Local Lipschitz continuity of solutions to a problem in the calculus of variations

Pierre Bousquet[*] and Francis Clarke[†]

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Dedicated to Arrigo Cellina
on the occasion of his 65th birthday

Abstract

This article studies the problem of minimizing $\int_\Omega F(Du) + G(x,u)$ over the functions $u \in W^{1,1}(\Omega)$ that assume given boundary values ϕ on $\partial\Omega$. The function F and the domain Ω are assumed convex. In considering the same problem with $G = 0$, and in the spirit of the classical Hilbert-Haar theory, Clarke has introduced a new type of hypothesis on the boundary function ϕ: the *lower* (or upper) *bounded slope condition.* This condition, which is less restrictive than the classical bounded slope condition of Hartman, Niremberg and Stampacchia, is satisfied if ϕ is the restriction to $\partial\Omega$ of a convex (or concave) function. We show that for a class of problems in which $G(x,u)$ is locally Lipschitz (but not necessarily convex) in u, the lower bounded slope condition implies the local Lipschitz regularity of solutions.

1 Introduction.

We study the regularity of solutions to the following problem (P) in the multiple integral calculus of variations:

$$\min_u \int_\Omega \{F(Du(x)) + G(x,u(x))\}\, dx \text{ subject to } u \in W^{1,1}(\Omega),\ \operatorname{tr} u = \phi,$$

where Ω is a domain in $\mathbb{R}^n$, u is scalar-valued, and $\operatorname{tr} u$ signifies the trace of u on $\Gamma := \partial\Omega$.

[*]Université Claude Bernard Lyon 1, <bousquet@math.univ-lyon1.fr>.

[†]Membre de l'Institut universitaire de France et professeur à l'Université Claude Bernard Lyon 1 <Clarke@math.univ-lyon1.fr>.

The aim is to deduce local Lipschitz regularity from properties of the boundary function ϕ. This is in the general spirit of the well-known Hilbert-Haar theory (see for example [5][10]), which requires that ϕ satisfy the *bounded slope condition* (BSC). The BSC of rank K is the assumption that, given any point $\gamma \in \Gamma$, there exist two affine functions

$$y \mapsto \langle \zeta_\gamma^-, y - \gamma \rangle + \phi(\gamma),\ y \mapsto \langle \zeta_\gamma^+, y - \gamma \rangle + \phi(\gamma)$$

agreeing with ϕ at γ, whose slopes satisfy $|\zeta_\gamma^-| \leq K$, $|\zeta_\gamma^+| \leq K$, and such that

$$\langle \zeta_\gamma^-, \gamma' - \gamma \rangle + \phi(\gamma) \leq \phi(\gamma') \leq \langle \zeta_\gamma^+, \gamma' - \gamma \rangle + \phi(\gamma) \quad \forall \gamma' \in \Gamma.$$

The classical Hilbert-Haar theorem asserts that if F is convex, $G = 0$, and ϕ satisfies the BSC, then there exists a (globally) Lipschitz minimizer for (P). The first proof of this statement is due to Miranda [9], although there are several special cases that are antecedents to this. The case in which G is different from 0 has been treated by Stampacchia [11] (and implicitly in [7]) under stronger smoothness assumptions on the data than used here. As regards other and more recent uses of the BSC, see notably Cellina [2] and other references cited therein.

The BSC is a restrictive requirement on flat parts of Γ, since it forces ϕ to be affine. Moreover, if Ω is smooth, then it forces ϕ to be smooth as well (see Hartman [6] for precise statements). Recently, Clarke [3] has introduced a new hypothesis on ϕ, the *lower bounded slope condition* (LBSC) of rank K : given any point γ on the boundary, there exists an affine function

$$y \mapsto \langle \zeta_\gamma, y - \gamma \rangle + \phi(\gamma)$$

with $|\zeta_\gamma| \leq K$ such that

$$\langle \zeta_\gamma, \gamma' - \gamma \rangle + \phi(\gamma) \leq \phi(\gamma') \quad \forall \gamma' \in \Gamma.$$

This requirement, which can be viewed as a one-sided BSC, enlarges considerably the class of boundary functions which it allows (compared to the BSC). The property has been studied by Bousquet in [1], where it is shown that ϕ satisfies the LBSC if and only if it is the restriction to Γ of a convex function. When Ω is uniformly convex, ϕ satisfies the LBSC if and only if it is the restriction to Γ of a semiconvex function.

It turns out that the LBSC has significant implications for the regularity of the solution u, although it implies less than the full, two-sided BSC. In fact, it is shown in [3] that in the case where $G = 0$, the one-sided BSC gives the crucial regularity property that one seeks: u is *locally* Lipschitz in Ω. This allows one to assert that u is a weak solution of the Euler equation, in the absence of the usual upper growth conditions on F. Furthermore, the local Lipschitz property allows one to invoke De Giorgi's regularity theory (when the data are sufficiently smooth) to obtain the continuous differentiability of the solution.

The goal of this article is to prove local Lipschitz regularity of the solution for a class of problems with G different from 0, under weak regularity hypotheses on the data of the problem, and when the LBSC is satisfied (rather than the BSC). The next section describes the hypotheses and gives a self-contained proof of the main theorem of the article. It is most closely related to the work of Stampacchia, but the method of proof differs in several important respects. A variant of the main theorem is developed in Section 3, and the final section discusses the issue of the continuity of the solution at the boundary.

