisfies the growth condition

$$|\xi|^p - d \leq L(x, u, \xi) \leq c|\xi|^p + d \tag{1}$$

for some c, $d \in \mathbb{R}$ and for every $(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. The constrained growth of the same order p from both above and below has subsequently

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been weakened ([14, 8]). However, this condition is almost necessary to ensure *local* regularity: there exists a Lagrangian $L(x,u,\xi)=F(\xi)$ which is uniformly elliptic, satisfies

$$-d+|\xi|^2 \leq F(\xi) \leq c|\xi|^4+d\ ,\ \xi\in\mathbb{R}^n.$$

and for which the solution fails to be continuous on Ω (see [10, 14]). In such an example, though, the boundary function ϕ is itself discontinuous.

In this article, we follow another approach more relevant to Lagrangians which do not satisfy any upper growth condition: when some regularity properties are imposed on ϕ, this induces regularity of the solution u. We proceed to recall the main results that illustrate this principle. Consider first the case when $L(x,u,\xi)=F(\xi)$ is strictly convex, coercive and Ω is convex. If ϕ satisfies the so-called bounded slope condition (equivalently, ϕ is the restriction to $\partial\Omega$ of a convex function and the restriction of a concave function), then the solution of (P) is Lipschitz continuous (see [18]). When ϕ is the restriction of a convex function on $\mathbb{R}^n$, Clarke [5] has proved that the solution u is locally Lipschitz on Ω. Finally, when ϕ is merely continuous, the solution u is continuous on $\overline{\Omega}$ (see [5, 1, 16]).

These results have been extended in two different ways. On the one hand, when F is convex and superlinear but not necessarily *strictly* convex, several minima may exist. However, there exist a minimum of the minima and a maximum of the minima which satisfy the same regularity properties as above (see [4, 15, 16, 1]).

On the other hand, when L depends also on x and u, minima may have interior singularities (see [8, 9]). However, when Ω is convex and if L can be written as $L(x,u,\xi)=F(\xi)+G(x,u)$, with F uniformly convex (see (2) below) and G Lipschitz continuous, then any bounded minimum is

1) Lipschitz continuous when ϕ satisfies the bounded slope condition (see [21]),

2) locally Lipschitz continuous when ϕ is the restriction of a convex function (see [2]).

All these results naturally lead to the following conjecture. Assume that $L(x,u,\xi)=F(\xi)+G(x,u)$ is smooth and uniformly convex with respect to ξ, that Ω is smooth (but not necessarily convex), and that ϕ is smooth (but not necessarily the restriction of a convex function). Then any solution of (P) is continuous on $\overline{\Omega}$.

We observe in particular that the problem of the continuity of u when the boundary condition ϕ is continuous and G different from 0 is open. Theorem 1 below settles this question when Ω is convex. When L also depends on x and u, the Lagrangian is not convex and the comparison principle used in [1, 4, 5, 16, 17] does not apply.

We emphasize the fact that all the results quoted above were limited to convex domains Ω. In Theorem 2 below, we remove this restriction when F depends only on the norm of the gradient: if ϕ is continuous, then any bounded solution is continuous as well. This result is new even when $G = 0$.

We proceed now to state more specifically the main results of the article.

2 The main results

We first detail our assumptions. We denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$. We introduce the following geometric condition on Ω :

Definition 1 *1) We say that Ω satisfies the exterior sphere condition if for any $\gamma \in \Gamma := \partial\Omega$, there exist $x \in \mathbb{R}^n$ and $r > 0$ such that*

i) $|\gamma - x| = r$,

ii) $B(x, r) \subset \mathbb{R}^n \setminus \Omega$.

2) We say that Ω satisfies the uniform exterior sphere condition if there exists $r > 0$ such that for any $\gamma \in \Gamma$, there exists $x \in \mathbb{R}^n$ satisfying conditions i) and ii) above.

Any convex set satisfies the uniform exterior sphere condition for any $r > 0$. Any set of class $C^{1,1}$ satisfies the uniform sphere condition for some $r > 0$. On the contrary, for any $\alpha \in (0, 1)$, there exists a set of class $C^{1,\alpha}$ which does not satisfy the exterior sphere condition, e.g. $\{(x, y) \in \mathbb{R}^2 : y < |x|^{1+\alpha}\}$. However, the exterior sphere condition is not limited to convex or $C^{1,1}$ sets. For instance, it is satisfied by $\{(x, y) \in \mathbb{R}^2 : y < x^2 \sin(1/x)\}$ and also by $\mathbb{R}^2 \setminus (B((0, 0), 1) \cup B((2, 0), 1))$. In terms of nonsmooth analysis, Ω satisfies the exterior sphere condition if and only if the *proximal normal cone* of Ω at $\gamma \in \Gamma$ is non trivial, for any γ (see [6, Chapter 1] and also [20]).

In the following, the open set Ω is always assumed to be bounded. We now detail the assumptions on the Lagrangian $F : \mathbb{R}^n \to \mathbb{R}$.

Definition 2 *Let $F : \mathbb{R}^n \to \mathbb{R}$. We say that*

1) F is superlinear when $\displaystyle\lim_{|\xi|\to\infty} \frac{F(\xi)}{|\xi|} = +\infty$.

2) F is coercive of order $p > 1$ when there exist $a > 0$, $b \in \mathbb{R}$ such that $F(\xi) \geq a|\xi|^p + b$, for any $\xi \in \mathbb{R}^n$.

3) F is uniformly convex if there exists $\mu > 0$ such that for all $\theta \in (0, 1)$ and $\xi,\ \xi' \in \mathbb{R}^n$:

$$\theta F(\xi) + (1 - \theta)F(\xi') \geq F(\theta\xi + (1 - \theta)\xi') + (\mu/2)\theta(1 - \theta)|\xi - \xi'|^2. \quad (2)$$

We remark that when F is of class C^2, F is uniformly convex if and only if, for every $\xi \in \mathbb{R}^n$, we have

$$\langle z, \nabla^2 F(\xi) z \rangle \geq \mu |z|^2 \quad \forall z \in \mathbb{R}^n. \tag{3}$$

When F is uniformly convex, F is coercive of order 2.

When F is convex, $\displaystyle\int_\Omega F(Dw)$ is well-defined (possibly as $+\infty$) for any $w \in W^{1,1}(\Omega)$. Moreover, $\displaystyle\int_\Omega F(Dw) < +\infty$ implies that $Dw \in L^p(\Omega)$ when F is coercive of order $p > 1$.

Assumption (H) Let $G : \Omega \times \mathbb{R} \to \mathbb{R}$. We introduce the following assumption (H) for G:

1) $G(x, u)$ is measurable in x and differentiable in u,

2) for every T in $\mathbb{R}$, there is a constant $\chi = \chi(T)$ such that for almost all $x \in \Omega$,

$$|G(x, u) - G(x, u')| \leq \chi |u - u'| \quad \forall\, u, u' \in [-T, T], \tag{4}$$

3) there exists a bounded function b such that $\displaystyle\int_\Omega G(x, b(x))\, dx$ is well-defined and finite.

By 2) and 3), $\displaystyle\int_\Omega G(x, u(x))\, dx$ is well-defined and finite for any $u \in L^\infty(\Omega)$.

In problem (P), the Dirichlet boundary condition is defined by $u \in W^{1,1}_\phi(\Omega)$, where $\phi \in W^{1,1}_{loc}(\mathbb{R}^n)$. Here, $W^{1,1}_\phi(\Omega)$ is the set of those $u \in W^{1,1}(\Omega)$ such that the map

$$\left\{ \begin{array}{l} u(x) \text{ if } x \in \Omega, \\ \phi(x) \text{ if } x \in \mathbb{R}^n \setminus \Omega \end{array} \right. \quad \text{belongs to } \; W^{1,1}_{loc}(\mathbb{R}^n). \tag{5}$$

Hence, the boundary condition is well-defined for *any* bounded open set Ω.

When F is convex and G satisfies (H), $I(u) = \displaystyle\int_\Omega F(Du) + G(x, u)$ is well-defined (possibly as $+\infty$) on $W^{1,1}_\phi(\Omega) \cap L^\infty(\Omega)$. We say that an admissible map u solves (P) relative to $L^\infty(\Omega)$ if u is itself bounded, $I(u) < +\infty$ and if we have $I(u) \leq I(v)$ for all bounded v that are admissible for (P).

We now state our first main result:

Theorem 1 *We assume that Ω is convex, F is uniformly convex and G satisfies (H). Then any solution u of (P) relative to $L^\infty(\Omega)$ is*

1) continuous on $\overline{\Omega}$ if ϕ is continuous, *2) Hölder continuous of order $\alpha = 1/(n+1)$ if ϕ is Lipschitz continuous.*

This generalizes [1, Theorem 2] and [16, Theorem 4.5] to the case when the Lagrangian also depends on (x, u). We remark that Stampacchia [21] has described structural assumptions on G which guarantee *a priori* the existence and boundedness of solutions of (P). Under these assumptions, one may extend Theorem 1 to the case when G is not differentiable in u (exactly as in [2], section 3).

