Topological singularities in $W^{s,p}(S^N, S^1)$

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Abstract

We are interested in the location of the singularities of maps $u \in W^{s,p}(S^N, S^1)$ when $1 \leq s, p$ and $1 < sp < 2$. To this end, we consider the distributional Jacobian. We show that the range of this operator on $W^{s,p}(S^N, S^1)$ is the closure in $W^{s-2,p} \cap W^{-1,sp}$ of the set of $N-2$ currents defined as the integration on smooth oriented $N-2$ dimensional boundaryless submanifolds.

1 Introduction

In this article, we are interested in the location of the singularities of maps u defined on S^N with values into S^1. Assume first that $u \in C^\infty(S^N \setminus A, S^1) \cap W^{1,1}(S^N, S^1)$. When A is 'small' (i.e. of finite $(N-2)$ Hausdorff measure), the set A can be recovered from u by computing the Jacobian of u. This quantity has been introduced in [8] in the context of liquid cristals, and also studied in [15] and [1]. It is defined as follows: let ω_0 be the 1 form in $\mathbb{R}^2$ given by

$$\omega_0(y) := y_1 dy_2 - y_2 dy_1.$$

Its restriction to the unit circle is exactly the standard volume form on S^1. The pullback of ω_0 by u is defined by

$$u^\sharp \omega_0 := u_1 du_2 - u_2 du_1 =: j(u).$$

This definition makes sense not only when u is smooth (that is when $A = \emptyset$) but also when u belongs merely to $W^{1,1}(S^N, S^1)$. In this case, the Jacobian $J(u)$ of u will be defined, in the distribution sense, as $1/2d(u^\sharp\omega_0)$, that is:

$$\langle J(u), \omega \rangle = \frac{1}{2}\langle d(u^\sharp \omega_0), \omega \rangle := -\frac{1}{2}\langle u^\sharp \omega_0, \delta\omega \rangle, \quad \forall \omega \in C^\infty(\Lambda^2 S^N).$$

Here, $\langle ., . \rangle$ denotes the inner product between forms of the same degree and δ is the formal adjoint of the differential operator d. Using the Hodge operator

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$\star$ (see precise definitions in section 2), the Jacobian of u can also be written as:

$$\langle J(u), \omega \rangle = -\frac{1}{2} \int_{S^N} (u^\sharp \omega_0) \wedge (\star \delta \omega).$$

First, note that when u is smooth with values into S^1 (that is when $A = \emptyset$), the Jacobian $J(u)$ is zero, since we have in local coordinates:

$$\begin{aligned} J(u) = \frac{1}{2} d(u_1 du_2 - u_2 du_1) &= \frac{1}{2}(du_1 \wedge du_2 - du_2 \wedge du_1) \\ = du_1 \wedge du_2 &= \sum_{i<j} (u_{1x_i} u_{2x_j} - u_{1x_j} u_{2x_j}) dx_i \wedge dx_j. \end{aligned}$$

The rank of the tangent map $T_x u$ is at most 1, so that all the minors of order 2 vanish. This shows that $J(u)$ is zero when u is smooth.

Consider now the case when $N = 2$ and A is a nonempty finite set of points. Then (see [8] and also [4]), we have:

$$\star J(u) = \pi \sum_{a \in A} \deg(u, a) \delta_a, \tag{1}$$

where δ_a is the Dirac mass in a and $\deg(u, a)$ is the degree of the restriction of u to a small well-oriented circle around a.

When $N \geq 3$, there is an analogue of (1) provided A is a finite union of $N-2$ dimensional connected oriented boundaryless manifolds. Let C be any small circle which links with such a manifold, say Γ. On C there is a natural orientation which is consistent with the orientation of Γ. For any $u \in C^\infty(S^N \setminus \Gamma, S^1)$, we can define the degree of the restriction of u to C. This degree is independent of the choice of C (see a more precise statement in section 2) and we denote it by $\deg(u, \Gamma)$.

Then the value of $J(u)$ is given by the following proposition (stated in [1]):

Proposition 1 *When A is a smooth oriented $N-2$ dimensional boundaryless manifold ($N \geq 3$), with connected components $A_1, ..., A_r$, we have*

$$\star J(u) := \pi \sum_{i=1}^{r} deg(u, A_i) \int_{A_i} \cdot \tag{2}$$

Here, $\int_{A_i} \cdot$ is the $N-2$ current defined on the set of smooth forms of degree $N-2$ by: $\zeta \mapsto \int_{A_i} \zeta$ and $deg(u, A_i)$ is the degree of u around A_i.

Note that there exist topological obstructions on A and the degrees. For instance, when $N = 2, \langle J(u), 1 \rangle = 0$ (by definition of $J(u)$) so that $\sum_{a \in A} \deg(u, a) = 0$.

The interest of $J(u)$ is the possibility to identify a singular set A which is still relevant for any map $u \in W^{1,1}(S^N, S^1)$. Indeed, let $\mathcal{R}_0$ be the following set:

$$\bullet N = 2 : \mathcal{R}_0 := \{u \in \bigcap_{1 \le r < 2} W^{1,r}(S^2, S^1); u \text{ is smooth outside a finite set of points}\}$$

$$\bullet N \ge 3 : \mathcal{R}_0 := \{u \in \bigcap_{1 \le r < 2} W^{1,r}(S^N, S^1); u \text{ is smooth outside a smooth oriented } N-2 \text{ dimensional boundaryless submanifold}\}.$$

The class $\mathcal{R}_0$ is dense in $W^{1,1}(S^N, S^1)$ (see [2]). Furthermore, J is a continuous map from $W^{1,1}(S^N, S^1)$ into $(W^{1,\infty}(\Lambda^2 S^N))^*$, the dual space of Lipschitz forms of degree 2 on S^N. Using these two results together, we get (see [10] for the case $N = 2$ and [1] for $N \ge 3$):

• $N = 2, \star J(u) = \pi \sum(\delta_{P_i} - \delta_{N_i})$ with $\sum_i d(P_i, N_i) \le C||du||_{L^1(\Lambda^1 S^2)}$.

• $N \ge 3, \star J(u) = \pi \partial S$ where S is an $N-1$ dimensional rectifiable current (in the sense of [12]) whose mass $||S||$ satisfies $||S|| \le C||du||_{L^1(\Lambda^1 S^N)}$.

There exists a converse to the previous properties (see [10] and [1]):

• $N = 2$, let $T := \sum(\delta_{P_i} - \delta_{N_i})$ with $\sum_i d(P_i, N_i) < \infty$. Then there exists $u \in W^{1,1}(S^N, S^1)$ such that $\star J(u) = \pi T$.

• $N \ge 3$, let T be the boundary of an $N-1$ dimensional rectifiable current with finite mass. Then there exists $u \in W^{1,1}(S^N, S^1)$ such that $\star J(u) = \pi T$.

To see that $J(u)$ does describe in some sense the singular set of u, the following result, due to Bethuel, is relevant:

$$u \in \overline{C^\infty(S^N, S^1)}^{W^{1,1}} \iff J(u) = 0. \tag{3}$$

The aim of this paper is twofold: we want to describe the range of $J(u)$ when u belongs to a fractional Sobolev space $W^{s,p}(S^N, S^1)$, and to generalise (3) to this context.

Let us first note that $C^\infty(S^N, S^1)$ is dense in $W^{s,p}(S^N, S^1)$ when $sp < 1$ (see [11]) or $sp \ge 2$ (see [7] when $N = 2$ and [3] when $N \ge 3$), and thus there is no 'good' notion of singular set in that case. Hence, in the following, we will assume that $1 \le sp < 2$. If $s \ge 1$, then $W^{s,p}(S^N, S^1) \subset W^{1,1}(S^N, S^1)$, so that $J(u)$ is defined as above. In particular, it is still true that $\star J(u)$ is the boundary of a rectifiable current with codimension 1 and finite mass. However, such a current is *not* in general the Jacobian of some $u \in W^{s,p}(S^N, S^1)$. A counterexample is given at the beginning of section 3.

Let $\mathcal{E}$ denote the set of $N-2$ currents of the form:

$$\bullet N = 2 : \quad \pi \sum_{i=1}^{r} (\delta_{B_i} - \delta_{C_i}) \ , \ r \in \mathbb{N}, \text{ where } B_i, C_i \text{ are points in } S^2,$$

$\bullet N \geq 3: \quad \pi \sum_{i=1}^{r} \int_{A_i} \cdot \ , \ r \in \mathbb{N}$, where A_i is a smooth oriented connected $N-2$ dimensional boundaryless submanifold.

Our main result is the following:

Theorem 1 *Let $s \geq 1, 1 \leq p < \infty, 1 < sp < 2$.*

a) If u belongs to $W^{s,p}(S^N, S^1)$, then $\star J(u)$ belongs to the closure of $\mathcal{E}$ in $W^{s-2,p}(\Lambda^{N-2}S^N) \cap W^{-1,sp}(\Lambda^{N-2}S^N)$. Moreover, we have

$$||J(u)||_{W^{s-2,p}(\Lambda^2 S^N)} \leq C||u||_{W^{s,p}(S^N)} \ , \ ||J(u)||_{W^{-1,sp}(\Lambda^2 S^N)} \leq C||u||^{1/s}_{W^{s,p}(S^N)}.$$

b) Conversely, if M belongs to the closure of $\mathcal{E}$ in $W^{s-2,p}(\Lambda^{N-2}S^N) \cap W^{-1,sp}(\Lambda^{N-2}S^N)$, then there exists $u \in W^{s,p}(S^N, S^1)$ such that $\star J(u) = M$. In addition, we may choose u such that

$$||u||_{W^{s,p}(S^N)} \leq C(||M||_{W^{s-2,p}(\Lambda^{N-2}S^N)} + ||M||^s_{W^{-1,sp}(\Lambda^{N-2}S^N)})$$

for some constant $C \geq 0$.

To prove this theorem, we will use a density result:

Theorem 2 *The set $\mathcal{R} := \mathcal{R}_0 \cap W^{s,p}(S^N, S^1)$ is dense in $W^{s,p}(S^N, S^1)$.*

This answers an open problem raised in [6]. Theorem 2 was already known for $s = 1$ (see [2]), and $s < 1$ (see [5], which generalizes previous results in [21], [13]). Our result covers the remaining case $1 < s$.

Finally, the analogue of (3) in the context of $W^{s,p}(S^N, S^1)$ spaces is

Theorem 3

$$u \in \overline{C^\infty(S^N, S^1)}^{W^{s,p}(S^N,S^1)} \Longleftrightarrow J(u) = 0.$$

In the case when $s < 1$, the Jacobian can still be defined, but with another formula (see [5]). The description of $J(u)$ in that case remains open. However, Theorem 3 still holds when $N = 2$ and $s < 1$ (see [20]).

The paper is organized as follows. In the next section, we describe the notations and give the precise definitions used throughout the article. In section 3, we prove Proposition 1 and the first part of Theorem 1. The proof relies on the regularity theory for the Laplace-Beltrami operator (briefly recalled in the last section) and the density of $\mathcal{R}$ (whose proof is postponed to section 5). Section 4 is dedicated to the proof of the second part of Theorem 1 and to the proof of Theorem 3.

2 Definitions

The unit sphere S^N is a smooth manifold of dimension N, embedded in $\mathbb{R}^{N+1}$, and it inheritates from $\mathbb{R}^{N+1}$ its Riemannian structure and its orientation (via its outer normal).

The Riemannian metric gives birth to an inner product on any tangent space T_xS^N to S^N at $x \in S^N$. We will denote it by $(.|.)$ (without mentioning the dependence on x). It can be extended to antisymetric multilinear forms on T_xS^N with the same notation. Then, we can define an inner product on l forms $(0 \leq l \leq N)$ as

$$\langle \alpha, \beta \rangle := \int_{S^N} (\alpha_x | \beta_x) d\mathcal{H}^N(x)$$

for any $\alpha, \beta \in C^\infty(\Lambda^l S^N)$, that is the set of smooth l forms on S^N. This inner product will be extended to measurable forms as soon as $x \to (\alpha_x | \beta_x)$ is an integrable function on S^N.