2 The main result.

We now specify the hypotheses on the data of the problem (P). The first one, in particular, justifies the use of trace.

$(H\Omega)$ Ω is an open bounded convex set.

We require that F be uniformly elliptic, and that G be locally Lipschitz in u. More precisely:

(HF) For some $\mu > 0$, F satisfies, for all $\theta \in (0,1)$ and $p, q \in \mathbb{R}^n$:

$$\theta F(p) + (1-\theta)F(q) \geq F(\theta p + (1-\theta)q) + (\mu/2)\theta(1-\theta)|p-q|^2.$$

We remark that when F is of class C^2, (HF) holds if and only if, for every $v \in \mathbb{R}^n$, we have

$$\left\langle z, \nabla^2 F(v) z \right\rangle \geq \mu |z|^2 \quad \forall\, z \in \mathbb{R}^n.$$

Under (HF), it is easy to see that $\int_\Omega F(Dw)\,dx$ is well-defined (possibly as $+\infty$) for any $w \in W^{1,1}(\Omega)$.

(HG) $G(x,u)$ is measurable in x and differentiable in u, and for every bounded interval U in $\mathbb{R}$ there is a constant L such that for almost all $x \in \Omega$,

$$|G(x,u) - G(x,u')| \leq L|u-u'| \quad \forall\, u, u' \in U.$$

We also postulate as part of (HG) that for some bounded function b, the integral $\int_\Omega G(x, b(x))\,dx$ is well-defined and finite. It follows that the same is true for all bounded measurable functions w.

In the presence of $(H\Omega), (HF)$, and (HG), it follows that

$$I(w) := \int_\Omega \{F(Dw(x)) + G(x, w(x))\}\, dx$$

is well-defined for all $w \in W^{1,1}(\Omega)$ for which w is bounded. We say that u solves (P) relative to $L^\infty(\Omega)$ if u is itself bounded, and if we have $I(u) \le I(w)$ for all bounded w that are admissible for (P).

The theorem to be proved is the following.

Theorem 2.1
Under the hypotheses $(H\Omega), (HF)$, and (HG), and when ϕ satisfies the Lower Bounded Slope Condition, any solution u of (P) relative to $L^\infty(\Omega)$ is locally Lipschitz in Ω.

In the context of the theorem, even when $G = 0$ and $F(v) = |v|^2$, a bounded solution u of (P) may fail to be globally Lipschitz; an example of this type is given in [1][3]. Let us also point out that the theorem has an alternate version in which the LBSC is replaced by the *upper* BSC; the conclusion is the same. Finally, we remark that Stampacchia [11] has described structural assumptions on G which guarantee *a priori* the existence and boundedness of solutions of (P); these will be described in the next section.

2.1 The lower barrier condition

The proof of the main result uses in part the well-known barrier technique. Our one-sided version of this is the following.

Theorem 2.2
Under hypotheses $(H\Omega), (HF)$, and (HG), let u be a bounded solution of problem (P) as described above, where ϕ satisfies the Lower Bounded Slope Condition of rank K. Then there exists $\bar{K} > 0$ with the following property: for any $\gamma \in \Gamma$ there exists a function w which is Lipschitz of rank $\bar{K}$, which agrees with ϕ at γ, and which satisfies $w \le u$ a.e. in Ω.

Proof We may suppose that ϕ is a globally defined convex function of Lipschitz rank K. Thus there is an element ζ with $|\zeta| \le K$ in the subdifferential of ϕ at γ:

$$\phi(x) - \phi(\gamma) \ge \langle \zeta, x - \gamma \rangle \quad \forall\, x \in \mathbb{R}^n.$$

By (HG) there is a Lipschitz constant L valid for $G(x, \cdot)$ over the interval

$$\left[-\|u\|_{L^\infty(\Omega)}, \|\phi\|_{L^\infty(\Gamma)} + K \operatorname{diam} \Omega \right],$$

for $x \in \Omega$ a.e. Fix any $T > (L+1)\exp(\operatorname{diam}\Omega)/\mu$, where μ is given by (HF).

The following construction is a refinement of that proposed by Hartmann and Stampacchia [7] (Lemma 10.1). Let ν be a unit outward normal vector to $\overline{\Omega}$ at γ, and define

$$w(x) := \phi(\gamma) + \langle \zeta, x-\gamma \rangle - T\{1 - \exp(\langle x-\gamma, \nu \rangle)\}.$$

We proceed to prove that w has the required properties. Clearly w agrees with ϕ at γ, and is Lipschitz of rank

$$\bar{K} := K + T\exp(\operatorname{diam}\Omega).$$

We need only show that the set

$$S := \{x \in \Omega : w(x) > u(x)\}$$

has measure 0.