When Ω is not convex, we do not know whether Theorem 1 remains true, except in one case, namely when F depends only on the norm of the gradient.

Theorem 2 *We assume that F is uniformly convex, that G satisfies (H), and there exists $f : [0, \infty) \to \mathbb{R}$ such that $F(\xi) = f(|\xi|)$ for any $\xi \in \mathbb{R}^n$. Then any solution u of (P) relative to $L^\infty(\Omega)$ is*

1) continuous on $\overline{\Omega}$ if ϕ is continuous and Ω satisfies an exterior sphere condition,

2) Hölder continuous of order $\alpha = 1/(n+1)$ if ϕ is Lipschitz continuous and Ω satisfies a uniform exterior sphere condition.

We remark that the map f appearing in the statement of Theorem 2 is necessarily convex and nondecreasing.

Theorem 2 is new even when $G = 0$. As a matter of fact, in that case, it is enough to assume that $F(\xi) = f(|\xi|)$ is convex and superlinear. Since F is not necessarily *strictly* convex, several minima may exist. Mariconda and Treu [15] have proved however that the set of minima has a minimum. This means that there exists a minimum u of (P) and $u \leq v$ a.e. for any other minimum v. Symmetrically, there exists a maximum of the minima. We then have:

Theorem 3 *We assume that $G = 0$, F is convex and there exists f : $[0, +\infty) \to \mathbb{R}$ such that $F(\xi) = f(|\xi|)$, $\xi \in \mathbb{R}^n$.*

1) If f is superlinear, $\phi : \mathbb{R}^n \to \mathbb{R}$ is continuous and Ω satisfies the exterior sphere condition, then the minimum and the maximum of the solutions of (P) are continuous on $\overline{\Omega}$.

2) If f is coercive of order $p > 1$, $\phi : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous and Ω satisfies a uniform exterior sphere condition, then the minimum and the maximum of the solutions of (P) are Hölder continuous of order α with

$$\alpha := \frac{p-1}{n+p-1}.$$

We have not been able to prove the continuity of *any* solution of (P). However, if we assume further that the projections of the faces of the epigraph of f on $\mathbb{R}^n$ are uniformly bounded, the difference of two minima is a Lipschitz continuous function. This implies that any solution is continuous on $\overline{\Omega}$ (see [16, Theorem 4.7] for details).

When $F = f(|\cdot|)$ is not superlinear, the continuity of ϕ is not enough to guarantee the continuity of u, as shown by the minimum area problem: $f(t) = \sqrt{1+t^2}$.

Even if f is coercive, regularity and/or geometric assumptions on the domain must be introduced somehow to obtain the continuity of the minima: we know from classical potential theory ($f(t) = t^2$) that the point at the tip of a sharp inward-pointing spine of a domain is not a regular boundary point. However, it is not clear what the exact regularity of Ω and ϕ required in Theorem 3 should be.

Remark 1 *In this article, $W_0^{1,p}(\Omega)$ is the set of those functions $u \in W^{1,p}(\Omega)$ that we may extend by 0 to get a map in $W^{1,p}(\mathbb{R}^n)$. Alternatively, $W_0^{1,p}(\Omega)$ could have been defined as the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$. The first definition seems to be more convenient to prove Proposition 1 below. In general, these two definitions are not equivalent, so that the sets of admissible functions for the problem (P) may differ. However, both definitions of $W_0^{1,p}(\Omega)$ coincide in the case of Lipschitz domains (in particular, when Ω is convex) or when Ω satisfies a uniform exterior sphere condition and $1 < p$. A similar remark holds true for the set $W_\phi^{1,p}(\Omega)$.*

In the following section, we introduce the main tool of the paper: a comparison principle, and several variants to treat the nonconvex term G. In section 4, we prove Theorem 1. Section 5 is devoted to the case when Ω is an annulus. This specific case is useful to prove Theorem 3 and Theorem 2 in sections 6 and 7 respectively.

3 Several variants of the comparison principle

The three lemmas below give important tools to derive the comparison principles needed in the proofs or our regularity results.

To begin with, we introduce the framework of this section. Let Ω be a bounded open set in $\mathbb{R}^n$ and $\phi \in W_{loc}^{1,1}(\mathbb{R}^n)$. We recall that $W_\phi^{1,1}(\Omega)$ is the set of those functions $u \in W^{1,1}(\Omega)$ such that $\overline{u}^\phi \in W_{loc}^{1,1}(\mathbb{R}^n)$ where

$$\overline{u}^\phi := \begin{cases} u(x) \text{ if } x \in \Omega, \\ \phi(x) \text{ if } x \notin \Omega. \end{cases}$$

We begin with the following

Lemma 1 *Let Ω_1, Ω_2 be two bounded open subsets of $\mathbb{R}^n$ such that $\Omega_1 \cap \Omega_2 \neq \emptyset$. Let $G_1 : \Omega_1 \times \mathbb{R} \to \mathbb{R}$, $G_2 : \Omega_2 \times \mathbb{R} \to \mathbb{R}$ satisfy (H). Let $\phi_1, \phi_2 \in W_{loc}^{1,1}(\mathbb{R}^n)$.*

We define for $i = 1, 2$,

$$I_i(u) := \int_{\Omega_i} F(Du(x)) + G_i(x, u(x))\, dx$$

For $i = 1, 2$, let $u_i \in W^{1,1}_{\phi_i}(\Omega_i)$ be a minimum of I_i on $W^{1,1}_{\phi_i}(\Omega_i)$ relative to $L^\infty(\Omega_i)$.

We assume that there exists $\alpha_0 \in \mathbb{R}$ such that $\overline{u_1}^{\phi_1} \leq \overline{u_2}^{\phi_2} + \alpha_0$ a.e. on $\mathbb{R}^n \setminus (\Omega_1 \cap \Omega_2)$. Let $\alpha \geq \alpha_0$ and $A := \{x \in \Omega_1 \cap \Omega_2 :\ u_1(x) > u_2(x) + \alpha\}$. When F is uniformly convex, we have

$$\mu \int_A |Du_1 - Du_2|^2 \leq \int_A (G_{1,u}(x, u_1) - G_{2,u}(x, u_2))(u_2 + \alpha - u_1), \qquad (6)$$

where μ is given by (2). Here, $G_{i,u}$, $i \in \{1, 2\}$, is the derivative of G_i with respect to u.

Proof: Let $\alpha \geq \alpha_0$ and $t \in [0, 1]$. We define

$$v_1(x) := \begin{cases} tu_1(x) + (1-t)\min(u_1(x), u_2(x) + \alpha) & \text{if } x \in \Omega_1 \cap \Omega_2, \\ u_1(x) & \text{if } x \in \Omega_1 \setminus \Omega_2. \end{cases} \qquad (7)$$

We claim that $v_1 \in W^{1,1}_{\phi_1}(\Omega_1)$. Indeed,

$$\begin{aligned} v_1(x) - u_1(x) &= (1-t)\begin{cases} \min(u_2(x) + \alpha - u_1(x), 0) & \text{if } x \in \Omega_1 \cap \Omega_2, \\ 0 & \text{if } x \in \Omega_1 \setminus \Omega_2 \end{cases} \\ &= (1-t)\min(\overline{u_2}^{\phi_2} + \alpha - \overline{u_1}^{\phi_1}, 0)(x) \end{aligned}$$

where

$$\overline{u_1}^{\phi_1} = \begin{cases} u_1 & \text{if } x \in \Omega_1, \\ \phi_1 & \text{if } x \in \mathbb{R}^n \setminus \Omega_1 \end{cases}, \quad \overline{u_2}^{\phi_2} = \begin{cases} u_2 & \text{if } x \in \Omega_2, \\ \phi_2 & \text{if } x \in \mathbb{R}^n \setminus \Omega_2 \end{cases}.$$

Since $\overline{u_1}^{\phi_1}$ and $\overline{u_2}^{\phi_2}$ belong to $W^{1,1}_{loc}(\mathbb{R}^n)$ and $\min(\overline{u_2}^{\phi_2} + \alpha - \overline{u_1}^{\phi_1}, 0) = 0$ on $\mathbb{R}^n \setminus \Omega_1$, we have $v_1 \in W^{1,1}_{\phi_1}(\Omega_1)$. Similarly, we define

$$v_2(x) := \begin{cases} tu_2(x) + (1-t)\max(u_2(x), u_1(x) - \alpha) & \text{if } x \in \Omega_1 \cap \Omega_2, \\ u_2(x) & \text{if } x \in \Omega_2 \setminus \Omega_1. \end{cases} \qquad (8)$$

Then $v_2 \in W^{1,1}_{\phi_2}(\Omega_2)$. Since u_1 and u_2 are minima, we have $I_1(u_1) \leq I_1(v_1)$ and $I_2(u_2) \leq I_2(v_2)$.