We follow [12] for the definitions of the exterior differential d, the codifferential δ and the Hodge operator. In particular, the Hodge operator $\star$ is a map from the l forms onto the $N-l$ forms $(0 \leq l \leq N)$ such that if $(e_1, ..., e_N)$ is an oriented orthonormal basis on T_xS^N, then

$$\star e_\alpha = \sigma(\alpha, \bar{\alpha}) e_{\bar{\alpha}}$$

where $\alpha = (\alpha_1 < ... < \alpha_l), e_\alpha = e_{\alpha_1} \wedge ... \wedge e_{\alpha_l}, \bar{\alpha}$ is the complement of α in $[|1, N|]$ in the natural increasing order and $\sigma(\alpha, \bar{\alpha})$ is the sign of the permutation which reorders $(\alpha, \bar{\alpha})$ in the natural increasing order. Then

$$\star\star = (-1)^{l(N-l)}$$

on l forms. We will use the fact that:

$$\langle \alpha, \beta \rangle = \int_{S^N} \alpha \wedge (\star\beta) \ , \ \forall \alpha, \beta \in C^\infty(\Lambda^l S^N).$$

The codifferential operator δ maps the smooth l forms $C^\infty(\Lambda^l S^N)$ into the smooth $l-1$ forms $C^\infty(\Lambda^{l-1} S^N)$. It is the formal adjoint of the differential operator d, that is:

$$\langle \delta\alpha, \beta \rangle = -\langle \alpha, d\beta \rangle, \ \ \forall \alpha \in C^\infty(\Lambda^l S^N), \ \beta \in C^\infty(\Lambda^{l-1} S^N).$$

The following property will be often used:

$$\delta = (-1)^{N(l+1)} \star d \star .$$

The Laplace-Beltrami operator on $C^\infty(\Lambda^l S^N)$ is

$$\Delta := d\delta + \delta d.$$

We need to define the degree of u around a smooth oriented connected $N-2$ dimensional boundaryless submanifold, say Γ. Fix $x_0 \in \Gamma$. There exists a connected neighborhood U of x_0 in Γ and two smooth vector fields v_1, v_2 on S^N such that $(v_1(x), v_2(x))$ is an orthonormal basis of $(T_x\Gamma)^\perp$ for any $x \in U$ (actually, this property could be assumed on the whole Γ since the normal bundle of an $N-2$ dimensional oriented boundaryless submanifold is trivial, see [16]). We may assume that $(v_1(x), v_2(x))$ is 'well-oriented', i.e. that, when $(e_1, .., e_{N-2})$ is a well-oriented basis of $T_x\Gamma$, then $(e_1, .., e_{N-2}, v_1(x), v_2(x))$ is a well-oriented basis of T_xS^N.

There exists $\eta > 0$ such that the endpoint $e(x, t_1, t_2)$ of the geodesic segment of length $r := (t_1^2 + {t_2}^2)^{1/2}$ which starts at x with the initial velocity vector $(t_1/r)v_1(x) + (t_2/r)v_2(x)$ is well defined for any $r < \eta$. Then, the map

$$e : (x, t_1, t_2) \in U \times B_{\mathbb{R}^2}(0, \eta) \mapsto e(x, t_1, t_2)$$

is a diffeomorphism from $U \times B_{\mathbb{R}^2}(0, \eta)$ onto a neighborhood U_η of U in S^N (see the Product Neighborhood Theorem, [18]). Now, for any $x \in U$, we can define the circle $C(x, r)$ centered in x and of radius $r < \eta$ as the set

$$C(x, r) := \{e(x, r\cos\theta, r\sin\theta) : \theta \in [0, 2\pi]\}.$$

We define the degree of u on $C(x, r)$ as the degree of the map $v : S^1 \to S^1, v(\cos\theta, \sin\theta) := u(e(x, r\cos\theta, r\sin\theta))$. Note that the parametrization $\theta \mapsto e(e, r\cos\theta, r\sin\theta)$ defines an orientation on $C(x, r)$, and that the degree of u on $C(x, r)$ is precisely the degree of u with respect to this orientation.

We next check that this degree does not depend on x and on small $r > 0$. Let $(x, r), (x', r') \in U \times [0, \eta)$. We want to show that there exists an orientation preserving homotopy which maps continuously $C(x, r)$ onto $C(x', r')$. Since Γ is connected, there exists a continuous map $l : [0, 1] \to \Gamma$ such that $l(0) = x$ and $l(1) = x'$. Then, we define:

$$H : (t, \theta) \in [0, 1] \times [0, 2\pi] \to e(l(t), [(1-t)r + tr']\cos\theta, [(1-t)r + tr']\sin\theta).$$

The map H is the desired homotopy. By connectedness, it does make sense to define the degree $\deg(u, \Gamma)$ of u as the degree of u restricted to $C(x, r)$ for any $x \in \Gamma$ and any r sufficently small.

Let $(U_i', V_i', \phi_i)_{i\in\{1,2\}}$ be an oriented atlas of S^N and $U_i \subset \overline{U}_i \subset U_i'$ be open sets such that $U_1 \cup U_2 = S^N$. We denote $V_i := \phi_i(U_i)$. Let $(\theta_i)_{i\in\{1,2\}}$ be a partition of unity subordinate to the covering $(U_i)_{i\in\{1,2\}}$. We will also introduce $\psi_i = \phi_i^{-1}$. We will denote by

$$g_{jk}(x) := (\frac{\partial}{\partial x_j}|\frac{\partial}{\partial x_k})$$

the coefficients of the metric tensor of g (in local coordinates $(x_1, ..., x_N) := \phi_i$) and $(g^{jk}(x)) = (g_{jk}(x))^{-1}$. By continuity and compacity, there exists

$C > 0$ such that

$$||d_x\phi_i|| \leq C, ||d_y\psi_i|| \leq C, \frac{1}{C}|\eta|^2 \leq \sum_{j,k} g_{jk}(x)\eta_j\eta_k \leq C|\eta|^2$$

for any $i = 1, 2, x \in U_i, y \in V_i, \eta = (\eta_1, ..., \eta_N) \in \mathbb{R}^N$.

The space of l currents is the topological dual of the space of l forms: $C^\infty(\Lambda^l S^N)$, the latter being equipped with the usual topology, see [23]. It will be denoted by $\mathcal{D}'(\Lambda^l S^N)$. Any integrable l form $\alpha \in L^1(\Lambda^l S^N)$ defines an l current by:

$$\langle T_\alpha, \beta\rangle := \int_{S^N} (\alpha_x|\beta_x) d\mathcal{H}^N(x) \quad , \ \forall \beta \in C^\infty(\Lambda^l S^N). \tag{4}$$

In the following, we will identify α and T_α. This identification is a guideline to define several operations on currents. For instance,

$$\langle \star T, \omega\rangle = (-1)^{l(N-l)}\langle T, \star\omega\rangle$$

for any $\omega \in C^\infty(\Lambda^l S^N)$. The exterior differential d as well as the codifferential δ are defined by duality on $\mathcal{D}'(\Lambda^l S^N)$.

The multiplication of a distribution on l forms $T \in \mathcal{D}'(\Lambda^l(M))$ and a smooth function θ is defined as:

$$\langle \theta T, \alpha\rangle := \langle T, \theta\alpha\rangle, \quad \forall \alpha \in C^\infty(\Lambda^l S^N).$$

The pushing forward of a distribution $T \in \mathcal{D}'(\Lambda^l(S^N))$ compactly supported in some U_i by the smooth diffeomormphism $\phi_i : U_i \to V_i$ is defined by

$$\langle \phi_{i\sharp}T, \alpha\rangle = \langle \star T, \phi_i^\sharp(\star_0\alpha)\rangle, \quad \forall \alpha \in C^\infty(\Lambda^l V_i),$$

where $\star_0$ is the Hodge operator in $\mathbb{R}^N$ (endowed with the Euclidean metric) and $\phi_i^\sharp(\star_0\alpha)$ denotes the pullback of $\star_0\alpha$ by ϕ_i.

To justify this definition, note that if $T = T_\omega$ were defined by an integrable l form ω, as in (4), then we would set $\phi_{i\sharp}T_\omega := T_{\phi_{i\sharp}\omega}$, that is for any $\alpha \in C^\infty(\Lambda^l V_i)$:

$$\begin{aligned} \langle \phi_{i\sharp}T_\omega, \alpha\rangle = \int_{V_i} (\phi_{i\sharp}\omega|\alpha)_0 \quad &= \quad \int_{V_i} (\phi_{i\sharp}\omega) \wedge (\star_0\alpha) \\ = \int_{U_i} \phi_i^\sharp\{(\phi_{i\sharp}\omega) \wedge (\star_0\alpha)\} \quad &= \quad \int_{U_i} \omega \wedge \phi_i^\sharp(\star_0\alpha) = \langle \star T_\omega, \phi_i^\sharp(\star_0\alpha)\rangle. \end{aligned}$$

(In the first line, we have denoted by $(\cdot|\cdot)_0$ the Euclidean inner product on $\mathbb{R}^N$).

Note also that since $\phi_{i\sharp}T$ is compactly supported in V_i (its support being included in $\phi_i(\text{supp}T)$), we can consider it as an element of $\mathcal{D}'(\Lambda^l\mathbb{R}^N)$.

The multiplication of a distribution by an element of the partition of unity is called *localization*. The pushing forward of a distribution by ϕ_i is called *rectification*. Finally, when a distribution is compactly supported in an open set $V \subset \mathbb{R}^N$, we will automatically identify it with a distribution on $\mathbb{R}^N$, in the usual way. This procedure corresponds to the one described in the case of 0 forms in [26].

Several spaces of functions, of forms, of distributions on forms appear in the statement of the theorems or in the proofs below. Sobolev spaces on l forms $(0 \leq l \leq N)$ $W^{k,p}(\Lambda^l S^N), k \in \mathbb{N}, p \geq 1$ are defined as in [19], Chapter 7 (or [12]), that is *via* charts defining an atlas on S^N. In [24], one can find an *intrinsic* definition of Sobolev spaces on forms (that is without references to local charts), which turns out to be rather convenient. When $1 < p < \infty$ and $k \in \mathbb{N}^*$, we define $W^{-k,p}(\Lambda^l S^N) := (W^{k,p'}(\Lambda^l S^N))^*$, where $p' = p/(p-1)$. Besov spaces of functions and of distributions on the boundary of an open set (which is the case of S^N) are defined in [26], and some properties of these sets are studied there. We will denote them $B^s_{p,q}(S^N), s \in \mathbb{R}, p, q \geq 1$. The corresponding definitions for p forms and distributions on p forms (which could be called *Besov currents*) remain to be given, thanks to a localization-rectification procedure.

Let $A(\mathbb{R}^N)$ be a vector subspace of $\mathcal{D}'(\mathbb{R}^N)$, equipped with a norm $||.||_{A(\mathbb{R}^N)}$. We make two hypotheses on $A(\mathbb{R}^N)$: the *multiplication property* and the *diffeomorphism property.* The multiplication property requires that for any $u \in A(\mathbb{R}^N)$ and any $\theta \in C^\infty_c(\mathbb{R}^N), \theta u \in A(\mathbb{R}^N)$ with $||\theta u||_{A(\mathbb{R}^N)} \leq C(\theta)||u||_{A(\mathbb{R}^N)}$. The diffeomorphism property requires that for any $u \in A(\mathbb{R}^N)$ compactly supported in some open set V and for any diffeomorphism ϕ between two open sets U and V in $\mathbb{R}^N$, the distribution $u \circ \phi$ belongs to $A(\mathbb{R}^N)$ and satisfies $||u \circ \phi||_{A(\mathbb{R}^N)} \leq C(\phi)||u||_{A(\mathbb{R}^N)}$.