The function $M(x) := \max[u(x), w(x)]$ belongs to $W^{1,1}(\Omega)$ (see for example [4] or [8]), and we have:

$$DM(x) = Dw(x), x \in S \text{ a.e.}, \; DM(x) = Du(x), x \in \Omega \backslash S \text{ a.e.}$$

It follows from the subgradient inequality for ζ that $M \in \phi + W_0^{1,1}(\Omega)$ (in deriving this, we also use the fact that $\langle x-\gamma, \nu \rangle \le 0$ for $x \in \Omega$). By the optimality of u (relative to M) we deduce

$$\int_S \{F(Du(x)) + G(x, u(x))\}\, dx \le \int_S \{F(Dw(x)) + G(x, w(x))\}\, dx.$$

The Lipschitz condition satisfied by G now leads to

$$\int_S \{F(Du(x)) - F(Dw(x))\}\, dx \le L \int_S \{w(x) - u(x)\}\, dx \qquad (1)$$

In deriving the next estimate (which concludes the proof), let us make the temporary assumption that F is smooth (C^2 or better). Then, by straightforward calculation, the function $\psi(x) := \nabla F(Dw(x))$ satisfies

$$\operatorname{div}\psi(x) = T\exp(\langle x-\gamma, \nu \rangle)\, \left\langle \nu, \nabla^2 F(Dw(x))\nu \right\rangle \ge L+1, \qquad (2)$$

in light of (HF), and because of how T was chosen. We proceed to deduce from (1) the following:

$$\begin{aligned} L \int_S \{w(x) - u(x)\}\, dx &\ge \int_S \{F(Du(x)) - F(Dw(x))\}\, dx \\ &\ge \int_S \langle \psi(x), Du(x) - Dw(x) \rangle \, dx \end{aligned}$$

(by the subdifferential inequality)

$$= \int_\Omega \langle \psi(x), D\min[u,w](x) - Dw(x)\rangle \, dx$$
$$= \int_\Omega (\operatorname{div}\psi(x))(w(x) - \min[u,w](x))\, dx$$

(integration by parts, noting that $\min[u,w] = w$ on Γ)

$$\geq (L+1)\int_\Omega \{w(x) - \min[u,w](x)\}\, dx$$

(in view of (2))

$$\geq (L+1)\int_S \{w(x) - u(x)\}\, dx.$$

This shows that S is of measure 0, since $w - u > 0$ in S.

In the general case in which F is not smooth, we consider a non-decreasing sequence $\{F_k\}_{k\in\mathbb{N}}$ of functions in $C^\infty(\mathbb{R}^n)$ converging to F uniformly on bounded sets, and such that the ellipticity condition in (HF) holds for F_k when p, q are restricted to a ball $\overline{B}(0, \overline{K}+1)$ containing all the values of Dw. Such a sequence exists by a mollification-truncation argument; see Morrey [10], Lemma 4.2.1. Then, arguing as above, we derive, for any $k \geq 1$,

$$\int_S \{F_k(Du(x)) - F_k(Dw(x))\}\, dx \geq (L+1)\int_S \{w(x) - u(x)\}\, dx.$$

The result now follows from the Monotone Convergence Theorem. □

2.2 Proof of Theorem 2.1.

Let λ and q be parameters satisfying

$$\lambda \in [1/2, 1),\ q > \bar{q} := \bar{K}\operatorname{diam}\Omega + ||\phi||_{L^\infty(\Gamma)},$$

and fix any point $z \in \Gamma$. We denote

$$\Omega_\lambda := \lambda(\Omega - z) + z.$$

Note that Ω_λ is a subset of Ω, since the latter is convex. We proceed to define the following function on Ω_λ:

$$u_\lambda(x) := \lambda u\left((x-z)/\lambda + z\right) - q(1-\lambda).$$

Then u_λ belongs to $W_0^{1,1}(\Omega_\lambda) + \phi_\lambda$, where

$$\phi_\lambda(y) := \lambda\phi((y-z)/\lambda + z) - q(1-\lambda).$$

For every $x \in \mathbb{R}^n$, we will denote $(x-z)/\lambda + z$ by x_λ.

We are now going to compare u_λ and u on $\Gamma_\lambda := \partial\Omega_\lambda$; this comparison via dilation was introduced in [3].

Lemma 1. We have $u_\lambda \leq u$ on Γ_λ.

The meaning of this inequality is that $(u_\lambda - u)^+ := \max(0, u_\lambda - u)$ belongs to $W_0^{1,1}(\Omega_\lambda)$, where here u signifies of course the restriction of u to Ω_λ. To prove the Lemma, recall first that in the preceding section we proved the existence, for any $\gamma \in \Gamma$, of a $\bar{K}$-Lipschitz function w_γ such that $w_\gamma(\gamma) = \phi(\gamma)$ and $w_\gamma \leq u$ a.e. in Ω (which implies $w_\gamma \leq \phi$ on Γ).

Introduce $l(y) := \sup_{\gamma\in\Gamma} w_\gamma(y)$. Then l is a $\bar{K}$-Lipschitz function which coincides with ϕ on Γ and which has $l \leq u$ a.e. on Ω. Thus $u - l \in W_0^{1,1}(\Omega)$. There exists therefore a sequence $v_m \in \mathrm{Lip}_0(\Omega)$ (the class of Lipschitz functions vanishing at the boundary) converging to $u-l$ in $W^{1,1}(\Omega)$ and almost everywhere in Ω. We can suppose moreover $v_m \geq 0$, by replacing v_m by $v_m^+ := \max(v_m, 0)$. We have used here the fact that if a sequence of functions k_m converges almost everywhere and in $W^{1,1}(\Omega)$ to k, then k_m^+ converges to k^+ in $W^{1,1}(\Omega)$.