Now, we write $I_1(u_1) + I_2(u_2) \leq I_1(v_1) + I_2(v_2)$. This gives

$$\begin{aligned} &\int_A F(Du_1) + G_1(x, u_1) + \int_A F(Du_2) + G_2(x, u_2) \\ &\qquad \leq \int_A F(tDu_1 + (1-t)Du_2) + G_1(x, tu_1 + (1-t)(u_2 + \alpha)) \\ &\qquad + \int_A F(tDu_2 + (1-t)Du_1) + G_2(x, tu_2 + (1-t)(u_1 - \alpha)). \end{aligned}$$

We recall that $A = \{x \in \Omega_1 \cap \Omega_2 : u_1(x) > u_2(x) + \alpha\}$. We write $F(Du_1) = tF(Du_1) + (1-t)F(Du_1)$ and similarly for $F(Du_2)$. By using the uniform convexity of F (see (2)), we get

$$\int_A F(Du_1) + F(Du_2) - F(tDu_1 + (1-t)Du_2) - F(tDu_2 + (1-t)Du_1) \geq \mu t(1-t) \int_A |Du_1 - Du_2|^2.$$

This gives

$$\mu t(1-t) \int_A |Du_1 - Du_2|^2 \leq \int_A G_1(x, tu_1 + (1-t)(u_2+\alpha)) - G_1(x, u_1) + \int_A G_2(x, tu_2 + (1-t)(u_1 - \alpha)) - G_2(x, u_2). \tag{9}$$

We divide (9) by $(1-t)$ and let $t \to 1$. In the right hand side, we may justify the use of the dominated convergence theorem with (H). We get

$$\mu \int_A |Du_1 - Du_2|^2 \leq \int_A (G_{1,u}(x, u_1) - G_{2,u}(x, u_2))(u_2 + \alpha - u_1).$$

This completes the proof of Lemma 1.

□

We may obtain a plain comparison principle in the following situation :

Lemma 2 *Let Ω_i, F, G_i, ϕ_i, u_i be as in Lemma 1. We further assume that $\Omega_1 \subset \Omega_2$ and*

$$G_2(x,u) = \begin{cases} G_{1,u}(x, u_1(x))u \text{ if } x \in \Omega_1, \\ 0 \text{ otherwise.} \end{cases}$$

Then $u_1 \leq u_2 + \alpha_0$ a.e. on Ω_1.

Proof: By (6) with $\alpha = \alpha_0$ and the fact that $G_{2,u}(x, u_2(x)) = G_{1,u}(x, u_1(x))$ when $x \in \Omega_1$, we get $\mu \int_A |Du_1 - Du_2|^2 \leq 0$. Hence

$$\int_{\Omega_1} |Du_1 - D[\min(u_1, u_2 + \alpha_0)]|^2 = 0.$$

Since $u_1 - \min(u_1, u_2 + \alpha_0) \in W_0^{1,1}(\Omega_1)$, this implies $u_1 = \min(u_1, u_2 + \alpha_0)$, so that $u_1 \leq u_2 + \alpha_0$ a.e. on Ω_1.

□

When $\Omega_1 = \Omega_2$, $\phi_1 = \phi_2$ and $\alpha_0 = 0$, Lemma 2 implies that a minimum u_1 of $I_1(u) = \int_{\Omega_1} F(Du) + G_1(x,u)$ on $W_{\phi_1}^{1,1}(\Omega_1)$ is also the unique minimum of $u \mapsto \int_{\Omega_1} F(Du) + g_1(x)u$, where $g_1(x) = G_{1,u}(x, u_1(x))$. We observe that in this last problem, the Lagrangian is now convex in (u, Du).

We now introduce some notation to state Lemma 3 below. Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be an affine map. We say that H is a rotation if there exists a 2 dimensional plane P such that H is a rotation on P and is equal to the identity on the orthogonal of P. In that case, we denote by $|Id - H|$ the distance (with respect to the Euclidean norm) between Id and the linear part of the affine map H. When H is a translation of vector τ, we denote by $|Id - H|$ the norm of τ. We have

Lemma 3 *Let Ω_i, G_i, ϕ_i, u_i, α_0 and F be as in Lemma 1. We further assume that there exists $H : \mathbb{R}^n \to \mathbb{R}^n$ such that*

- *H is a translation or a rotation,*
- *$\Omega_2 = H^{-1}(\Omega_1)$, $G_2(x, u) = G_1(H(x), u)$, $\phi_2 = \phi_1 \circ H$ and $u_2 = u_1 \circ H$.*

Then there exists $q > 0$ such that

$$u_1(x) - u_1(Hx) \leq q|Id - H| + \alpha_0 \quad a.e. \quad x \in \Omega_1 \cap \Omega_2. \tag{10}$$

The constant q depends on n, Ω_1 and χ/μ where χ is such that (4) holds true for $T = |u_1|_{L^\infty(\Omega_1)}$, namely the sup norm of u_1 on Ω_1.

Proof: Let $\alpha \geq \alpha_0$. We define $v := u_2 + \alpha - u_1$ on $\Omega_1 \cap \Omega_2$ and $w := \min(\overline{u_2}^{\phi_2} + \alpha - \overline{u_1}^{\phi_1}, 0) = \begin{cases} \min(v, 0) \text{ on } \Omega_1 \cap \Omega_2, \\ 0 \text{ on } \mathbb{R}^n \setminus (\Omega_1 \cap \Omega_2) \end{cases}$. We have

$$\int_A (G_{1,u}(x, u_1) - G_{2,u}(x, u_2))(u_2 + \alpha - u_1) = \int_{\Omega_1} G_{1,u}(x, u_1)w - \int_{\Omega_2} G_{2,u}(x, u_2)w$$

$$= \int_{\Omega_1} G_{1,u}(x, u_1(x))w(x)\, dx - \int_{\Omega_1} G_{1,u}(x, u_1(x))w(H^{-1}(x))\, dx. \tag{11}$$

If H is a translation, we define $\gamma(t) = tId + (1 - t)H^{-1}$. If H is a rotation of angle θ, we define $\gamma(t) = H_{(t-1)\theta}$ (where we denote by H_λ, $\lambda \in \mathbb{R}$ the rotation of angle λ in the same plane P as H and with the same center).

Then by (11)

$$\int_A (G_{1,u}(x, u_1) - G_{2,u}(x, u_2))(u_2 + \alpha - u_1)$$
$$= \int_{\Omega_1} G_{1,u}(x, u_1)\, dx \int_0^1 Dw(\gamma(t)(x))(\dot{\gamma}(t)(x))\, dt$$
$$\leq C\, \chi |Id - H| \int_{\Omega_1} dx \int_0^1 |Dw(\gamma(t)(x))|\, dt$$

where $C = C(n, \Omega_1)$.

Now, for any $t \in (0,1)$,

$$\int_{\Omega_1} |Dw(\gamma(t)(x))|\, dx \le \int_{\mathbb{R}^n} |Dw(y)|\, dy = \int_A |Du_1 - Du_2|.$$

Hence,

$$\int_A (G_{1,u}(x,u_1) - G_{2,u}(x,u_2))(u_2 + \alpha - u_1) \le C\,\chi |Id - H| \int_A |Du_1 - Du_2|.$$

In view of (6), we then get

$$\mu \int_A |Du_1 - Du_2|^2 \le C\,\chi |Id - H| \int_A |Du_1 - Du_2|,$$

so that

$$|Du_1 - Du_2|_{L^2(A)} \le C\frac{\chi}{\mu}|A|^{1/2}|Id - H|. \tag{12}$$

By Hölder's inequality,

$$|u_1 - u_2 - \alpha|_{L^1(A)} \le |A|^{1-1/2^*}|u_1 - u_2 - \alpha|_{L^{2^*}(A)},$$

with $2^* = 2n/(n-2)$ if $n > 2$ and 2^* any number > 2 if $n = 2$. We then apply Sobolev's lemma to the function $w \in W_0^{1,1}(\Omega_1)$ to obtain

$$|u_1 - u_2 - \alpha|_{L^1(A)} \le S|A|^{1-1/2^*}|Du_1 - Du_2|_{L^2(A)}$$

for some constant $S > 0$ which depends only on n (and on Ω_1 when $n = 2$). In view of (12), this implies

$$|u_1 - u_2 - \alpha|_{L^1(A)} \le CS\frac{\chi}{\mu}|Id - H||A|^{\gamma} \tag{13}$$

with $\gamma := 1 - 1/2^* + 1/2 = 1 + 1/n > 1$.

If $H = Id$, (10) is obvious. Otherwise we define $q_0 := \alpha_0/|Id - H|$ and for any $q \ge q_0$, we denote by $A(q) := \{x \in \Omega_1 \cap \Omega_2 : u_1(x) \ge u_2(x) + q|Id - H|\}$, $\rho(q) = |A(q)|$. By Fubini Theorem and (13), we get for $q \ge q_0$

$$\int_q^{+\infty} \rho(q')\, dq' = \frac{1}{|Id - H|}\int_{A(q)} (u_1 - u_2 - q|Id - H|) \le CS\frac{\chi}{\mu}\rho(q)^{\gamma}.$$

Then (see [13, Lemma 7.2]) $\rho(q) = 0$ for $q \ge q_0 + (n+1)CS\frac{\chi}{\mu}|\Omega_1|^{1/n}$. Hence, for a.e. $x \in \Omega_1$,

$$u_1(x) \le u_1(Hx) + (q_0 + (n+1)CS\frac{\chi}{\mu}|\Omega_1|^{1/n})|Id - H| = u_1(Hx) + \alpha_0 + q|Id - H|$$

for some $q = q(n, \Omega_1, \frac{\chi}{\mu})$. This completes the proof of Lemma 3.