Now, it is possible to define $A(\Lambda^l \mathbb{R}^N)$ as the product of l copies of $A(\mathbb{R}^N)$, endowed with the product topology (and a norm defining it). This definition follows the definition of $\mathcal{D}'(\Lambda^l \mathbb{R}^N)$, the set of distributions on l forms, which can be identified with the product of l copies of $\mathcal{D}'(\mathbb{R}^N)$. Then $A(\Lambda^l \mathbb{R}^N)$ still satisfies the multiplication property and the diffeomorphism property (where the multiplication and the composition are now understood in the sense of l currents $\mathcal{D}'(\Lambda^l \mathbb{R}^N)$, exactly as we have done above in the case of S^N).

Finally, we define $A(\Lambda^l S^N)$ as the set of those elements T in $\mathcal{D}'(\Lambda^l S^N)$ such that for $i = 1, 2, \phi_{i\sharp}(\theta_i T) \in A(\Lambda^l \mathbb{R}^N)$. (Recall that $\phi_{i\sharp}(\theta_i T)$ is extended by 0 on $\mathbb{R}^N \setminus V_i$). A norm on $A(\Lambda^l S^N)$ is then given by

$$\sum_i ||\phi_{i\sharp}(\theta_i T)||_{A(\Lambda^l \mathbb{R}^N))}.$$

Different atlases and partitions of unity yield equivalent norms.

The Besov spaces $B^s_{p,q}(\mathbb{R}^N)$ (see [26]) satisfy the multiplication property and the diffeomorphism property, so that we can define $B^s_{p,q}(\Lambda^l S^N)$, the Besov space of l forms on S^N.

Among the Besov spaces, only the fractional Sobolev spaces and their duals will be of interest to us. When s is not an integer, we set $W^{s,p}(\Lambda^l S^N) := B^s_{p,p}(\Lambda^l S^N)$.

For the following, it is also convenient to have intrinsic definitions of $W^{s,p}(S^N)$ when $s \in]1,2[$. We can see that $u \in W^{s,p}(S^N)$ if and only if $u \in W^{1,p}(S^N)$ and $D_{\sigma,p} du \in L^p(S^N)$ where $\sigma := s-1$ and

$$D_{\sigma,p}\alpha(x) := \{\int_{S^N} \frac{|\alpha_x - \alpha_y|^p}{d(x,y)^{N+\sigma p}}\, dy\}^{1/p} \quad \forall \alpha \in L^p(\Lambda^1 S^N),$$

with $|\alpha_x - \alpha_y|$ defined by

$$|\alpha_x - \alpha_y| := \sum_{i:x,y\in U_i} |\alpha_x - \alpha_y|_i \tag{5}$$

and if $x, y \in U_i$,

$$|\alpha_x - \alpha_y|_i = \sum_{k=1}^{N} |\alpha^k(x) - \alpha^k(y)|$$

where $\alpha =: \sum_k \alpha^k dx_k$ in the local coordinates $(x_1, ..., x_N) := \phi_i$ on U_i. Then, for any $\alpha \in W^{\sigma,p}(\Lambda^1 S^N)$, we define

$$||\alpha||_{W^{\sigma,p}(\Lambda^1 S^N)} := ||\alpha||_{L^p(\Lambda^1 S^N)} + ||D_{\sigma,p} du||_{L^p(S^N)}.$$

Now, a norm on $W^{s,p}(S^N)$ is given by

$$||u||_{W^{s,p}(S^N)} := ||u||_{L^p(S^N)} + ||du||_{W^{\sigma,p}(\Lambda^1 S^N)}.$$

We will also use the notation $D_{\sigma,p}$ for functions $u \in L^p(S^N)$:

$$D_{\sigma,p}u(x) := \{\int_{S^N} \frac{|u(x) - u(y)|^p}{d(x,y)^{N+\sigma p}}\, dy\}^{1/p}$$

or for 1 forms with values into some $\mathbb{R}^d$ (if $\alpha := (\alpha_1, .., \alpha_d)$, the quantity $|\alpha_x - \alpha_y|$ becomes $\displaystyle\sum_{i:x,y\in U_i} \sum_{k=1}^{N} \sum_{j=1}^{d} |\alpha^k_j(x) - \alpha^k_j(y)|_i$).

The following remarks will be useful: The operator d is a bounded linear operator from $W^{s,p}(\Lambda^l S^N)$ into $W^{s-1,p}(\Lambda^{l+1} S^N)$, for $1 < p < \infty, s \in \mathbb{Z}$ or $1 \leq p < \infty, s \notin \mathbb{Z}$. The multiplication property implies that if $T \in W^{s,p}(\Lambda^l S^N)$ and $\theta \in C^\infty(S^N)$, then $\theta T \in W^{s,p}(\Lambda^l S^N)$. Any embedding between two Besov spaces on $\mathbb{R}^N$ has its counterpart for Besov currents on S^N.

3 Proof of Theorem 1, first part

In this section, we want to prove Theorem 1 a). First, we are going to justify its interest by presenting an example of some $T \in \star J(W^{1,1}(S^N, S^1))$ which does not belong to $\star J(W^{s,p}(S^N, S^1))$. We consider the case $s = 1, p \in]1, 2[$ and $N = 2$. In that case, we know that

$$\star J(W^{1,1}(S^2, S^1)) := \{\pi \sum_i (\delta_{P_i} - \delta_{N_i}) : \sum_i d(P_i, N_i) < \infty\}.$$

Moreover, it is easy to see that $J(W^{1,p}(S^2, S^1)) \subset W^{-1,p}(\Lambda^2 S^2)$ (see details below).

Let $d_i := 1/i^{1/\alpha}$ where $\alpha \in]1 - 1/p', 1[$. Let $N_i := (\sqrt{1 - d_i^2}, 0, d_i)$ and $P_i := (\sqrt{1 - 4d_i^2}, 0, 2d_i)$. Set $T := \sum_i (\delta_{P_i} - \delta_{N_i})$. For any $n \geq 1$, we define $u_n(x, y, z) = z^\alpha$ if $z > 1/n$ and $1/n^\alpha$ elsewhere. Then, u_n is Lipschitz on S^2. The sequence $(||u_n||_{W^{1,p'}(S^2)})_n$ is bounded (here, we use $(1 - \alpha)p' < 1$). Hence, if T were in $W^{-1,p}(S^2)$, then the sequence $(|T(u_n)|)_n$ would be bounded too. We now show that this is not the case.

First, we note that if $0 < z_1 < z_2$, then

$$z_2^\alpha - z_1^\alpha \geq \alpha(z_2 - z_1)^\alpha (\frac{z_2 - z_1}{z_2})^{1-\alpha}.$$

This implies that, if $d_i \geq 1/n$, then

$$u_n(P_i) - u_n(N_i) \geq \alpha 2^{\alpha-1} d_i^\alpha,$$

so that

$$T(u_n) \geq \alpha 2^{\alpha-1} \sum_{i : d_i \geq 1/n} d_i^\alpha = \alpha 2^{\alpha-1} \sum_{i \leq n^\alpha} 1/i.$$

The right side goes to $+\infty$, as claimed. This completes the proof of the fact that $J(W^{1,p}(S^2, S^1))$ is strictly contained in $J(W^{1,1}(S^2, S^1))$.

To prove Theorem 1, we will first calculate J on the set $\mathcal{R}$ (Proposition 1): the result is well known but to our knowledge, no proof has been published yet. Then, we will show that J is continuous from $W^{s,p}(S^N, S^1)$ into $W^{s-2,p}(\Lambda^2 S^N) \cap W^{-1,sp}(\Lambda^2 S^N)$. Finally, we will use the density of $\mathcal{R}$ into $W^{s,p}(S^N, S^1)$ (the proof of which is postponed to section 6) to get the result.

Proof of Proposition 1. In the case when $N = 2$, a proof can be found in [4]. Hence, we restrict our attention to the case $N \geq 3$. Let Γ be a smooth oriented $N - 2$ dimensional boundaryless submanifold of S^N. Let u be a smooth map on $S^N \setminus \Gamma$, and we assume that u belongs to $W^{1,1}(S^N, S^1)$. We want to prove that:

$$\langle J(u), \zeta \rangle = \pi \sum_{i=1}^{r} \deg(u, \Gamma_i) \int_{\Gamma_i} \star\zeta \ , \quad \forall \zeta \in C^\infty(\Lambda^2 S^N), \tag{6}$$

where $\Gamma_1, .., \Gamma_r$ are the connected components of Γ. As stated in section 2, there exist two smooth vector fields v_1, v_2 on S^N such that $(v_1(x), v_2(x))$ is an orthonormal basis of $(T_x\Gamma)^\perp$ for any $x \in \Gamma$. In addition, we may assume that (v_1, v_2) is well-oriented. There exists $\eta > 0$ such that the endpoint $e(x, t_1, t_2)$ of the geodesic segment of length $r := (t_1^2 + {t_2}^2)^{1/2}$ which starts at x with the initial velocity vector $(t_1/r)v_1(x) + (t_2/r)v_2(x)$ is well defined for any $r < \eta$ and the map

$$e : (x, t_1, t_2) \in \Gamma \times B_{\mathbb{R}^2}(0, \eta) \mapsto e(x, t_1, t_2)$$

is a diffeomorphism from $\Gamma \times B_{\mathbb{R}^2}(0, \eta)$ onto a neighborhood Δ_η of Γ. Each point $x \in \Gamma$ belongs to the domain U of a well-oriented chart $\phi_0 : U \subset S^N \to V \subset \mathbb{R}^N$ which satisfies:

$$\phi_0(U \cap \Gamma) = V \cap (\mathbb{R}^{N-2} \times \{(0, 0)\}).$$

We can assume that $U \subset \Delta_\eta$. We define:

$$\phi : x \in U \mapsto (\phi_0(x'), t_1, t_2) \in \mathbb{R}^{N-2} \times B_{\mathbb{R}^2}(0, \eta)$$

where $x' \in \Gamma, (t_1, t_2) \in B_{\mathbb{R}^2}(0, \eta)$ are defined by $e(x', t_1, t_2) = x$. Then ϕ is still a diffeomorphism from U onto $\phi(U)$ and we can assume (by shrinking U if necessary) that V has the form $]-\sigma, \sigma[^N$. The interest of this modification is that ϕ^{-1} maps the circle $C(\phi(x'), r) := \{(\phi(x'), r\cos\theta, r\sin\theta) : \theta \in [0, 2\pi]\}$ onto the circle in S^N : $\{e(x', r\cos\theta, r\sin\theta) : \theta \in [0, 2\pi]\}$. This remark will be useful below.

Let $\zeta \in C^\infty(\Lambda^2 S^N)$. Using a partition of unity, we may assume that ζ is compactly supported in the domain U of a chart ϕ of the type above.