We define the functions

$$u_m(x) := v_m(x) + l(x)\ ,\ \ u_{m,\lambda}(x) := \lambda u_m((x-z)/\lambda + z) - q(1-\lambda).$$

(Note that u_m is defined on Ω, and $u_{m,\lambda}$ on Ω_λ.) These regularizations of u and u_λ will allow us to complete the proof of the Lemma.

We have $u_m \in C^0(\bar{\Omega}), l \leq u_m$ on Ω and $u_m = \phi = l$ on Γ. We claim that $u_{m,\lambda}(\gamma) \leq u_m(\gamma)$ for every $m \geq 0, \gamma \in \Gamma_\lambda$. Suppose for a moment this claim were true. Then we could assert that

$$(u_{m,\lambda} - u_m)^+ \in W_0^{1,1}(\Omega_\lambda).$$

Now, $u_{m,\lambda}$ tends to u_λ in $W^{1,1}(\Omega_\lambda)$ and almost everywhere, as does u_m to u. It would follow therefore that $(u_\lambda - u)^+ \in W_0^{1,1}(\Omega_\lambda)$, which is what we wish to prove.

So it suffices to prove the claim. Fix some $\gamma \in \Gamma_\lambda$. Then,

$$\begin{aligned}
u_{m,\lambda}(\gamma) - u_m(\gamma) &= \lambda u_m(\gamma_\lambda) - u_m(\gamma) - q(1-\lambda) \\
&\leq \lambda\phi(\gamma_\lambda) - l(\gamma) - q(1-\lambda) \\
&= \lambda l(\gamma_\lambda) - l(\gamma) - q(1-\lambda) \\
&\leq (l(\gamma_\lambda) - l(\gamma)) + (1-\lambda)(||l||_{L^\infty(\Gamma)} - q) \\
&\leq \bar{K}|\gamma - \gamma_\lambda| + (1-\lambda)(||l||_{L^\infty(\Gamma)} - q) \\
&\leq (1-\lambda)(\bar{K}\,\mathrm{diam}\,\Omega + ||l||_{L^\infty(\Gamma)} - q) \\
&\leq 0,
\end{aligned}$$

since $\gamma_\lambda \in \Gamma$, $||l||_{L^\infty(\Gamma)} = \|\phi\|_{L^\infty(\Gamma)}$, and because q has been chosen to be greater than $\bar{q}$. This proves the claim and completes the proof of Lemma 1.

The next step of the proof is to show that the set

$$A := \{y \in \Omega_\lambda : u_\lambda(y) > u(y)\}$$

has measure zero. Let $w(x) := \min(u, u_\lambda)$, which belongs to $W_0^{1,1}(\Omega_\lambda) + \phi_\lambda$ in light of Lemma 1, and define

$$\tilde{w}^\lambda(x) := \frac{1}{\lambda} w(\lambda(x - z) + z) + q(\lambda^{-1} - 1),$$

an element of $W_0^{1,1}(\Omega) + \phi$. Fix any $\theta \in (0, 1)$. Then $v := \theta\tilde{w}^\lambda + (1 - \theta)u$ lies in $W_0^{1,1}(\Omega) + \phi$, so that $I(u) \leq I(v)$, which yields after an evident change of variables

$$\int_{\Omega_\lambda} \Big\{F(Du_\lambda) + G(\frac{y - z}{\lambda} + z, \frac{u_\lambda + q(1 - \lambda)}{\lambda})\Big\}\, dy \leq$$
$$\int_{\Omega_\lambda} \Big\{F(\theta Dw + (1 - \theta)Du_\lambda) + G(\frac{y - z}{\lambda} + z, \theta\frac{w + q(1 - \lambda)}{\lambda} +$$
$$(1 - \theta)\frac{u_\lambda + q(1 - \lambda)}{\lambda})\Big\}\, dy.$$

We also note that the right side is finite, since $\int_\Omega F(Du)\, dx$ is finite, and in light of the convexity of F. This implies

$$\int_{A} \Big\{F(Du_\lambda) + G(\frac{y - z}{\lambda} + z, \frac{u_\lambda + q(1 - \lambda)}{\lambda})\Big\}\, dy \leq$$
$$\int_{A} \Big\{F(\theta Dw + (1 - \theta)Du_\lambda) + G(\frac{y - z}{\lambda} + z, \theta\frac{w + q(1 - \lambda)}{\lambda} +$$
$$(1 - \theta)\frac{u_\lambda + q(1 - \lambda)}{\lambda})\Big\}\, dy,$$

whence (since $w = u$ on A)

$$\int_A \{F(Du_\lambda) - F(\theta Du + (1 - \theta)Du_\lambda)\}\, dy \leq$$
$$\int_A \Big\{G(\frac{y - z}{\lambda} + z, \theta\frac{u + q(1 - \lambda)}{\lambda} + (1 - \theta)\frac{u_\lambda + q(1 - \lambda)}{\lambda})$$
$$- G(\frac{y - z}{\lambda} + z, \frac{u_\lambda + q(1 - \lambda)}{\lambda})\Big\}\, dy. \quad (3)$$