$\square$

Remark 2 *When the map H is a rotation, if we assume further that $\Omega := \Omega_1 = \Omega_2$ is Lipschitz continuous, $\phi_1, \phi_2 \in C^0(\mathbb{R}^n) \cap W^{1,1}_{loc}(\mathbb{R}^n)$, then it is enough to assume that $\phi_1 \leq \phi_2 + \alpha_0$ on $\partial\Omega_1$.*

To prove this, we introduce $\psi_1 := \min(\phi_1, \phi_2 + \alpha_0)$ which belongs to $C^0(\mathbb{R}^n) \cap W^{1,1}_{loc}(\mathbb{R}^n)$. Since $\psi_1 - \phi_1 = 0$ on $\partial\Omega$, $(\psi_1 - \phi_1)|_\Omega \in \overline{C^\infty_c(\Omega)}^{W^{1,1}} = W^{1,1}_0(\Omega)$ (here, we use that Ω is Lipschitz). Hence, $W^{1,1}_{\phi_1}(\Omega) = W^{1,1}_{\psi_1}(\Omega)$. Since $\psi_1 \leq \phi_2 + \alpha_0$ on $\mathbb{R}^n$, we may apply Lemma 3 with ψ_1 and ϕ_2 to obtain $u_1 \leq u_2 + \alpha_0$ on Ω.

Here is the corresponding version of Lemma 1 when $G_1 = G_2 = 0$ and F is merely convex.

Lemma 4 *Let Ω_1, Ω_2 be two bounded open subsets of $\mathbb{R}^n$ such that $\Omega_1 \cap \Omega_2 \neq \emptyset$. Let $\phi_1, \phi_2 \in W^{1,1}_{loc}(\mathbb{R}^n)$. We define for $i = 1$, 2,*

$$I_i(u) := \int_{\Omega_i} F(Du(x))\, dx$$

For $i = 1, 2$, let $u_i \in W^{1,1}_{\phi_i}(\Omega_i)$ be a minimum of I_i on $W^{1,1}_{\phi_i}(\Omega_i)$, $i = 1$, 2.

We assume that there exists $\alpha_0 \in \mathbb{R}$ such that $\overline{u_1}^{\phi_1} \leq \overline{u_2}^{\phi_2} + \alpha_0$ a.e. on $\mathbb{R}^n \setminus (\Omega_1 \cap \Omega_2)$.

When F is convex and either u_1 is the minimum of the minima or u_2 is the maximum of the minima, we have

$$u_1 \leq u_2 + \alpha_0 \quad on \quad \Omega_1 \cap \Omega_2. \tag{14}$$

Proof : Assume for instance that u_1 is the minimum of the minima. Define v_1 and v_2 as in the proof of Lemma 1 (see (7), (8)) with $t = 0$ and $\alpha = \alpha_0$. We get

$$\int_A F(Du_1) \leq \int_A F(Du_2) \quad \text{and} \quad \int_A F(Du_2) \leq \int_A F(Du_1),$$

where the set A is defined by $A := \{x \in \Omega_1 \cap \Omega_2 : u_1(x) > u_2(x) + \alpha_0\}$. This implies $I_1(u_1) = I_1(\min(u_1, v_1))$. Hence, $\min(u_1, v_1)$ is another solution on $W^{1,1}_{\phi_1}(\Omega_1)$. Since u_1 is the minimum of the solutions, we get $u_1 \leq \min(u_1, v_1)$, so that $u_1 \leq u_2 + \alpha_0$ on $\Omega_1 \cap \Omega_2$. This implies Lemma 4.

□

In the following proposition, we consider a solution u of (P); that is, u is a minimum of $I(u) = \displaystyle\int_\Omega F(Du) + G(x, u)$ on $W^{1,1}_\phi(\Omega)$. We prove that if u is continuous at the boundary, then u is continuous on $\overline{\Omega}$.

Proposition 1 *Let $\phi \in W^{1,1}_{loc}(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$. Assume either that* *a) F is uniformly convex, G satisfies (H), u is a solution of (P) on $W^{1,1}_{\phi}(\Omega)$ relative to $L^{\infty}(\Omega)$,*

b) or F is convex and superlinear, $G = 0$ and u is the minimum or the maximum of the minima of I on $W^{1,1}_{\phi}(\Omega)$.

Then for any Lebesgue points x and y of u, we have

$$|u(x) - u(y)| \leq C\left\{|x-y| + \alpha(|x-y|)\right\}, \tag{15}$$

where $\alpha(t) := \max\Big(\sup_{\substack{z\in J, b\in\mathbb{R}^n\setminus\Omega\\ |z-b|\leq t}} |u(z)-\phi(b)|, \sup_{\substack{b,b'\in\mathbb{R}^n\setminus\Omega\\ |b-b'|\leq t}} |\phi(b)-\phi(b')|\Big)$. *Here, J is the set of Lebesgue points of u in Ω.*

The main tool of the proof is Lemma 3 when the map H is a translation. The method of translations has already been used to obtain a similar result when $G = 0$ and Ω is convex (see [18]) or when G is convex and $\Omega\cap(\Omega - h)$ is regular for all $h \in \mathbb{R}^n$ (see [17]). This last assumption is quite restrictive: For instance, if $\Omega := B(0,3)\setminus\overline{B}(0,1) \subset \mathbb{R}^2$ and $h = (2,0)$, the set $\Omega\cap(\Omega-h)$ is not Lipschitz. This is also the reason why we have introduced the space $W^{1,1}_{\phi}(\Omega)$, with $\phi \in W^{1,1}_{loc}(\mathbb{R}^n)$. This set is well defined for any open set Ω and seems to be more convenient to get (15) without any further assumption.

When G is not convex, there exists a corresponding version of Proposition 1 in the framework of nonlinear elliptic equations, when the set of admissible maps is a subset of Lipschitz continuous functions, see [13, Lemma 10.0]. The proof below shares also some ideas with the proof of [16, Theorem 4.5].

Proof of Proposition 1: a) We first consider the case when F is uniformly convex and G satisfies (H). Let $\bar{x}$, $\bar{y}$ be two Lebesgue points of u and $\tau := \bar{y} - \bar{x}$. We introduce $\Omega_\tau := \Omega - \tau$, $\phi_\tau(x) := \phi(x+\tau)$, $G_\tau(x,v) = G(x+\tau, v)$. It follows from an obvious change of variables that $u_\tau(x) = u(x+\tau)$ is a minimum of

$$I_\tau(v) = \int_{\Omega_\tau} F(Dv(x)) + G(x, v(x))\, dx$$

on $W^{1,1}_{\phi_\tau}(\Omega_\tau)$ relative to $L^{\infty}(\Omega_\tau)$.

In view of the definition of $\alpha(|\tau|)$ and the fact that

$$\mathbb{R}^n \setminus (\Omega\cap\Omega_\tau) = (\mathbb{R}^n\setminus(\Omega\cup\Omega_\tau)) \cup (\Omega\setminus\Omega_\tau)\cup(\Omega_\tau\setminus\Omega),$$

we have $\overline{u}^{\phi} \leq \overline{u_\tau}^{\phi_\tau} + \alpha(|\tau|)$ a.e. on $\mathbb{R}^n\setminus(\Omega\cap\Omega_\tau)$. Indeed, for any $x \in \mathbb{R}^n\setminus(\Omega\cup\Omega_\tau)$, $\phi(x) - \phi(x+\tau) \leq \alpha(|\tau|)$. Hence, $\overline{u}^{\phi}(x) - \overline{u_\tau}^{\phi_\tau}(x) \leq \alpha(|\tau|)$ for a.e. $x \in \mathbb{R}^n\setminus(\Omega\cup\Omega_\tau)$. Similarly, a.e $x \in \Omega\setminus\Omega_\tau$ belongs to J, so that $u(x) - \phi(x+\tau) \leq \alpha(|\tau|)$; that is, $\overline{u}^{\phi}(x) - \overline{u_\tau}^{\phi_\tau}(x) \leq \alpha(|\tau|)$. The same inequality holds true for a.e. $x \in \Omega_\tau\setminus\Omega$.

By Lemma 3 (see (10)), there exists $q > 0$ such that

$$u(x) - u(x+\tau) \leq q|\tau| + \alpha(|\tau|), \ \textit{a.e.}\ x \in \Omega \cap (\Omega - \tau). \tag{16}$$

The constant q depends on n, Ω, and χ/μ, with μ given by (2) and $\chi = \chi(|u|_{L^\infty(\Omega)})$ given by (4). Since $\bar{x}$, $\bar{y}$ are Lebesgue points of u, inequality (16) holds true for $x = \bar{x}$. This implies (15) when F is uniformly convex.

b) We now consider the case when F is merely convex and superlinear and $G = 0$.