In particular, $\operatorname{supp}\zeta$ intersects only one connected component of Γ, say Γ_1. Let us introduce some notations. We will decompose any $x \in \mathbb{R}^N$ as $x = (x', y, z) \in \mathbb{R}^{N-2} \times \mathbb{R} \times \mathbb{R}$. For small $\epsilon > 0$ and $\delta \in]0, \pi/2[$, we define:

$$\Delta_\epsilon := \phi^{-1}(\{(x', y, z) \in V : |(y, z)| < \epsilon\}),$$

$$\Sigma_\epsilon := \phi^{-1}(\{(x', y, z) \in V : |(y, z)| = \epsilon\}),$$

$$\Sigma_{\epsilon,\delta} := \phi^{-1}(\{(x', \epsilon\cos\theta, \epsilon\sin\theta) \in V : \theta \in]\delta, 2\pi - \delta[\}),$$

$$A := \phi^{-1}(\{(x', y, z) \in V : z = 0, y \geq 0\}).$$

The set $U_0 := U \setminus A$ is simply connected (since it is homeomorphic to a star-shaped open set in $\mathbb{R}^N$). The map u is smooth on U_0 and takes its values into S^1. So, there exists some smooth function $\kappa : U_0 \to \mathbb{R}$ such that

$$u = (\cos\kappa, \sin\kappa) \quad \text{on} \quad U_0.$$

Moreover, $|\nabla\kappa| = |\nabla u|$, so that κ is Lipschitz continuous on $U_0 \cap \Sigma_\epsilon$, its Lipschitz constant depending only on ϵ. This implies that $\kappa\circ\phi^{-1}(x', \epsilon\cos\delta, \epsilon\sin\delta)$ has a limit $\kappa\circ\phi^{-1}(x', \epsilon, 0^+)$ when $\delta \to 0^+$, the convergence being uniform with respect to $x' \in]-\sigma, \sigma[^{N-2}$. Similarly, $\kappa\circ\phi^{-1}(x', \epsilon\cos\delta, \epsilon\sin\delta)$ converges to $\kappa\circ\phi^{-1}(x', \epsilon, 2\pi^-)$ when $\delta \to 2\pi^-$, uniformly with respect to x'. Furthermore, the quantity $\kappa\circ\phi^{-1}(x', \epsilon, 2\pi^-) - \kappa\circ\phi^{-1}(x', \epsilon, 0^+)$ is exactly $2\pi\text{deg}\,(u, \Gamma_1)$ since

$$\phi^{-1}(\{(x', \epsilon\cos\theta, \epsilon\sin\theta) : \theta \in [0, 2\pi]\})$$

is the circle perpendicular to Γ_1 at x with radius ϵ. The definition of the Jacobian and the dominated convergence theorem imply that:

$$\langle J(u), \zeta\rangle = \lim_{\epsilon\to 0}\frac{1}{2}\int_{S^N\setminus\Delta_\epsilon} j(u)\wedge(d\star\zeta) = \lim_{\epsilon\to 0}\frac{1}{2}\int_{U\setminus\Delta_\epsilon} j(u)\wedge(d\star\zeta).$$

Using the formula $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^{\deg\alpha}\alpha\wedge d\beta$ for two forms α, β, we have:

$$\begin{aligned}\int_{U\setminus\Delta_\epsilon} j(u)\wedge(d\star\zeta) &= -\int_{U\setminus\Delta_\epsilon} d(j(u)\wedge(\star\zeta)) + \int_{U\setminus\Delta_\epsilon} d(j(u))\wedge(\star\zeta) \\ &= \int_{\partial(U\setminus\Delta_\epsilon)} j(u)\wedge(\star\zeta).\end{aligned}$$

The second line follows from the Stokes' formula and the fact that $d(j(u)) = 0$ pointwise on $U \setminus \Delta_\epsilon$.

On U_0, we have $j(u) = d\kappa$. Whence (note that $\Sigma_{\epsilon,0} = \partial\Delta_\epsilon \setminus A$),

$$\int_{\partial(U\setminus\Delta_\epsilon)} j(u)\wedge(\star\zeta) = \lim_{\delta\to 0}\int_{\Sigma_{\epsilon,\delta}} d\kappa\wedge(\star\zeta).$$

Write once again:

$$\begin{aligned}\int_{\Sigma_{\epsilon,\delta}} d\kappa\wedge(\star\zeta) &= \int_{\Sigma_{\epsilon,\delta}} d(\kappa(\star\zeta)) - \int_{\Sigma_{\epsilon,\delta}} \kappa d(\star\zeta) \\ &= \int_{\partial\Sigma_{\epsilon,\delta}} \kappa(\star\zeta) - \int_{\Sigma_{\epsilon,\delta}} \kappa d(\star\zeta).\end{aligned}$$

We have:

$$\int_{\partial\Sigma_{\epsilon,\delta}} \kappa(\star\zeta) = \int_{S_{\epsilon,\delta}} \kappa(\star\zeta) + \int_{S_{\epsilon,2\pi-\delta}} \kappa(\star\zeta),$$

where

$$S_{\epsilon,\delta} := \phi^{-1}(\{(x', \epsilon\cos\delta, \epsilon\sin\delta) \in V\})$$

is oriented by $\Sigma_{\epsilon,\delta}$. Let us write explicitly the first quantity $\displaystyle\int_{S_{\epsilon,\delta}} \kappa(\star\zeta)$:

$$-\int_{]-\sigma,\sigma[^{N-2}} \kappa\circ\phi^{-1}(x', \epsilon\cos\delta, \epsilon\sin\delta)\phi_\sharp(\star\zeta)(x', \epsilon\cos\delta, \epsilon\sin\delta)\, dx'.$$

As explained above, the quantity under the sign $\displaystyle\int$ converges uniformly with respect to $x' \in]-\sigma,\sigma[^{N-2}$ when $\delta \to 0$ (and ϵ is fixed) to

$$\kappa\circ\phi^{-1}(x', \epsilon, 0^+)\phi_\sharp(\star\zeta)(x', \epsilon, 0).$$

So, we have:

$$\lim_{\delta\to 0}\int_{\partial\Sigma_{\epsilon,\delta}} \kappa(\star\zeta) = \int_{]-\sigma,\sigma[^{N-2}} \phi_\sharp(\star\zeta)(x', \epsilon, 0)(\kappa(x', \epsilon, 2\pi^-) - \kappa(x', \epsilon, 0^+))\, dx'$$

$$= 2\pi\deg(u, \Gamma_1)\int_{]-\sigma,\sigma[^{N-2}} \phi_\sharp(\star\zeta)(x', \epsilon, 0)\, dx'.$$

Before letting ϵ go to 0, it remains to estimate

$$\int_{\Sigma_{\epsilon,\delta}} \kappa d(\star\zeta).$$

This quantity is not greater than $||d\zeta||_{L^\infty(U)}||\kappa||_{L^1(\Sigma_\epsilon)}$, and

$$||\kappa||_{L^1(\Sigma_\epsilon)} \le C\int_{]-\sigma,\sigma[^{N-2}} dx' \int_0^{2\pi} \epsilon\kappa\circ\phi^{-1}(x', \epsilon\cos\theta, \epsilon\sin\theta)d\theta.$$

We claim that this last quantity goes to 0. Let us admit this claim for a moment and complete the proof. We have

$$\int_{\partial(U\setminus\Delta_\epsilon)} j(u)\wedge(\star\zeta) = 2\pi\deg(u, \Gamma_1)\int_{]-\sigma,\sigma[^{N-2}} \phi_\sharp(\star\zeta)(x', \epsilon, 0)\, dx' + o(1).$$

When ϵ goes to 0, we obtain:

$$\begin{aligned}\langle J(u), \zeta\rangle &= \pi\deg(u, \Gamma_1)\int_{]-\sigma,\sigma[^{N-2}} \phi_\sharp(\star\zeta)(x', 0, 0)\, dx' \\ &= \pi\deg(u, \Gamma_1)\int_{\Gamma_1} \star\zeta,\end{aligned}$$

which was required.

Let us now prove the claim. It amounts to proving the following result.

Lemma 1 *Let $v \in W^{1,1}(\mathbb{R}^N)$. Let $\Xi_\epsilon := \{(x', y, z) : |(y,z)| = \epsilon\}$. Then, $||v||_{L^1(\Xi_\epsilon)}$ goes to 0 when ϵ goes to 0.*

Proof: Let $Z_\epsilon := \{(x', y, z) : |(y,z)| < \epsilon\}$. The Stokes' formula implies (with ν the outing unit normal to Ξ_ϵ):

$$\int_{\Xi_\epsilon} |v| = \int_{\Xi_\epsilon} |v|\nu.\nu = \int_{Z_\epsilon} \mathrm{div}\,(|v|\nu) = \int_{Z_\epsilon} |v| \mathrm{div}\,\nu + \nabla|v|.\nu$$

$$= \int_{Z_\epsilon} \frac{|v|}{(y^2+z^2)^{1/2}} + \nabla|v|.\nu \leq \int_{Z_\epsilon} \frac{|v|}{(y^2+z^2)^{1/2}} + |\nabla v|.$$

So, it is enough to show that $|v|/(y^2+z^2)^{1/2}$ is summable on Z_1. This follows from the above computation with $\epsilon = 1$. This completes the proof of Proposition 1.

□

We now show the following:

Proposition 2 *The operator J is continuous from $W^{s,p}(S^N, S^1)$ into*

$$W^{s-2,p}(S^N) \cap W^{-1,sp}(S^N).$$

This proposition relies on the multiplication properties of the fractional Sobolev spaces. To show some of them, we will have a frequent use of the following lemma (where $\sigma := s - 1 \in]0,1[$).

Lemma 2 *([17]) Let $w \in W^{1,p}(S^N)$. Then there exists some constant $C \geq 0$ such that for almost every $x \in S^N$, we have*

$$D_{\sigma,p} w(x) \leq C(\mathcal{M}|w - w(x)|^p(x))^{(1-\sigma)/p}(\mathcal{M}|dw|^p(x))^{\sigma/p}.$$

Here, $\mathcal{M}$ denotes the maximal function

$$\mathcal{M}|dw|^p(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |dw|^p(y)\,dy.$$

Corollary 1 *There exists $C > 0$ such that:*
a) For any $w \in W^{1,sp}(S^N, B_{\mathbb{R}^2}(0,3))$ and $z \in L^{sp}(S^N)$, we have:

$$||zD_{\sigma,p}w||_{L^p(S^N)} \leq C||z||_{L^{sp}(S^N)}||dw||^{\sigma}_{L^{sp}(\Lambda^1 S^N)}.$$

b) For any $w \in W^{1,sp}(S^N, B_{\mathbb{R}^2}(0,3))$ and $\alpha \in L^{sp}(\Lambda^1 S^N) \cap W^{\sigma,p}(\Lambda^1 S^N)$, we have:

$$\begin{aligned} ||w\alpha||_{W^{\sigma,p}(\Lambda^1 S^N)} &\leq ||w\alpha||_{L^p(\Lambda^1 S^N)} + ||\alpha||_{L^{sp}(\Lambda^1 S^N)}||D_{\sigma,p}w||_{L^{sp/\sigma}(S^N)} \\ &\quad + ||wD_{\sigma,p}\alpha||_{L^p(S^N)} \\ &\leq C||\alpha||_{W^{\sigma,p}(\Lambda^1 S^N)} + C||\alpha||_{L^{sp}(\Lambda^1 S^N)}||dw||^{\sigma}_{L^{sp}(\Lambda^1 S^N)}. \end{aligned}$$

c) For any $w \in W^{s,p}(S^N, B_{\mathbb{R}^2}(0,3))$ and $\alpha \in L^{sp}(\Lambda^1 S^N) \cap W^{\sigma,p}(\Lambda^1 S^N)$, we have:

$$||w\alpha||_{W^{\sigma,p}(\Lambda^1 S^N)} \leq C||\alpha||_{W^{\sigma,p}(\Lambda^1 S^N)} + C||\alpha||_{L^{sp}(\Lambda^1 S^N)}||w||^{\sigma/s}_{W^{s,p}(S^N)}.$$

Proof: Part a) follows from Hölder's inequality and the boundedness of $\mathcal{M}$ on L^s:

$$||zD_{\sigma,p}w||_{L^p(S^N)} \leq ||z||_{L^{sp}(S^N)}||D_{\sigma,p}w||_{L^{s'p}(S^N)} \text{ , with } s' = s/(s-1)$$

$$\leq C||z||_{L^{sp}(S^N)}||w||^{1-\sigma}_{L^\infty(S^N)}||\mathcal{M}|dw|^p||^{\sigma/p}_{L^s(S^N)}$$

$$\leq C||z||_{L^{sp}(S^N)}||dw||^{\sigma}_{L^{sp}(\Lambda^1 S^N)}.$$

We now prove part b).

$$||w\alpha||_{W^{\sigma,p}(\Lambda^1 S^N)} \leq ||w\alpha||_{L^p(\Lambda^1 S^N)} + ||D_{\sigma,p}(w\alpha)||_{L^p(S^N)}$$

$$\leq ||w\alpha||_{L^p(\Lambda^1 S^N)} + |||\alpha|D_{\sigma,p}w||_{L^p(S^N)} + ||wD_{\sigma,p}\alpha||_{L^p(S^N)}$$

$$\leq ||w\alpha||_{L^p(\Lambda^1 S^N)} + ||\alpha||_{L^{sp}(\Lambda^1 S^N)}||D_{\sigma,p}w||_{L^{s'p}(S^N)} + ||wD_{\sigma,p}\alpha||_{L^p(S^N)}$$

$$\leq ||w||_{L^\infty(S^N)}||\alpha||_{W^{\sigma,p}(\Lambda^1 S^N)} + C||\alpha||_{L^{sp}(\Lambda^1 S^N)}||dw||^{\sigma}_{L^{sp}(\Lambda^1 S^N)}$$

(this is the same calculation as in part a).