Now let $W(x) := \max(u(x), u_\lambda(x))$ for $x \in \Omega_\lambda$, and $W(x) := u(x)$ for $x \in \Omega\backslash\Omega_\lambda$. Then $W \in W_0^{1,1}(\Omega) + \phi$ since $u_\lambda \leq u$ on Γ_λ. With $v := \theta W + (1 - \theta)u$, we have $I(u) \leq I(v)$, which yields

$$\int_A \{(F(Du) + G(y, u))\}\, dy \leq$$
$$\int_A \{(F(\theta Du_\lambda + (1 - \theta)Du) + G(y, \theta u_\lambda + (1 - \theta)u))\}\, dy,$$

so that

$$\int_A \{(F(Du) - F(\theta Du_\lambda + (1-\theta)Du))\}\, dy \leq \int_A \{(G(y, \theta u_\lambda + (1-\theta)u) - G(y,u))\}\, dy. \quad (4)$$

Summing (3) and (4), we get

$$\begin{aligned}\int_A \Big\{ &(1-\theta)F(Du_\lambda) + \theta F(Du) - F(\theta Du + (1-\theta)Du_\lambda) \\ &+ \theta F(Du_\lambda) + (1-\theta)F(Du) - F(\theta Du_\lambda + (1-\theta)Du) \Big\}\, dy \\ \leq \int_A \Big\{ &G(\frac{y-z}{\lambda} + z, \theta \frac{u + q(1-\lambda)}{\lambda} + (1-\theta)\frac{u_\lambda + q(1-\lambda)}{\lambda}) \\ &- G(\frac{y-z}{\lambda} + z, \frac{u_\lambda + q(1-\lambda)}{\lambda}) \\ &+ G(y, \theta u_\lambda + (1-\theta)u) - G(y,u) \Big\}\, dy. \quad (5)\end{aligned}$$

Thanks to (HF) we see that the left side of the last inequality is no less than

$$\mu\theta(1-\theta)\int_A |Du_\lambda - Du|^2 dy.$$

Substituting into (5), dividing by θ, and letting θ go to 0, we find

$$\mu \int_A |Du_\lambda - Du|^2\, dy \leq \int_A (\frac{1}{\lambda} g(\frac{y-z}{\lambda} + z) - g(y))(u_\lambda - u)\, dy,$$

where we have denoted by $g(y)$ the function $-G_u(y, u(y))$, which belongs to $L^\infty(\Omega)$. Write for any $y \in \Omega_\lambda$:

$$\frac{1}{\lambda} g(\frac{y-z}{\lambda} + z) - g(y) = (\frac{1}{\lambda} - \frac{1}{\lambda^n}) g(\frac{y-z}{\lambda} + z) + h(y)$$

where we define $h(y) := 1/\lambda^n g((y-z)/\lambda + z) - g(y)$ for $y \in \Omega_\lambda$ and $h(y) = 0$ for $y \in \mathbb{R}^n \backslash \Omega_\lambda$. Then

$$\mu \int_A |Du_\lambda - Du|^2 dy \leq \int_A [(\frac{1}{\lambda^n} - \frac{1}{\lambda}) g_0 + h(y)](u_\lambda - u)\, dy, \quad (6)$$

where $g_0 := ||g||_\infty$.

Lemma 2. There exists $f = (f_1, \ldots, f_n) \in L^\infty(\mathbb{R}^n)^n$ such that for each $j \in \{1, \ldots, n\}$, for almost every $(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n) \in \mathbb{R}^{n-1}$, the function

$$y_j \mapsto f_j(y_1, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, y_n)$$

is absolutely continuous on $\mathbb{R}$ with

$$\frac{\partial f_j}{\partial y_j} \in L^\infty(\mathbb{R}^n)$$

and

$$\operatorname{div} f := \sum_{j=1}^{n} \frac{\partial f_j}{\partial y_j} = h \text{ a.e. in } \Omega_\lambda \tag{7}$$

and such that $||f||_{L^\infty(\Omega)} \leq C_0(1-\lambda)$ for some constant C_0 which depends only on g_0 and Ω (λ being restricted to $[1/2, 1)$).

Proof of Lemma 2. We extend g by setting it equal to 0 outside Ω. There exists $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ such that

$$\Omega \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_j \geq c_j \ \forall j = 1, \ldots, n\}.$$

Define $f_1(y_1, \ldots, y_n)$ to be

$$\int_{c_1}^{y_1} \left[\frac{1}{\lambda} g(\frac{y_1' - z_1}{\lambda} + z_1, y_2, \ldots, y_n) - g(y_1', y_2, \ldots, y_n)\right] dy_1'$$

and similarly set $f_j(y_1, \ldots, y_n)$ equal to

$$\int_{c_j}^{y_j} \Big[\frac{1}{\lambda^j} g(\frac{y_1 - z_1}{\lambda} + z_1, \ldots, \frac{y_j' - z_j}{\lambda} + z_j, y_{j+1}, \ldots, y_n)\Big] dy_j' -$$

$$\int_{c_j}^{y_j} \Big[\frac{1}{\lambda^{j-1}} g(\frac{y_1 - z_1}{\lambda} + z_1, \ldots, \frac{y_{j-1} - z_{j-1}}{\lambda} + z_{j-1}, y_j', y_{j+1}, \ldots, y_n)\Big] dy_j'$$