Assume for instance that u is the minimum of the minima. Once again, we fix $\bar{x}, \bar{y} \in J$ and define as above τ, Ω_τ, ϕ_τ, u_τ (which is the minimum of the minima of I_τ on $W^{1,1}_{\phi_\tau}(\Omega_\tau)$.) Then, we still have $\overline{u}^\phi \leq \overline{u_\tau}^{\phi_\tau} + \alpha(|\tau|)$, a.e. on $\mathbb{R}^n \setminus (\Omega \cap \Omega_\tau)$. By Lemma 4 (see (14)),

$$u(x) \leq u_\tau(x) + \alpha(|\tau|) \ \text{ a.e. } x \in \Omega \cap \Omega_\tau.$$

We may now complete the proof of Proposition 1 b) as above.

□

Remark 3 *Assume that (15) is satisfied, $\phi \in W^{1,1}_{loc}(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ and u is continuous at the boundary in the following sense:*

$$\lim_{\substack{x \to \gamma \\ x \in J}} u(x) = \phi(\gamma) \quad \gamma \in \Gamma, \tag{17}$$

(as in Proposition 1, J is the set of Lebesgue points of u in Ω). Then u is continuous on $\overline{\Omega}$.

To prove Remark 3, one may assume without loss of generality that ϕ is uniformly continuous (modifying ϕ outside a neighborhood of Ω does not modify the set $W^{1,1}_\phi(\Omega)$). By (17) and the compacity of $\overline{\Omega}$, we have

$$\lim_{r \to 0} \sup_{\substack{z \in J, b \in \mathbb{R}^n \setminus \Omega \\ |z-b| \leq r}} |u(z) - \phi(b)| = 0. \tag{18}$$

Then $\lim_{t \to 0} \alpha(t) = 0$, where

$$\alpha(t) = \max\left(\sup_{\substack{z \in J, b \in \mathbb{R}^n \setminus \Omega \\ |z-b| \leq t}} |u(z) - \phi(b)|, \sup_{\substack{b, b' \in \mathbb{R}^n \setminus \Omega \\ |b-b'| \leq t}} |\phi(b) - \phi(b')| \right).$$

By (15), this implies that u has a representative which is uniformly continuous on $\overline{\Omega}$.

4 Proof of Theorem 1

We first prove the continuity of a solution u at the boundary. To do so, we use the technique of barriers. Here, the barriers are the solutions of auxiliary variational problems on domains which are larger than Ω. Fix $\gamma \in \Gamma$. Since Ω is convex, there exists an open hypercube Λ such that:

- $\Lambda \supset \Omega$,
- γ is the center of an $n-1$ dimensional face of Λ.

We now introduce a function ψ, the definition of which depends on the regularity properties of ϕ.

When ϕ is merely continuous, let $\epsilon > 0$ and

$$\psi(x) := \phi(\gamma) + \epsilon + a|x-\gamma|^2, \tag{19}$$

where

$$a := \max\left(\frac{\chi}{n\mu}, \max_{x\in\overline{\Lambda}, x\neq\gamma} \frac{\phi(x)-\phi(\gamma)-\epsilon}{|x-\gamma|^2}\right).$$

The constant μ is given by (2) while $\chi = \chi(|u|_{L^\infty(\Omega)})$ is given by (4). Since ϕ is continuous, $a < \infty$.

When ϕ is Lipschitz continuous, let Q be the Lipschitz rank of ϕ and

$$\psi(x) = \phi(\gamma) + Q|x-\gamma| + a|x-\gamma|^2 \tag{20}$$

where $a := \frac{\chi}{n\mu}$.

In both cases, $\psi \geq \phi$ on $\overline{\Lambda}$, and ψ is smooth and convex on Λ. Moreover, in the sense of quadratic forms,

$$\nabla^2\psi(x) \geq 2a\, Id \quad \forall\, x \in \Lambda. \tag{21}$$

We now define

$$g(x) := \begin{cases} G_u(x, u(x)) \text{ if } x \in \Omega, \\ 0 \text{ otherwise.} \end{cases}$$

Then $|g|_{L^\infty(\Omega)} \leq \chi$.

Consider the following auxiliary problem (P_0):

$$\text{Minimize } \ v \mapsto \int_\Lambda F(Dv(x)) + g(x)v(x)\, dx \quad v \in W^{1,1}_\psi(\Lambda).$$

We observe that the domain Λ is a convex polyhedron and the boundary condition ψ is convex. Such variational problems have been thoroughly studied in [21, 2]. We proceed to recall some *a priori* bounds satisfied by the solution of (P_0).

In view of the uniform convexity of F, the direct method in the calculus of variations gives a (unique) solution $v \in W^{1,1}_\psi(\Lambda)$. Moreover, there exists T_0 such that (see [21, Theorem 6.1, Theorem 6.2])

$$|v|_{W^{1,2}(\Lambda)} + |v|_{L^\infty(\Lambda)} \leq T_0.$$

The constant T_0 depends on Λ, F, χ, $|\phi|_\infty$, and the Lipschitz rank Q_0 of ψ on Λ.

The function ψ is convex, so that it satisfies the *lower bounded slope condition* introduced in [5]. This implies (see the proof of [2, Theorem 2.1]) that there exists $C_0 > 0$ such that

$$v(x) \leq v(y) + C_0 \frac{|x-y|}{|x - \pi_{\partial\Lambda}(x|y)|} \quad x, y \in \Lambda, \tag{22}$$

where $C_0 = C_0(\Lambda, F, \chi, |\phi|_\infty, Q_0)$ and $\pi_{\partial\Lambda}(x|y)$ is the unique point of $\partial\Lambda$ of the form $x + t(y-x)$ with $t > 0$.

Inequality (22) implies (as in the proof of [5, Lemma 2.11], see also the proof of (34) below) that there exists $C > 0$ such that

$$|v(x) - \psi(\gamma)| \leq C|x-\gamma|^{1/(n+1)} \quad x \in \Lambda, \tag{23}$$

for some constant $C = C(\Lambda, F, \chi, |\phi|_\infty, Q_0)$.

On the other hand, we claim that

$$v(x) \geq \psi(x) \ , \ x \in \Lambda. \tag{24}$$

Indeed, since $\max(v, \psi) \in W^{1,1}_\psi(\Lambda)$, we have

$$\begin{aligned} &\int_\Lambda F(Dv(x)) + g(x)v(x)\,dx \\ &\qquad \leq \int_\Lambda F(D\max(v(x),\psi(x))) + g(x)\max(v(x),\psi(x))\,dx. \end{aligned} \tag{25}$$

This gives

$$\int_B (F(Dv(x)) - F(D\psi(x)))\,dx \leq \chi \int_B (\psi(x) - v(x))\,dx \tag{26}$$

where $B := \{x \in \Lambda : \psi(x) > v(x)\}$. We claim that the left hand side is not lower than

$$2\chi \int_B (\psi(x) - v(x))\,dx.$$

Indeed, let (F_k) be a nondecreasing sequence of functions in $C^\infty(\mathbb{R}^n)$ converging uniformly on bounded sets to F, and such that the convexity assumption (2) holds for F_k with the same constant μ when ξ, ξ' are restricted to a ball

$B(0, Q_0+1)$ containing all the values of $D\psi_0$ (such a sequence exists, see [19, Lemma 4.2.1]). For each k, we have

$$\int_B (F_k(Dv(x)) - F_k(D\psi(x)))\,dx \geq \int_B DF_k(D\psi(x))(Dv(x) - D\psi(x))\,dx.$$

We have

$$\int_B DF_k(D\psi(x))(Dv(x) - D\psi(x))\,dx$$
$$= -\int_\Lambda DF_k(D\psi(x))(D(\max(\psi, v) - v)(x))\,dx.$$

Stokes' formula then implies

$$\int_B DF_k(D\psi(x))(Dv(x) - D\psi(x))\,dx = \int_B \operatorname{div}\left[DF_k(D\psi(x))\right](\psi(x) - v(x))\,dx$$
$$= \int_B \sum_{i,j=1}^n \frac{\partial^2 F_k}{\partial p_i \partial p_j}(D\psi(x))\frac{\partial^2 \psi}{\partial x_i \partial x_j}(x)(\psi(x) - v(x))\,dx$$
$$\geq 2a \int_B \Delta F_k(D\psi(x))(\psi(x) - v(x))\,dx \geq 2n\mu a \int_B \psi - v \quad \text{by (3) and (21).}$$

Letting $k \to \infty$, we get by the monotone convergence theorem

$$\int_B F(Dv(x)) - F(D\psi(x)) \geq 2n\mu a \int_B (\psi(x) - v(x))\,dx \geq 2\chi \int_B (\psi(x) - v(x))\,dx. \tag{27}$$

Inequalities (26) and (27) imply that $\psi \leq v$ a.e. on Λ.