Part c) follows from part a) thanks to the inequality:

$$||u||_{W^{1,sp}(S^N)} \leq C||u||^{1/s}_{W^{s,p}(S^N)}||u||^{1-1/s}_{L^\infty(S^N)}. \tag{7}$$

(see [22], Theorem 2.2.5). This completes the proof of the corollary.

□

Let $u = (u^1, u^2) \in W^{s,p}(S^N, S^1)$. Then $du^2 \in W^{\sigma,p}(\Lambda^1 S^N) \cap L^{sp}(\Lambda^1 S^N)$. Corollary 1 c) shows that $u^1 du^2 \in L^{sp}(\Lambda^1 S^N) \cap W^{\sigma,p}(\Lambda^1 S^N)$. Hence, $j(u)$ lies in this space so that finally, $J(u) = dj(u) \in W^{-1,sp}(\Lambda^2 S^N) \cap W^{s-2,p}(\Lambda^2 S^N)$.

If a sequence (u_n) converges in $W^{s,p}(S^N, S^1)$ to some u, let us prove that $J(u_n)$ converges to $J(u)$ in $W^{-1,sp}(\Lambda^2 S^N)$ and in $W^{s-2,p}(\Lambda^2 S^N)$.

First, we show that $u_n^\sharp\omega_0$ converges to $u^\sharp\omega_0$ in $L^{sp}(\Lambda^1 S^N)$. This will imply the convergence of $J(u_n)$ to $J(u)$ in $W^{-1,sp}(\Lambda^2 S^N)$ since d is continuous from $L^{sp}(\Lambda^1 S^N)$ into $W^{-1,sp}(\Lambda^2 S^N)$. Now,

$$||u_n^1 du_n^2 - u^1 du^2||_{L^{sp}(\Lambda^1 S^N)} \leq ||(u_n^1 - u^1)du_n^2||_{L^{sp}(\Lambda^1 S^N)} + ||du_n^2 - du^2||_{L^{sp}(\Lambda^1 S^N)}$$

since $|u| = 1$. The second term goes to 0 because of the continuous embedding $W^{s,p}(\Lambda^1 S^N, S^1) \subset W^{1,sp}(\Lambda^1 S^N, S^1)$. Up to a subsequence, we can assert the existence of a $k \in L^1(S^N)$ such that $|du_n|^{sp} \leq k$ almost everywhere, and the convergence almost everywhere of u_n^1 to u^1. The dominated convergence theorem implies that for this subsequence, the first term in the right hand side goes to 0. Actually, this argument is valid for any subsequence of the original sequence u_n, that is, from any subsequence of the sequence $||(u_n^1 - u^1)du_n^2||_{L^{sp}(\Lambda^1 S^N)}$, we can extract a subsequence which converges to 0. This shows that the whole original sequence goes to 0. Similarly, $||u_n^2 du_n^1 - u^2 du^1||_{L^{sp}(\Lambda^1 S^N)}$ converges to 0. So $J(u_n)$ converges to $J(u)$ in $W^{-1,sp}(\Lambda^2 S^N)$.

We have now to prove that $u_n^\sharp \omega_0$ converges to $u^\sharp \omega_0$ in $W^{\sigma,p}(\Lambda^1 S^N)$ (this will imply the convergence of $J(u_n)$ to $J(u)$ in $W^{s-2,p}(\Lambda^2 S^N)$). Thanks to Corollary 1 a) and c), we have:

$$||u_n^1 du_n^2 - u^1 du^2||_{W^{\sigma,p}(\Lambda^1 S^N)} \leq ||(u_n^1 - u^1)du^2||_{W^{\sigma,p}(\Lambda^1 S^N)}$$
$$+||u_n^1(du_n^2 - du^2)||_{W^{\sigma,p}(\Lambda^1 S^N)}$$
$$\leq ||(u_n^1 - u^1)D_{\sigma,p}(du^2)||_{L^p(S^N)} + |||du^2|D_{\sigma,p}(u_n^1 - u^1)||_{L^p(S^N)}$$
$$+||u_n^1(du_n^2 - du^2)||_{W^{\sigma,p}(\Lambda^1 S^N)} + ||(u_n^1 - u^1)du^2||_{L^p(\Lambda^1 S^N)}$$
$$\leq ||(u_n^1 - u^1)D_{\sigma,p}(du^2)||_{L^p(S^N)} + C||du^2||_{L^{sp}(\Lambda^1 S^N)}||du_n^1 - du^1||^{\sigma}_{L^{sp}(\Lambda^1 S^N)}$$
$$+C||du_n^2 - du^2||_{W^{\sigma,p}(\Lambda^1 S^N)} + C||du_n^2 - du^2||_{L^{sp}(\Lambda^1 S^N)}||u_n^1||^{\sigma/s}_{W^{s,p}(S^N)}$$
$$+||(u_n^1 - u^1)du^2||_{L^p(\Lambda^1 S^N)}.$$

The right hand side goes to 0 (use the dominated convergence theorem for the terms $||(u_n^1 - u^1)D_{\sigma,p}(du_2)||_{L^p(S^N)}$ and $||(u_n^1 - u^1)du^2||_{L^p(\Lambda^1 S^N)}$).

This completes the proof of the continuity of J, which implies Theorem 1 a, in view of the calculation of J on $\mathcal{R}$ (at the beginning of this section) and the density of $\mathcal{R}$ (see section 5).

□

4 Proof of Theorem 1, part 2

The second part of Theorem 1 is a consequence of the following lemma:

Lemma 3 *Let Γ be a smooth oriented $(N-2)$ dimensional boundaryless submanifold of $S^N, N \geq 3$. Let $\Gamma_1, .., \Gamma_r$ be its connected components and $a_1, .., a_r$ be integers. We define the 2 current T as:*

$$\langle T, \omega \rangle := \sum_{i=1}^{r} a_i \int_{\Gamma_i} \star\omega, \quad \forall \omega \in C^\infty(\Lambda^2 S^N). \tag{8}$$

Then there exists $u \in C^\infty(S^N \setminus \Gamma, S^1) \cap W^{s,p}(S^N, S^1)$ such that

$$J(u) = \pi T.$$

Moreover, we may choose u such that

$$||u||_{W^{s,p}(S^N)} \le C(||T||^s_{W^{-1,sp}(\Lambda^2 S^N)} + ||T||_{W^{s-2,p}(\Lambda^2 S^N)}) \tag{9}$$

for some $C > 0$ independent of Γ and of the a_i's.

Remark 1 *We have stated the lemma for the case $N \ge 3$. A similar statement holds for $N = 2$, with $\Gamma := \{A_1, .., A_r\} \subset S^N, a_1, .., a_r \in \mathbb{Z}$ such that $\sum_{i=1}^r a_i = 0$ and $\langle T, \omega\rangle := \sum_{i=1}^r a_i \star \omega(A_i)$. With minor modifications, our proof applies also to the case $N = 2$. We treat below only the case $N \ge 3$.*

Note that (9) is meaningful, since T belongs to both $W^{-1,sp}(\Lambda^2 S^N)$ and $W^{s-2,p}(\Lambda^2 S^N)$. Indeed, for any $\alpha \in W^{1,q}(\Lambda^2 S^N) \cap W^{2-s,p'}(\Lambda^2 S^N)$ (with $q = sp/(sp-1)$ and $p' = p/(p-1)$), we have (as a consequence of the trace theory and the fact that $q > 2$ and $2 - s - 2/p' > 0$):

$$|\int_\Gamma \star\alpha| \le C||\star\alpha||_{L^1(\Lambda^{N-2}\Gamma)} \le C||\star\alpha||_{W^{1-2/q,q}(\Lambda^{N-2}\Gamma)} \le C||\alpha||_{W^{1,q}(\Lambda^2 S^N)}$$

$$\text{and} \quad |\int_\Gamma \star\alpha| \le C||\star\alpha||_{L^1(\Lambda^{N-2}\Gamma)} \le C||\star\alpha||_{W^{2-s-2/p',p'}(\Lambda^{N-2}\Gamma)} \le C||\alpha||_{W^{2-s,p'}(\Lambda^2 S^N)}.$$

We admit Lemma 3 for an instant and we prove Theorem 1 b). Let T be in the closure of the set of 2 currents $\star\mathcal{E}$ associated to a smooth connected $N-2$ dimensional boundaryless submanifold as in (8). Then, there exists a sequence $(T_n)_{n\in\mathbb{N}}$ satisfying the hypotheses of the lemma, converging in $W^{-1,sp}(\Lambda^2 S^N) \cap W^{s-2,p}(\Lambda^2 S^N)$ to T. The above lemma implies the existence of a sequence $(u_n)_{n\in\mathbb{N}}$, such that $J(u_n) = T_n$ and satisfying (9) with T replaced by T_n. The sequence (u_n) is bounded in $W^{s,p}(S^N, S^1) \subset W^{1,sp}(S^N, S^1)$. Then, up to a subsequence, we can assume that (u_n) converges a.e. to some $u \in W^{1,sp}(S^N, S^1)$, and since $|u_n| \le 1$ a.e., the dominated convergence theorem shows that $(u_n)_{n\in\mathbb{N}}$ converges to u in L^q. We can also assume that $(du_n)_{n\in\mathbb{N}}$ weakly converges to du in $L^{sp}(\Lambda^1 S^N)$. Thus $(J(u_n))_{n\in\mathbb{N}}$ converges in $\mathcal{D}'(\Lambda^2 S^N)$ to $J(u)$. Hence $J(u) = \pi T$ and u satisfies (9).

Proof of Lemma 3: Let $M := S^N \setminus \Gamma$. Then M is a smooth open subset of S^N.

step 1: We first introduce $v \in W^{1,sp}(\Lambda^{N-2} S^N) \cap W^{s,p}(\Lambda^{N-2} S^N)$ such that $\delta dv = \star T = \gamma$ where γ denotes the $N-2$ current

$$\langle\gamma, \alpha\rangle = \sum_i a_i \int_{\Gamma_i} \alpha \;\;,\;\; \forall\alpha \in C^\infty(\Lambda^{N-2} S^N).$$

Such a v exists. Indeed, Γ has no boundary, so that in the sense of distributions $\delta\gamma = 0$. This implies that γ vanishes on closed forms and thus on harmonic fields. Hence, denoting by $v := G(\gamma)$, (where G is the Green operator, see section 6), we have $\gamma = \delta dv + d\delta v = \delta dv$ since $0 = G(\delta\gamma) = \delta G(\gamma) = \delta v$. Moreover, as a consequence of the properties of the Green operator, the following estimates hold: there exists $C \geq 0$ such that:

$$||v||_{W^{s,p}(\Lambda^{N-2}S^N)} \leq C||\gamma||_{W^{s-2,p}(\Lambda^{N-2}S^N)} \leq C||T||_{W^{s-2,p}(\Lambda^2 S^N)}$$

$$||v||_{W^{1,sp}(\Lambda^{N-2}S^N)} \leq C||\gamma||_{W^{-1,sp}(\Lambda^{N-2}S^N)} \leq C||T||_{W^{-1,sp}(\Lambda^2 S^N)}.$$

Note that v is a measurable function, which is harmonic on M, and in particular smooth.

step 2: There exists an $N-1$ current A such that $\delta A = \gamma$; moreover, we may assume that for each i, there exists an $N-1$ dimensional rectifiable set A_i and a measurable $N-1$ form τ_i satisfying $|\tau_i| = 1$ a.e. such that

$$\langle A, \omega\rangle := \sum_i a_i \int_{A_i} (\omega|\tau_i)\, d\mathcal{H}^{N-1} \quad , \ \forall \omega \in C^\infty(\Lambda^{N-1}S^N).$$

Here, we use the fact that every rectifiable current in $\mathbb{R}^N$ with finite mass, bounded support and no boundary is the boundary of an integrable current with finite mass (see [1], Remark 2.6.).