This implies

$$\frac{\partial f_j}{\partial y_j} \in L^\infty(\mathbb{R}^n)$$

and that f satisfies (7). Upon making the change of variables $y_j'' = (y_j' - z_j)/\lambda + z_j$ in the first integral defining f_j, we find readily

$$\begin{aligned}
|f_j(y_1, ..., y_n)| &\leq g_0 \frac{1}{\lambda^{j-1}} \{|\frac{y_j - z_j}{\lambda} + z_j - y_j| + |\frac{c_j - z_j}{\lambda} + z_j - c_j|\} \\
&= g_0 \frac{1-\lambda}{\lambda^j} \{|y_j - z_j| + |c_j - z_j|\} \\
&\leq g_0 \frac{1-\lambda}{\lambda^j} \{\operatorname{diam} \Omega + |c| + \max_{z \in \Omega} |z|\},
\end{aligned}$$

if the y_i lie in Ω. This completes the proof of the Lemma.

Thanks to Lemma 2, we can write (6) as

$$\mu \int_A |Du_\lambda - Du|^2 dy \leq \int_A \Big\{ \langle f, Du - Du_\lambda \rangle + g_0 (\frac{1}{\lambda^n} - \frac{1}{\lambda})(u_\lambda - u) \Big\} \, dy. \quad (8)$$

We have used the divergence theorem for this, based on the fact that

$$\int_A (u_\lambda - u) \operatorname{div} f \, dy = \int_{\Omega_\lambda} (u_\lambda - u)^+ \operatorname{div} f \, dy$$

and since $(u_\lambda - u)^+ \in W_0^{1,1}(\Omega_\lambda)$. Now Poincaré's inequality, applied to $(u_\lambda - u)^+$, yields the existence of a constant C_P which depends only on Ω such that

$$\int_A (u_\lambda - u) \, dy \leq C_P \int_A |Du_\lambda - Du| \, dy.$$

Then (8) implies

$$\mu \int_A |Du_\lambda - Du|^2 dy \leq \Big[||f||_{L^\infty(\Omega)} + g_0 C_P (\frac{1}{\lambda^n} - \frac{1}{\lambda}) \Big] \int_A |Du_\lambda - Du| \, dy.$$

Applying the Cauchy-Schwartz inequality on the right side, we get

$$\mu \|Du_\lambda - Du\|_{L^2(A)} \leq \Big[||f||_{L^\infty(\Omega)} + g_0 C_P (\frac{1}{\lambda^n} - \frac{1}{\lambda}) \Big] |A|^{1/2}. \quad (9)$$

Let 1^* be the Sobolev conjugate of 1 defined by $1/1^* = 1 - 1/n$. We now observe that

$$\|u_\lambda - u\|_{L^1(A)} \leq \|u_\lambda - u\|_{L^{1^*}(A)} |A|^{1-1/1^*}$$

(by Hölder's inequality)

$$\leq S_1 \|Du_\lambda - Du\|_{L^1(A)} |A|^{1/n}$$

(for some S_1 depending only upon α, n and Ω, by the Gagliardo-Nirenberg-Sobolev Lemma, with $w = (u_\lambda - u)^+$)

$$\leq S_1 \|Du_\lambda - Du\|_{L^2(A)} |A|^{1/2+1/n},$$

by Hölder's inequality. Then, using this in (9), we get

$$\|u_\lambda - u\|_{L^1(A)} \leq C \Big[||f||_{L^\infty(\Omega)} + (\frac{1}{\lambda^n} - \frac{1}{\lambda}) \Big] |A|^\gamma.$$

with $\gamma := 1 + 1/n > 1$ and for some constant C which depends only on Ω, g_0, and μ. By Lemma 2 we have $||f||_{L^\infty(\Omega)} \leq C_0(1 - \lambda)$. Moreover,

$(1/\lambda^n - 1/\lambda)$ is bounded above by $C_1(1-\lambda)$ (where C_1 depends only on n; recall that $\lambda \geq 1/2$). Thus

$$\|u_\lambda - u\|_{L^1(A)} \leq C_2(1-\lambda)|A|^\gamma.$$

with $C_2 := C(C_0 + C_1)$.

Now let us denote A by $A(q)$ to display its dependence on q. Put $\rho(q) := |A(q)|$. Then ρ is a nonnegative, nonincreasing function such that $\rho(q) \to 0$ when $q \to +\infty$. Moreover, we have for any $q > \bar{q}$, thanks to Fubini's theorem,

$$\int_q^{+\infty} \rho(t)\,dt = \frac{1}{1-\lambda}\int_{A(q)} |u_\lambda - u|\,dy \leq C_2\rho(q)^\gamma. \tag{10}$$

(The proof also uses the observation that, setting $u_\lambda^q(x) := \lambda u((x-z)/\lambda + z) - q(1-\lambda)$, the equality $u_\lambda^{q+s} = u_\lambda^q - s(1-\lambda)$ holds.)