Since $\psi \geq \phi$ on $\overline{\Lambda}$, we have $\overline{v}^\psi \geq \overline{u}^{\tilde{\phi}}$ on $\mathbb{R}^n \setminus \Omega$, where $\tilde{\phi} := \min(\phi, \psi)$. Remark that $W^{1,1}_\phi(\Omega) = W^{1,1}_{\tilde{\phi}}(\Omega)$. Hence, by Lemma 2, we have $u \leq v$ on Ω.

When ϕ is Lipschitz continuous, we have (see (23) and (20)),

$$u(x) \leq v(x) \leq \psi(\gamma) + C|x-\gamma|^{1/(n+1)} = \phi(\gamma) + C|x-\gamma|^{1/(n+1)} \quad a.e \;\; x \in \Omega. \tag{28}$$

Symmetrically, we have

$$u(x) \geq \phi(\gamma) - C|x-\gamma|^{1/(n+1)} \quad a.e \;\; x \in \Omega,$$

so that $|u(x) - \phi(\gamma)| \leq C|x-\gamma|^{1/(n+1)}$ for a.e. $x \in \Omega$. Since γ is arbitrary and ϕ is Lipschitz continuous, Theorem 1 2) is a consequence of Proposition 1 and Remark 3.

When ϕ is merely continuous, we have (see (23) and (19))

$$u(x) \leq v(x) \leq \psi(\gamma) + C|x-\gamma|^{1/(n+1)} = \phi(\gamma) + \epsilon + C|x-\gamma|^{1/(n+1)}, \;\; a.e \;\; x \in \Omega$$

(note that C now also depends on ϵ). We then get

$$\limsup_{x\to\gamma} u(x) \leq \limsup_{x\to\gamma} v(x) = \psi(\gamma) = \phi(\gamma) + \epsilon.$$

Since this is true for any $\epsilon > 0$, this implies $\limsup_{x\to\gamma} u(x) \leq \phi(\gamma)$. Symmetrically, $\liminf_{x\to\gamma} u(x) \geq \phi(\gamma)$. Hence, we have $\lim_{x\to\gamma} u(x) = \phi(\gamma)$.

Proposition 1 and Remark 3 then imply Theorem 1 1).

□

5 The case when Ω is an annulus

In this section, $\Omega := B(0,R) \setminus \overline{B}(0,r)$, $f : [0,\infty) \to \mathbb{R}$ is a nondecreasing convex function and $G : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies (H). Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous map. Then the problem (P) now reads

$$\min_u \int_\Omega f(|Du(x)|) + G(x, u(x))\, dx \quad , \quad u \in W^{1,1}_\phi(\Omega).$$

Proposition 2 *a) If f is uniformly convex, then any solution of (P) relative to $L^\infty(\Omega)$ is Hölder continuous on $\overline{\Omega}$ of order $\alpha := 1/(n+1)$.*

b) If f is superlinear and $G = 0$, then the minimum or the maximum of the minima is continuous on $\overline{\Omega}$.

c) If f is coercive of order $p > 1$ and $G = 0$, then the minimum or the maximum of the minima is Hölder continuous on $\overline{\Omega}$ of order $\alpha := (p-1)/(n+p-1)$.

We begin with the following lemma:

Lemma 5 *Let $p \geq 1$, $0 < r < R$ and $u \in W^{1,p}(B(0,R) \setminus \overline{B}(0,r))$.*

We assume that there exists $Q > 0$ such that for a.e. $\rho \in (r,R)$, for a.e. x, $y \in \partial B(0,\rho)$, we have

$$|u(x) - u(y)| \leq QR \left| \frac{x}{|x|} - \frac{y}{|y|} \right|. \tag{29}$$

Then u has a representative $u_0 \in W^{1,p}(B(0,R) \setminus \overline{B}(0,r))$ such that

i) when $p = 1$, u_0 is continuous on $\overline{B}(0,R) \setminus B(0,r)$.

ii) when $p > 1$, u_0 is Hölder continuous of order $\alpha := (p-1)/(n+p-1)$.

Proof : We first consider the case $p > 1$.

Step 1 Let $u \in W^{1,\infty}((-1,1) \times \mathbb{R}^{n-1})$ be such that $u(t,x) = 0$ for $|x| > 1$, $t \in (-1,1)$. We claim that for any $a, b \in (-1,1)$, for any $x \in \mathbb{R}^{n-1}$, we have

$$|u(a,x) - u(b,x)| \leq C|D_x u|_{L^\infty}^{(n-1)/(n+p-1)} |D_t u|_{L^p}^{p/(n+p-1)} |a-b|^{(p-1)/(n+p-1)}. \tag{30}$$

Here, $D_t u$ is the partial derivative of u with respect to the first coordinate, while $D_x u$ is the $(n-1)$ vector of the partial derivatives of u with respect to the other coordinates. To prove (30), we define for $\epsilon > 0$

$$u_\epsilon(a,x) := \frac{1}{\alpha_{n-1}\epsilon^{n-1}} \int_{B^{n-1}(x,\epsilon)} u(a,y)\,dy,$$

where α_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$ and $B^{n-1}(x,\epsilon)$ is the ball of center x and radius ϵ in $\mathbb{R}^{n-1}$. On the one hand,

$$|u_\epsilon(a,x) - u(a,x)| \le \epsilon |D_x u|_{L^\infty}. \tag{31}$$

On the other hand,

$$|u_\epsilon(a,x) - u_\epsilon(b,x)| \le \frac{1}{\alpha_{n-1}\epsilon^{n-1}} \int_{B^{n-1}(x,\epsilon)} dy \left| \int_a^b D_t u(t,y)\,dt \right| \tag{32}$$

$$\le C \frac{|a-b|^{(p-1)/p} |D_t u|_{L^p}}{\epsilon^{(n-1)/p}} \tag{33}$$

where C depends only on n and p. From (31) and (33), it follows that

$$|u(a,x) - u(b,x)| \le 2\epsilon |D_x u|_{L^\infty} + C \frac{|a-b|^{(p-1)/p} |D_t u|_{L^p}}{\epsilon^{(n-1)/p}}.$$

Since $u(\cdot,x) = 0$ for $|x| > 1$, one may assume that $|D_x u|_{L^\infty} > 0$ (otherwise $u = 0$ and there is nothing to prove). By taking

$$\epsilon := \frac{|a-b|^{(p-1)/(p+n-1)} |D_t u|_{L^p}^{p/(p+n-1)}}{|D_x u|_{L^\infty}^{p/(p+n-1)}},$$

we get

$$|u(a,x) - u(b,x)| \le C |a-b|^{(p-1)/(p+n-1)} |D_t u|_{L^p}^{p/(p+n-1)} |D_x u|_{L^\infty}^{(n-1)/(p+n-1)}.$$

This proves Step 1.

Step 2 Let $u \in W^{1,\infty}((-1,1)^n)$. We claim that for any $a, b \in (-1,1)$, for any $x \in (-1,1)^{n-1}$, we have

$$|u(a,x) - u(b,x)|$$
$$\le C(|u|_{L^\infty} + |D_x u|_{L^\infty})^{(n-1)/(n+p-1)} |D_t u|_{L^p}^{p/(n+p-1)} |a-b|^{(p-1)/(n+p-1)}. \tag{34}$$

Let $h \in C_c^\infty(\mathbb{R}^{n-1})$ be such that $0 \le h \le 1$ and $h(x) = 1$ on $(-1,1)^{n-1}$, $h(x) = 0$ if $x \notin (-2,2)^{n-1}$. For $a \in (-1,1)$, we extend $u(a,\cdot)$ on $\mathbb{R}^{n-1}$ by reflection in $\mathbb{R}^{n-1}$ (as in [3, Lemma 9.2, Remark 10]). We then multiply the resulting map by h. We get a map $v \in W^{1,\infty}((-1,1)\times\mathbb{R}^{n-1})$ which satisfies:

$$\begin{aligned}|D_t v|_{L^p((-1,1)\times\mathbb{R}^{n-1})} &\leq C|D_t u|_{L^p((-1,1)^n)},\\ |D_x v|_{L^\infty((-1,1)\times\mathbb{R}^{n-1})} &\leq C\left(|u|_{L^\infty((-1,1)^n)} + |D_x u|_{L^\infty((-1,1)^n)}\right).\end{aligned}$$

We now apply Step 1 to v (remark that $v(a,x) = 0$ if $x \notin (-2,2)^{n-1}$) and (34) follows.