We consider the 1 current $\star A$ defined by

$$\langle \star A, \alpha\rangle := (-1)^{N-1}\langle A, \star\alpha\rangle, \quad \forall \alpha \in C^\infty(\Lambda^1 S^N)$$

and set

$$C := \star dv - \star A.$$

We note that $dC := d\star(dv - A) = (-1)^{N-2} \star \delta(dv - A) = \star(\gamma - \gamma) = 0$. Then, thanks to a BV version of the Poincaré Lemma on manifolds (see Lemma 4 below), there exists some $\phi \in BV(S^N)$ such that (in the sense of distributions)

$$d\phi = C.$$

Lemma 4 *Let C be a 1 current on S^N such that $dC = 0$. We suppose that C is associated to a Radon measure on S^N, which means that*

$$sup\langle C, \alpha\rangle < +\infty$$

where the supremum is taken over all $\alpha \in C^\infty(\Lambda^1 S^N)$ satisfying

$$||\alpha||_{L^\infty(\Lambda^1 S^N)} \leq 1.$$

Then there exists $\phi \in BV(S^N)$ such that $d\phi = C$ (in the sense of distributions).

Proof: As usual, we regularize C, we apply the classical Poincaré Lemma to this smooth C and we then pass to the limit. We recall the following

Lemma 5 *([25]) For any p current D associated to a Radon measure on S^N and any $\epsilon > 0$, there exists $\omega_\epsilon \in C^\infty(\Lambda^{N-p}S^N)$ such that $\mathcal{R}_\epsilon(D)$ defined by*

$$\langle \mathcal{R}_\epsilon(D), \alpha\rangle = \int_{S^N} \omega_\epsilon \wedge \alpha \quad , \quad \forall \alpha \in C^\infty(\Lambda^p S^N)$$

satisfies:
i) $M(\mathcal{R}_\epsilon(D)) \leq (1+\epsilon)M(D)$ where $M(D) := \sup\langle D, \alpha\rangle$ over the $\alpha \in C^\infty(\Lambda^p S^N)$ satisfying $||\alpha||_{L^\infty(\Lambda^p S^N)} \leq 1$,
ii) if $\delta D = 0$ then $\delta \mathcal{R}_\epsilon(D) = 0$,
iii) $\mathcal{R}_\epsilon(D) \to D$ in $\mathcal{D}'(\Lambda^p S^N)$ when $\epsilon \to 0$.

Let $\beta_\epsilon \in C^\infty(\Lambda^{N-1}S^N)$ be such that

$$\langle \mathcal{R}_\epsilon(\star C), \alpha\rangle = \int_{S^N} (\beta_\epsilon | \alpha) d\mathcal{H}^N \ , \quad \forall \alpha \in C^\infty(\Lambda^{N-1}S^N).$$

Put it otherwise, β_ϵ is defined by $(-1)^{N-1} \star \beta_\epsilon := \omega_\epsilon$ where ω_ϵ is the 1 form appearing in the statement of Lemma 5 for $D := \star C$. Since $dC = 0$, we have $\delta\beta_\epsilon = 0$. Hence, by the classical version of the Poincaré Lemma, there exists a smooth function $\phi_\epsilon : S^N \to \mathbb{R}$ such that $\int_{S^N} \phi_\epsilon = 0$ and $d\phi_\epsilon = (-1)^{N-1} \star \beta_\epsilon$.

Then, using the Poincaré Sobolev inequality for $W^{1,1}$ functions,

$$\begin{aligned} ||\phi_\epsilon||_{L^1(S^N)} &\leq c\,||d\phi_\epsilon||_{L^1(\Lambda^1 S^N)} \\ &\leq c \sup_{||h||_{L^\infty(\Lambda^1 S^N)} \leq 1} \langle d\phi_\epsilon, h\rangle \leq c \sup_{||\alpha||_{L^\infty(\Lambda^{N-1} S^N)} \leq 1} \langle \beta_\epsilon, \alpha\rangle \\ &\leq c(1+\epsilon) \sup_{||h||_{L^\infty(\Lambda^1 S^N)} \leq 1} \langle C, h\rangle. \end{aligned}$$

Hence, the sequence (ϕ_ϵ) is bounded in $W^{1,1}(S^N)$. Then, up to a subsequence, ϕ_ϵ converges in $BV(S^N)$ to a function of bounded variations ϕ. In particular, we have in the sense of distributions,

$$d\phi = \lim_{\epsilon \to 0} d\phi_\epsilon = \lim_{\epsilon \to 0} (-1)^{N-1} \star \beta_\epsilon = C.$$

step 3: Recall that, for any $f \in BV(S^N)$, df is the sum of three 1 currents of measure type: the absolutely continuous part $d_a f \llcorner \mathcal{H}^N$, the Cantor part $d_C f$ which is singular with respect to the Lebesgue measure and does not charge any $\mathcal{H}^{N-1}$-finite set and the jump part $d_j f$ which is concentrated on a rectifiable set of codimension 1. Furthermore, $d_j f$ can be written as $[f]\nu_f \mathcal{H}^{N-1} \llcorner Sf$, where the $N-1$ rectifiable set Sf is the set of point of approximate discontinuity of f, ν_f is an $N-1$ form defining the orientation of Sf a.e. and the jump $[f]$ is the difference between the trace f^+ and f^- of f on the two sides of Sf (see [12] for details).

Here, we have

$$d\phi = d_a\phi + d_C\phi + d_j\phi = \star dv - \star A,$$

so that $d_C\phi = 0, d_a\phi = \star dv$ and $d_j\phi = -\star A$.

Since $d_j\phi = (\phi^+ - \phi^-)\nu_\phi \mathcal{H}^{N-1} \llcorner S\phi$, we see that $S\phi = \cup_i A_i \;\; \mathcal{H}^{N-1}$ a.e. and that $\phi^+ - \phi^-$ is an integer $\mathcal{H}^{N-1}$ a.e. $x \in S\phi$.

step 4: Let us consider: $u := (-1)^N exp(2i\pi\phi)$.

Hence, thanks to the chain rule for BV functions (see [12]), u is a BV function with

$$d_a u = (-1)^N 2\pi i u d_a\phi = (-1)^N 2\pi i u \star dv \quad , \quad d_C u = 0$$

and $Su \subset S\phi$, with $(-1)^N(u^+ - u^-) = exp(2i\pi\phi^+) - exp(2i\pi\phi^-) = 0 \; \mathcal{H}^{N-1}$ a.e. $x \in Su$. Hence, $d_j u = 0$.

Thus $du = d_a u$ is absolutely continuous with respect to the Lebesgue measure.

step 5: Up to now, u is a smooth function on M. Moreover, since u is S^1 valued, $|du| \leq C|dv|$ so that $||du||_{L^{sp}(\Lambda^1 S^N)} \leq C||dv||_{L^{sp}(\Lambda^{N-2} S^N)} \leq C||T||_{W^{-1,sp}(\Lambda^2 S^N)}$.

Let us now prove that

$$||du||_{W^{\sigma,p}(\Lambda^1 S^N)} \leq C(||T||_{W^{\sigma-1,p}(\Lambda^2 S^N)} + ||T||^s_{W^{-1,sp}(\Lambda^2 S^N)}).$$

Thanks to Corollary 1 b), we have (taking into account the fact that $|u| \leq 1$),

$$||du||_{W^{\sigma,p}(\Lambda^1 S^N)} \leq C||u \star dv||_{W^{\sigma,p}(\Lambda^1 S^N)}$$

$$\leq C(||dv||_{W^{\sigma,p}(\Lambda^{N-1} S^N)} + ||du||^{s-1}_{L^{sp}(\Lambda^1 S^N)} ||dv||_{L^{sp}(\Lambda^{N-1} S^N)})$$

$$\leq C(||dv||_{W^{\sigma,p}(\Lambda^{N-1} S^N)} + ||dv||^{s-1}_{L^{sp}(\Lambda^{N-1} S^N)} ||dv||_{L^{sp}(\Lambda^{N-1} S^N)})$$

$$\leq C(||T||_{W^{\sigma-1,p}(\Lambda^2 S^N)} + ||T||^s_{W^{-1,sp}(\Lambda^2 S^N)}).$$

Hence, $u \in W^{s,p}(\Lambda^1 S^N)$.

This ends the proof of Lemma 3, in view of the fact that:

$$J(u) = 1/2 du^\sharp \omega_0 = (-1)^N \pi d \star dv = \pi \star \delta dv = \pi \star \gamma = \pi T.$$

□

Proof of Theorem 3. If $u \in \overline{C^\infty(S^N, S^1)}^{W^{s,p}(S^N,S^1)}$, then there exists a sequence of smooth maps u_n converging to u in $W^{s,p}(S^N, S^1)$. Using the continuity of J from $W^{s,p}(S^N, S^1)$ into $\mathcal{D}'(\Lambda^2 S^N)$ and the fact that J vanishes on $C^\infty(S^N, S^1)$, we get $J(u) = 0$.

Conversely, if $J(u) = 0$ for some $u \in W^{s,p}(S^N, S^1)$, then there exists $\phi \in W^{s,p}(S^N) \cap W^{1,sp}(S^N)$ such that $j(u) = d\phi$. Indeed, there exists $k \in \mathbb{N}$ such that $G^k(j(u))$ (the k^{th} iterate of the Green operator) is C^1 on S^N (thanks to the Sobolev embeddings and in view of the regularization properties of the Green operator, see section 6). Moreover, $dG^k(j(u)) = G^k(dj(u)) = 0$. Then, by the smooth version of the Poincaré Lemma, there exists some $\psi \in C^1(S^N)$ such that $G^k(j(u)) = d\psi$. Then

$$j(u) = \Delta^k G^k(j(u)) = \Delta^k d\psi = d\Delta^k \psi.$$

Then, we set $\phi := \Delta^k \psi$. By construction and thanks to the regularization properties of the Green operator, ϕ is in $W^{s,p}(S^N) \cap W^{1,sp}(S^N)$.

So,

$$\begin{aligned} d(ue^{-i\phi}) &= e^{-i\phi}(du - iud\phi) = ue^{-i\phi}(\bar{u}du - iu^\sharp\omega_0) \\ &= ue^{-i\phi}(u_1 du_1 + u_2 du_2) = 1/2ue^{-i\phi} d(u_1^2 + u_2^2) \\ &= 1/2ue^{-i\phi} d1 = 0. \end{aligned}$$

Hence, there exists $C \in \mathbb{R}$ (since $|ue^{-i\phi}| = 1$) such that $u = e^{i(\phi + C)}$. Moreover, there exists a sequence of smooth functions $(\phi_n) \subset C^\infty(S^N)$ converging to ϕ in $W^{1,sp}(S^N) \cap W^{s,p}(S^N)$. Then, $u_n := e^{i\phi_n}$ converges to u in $W^{s,p}(S^N, S^1)$, see [9] and [17]. Finally, $u \in \overline{C^\infty(S^N, S^1)}^{W^{s,p}(S^N,S^1)}$.