We now require the following result (cf. Hartman and Stampacchia [7]):

Lemma 3 Let ρ be a nonnegative, nonincreasing function on $[0,+\infty)$ such that $\rho(t) \to 0$ as $t \to +\infty$ and

$$\int_q^{+\infty} \rho(t)\,dt \leq c\rho(q)^\gamma,\ q > \bar{q},$$

where $c > 0, \gamma > 1$ are constants. Then $\rho(t) = 0$ for

$$t > c\gamma\rho(\bar{q})^{\gamma-1}/(\gamma-1) + \bar{q}.$$

To see this, note that the function $H(q) := \int_q^{+\infty} \rho(t)\,dt$ is absolutely continuous and satisfies

$$H'(q) = -\rho(q) \leq -[H(q)/c]^{1/\gamma}.$$

Then

$$G(q) := \gamma H^{(\gamma-1)/\gamma}(q)/(\gamma-1) + q/c^{1/\gamma}$$

has $G'(q) \leq 0$ for $q > \bar{q}$, as long as $H > 0$. For such q we may therefore write

$$0 \leq \gamma H^{1-1/\gamma}(q)/(\gamma-1) \leq \gamma H^{1-1/\gamma}(\bar{q})/(\gamma-1) - (q-\bar{q})/c^{1/\gamma}.$$

Consequently, $H(q) = 0$ for every

$$q \geq q_0 := \gamma c^{1/\gamma} H^{1-1/\gamma}(\bar{q})/(\gamma-1) + \bar{q},$$

in which case $\rho(t) = 0$ for $t > q_0$. The Lemma follows from the fact that $H(\bar{q}) \leq c\rho(\bar{q})^\gamma$.

Applying this Lemma to (10), we deduce that for any choice of q_0 satisfying

$$q_0 > C_2|\Omega|^{\gamma-1}\gamma/(\gamma-1) + \bar{q},$$

we have $|A(q)| = 0$ if $q \geq q_0$. We may summarize the current state of the proof as follows: for any choice of $z \in \Gamma$, we have, almost everywhere on $\Omega_\lambda := \lambda(\Omega - z) + z$, the inequality

$$u_\lambda(x) := \lambda u((x-z)/\lambda + z) - q_0(1-\lambda) \leq u(x).$$

Note that q_0 does not depend on λ, so that this assertion is true for any $\lambda \in [1/2, 1)$.

The final step in the proof is to deduce from this that u is locally Lipschitz in Ω. Let $x_0 \in \Omega$, and let $x, y \in B(x_0, d_\Gamma(x_0)/8)$ be two Lebesgue points for u; thus x, for example, satisfies

$$\lim_{\epsilon \to 0} \frac{1}{|B(x,\epsilon)|} \int_{B(x,\epsilon)} u(\omega)\, d\omega = u(x).$$

Let $z := \pi_\Gamma(y|x)$ be the unique point of Γ of the form $y + t(x-y)$ with $t \geq 0$. There exists $\lambda \in [1/2, 1)$ such that $y = (x-z)/\lambda + z$. Then $x \in \Omega_\lambda$. Let $\epsilon > 0$ such that $B(x,\epsilon) \subset \Omega_\lambda$. We have proved that for almost every $\omega \in B(x,\epsilon)$, we have

$$\lambda u((\omega - z)/\lambda + z) \leq u(\omega) + q_0(1-\lambda).$$

Integrating this relation over $B(x,\epsilon)$ and dividing by $|B(x,\epsilon)|$, we get

$$\frac{\lambda}{|B(x,\epsilon)|} \int_{B(x,\epsilon)} u((\omega-z)/\lambda+z)\, d\omega \leq \frac{1}{|B(x,\epsilon)|} \int_{B(x,\epsilon)} u(\omega)\, d\omega + q_0(1-\lambda)$$

which, by a change of variables, is equivalent to

$$\frac{\lambda}{|B(y,\frac{\epsilon}{\lambda})|} \int_{B(y,\frac{\epsilon}{\lambda})} u(\omega)\, d\omega \leq \frac{1}{|B(x,\epsilon)|} \int_{B(x,\epsilon)} u(\omega)\, d\omega + q_0(1-\lambda).$$

When $\epsilon \to 0$, we get $\lambda u(y) \leq u(x) + q_0(1-\lambda)$, so that

$$u(y) \leq u(x) + Q\frac{|x-y|}{|y - \pi_\Gamma(y|x)|}, \tag{11}$$

with $Q := q_0 + ||u||_{L^\infty(\Omega)}$.

This inequality holds for almost all $x, y \in B(x_0, d_\Gamma(x_0)/8)$, since Lebesgue points for u constitute a set of full measure. It follows that u admits a locally Lipschitz representative for which (11) holds everywhere in Ω, and the theorem is proved.

Corollary. The solution u satisfies

$$|Du(x)| \le \frac{Q}{d_\Gamma(x)},\ x \in \Omega \text{ a.e.}, \tag{12}$$

where Q depends on $\|u\|_{L^\infty(\Omega)}$ and the data of the problem (P).