We remark that (34) remains true when $u \in L^\infty \cap W^{1,p}((-1,1)^n)$ is such that $D_x u \in L^\infty((-1,1)^n)$. This follows from an approximation argument : let $\eta \in C_c^\infty(B^n)$ be such that $\eta \geq 0$, $\int_{\mathbb{R}^n}\eta = 1$ and define $\eta_\epsilon := \eta(\cdot/\epsilon)/\epsilon^n$. On $\Omega_i := (-1+1/i, 1-1/i)^n$, $i \geq 1$, the map $u * \eta_\epsilon$ is well defined for $\epsilon < 1/i$. Fix $i \geq 1$. Apply (34) to $u * \eta_\epsilon$ on Ω_i : for any $a, b \in (-1+1/i, 1-1/i)$, $x \in (-1+1/i, 1-1/i)^{n-1}$,

$$\begin{aligned}|u*\eta_\epsilon(a,x) - u*\eta_\epsilon(b,x)|^{n+p-1} &\leq C\left(|u*\eta_\epsilon|_{L^\infty(\Omega_i)} + |D_x(u*\eta_\epsilon)|_{L^\infty(\Omega_i)}\right)^{n-1}\\ &\qquad |D_t(u*\eta_\epsilon)|^p_{L^p(\Omega_i)}|a-b|^{p-1}\\ &\leq C\left(|u|_{L^\infty((-1,1)^n)} + |D_x u|_{L^\infty((-1,1)^n)}\right)^{n-1}\\ &\qquad |D_t u|^p_{L^p((-1,1)^n)}|a-b|^{p-1}.\end{aligned}$$

The map $(a,x) \mapsto u * \eta_\epsilon(a,x)$ is Hölder continuous in a and Lipschitz continuous in x. Hence, it is Hölder continuous in (a,x). Moreover, its Hölder norm does not depend on ϵ. This implies that (up to a subsequence) when $\epsilon \to 0$, $u*\eta_\epsilon$ uniformly converges on Ω_i to a representative of u which satisfies (34) on Ω_i. By letting $i \to \infty$, we get the result.

Step 3 Let $u \in L^\infty \cap W^{1,p}(B(0,R)\setminus B(0,r))$ satisfy (29). We claim that u has a representative u_0 such that for $a, b \in (r,R)$ and $\theta \in S^{n-1}$, we have

$$\begin{aligned}|u_0(a\theta) - u_0(b\theta)| \leq C(\frac{R}{r})^{(n-1)/(n+p-1)}(\frac{|u|_{L^\infty}}{R} + Q)^{(n-1)/(n+p-1)}&\\ |D_t u|^{p/(n+p-1)}_{L^p}|a-b|^{(p-1)/(n+p-1)}&\end{aligned} \tag{35}$$

where $D_t u$ is the radial derivative of u.

Let $\Xi : U \to (-1,1)^{n-1}$ be a biLipschitz diffeomorphism from an open subset U of S^{n-1} onto $(-1,1)^{n-1}$. We define $v(a,x) := u(a\Xi^{-1}(x))$ for $(a,x) \in (-1,1)\times(-1,1)^{n-1}$. Then $v \in W^{1,p}((r,R)\times(-1,1)^{n-1})$ and satisfies for almost every $a \in (r,R)$, for almost every $x, y \in (-1,1)^{n-1}$,

$$|v(a,x) - v(a,y)| = |u(a\Xi^{-1}(x)) - u(a\Xi^{-1}(y))| \leq C_0 Q R|x-y|$$

where C_0 is a Lipschitz rank of Ξ^{-1}. By Step 2, v has a representative v_0 which is Hölder continuous and satisfies

$$|v_0(a,x) - v_0(b,x)|^{n+p-1} \leq C(|v|_{L^\infty} + C_0 QR)^{n-1}|D_t v|^p_{L^p}|a-b|^{p-1}. \tag{36}$$

By the change of variables $\theta = \Xi^{-1}(x)$, there exists $C_1 > 0$ such that

$$\begin{aligned} |D_t v|^p_{L^p((r,R)\times(-1,1)^{n-1})} &\le C_1 \int_{S^{n-1}} d\theta \int_r^R |D_t u(t\theta)|^p \, dt \\ &\le \frac{C_1}{r^{n-1}} \int_{B(0,R)\setminus B(0,r)} |D_t u(x)|^p \, dx. \end{aligned}$$

We define $u_0(x) := v_0(|x|, \Xi(x/|x|))$. It follows from (36) that there exists $C_2 > 0$ such that for $\theta \in U$ and $a, b \in (r, R)$,

$$\begin{aligned} |u_0(a\theta) - u_0(b\theta)|^{n+p-1} &\le \frac{C_2}{r^{n-1}} (|u|_{L^\infty} + QR)^{n-1} |D_t u|^p_{L^p} |a-b|^{p-1} \\ &\le C_2 (\frac{R}{r})^{n-1} \left(\frac{|u|_{L^\infty}}{R} + Q \right)^{n-1} |D_t u|^p_{L^p} |a-b|^{p-1}. \end{aligned}$$

Since U is arbitrary, this completes Step 3. In view of (29), this proves that when $p > 1$, u has a representative which is Hölder continuous of order $\alpha = (p-1)/(n+p-1)$.

When $p = 1$, the proof is essentially the same except that instead of using Hölder's inequality in (32), we simply write

$$\left| \frac{1}{\alpha_{n-1}\epsilon^{n-1}} \int_{B^{n-1}(x,\epsilon)} dy \int_a^b D_t u(t,y) \, dt \right| \le \frac{1}{\alpha_{n-1}\epsilon^{n-1}} |D_t u|_{L^1((a,b)\times\mathbb{R}^{n-1})}.$$

Then (34) remains true with $p = 1$ if one replaces $|D_t u|^{p/(n+p-1)}_{L^p} |a-b|^{(p-1)/(n+p-1)}$ by $|D_t u|^{1/n}_{L^1((a,b)\times(-1,1)^{n-1})}$. For a map $u \in L^\infty \cap W^{1,1}((-1,1)^n)$ such that $D_x u \in L^\infty((-1,1)^n)$, we claim that u has a continuous representative satisfying (34). Indeed, the map $u * \eta_\epsilon$ defined at the end of Step 2 satisfies on Ω_i, $i \ge 1$, for any $a, b \in (-1+1/i, 1-1/i)$, $x \in (-1+1/i, 1-1/i)^{n-1}$,

$$\begin{aligned} |u * \eta_\epsilon(a,x) - u * \eta_\epsilon(b,x)|^n &\le C \left(|u * \eta_\epsilon|_{L^\infty(\Omega_i)} + |D_x(u * \eta_\epsilon)|_{L^\infty(\Omega_i)} \right)^{n-1} \\ &\quad |D_t(u * \eta_\epsilon)|_{L^1((a,b)\times(-1+1/i,1-1/i)^{n-1})} \\ &\le C(|u|_{L^\infty((-1,1)^n)} + |D_x u|_{L^\infty((-1,1)^n)})^{n-1} J_{|b-a|}(u) \end{aligned}$$

where $J_\delta(u) = \sup\limits_{\substack{-1<a\le b<1 \\ b-a<\delta}} \int_{(a,b)\times(-1,1)^{n-1}} |D_t u|$. Since $\lim_{\delta\to 0} J_\delta(u) = 0$, the map $(a,x) \mapsto u * \eta_\epsilon(a,x)$ is uniformly continuous in a and Lipschitz continuous in x. Hence, it is uniformly continuous in (a,x). Moreover, its modulus of continuity does not depend on ϵ. This implies that when $\epsilon \to 0$, $u * \eta_\epsilon$ uniformly converges on Ω_i to a representative of u which satisfies on Ω_i

$$|u(a,x) - u(b,x)|^n \le C(|u|_{L^\infty} + |D_x u|_{L^\infty((-1,1)^n)})^{n-1} J_{|b-a|}(u).$$

We now let $i \to \infty$. This completes the analogue of Step 2 when $p = 1$. The analogue of Step 3 follows as in the case $p > 1$. This completes the proof of Lemma 5.

$\square$

We now prove Proposition 2 b) and c). Let u be the minimum of the minima of I. We denote by Q the Lipschitz rank of ϕ.

Let P be a vector subspace in $\mathbb{R}^n$ of dimension 2. Then, for any $\theta \in \mathbb{R}$, we denote by $H_{P,\theta} : \mathbb{R}^n \to \mathbb{R}^n$ the map which is the rotation of angle θ and center 0 on P and the identity on the orthogonal of P.

We define $v := u \circ H_{P,\theta}$. Then $v \in W^{1,1}(B(0,R) \setminus \overline{B}(0,r))$. An obvious change of variables yields $I(v) = I(u)$, which implies that v is the minimum of the minima with respect to the boundary condition $\psi = \phi \circ H_{P,\theta}$. We now apply Lemma 4 (see also Remark 2) to $\Omega_1 = \Omega_2 = \Omega = B(0,R) \setminus \overline{B}(0,r)$, $u_1 = u$, $\phi_1 = \phi$ and $u_2 = v$, $\phi_2 = \psi$ with $\alpha_0 = |\phi - \psi|_{L^\infty(\partial\Omega)}$. We get $u \leq v + |\phi - \psi|_{L^\infty(\partial\Omega)}$ on Ω. Symmetrically, $v \leq u + |\phi - \psi|_{L^\infty(\partial\Omega)}$. Hence,

$$|u - v|_{L^\infty(\Omega)} \leq RQ|1 - e^{i\theta}|. \tag{37}$$

For a.e. $\rho \in (r,R)$, a.e. $x \in \partial B(0,\rho)$ is a Lebesgue point of u. Fix such a ρ and 2 Lebesgue points x and y of $\partial B(0,\rho)$. Consider the plane P which contains 0, x and y. We denote by $H_{P,\theta}$ the rotation of center 0 which maps x to y. By (37), for a.e. $z \in B(0,R) \setminus \overline{B}(0,r)$, $|u(z) - u(H_{P,\theta}(z))| \leq RQ|1 - e^{i\theta}|$. This implies

$$|u(x) - u(y)| \leq RQ|1 - e^{i\theta}|.$$

Hence, u satisfies (29). When F is coercive of order $p > 1$, u belongs to $W^{1,p}(\Omega)$. Thus, we may apply Lemma 5, which proves Proposition 2 b) and c).