□

5 The set $\mathcal{R}$ is dense in $W^{s,p}(S^N, S^1)$

The aim of this section is to prove Theorem 2. Let $s \geq 1, p \geq 1$ such that $1 \leq sp < 2$. The case $s = 1, p < 2$ of Theorem 2 has been proved in [2]. Then, we limit ourselves to the case $s \in]1, 2[, p \geq 1$, following the strategy of the proof of Lemma 23 in [4]. Recall that

$$\mathcal{R} := \{u \in \bigcap_{1 \leq r < 2} W^{1,r}(S^N, S^1) \cap W^{s,p}(S^N, S^1) : u \text{ is smooth outside}$$
$$\text{a smooth oriented } N-2 \text{ dimensional boundaryless submanifold}\}.$$

When $N = 2, u$ is assumed to be smooth outside a finite set of points A in S^2.

We first introduce some notations. Let $f_a : \mathbb{R}^2 - \{a\} \to S^1$, be the function defined by:

$$f_a(X) := \frac{X - a}{|X - a|}$$

and $j_a : S^1 \to S^1$ the inverse of f_a when restricted to S^1.

For any $a \in B_{\mathbb{R}^2}(0, 1/10)$ and any $w : S^N \to \mathbb{R}^2$ we denote by w^a the map

$$w^a(x) := \frac{w(x) - a}{|w(x) - a|}$$

which is defined on $\{x \in S^N : w(x) \neq a\}$. We have

$$df_a(X) = \frac{Id}{|X - a|} - \frac{(X - a) \otimes (X - a)}{|X - a|^3}$$

where $(X-a) \otimes (X-a)$ denotes the 2×2 tensor $[(X-a) \otimes (X-a)]_{ij} = (X - a)_i (X - a)_j$, and for any smooth $w : S^N \to \mathbb{R}^2$ (or any $w \in W^{1,p}(S^N, S^1)$),

$$Dw^a(X) := \frac{Dw(X)}{|w(X) - a|} + \frac{(w(X) - a) \otimes (w(X) - a)}{|w(X) - a|^3} \cdot Dw(X)$$

for almost every $X \in \{X' \in S^N : w(X') \neq a\}$. Besides the fact that

$$|df_a(X)| \leq \frac{C}{|X - a|}, \tag{10}$$

we will also use the following Lipschitz property of df_a :

Lemma 6 *There exists $C \geq 0$ such that for any $X, Y \in \mathbb{R}^2 - \{a\}$,*

$$|df_a(X) - df_a(Y)| \leq C \frac{|X - Y|}{|X - a||Y - a|}. \tag{11}$$

Proof: First, remark that $df_a(X) = df_0(X - a)$ so that we can assume $a = 0$. Second, $df_0(\lambda X) = (1/\lambda) df_0(X)$ so that we can suppose $|X| = 1$. Finally, $df_0(R_\theta X) = R_\theta df_0(X) R_\theta^{-1}$ where R_θ is the rotation of angle θ. Hence, we may assume that $X = (1, 0), Y = (r \cos\theta, r \sin\theta)$. Then,

$$|df_a(X) - df_a(Y)| \leq C \frac{\max(|\sin\theta|, |r - \cos^2\theta|)}{r}.$$

We estimate the ratio $|\sin\theta|/|1 - re^{i\theta}|$; the ratio $|r - \cos^2\theta|/|1 - re^{i\theta}|$ is easier to handle. We have:

$$|1 - re^{i\theta}| = \sqrt{(1 - r)^2 + 2r(1 - \cos\theta)} = |1 - r| \sqrt{1 + 2r \frac{2\sin^2(\theta/2)}{(1 - r)^2}}.$$

Then

$$\frac{|\sin\theta|}{|1 - re^{i\theta}|} \leq \frac{\mu}{\sqrt{1 + r\mu^2}} \quad \text{with} \quad \mu = \frac{2|\sin(\theta/2)|}{|1 - r|}.$$

We have $\mu \leq 4$ if $r \leq 1/2$ and

$$\frac{\mu}{\sqrt{1 + r\mu^2}} \leq \frac{\mu}{\sqrt{1 + \mu^2/2}}$$

if $r > 1/2$. In any case $|\sin\theta|/|1 - re^{i\theta}|$ is bounded independently of θ, r. The proof of Lemma 6 is complete.

□

The proof of Lemma 22 in [4] shows that

Claim 1 *For any smooth function $v : S^N \to B_{\mathbb{R}^2}(0,1)$, for a.e. $a \in B_{\mathbb{R}^2}(0,1/10)$, the function v^a is smooth on $S^N \setminus v^{-1}(a)$ and belongs to $W^{1,r}$ for any $r < 2$.*

On $W^{s,p}(S^N, S^1)$, we choose the norm:

$$||u||_{W^{s,p}(S^N)} = ||u||_{L^p(S^N)} + ||du||_{L^p(\Lambda^1 S^N)} + ||D_{\sigma,p} du||_{L^p(S^N)},$$

with $\sigma = s - 1$.

We will use the fact that

$$|d(u_1 + u_2)_x - d(u_1 + u_2)_y| \leq |du_{1x} - du_{1y}| + |du_{2x} - du_{2y}|,$$

(this is an easy consequence of the definition of $|\cdot|$, see section 2).

Let $u \in W^{s,p}(S^N, S^1)$. There exists a sequence of smooth functions $v_\epsilon : S^N \to B_{\mathbb{R}^2}(0,1)$ which converges to u in $W^{s,p}(S^N, \mathbb{R}^2)$. We can suppose further that v_ϵ converges to u $\mathcal{H}^N$ a.e. and that dv_ϵ converges to du $\mathcal{H}^N$ a.e. Using the continuous embedding $W^{s,p}(S^N) \cap L^\infty(S^N) \subset W^{1,sp}(S^N)$ (see (7)), we may also assume that the sequence (v_ϵ) converges to u in $W^{1,sp}(S^N)$. Note also that $j_a(u^a) = u$. We then set

$$u_\epsilon^a := j_a(v_\epsilon^a).$$

The proof of Lemma 22 in [4] shows that

Claim 2 *The quantity* $\displaystyle\int_{B_{\mathbb{R}^2}(0,1/10)} ||u_\epsilon^a - u||^p_{W^{1,p}(S^N)}\, da$ *converges to* 0 *when* ϵ *goes to* 0.

One of the main tool of the proof (that we omit here) is that when $p < 2$, there exists some $C \geq 0$ such that

$$\int_{B_{\mathbb{R}^2}(0,1/10)} \frac{da}{|X - a|^p} \leq C, \quad \forall\ |X| \leq 1.$$

The new result, which enables us to generalise the density theorem to the case $s > 1$ is the following claim.

Claim 3 *The quantity* $\displaystyle\int_{B_{\mathbb{R}^2}(0,1/10)} ||D_{\sigma,p}(du_\epsilon^a - du)||^p_{L^p(S^N)}\, da$ *converges to* 0 *when* ϵ *goes to* 0.

We admit Claim 3 for an instant and we complete the proof of Theorem 2. Let $l_\epsilon(a) := ||u_\epsilon^a - u||^p_{W^{s,p}(S^N)}$. We know that $l_\epsilon := \int_{B_{\mathbb{R}^2}(0,1/10)} l_\epsilon(a)\, da$ tends to 0 when ϵ goes to 0 thanks to Claim 2 and Claim 3. Since (Chebychev's inequality)

$$|\{a \in B_{\mathbb{R}^2}(0,1/10) : l_\epsilon(a) \geq \sqrt{l_\epsilon}\}| \leq \sqrt{l_\epsilon} \quad (\text{if} \quad l_\epsilon \neq 0),$$

we see that for each $\epsilon > 0$, there exists a regular value of v_ϵ, say a_ϵ, such that

$$l_\epsilon(a_\epsilon) \leq \sqrt{l_\epsilon}. \tag{12}$$

(By Sard's Theorem, almost every a is a regular value of v_ϵ.) For such an a_ϵ, $u_\epsilon^{a_\epsilon}$ belongs to $W^{s,p}(S^N, S^1)$ and is smooth except on the smooth oriented $N-2$ dimensional boundaryless submanifold $v_\epsilon^{-1}(a_\epsilon)$ (respectively, a finite set of points when $N = 2$). Hence, $u_\epsilon^{a_\epsilon}$ belongs to $\mathcal{R}$ and converges to u in $W^{s,p}(S^N)$.

We now prove Claim 3. We will denote $g_a := j_a \circ f_a : \mathbb{R}^2 - \{a\} \to S^1 \subset \mathbb{R}^2$. Note that $|dg_a(u(x)) - dg_a(u(y))|$ is well defined for almost every $x, y \in S^N$ via any norm on the set of linear maps from $\mathbb{R}^2$ into $\mathbb{R}^2$. Moreover,

$$D_{\sigma,p}(\alpha + \beta) \leq D_{\sigma,p}(\alpha) + D_{\sigma,p}(\beta)\ , \qquad \forall \alpha, \beta \in L^p(\Lambda^1 S^N, \mathbb{R}^2).$$

We find that for any regular value a of v_ϵ:

$$||D_{\sigma,p}(d(g_a \circ u) - d(g_a \circ v_\epsilon))||_{L^p(S^N)} = ||D_{\sigma,p}(dg_a(u) \circ du - dg_a(v_\epsilon) \circ dv_\epsilon)||_{L^p(S^N)}$$

$$= ||D_{\sigma,p}\{(dg_a(u) - dg_a(v_\epsilon)) \circ dv_\epsilon + dg_a(u) \circ (du - dv_\epsilon)\}||_{L^p(S^N)}$$

$$\leq ||D_{\sigma,p}\{(dg_a(u) - dg_a(v_\epsilon)) \circ dv_\epsilon\}||_{L^p(S^N)} + ||D_{\sigma,p}\{dg_a(u) \circ (du - dv_\epsilon)\}||_{L^p(S^N)}$$

$$\leq |||dv_\epsilon| D_{\sigma,p}(dg_a(u) - dg_a(v_\epsilon))||_{L^p(S^N)} + |||dg_a(u) - dg_a(v_\epsilon)| D_{\sigma,p}(dv_\epsilon)||_{L^p(S^N)}$$

$$+ |||du - dv_\epsilon| D_{\sigma,p}(dg_a(u))||_{L^p(S^N)} + |||dg_a(u)| D_{\sigma,p}(du - dv_\epsilon)||_{L^p(S^N)}.$$

The fourth term is lower than $||dg_a(u)||_\infty ||D_{\sigma,p}(du - dv_\epsilon)||_{L^p}$ which goes to 0 (recall that u is S^1 valued so that $||dg_a(u)||_\infty$ is lower than a constant independent from a). Let us denote by A_1, A_2, A_3 the three terms still to be estimated. We have

$$A_2^p \leq C \int_{|v_\epsilon|<1/2} |D_{\sigma,p}(dv_\epsilon)|^p \left(\frac{1}{|u-a|^p} + \frac{1}{|v_\epsilon - a|^p}\right)$$

$$+ C \int_{|v_\epsilon| \geq 1/2} |D_{\sigma,p}(dv_\epsilon)|^p |dg_a(u) - dg_a(v_\epsilon)|^p =: C(B_1^p + B_2^p).$$

Since dv_ϵ converges to du in $W^{\sigma,p}(\Lambda^1 S^N)$, we find that $||D_{\sigma,p}(dv_\epsilon - du)||_{L^p(S^N)}$ goes to 0. Thus, there exists some $k_0 \in L^p(S^N)$ such that (up to a subsequence) $|D_{\sigma,p}(dv_\epsilon - du)| \leq k_0$. Hence, $D_{\sigma,p}(dv_\epsilon) \leq D_{\sigma,p}(dv_\epsilon - du) + D_{\sigma,p}(du)$ is lower than the L^p function $k := k_0 + D_{\sigma,p}(du)$. On the set where $|v_\epsilon| \geq 1/2, u$ and v_ϵ remain far from $B_{\mathbb{R}^2}(0,1/10)$, so that $|dg_a(u) - dg_a(v_\epsilon)|$ remains bounded. Since $dg_a(v_\epsilon) \to dg_a(u)$ a.e., the dominated convergence theorem implies that $\displaystyle\int_{B_{\mathbb{R}^2}(0,1/10)} B_2^p \, da \to 0$ when $\epsilon \to 0$.