3 A variant of the theorem.

The hypothesis (HG) used in the proof of Theorem 2.1 included the differentiability of G with respect to u. A natural approach to removing that condition is to approximate G by a smooth function G_i via mollification, apply the theorem in the differentiable case to the solution u_i of the perturbed problem (P_i), and then to pass to the limit. However, this line of argument requires an existence theorem for the perturbed problem, and one must also verify that the resulting Lipschitz condition for its solution u_i depends in a suitably stable way upon the data.

As regards existence, the required elements are provided for the most part in the results of Stampacchia [11], which can be adapted for the purpose described above. Following [11] (but without assuming differentiability) we introduce the hypothesis

$(HG)'$ G is measurable and we have

$$G(x,v) \ge -q|v|^2 - Q(x)|v|^\delta - R(x),$$

where $R \in L^1(\Omega), \delta \in (0,2), Q \in L^t(\Omega)$, with $1/t = 1 - \delta/2 + \delta/n$ and $q < \Lambda\mu/2$, where

$$\Lambda := \inf_{u \in W_0^{1,2}(\Omega)} \frac{\int_\Omega |Du|^2 dx}{\int_\Omega |u|^2 dx}.$$

Further, G is locally Lipschitz in u in the following sense: there exists $M > 0$ such that for any $u, u' \in \mathbb{R}$ and almost all $x \in \Omega$, one has

$$|G(x,u) - G(x,u')| \le M|u-u'|(1 + |u|^\beta + |u'|^\beta),$$

with $0 \le \beta < 2^* - 1$, where $1/2^* = 1/2 - 1/n$ if $n > 2$, and 2^* is any number greater than 2 if $n = 2$. Finally, we assume there is a function $\bar{u} \in W^{1,2}(\Omega)$ admissible for (P) such that $I(\bar{u}) < +\infty$.

We say that u solves (P) relative to $W^{1,2}(\Omega)$ if u is itself in that class, and if we have $I(u) \le I(w)$ for all $w \in \phi + W_0^{1,2}(\Omega)$.

Theorem 3.1
Under hypotheses $(H\Omega)$, (HF), and $(HG)'$, there exists a solution to problem (P) relative to $W^{1,2}(\Omega)$. Any such solution u is bounded, and is a solution of (P) relative to $L^\infty(\Omega)$; further, if ϕ satisfies the Lower Bounded Slope Condition, then u is locally Lipschitz in Ω.

The fact that a solution u_0 exists is provided by Theorem 8.1 in [11]. As indicated above, the next step in the proof is to approximate G by a smooth function G_i; a term $|u - u_0(x)|^2$ is added to assure convergence of the solution u_i of the perturbed problem to u_0. The existence theorem in [11] must be detailed more completely in order to observe the stability of the estimates with respect to the type of perturbations present (in particular, the provenance of the bound on $\|u_i\|_{L^\infty(\Omega)}$ must be carefully traced). Then Theorem 2.1 is applied to deduce the Lipschitz condition (12), which carries over in the limit to u_0. We omit the essentially routine details of this proof.

4 Continuity at the boundary.

The proof of Theorem 2.1 provided a Lipschitz constant for the solution u (see (12)) that goes to infinity at the boundary. We know by example that in general u fails to be globally Lipschitz, so this must be expected. But there remains the question of whether u is continuous at the boundary. Such a continuity conclusion cannot result from (12) alone, but it turns out that the directional nature of the Lipschitz condition (11), together with the barrier provided by Theorem 2.2, provides the extra information needed to obtain boundary continuity in a number of special cases. The arguments of [3] go through with no change, so we content ourselves here with recording the results. Note that the issue of continuity at the boundary does not arise in the classical setting with BSC, since then the solution is globally Lipschitz on Ω.

The theorems below introduce the hypothesis that u belongs to $W^{1,p}(\Omega)$. Under the hypotheses of either Theorem 2.1 or 3.1, this is easily seen to hold whenever F satisfies, for certain positive constants σ and N,

$$F(v) \geq \sigma |v|^p - N \quad \forall\, v \in \mathbb{R}^n.$$

Our hypothesis (HF) already guarantees that this holds for $p = 2$.

Theorem 4.1
In addition to the hypotheses of either Theorem 2.1 or 3.1, assume that Γ is a polyhedron. Then any solution u of (P) is Hölder continuous on $\overline{\Omega}$ of order $1/(n+2)$. If moreover $u \in W^{1,p}(\Omega)$ with $p > 2$, then u satisfies on $\overline{\Omega}$ a Hölder condition of order

$$a := \frac{p-1}{n+2p-2}.$$

Theorem 4.2
In addition to the hypotheses of either Theorem 2.1 or 3.1, assume that Γ *is* $C^{1,1}$ *and that* u *is a solution of* (P) *lying in* $W^{1,p}(\Omega)$*, with* $p > (n+1)/2$*. Then* u *satisfies on* $\overline{\Omega}$ *a Hölder condition of order*

$$b := \frac{2p - n - 1}{4p + n - 3}.$$

Under merely the hypotheses of Theorem 2.1 or 3.1, it is an open question whether a solution u of (P) must be continuous at the boundary.

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