We now prove Proposition 2 a). Let u be a solution of (P) relative to $L^\infty(\Omega)$. As above, we introduce $H_{P,\theta}$, ψ and v. We apply Lemma 3 with $H = H_{P,\theta}$, $\phi_1 = \phi$, $\phi_2 = \psi$, and $\alpha_0 = |\phi - \psi|_{L^\infty(\partial\Omega)}$ to obtain

$$u \leq u \circ H_{P,\theta} + q|1 - e^{i\theta}| + |\phi - \psi|_{L^\infty(\partial\Omega)}$$

for some $q = q(n, \Omega, \chi/\mu)$ (as usual, $\chi = \chi(|u|_{L^\infty(\Omega)})$ is given by (4) and μ is given by (2)). This implies $u \leq u \circ H_{P,\theta} + C|1 - e^{i\theta}|$ for some constant $C = C(n, \Omega, \chi/\mu, Q)$. Since F is uniformly convex and $G(x, u(x)) \in L^\infty(\Omega)$, the function u belongs to $W^{1,2}(\Omega)$ (recall that uniform convexity implies coercivity of order 2). In view of Lemma 5, this completes the proof of Proposition 2 a).

$\square$

Remark 4 *1) When ϕ is merely Hölder continuous, a proof similar to the one above shows that u is Hölder continuous as well. However, we have not been able to extend this result to more general open subsets of $\mathbb{R}^n$.*

2) As a consequence of the proof of Lemma 5, we observe that the Hölder norm of u in Proposition 2 c) only depends on n, p, $|\phi|_{W^{1,\infty}}$ and r, R as well as $|u|_{L^\infty}$ and $|Du|_{L^p}$. In Proposition 2 a), the Hölder norm of u also depends on χ/μ.

6 Proof of Theorem 3

We first consider the case when ϕ is continuous on $\mathbb{R}^n$, f is superlinear and Ω satisfies an exterior sphere condition.

Fix $\gamma \in \Gamma$. Then there exists $r > 0$ and $x \in \mathbb{R}^n$ such that

$$|x - \gamma| = r \ , \ \overline{B}(x, r) \subset \mathbb{R}^n \setminus \Omega. \tag{38}$$

Since Ω is bounded, there exists $R > 0$ such that $B(x, R) \supset \Omega$. It then follows that $\Omega \subset \Lambda$ where $\Lambda := B(x, R) \setminus \overline{B}(x, r)$. Let $\epsilon > 0$. Then we define the function ψ as follows:

$$\psi(x) = \phi(\gamma) + \epsilon + Q|x - \gamma|,$$

where

$$Q := \max_{x \in \overline{\Lambda}, x \neq \gamma} \frac{\phi(x) - \phi(\gamma) - \epsilon}{|x - \gamma|}.$$

Then ψ is a Lipschitz convex function, $\psi \geq \phi$ on $\overline{\Lambda}$ and $\psi(\gamma) = \phi(\gamma) + \epsilon$.

We consider the problem (P_0) of minimizing $v \mapsto \displaystyle\int_\Lambda f(|Dv|)$ on $v \in W^{1,1}_\psi(\Lambda)$. Since $f(|\cdot|)$ is convex and superlinear, one may consider the maximum of the minima v of (P_0).

By Proposition 2 b), v is continuous on $\overline{\Lambda}$. In particular, we have

$$\lim_{\substack{x \in \Omega \\ x \to \gamma}} v(x) = \psi(\gamma) = \phi(\gamma) + \epsilon.$$

Since ψ is convex, we have $v \geq \psi$ (see [1, Lemma 2.6]). Since $\phi \leq \psi$ on $\overline{\Lambda}$, we obtain by Lemma 4,

$$v \geq u \ \text{ on } \ \Omega.$$

This implies

$$\limsup_{\substack{x \in J \\ x \to \gamma}} u(x) \leq \lim_{\substack{x \in J \\ x \to \gamma}} v(x) = \phi(\gamma) + \epsilon,$$

where J is the set of Lebesgue points of u in Ω. Since ϵ is arbitrary, this gives $\limsup_{\substack{x \in J \\ x \to \gamma}} u(x) \leq \phi(\gamma)$. Symmetrically, we have $\liminf_{\substack{x \in J \\ x \to \gamma}} u(x) \geq \phi(\gamma)$. Whence $\lim_{\substack{x \in J \\ x \to \gamma}} u(x) = \phi(\gamma)$. In view of Remark 3, this proves that u is continuous on $\overline{\Omega}$.

When we assume further that f is coercive of order $p > 1$, ϕ is Lipschitz continuous and Ω satisfies a uniform exterior sphere condition, we repeat the above proof except that we define ψ in the following way

$$\psi(x) := \phi(\gamma) + Q|x - \gamma|,$$

where Q is now the Lipschitz rank of ϕ. Then ψ is Lipschitz continuous, convex and satisfies $\psi \geq \phi$ on $\mathbb{R}^n$, $\psi(\gamma) = \phi(\gamma)$. We observe that the radius r in (38) can be chosen independently of γ. By Proposition 2 c), v is Hölder continuous of order $\alpha = (p-1)/(n+p-1)$. Moreover, $|v|_{L^\infty}$ and $|Dv|_{L^p}$ are not larger than a constant which depends only on f, $|\psi|_{W^{1,\infty}(\Lambda)}$ and R. For $|v|_{L^\infty}$, it is implied by the maximum principle (see [15, Theorem 4.1]). For $|Dv|_{L^p}$, it is a consequence of the coercivity of f and the fact that ψ is Lipschitz continuous and admissible for (P): for some $a > 0, b \in \mathbb{R}$,

$$\int_{B(0,R)\setminus B(0,r)} (a|Dv|^p - b) \leq I(v) \leq I(\psi) \leq \int_{B(0,R)} f(|D\psi|_\infty).$$

By Remark 4 2), this implies that the Hölder norm of v is not greater than a constant C which depends only on n, p, f, $|\phi|_{L^\infty(\Omega)}$, Q, r and R. In particular, for any $x \in \Omega$,

$$u(x) \leq v(x) \leq \psi(\gamma) + C|x-\gamma|^\alpha = \phi(\gamma) + C|x-\gamma|^\alpha.$$

Symmetrically, we have $u(x) \geq \phi(\gamma) + C|x-\gamma|^\alpha$. By Proposition 1 (see (15)), u is Hölder continuous of order α. This completes the proof of Theorem 3 2).

$\square$

7 Proof of Theorem 2

To prove Theorem 2, it is enough to put together the tools of the proofs of Theorem 1 and Theorem 3. More specifically, let u be a solution of (P) relative to $L^\infty(\Omega)$. Fix $\gamma \in \Gamma$. As in the proof of Theorem 3, one may find $0 < r < R$ and $x \in \mathbb{R}^n$ such that $\Omega \subset \Lambda$ where $\Lambda := B(x,R) \setminus \overline{B}(x,r)$.

We now introduce the same function ψ as in the proof of Theorem 1 (the definition of ψ depends on the regularity properties of ϕ, see (19) and (20)).

We also define

$$g(x) := \begin{cases} G_u(x, u(x)) \text{ if } x \in \Omega \\ 0 \text{ otherwise.} \end{cases}$$

Consider the following auxiliary problem (P_0):

$$\text{Minimize } \; v \mapsto \int_\Lambda f(|Dv(x)|) + g(x)v(x)\,dx \quad , \quad v \in W^{1,1}_\psi(\Lambda).$$

In view of the uniform convexity of f, the direct method in the calculus of variations gives a (unique) solution $v \in W^{1,1}_{\psi}(\Lambda)$. Moreover, there exists $T_0 = T_0(|g|_{L^\infty}/\mu, |\psi|_{W^{1,\infty}}, r, R)$ such that

$$|v|_{W^{1,2}(\Lambda)} + |v|_{L^\infty(\Lambda)} \leq T_0.$$

By Proposition 2 a), there exists $C > 0$ such that

$$|v(x) - \psi(\gamma)| \leq C|x - \gamma|^\alpha \quad x \in \Lambda. \tag{39}$$

By Remark 4 2), the constant C depends only on n, $|g|_{L^\infty}/\mu$, $|\psi|_{W^{1,\infty}}$, r and R. As in the proof of Theorem 1, we have $u \leq v$ on Ω. The end of the proof is now exactly the same as the proof of Theorem 1.

□

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