Furthermore,

$$\begin{aligned}\int_{B_{\mathbb{R}^2}(0,1/10)} B_1^p &\leq C \int_{|v_\epsilon|<1/2} k^p \int_{B_{\mathbb{R}^2}(0,1/10)} \left(\frac{1}{|u-a|^p} + \frac{1}{|v_\epsilon - a|^p}\right) da \\ &\leq C \int_{|v_\epsilon|<1/2} k^p\end{aligned}$$

which goes to 0 since $|\{|v_\epsilon| < 1/2\}|$ goes to 0 as $\epsilon \to 0$. Using Corollary 1 a (with $z := |du - dv_\epsilon|$ and $w := dg_a(u)$), we see that

$$A_3 \leq C||d^2 g_a(u)||^\sigma_{L^\infty(S^N, \mathcal{L}(\mathbb{R}^2\times\mathbb{R}^2, \mathbb{R}^2))} ||du||^\sigma_{L^{sp}(\Lambda^1 S^N)} ||du - dv_\epsilon||_{L^{sp}(\Lambda^1 S^N)}.$$

Thus, $A_3 \to 0$ as $\epsilon \to 0$.

The term A_1 involves the most tricky computations. Let us introduce a smooth function $\psi : [0,\infty[\to [0,1]$ such that

$$\psi(t) = \begin{cases} 0 \text{ if } & t \leq 1/4, \\ 1 \text{ if } & t \geq 1/2. \end{cases}$$

We decompose $dg_a(v_\epsilon)$ as

$$dg_a(v_\epsilon) := dg_a(v_\epsilon)\psi(|v_\epsilon|) + dg_a(v_\epsilon)(1 - \psi(|v_\epsilon|)).$$

This decomposition yields

$$\begin{aligned} A_1 &= |||dv_\epsilon| D_{\sigma,p}(dg_a(u) - dg_a(v_\epsilon))||_{L^p(S^N)} \\ &= |||dv_\epsilon| D_{\sigma,p}\{dg_a(u) - dg_a(v_\epsilon)\psi(|v_\epsilon|) - dg_a(v_\epsilon)(1-\psi(|v_\epsilon|))\}||_{L^p(S^N)} \\ &\leq |||dv_\epsilon| D_{\sigma,p}\{dg_a(u) - dg_a(v_\epsilon)\psi(|v_\epsilon|)\}||_{L^p(S^N)} \\ &\quad + |||dv_\epsilon| D_{\sigma,p}\{dg_a(v_\epsilon)(1-\psi(|v_\epsilon|))\}||_{L^p(S^N)} \\ &=: K_1 + K_2. \end{aligned}$$

Using Corollary 1 a) with $z = |dv_\epsilon|$ and $w = dg_a(u) - dg_a(v_\epsilon)\psi(|v_\epsilon|)$, and the fact that dg_a is bounded near S^1, we obtain

$$\begin{aligned} K_1 &\leq C||dv_\epsilon||_{L^{sp}(\Lambda^1 S^N)} ||d\{dg_a(u) - dg_a(v_\epsilon)\psi(|v_\epsilon|)\}||^\sigma_{L^{sp}(S^N)} \\ &\leq C||dv_\epsilon||_{L^{sp}(\Lambda^1 S^N)} \{||d^2 g_a(u) \circ du - d^2 g_a(v_\epsilon) \circ dv_\epsilon \psi(|v_\epsilon|)||^\sigma_{L^{sp}(S^N)} \\ &\quad + |||dg_a(v_\epsilon)||d(\psi \circ |v_\epsilon|)|||^\sigma_{L^{sp}(S^N)}\}. \end{aligned}$$

The dominated convergence theorem shows that this quantity goes to 0 when ϵ goes to 0.

Next, we turn our attention to K_2.

$$\begin{aligned} K_2^p &:= \| |dv_\epsilon| D_{\sigma,p}\{dg_a(v_\epsilon)(1-\psi(|v_\epsilon|))\}\|^p_{L^p(S^N)} \\ &\leq \int_{|v_\epsilon(x)|<1/2} |dv_\epsilon(x)|^p (D_{\sigma,p}\{dg_a(v_\epsilon)(1-\psi(|v_\epsilon|))\})^p \, dx \\ &\quad + \int\int_{|v_\epsilon(y)|<1/2} |dv_\epsilon(x)|^p \frac{|D|^p}{|d(x,y)|^{N+\sigma p}} \, dy \, dx \end{aligned}$$

with

$$D := dg_a(v_\epsilon(x))(1-\psi(|v_\epsilon(x)|)) - dg_a(v_\epsilon(y))(1-\psi(|v_\epsilon(y)|)).$$

Writing $|dv_\epsilon(x)|^p \leq 2^p(|dv_\epsilon(x)-dv_\epsilon(y)|^p + |dv_\epsilon(y)|^p)$, we get that

$$\int_{|v_\epsilon(y)|<1/2}\int |dv_\epsilon(x)|^p \frac{|D|^p}{|d(x,y)|^{N+\sigma p}} \, dx \, dy$$

is lower than $C(\xi+\zeta)$, where

$$\xi := \int_{|v_\epsilon(y)|<1/2}\int |dv_\epsilon(x)-dv_\epsilon(y)|^p \frac{|D|^p}{|d(x,y)|^{N+\sigma p}},$$

$$\zeta := \int_{|v_\epsilon(y)|<1/2}\int |dv_\epsilon(y)|^p \frac{|D|^p}{|d(x,y)|^{N+\sigma p}}.$$

Recalling that

$$|D| \leq C(\frac{1}{|v_\epsilon(x)-a|^p} + \frac{1}{|v_\epsilon(y)-a|^p}),$$

we obtain

$$\int_{B_{\mathbb{R}^2}(0,1/10)} \xi(a)\, da \leq C \int_{|v_\epsilon(y)|<1/2} |D_{\sigma,p} dv_\epsilon(y)|^p \, dy$$

which is lower than $\int_{|v_\epsilon(y)|<1/2} k^p(y)\, dy$. This last quantity converges to 0. Concerning ζ, we have:

$$\zeta = \zeta(a) = \int_{|v_\epsilon(x)|<1/2} |dv_\epsilon(x)|^p (D_{\sigma,p}\{dg_a(v_\epsilon)(1-\psi(|v_\epsilon|))\})^p \, dx.$$

It remains to show that $\int_{B_{\mathbb{R}^2}(0,1/10)} \zeta(a)\, da \to 0$.

For any $X, Y \in B_{\mathbb{R}^2}(0,1) \setminus \{a\}$, we have:

$$dg_a(X) - dg_a(Y) = (dj_a(f_a(X)) - dj_a(f_a(Y))) \circ df_a(X)$$

$$+(dj_a(f_a(Y))) \circ (df_a(X) - df_a(Y)).$$

Using Lemma 6 combined with the inequality

$$|f_a(X) - f_a(Y)| = |\frac{X-a}{|X-a|} - \frac{Y-a}{|Y-a|}| \leq 2\frac{|X-a||Y-X|}{|X-a||Y-a|} = 2\frac{|X-Y|}{|Y-a|},$$

we find that

$$\begin{aligned} |dg_a(X) - dg_a(Y)| &\leq C\frac{|f_a(X) - f_a(Y)|}{|X-a|} + C\frac{|X-Y|}{|X-a||Y-a|} \\ &\leq C\frac{|X-Y|}{|X-a||Y-a|}. \end{aligned} \tag{13}$$

Moreover,

$$|(1-\psi(|v_\epsilon(x)|))dg_a(v_\epsilon(x)) - (1-\psi(|v_\epsilon(y)|))dg_a(v_\epsilon(y))| \leq$$

$$2|dg_a(v_\epsilon(x)) - dg_a(v_\epsilon(y))| + |dg_a(v_\epsilon(y))||\psi(|v_\epsilon(x)|) - \psi(|v_\epsilon(y)|)|.$$

Thanks to the mean value inequality applied to ψ, we have:

$$|\psi(|v_\epsilon(x)|) - \psi(|v_\epsilon(y)|)| \leq C||v_\epsilon(x)| - |v_\epsilon(y)|| \leq C|v_\epsilon(x) - v_\epsilon(y)|,$$

so that:

$$|dg_a(v_\epsilon(y))||\psi(|v_\epsilon(x)|) - \psi(|v_\epsilon(y)|)| \leq C\frac{|v_\epsilon(x) - v_\epsilon(y)|}{|v_\epsilon(y) - a|}$$

$$\leq C\frac{|v_\epsilon(x) - v_\epsilon(y)|}{|v_\epsilon(x) - a||v_\epsilon(y) - a|}.$$

Thanks to (13) with $X := v_\epsilon(x), Y := v_\epsilon(y)$, we have:

$$|dg_a(v_\epsilon(x)) - dg_a(v_\epsilon(y))| \leq C\frac{|v_\epsilon(x) - v_\epsilon(y)|}{|v_\epsilon(x) - a||v_\epsilon(y) - a|}.$$

Finally,

$$|(1-\psi(|v_\epsilon(x)|))dg_a(v_\epsilon(x)) - (1-\psi(|v_\epsilon(y)|))dg_a(v_\epsilon(y))| \leq$$

$$C\frac{|v_\epsilon(x) - v_\epsilon(y)|}{|v_\epsilon(x) - a||v_\epsilon(y) - a|}.$$

Hence,

$$D_{\sigma,p}\{dg_a(v_\epsilon)(1-\psi(|v_\epsilon|))\}(x)^p$$

$$\leq C\int_{S^N}\frac{|v_\epsilon(y) - v_\epsilon(x)|^p}{d(x,y)^{N+\sigma p}|v_\epsilon(x) - a|^p|v_\epsilon(y) - a|^p}\,dy.$$

So,

$$\zeta(a) \leq C\int_{|v_\epsilon(x)|<1/2}\int_{S^N} dx\,dy|dv_\epsilon(x)|^p\frac{|v_\epsilon(y) - v_\epsilon(x)|^p}{d(x,y)^{N+\sigma p}|v_\epsilon(x) - a|^p|v_\epsilon(y) - a|^p}. \tag{14}$$

In the sequel, we will use the following lemma:

Lemma 7 *For any $X, Y \in B_{\mathbb{R}^2}(0,1)$, we have*

$$\int_{B_{\mathbb{R}^2}(0,1/10)} \frac{da}{|X-a||Y-a|} \leq C(1+|\ln|X-Y||)$$

and $\displaystyle\int_{B_{\mathbb{R}^2}(0,1/10)} \frac{da}{|X-a|^p|Y-a|^p} \leq \frac{C}{|X-Y|^{2p-2}}$ *when* $p>1$.

Proof: Suppose first that $p>1$. Using the change of variables $a' = -X + a$ and then $a'' = a'/|Z|$ with $Z := Y - X$, we have

$$\int_{B_{\mathbb{R}^2}(0,1/10)} \frac{da}{|X-a|^p|Y-a|^p} = \int_{B_{\mathbb{R}^2}(-X,1/10)} \frac{da'}{|a'|^p|Z-a'|^p}$$

$$= \frac{1}{|Z|^{2(p-1)}} \int_{B_{\mathbb{R}^2}(-X/|Z|,1/(10|Z|))} \frac{da''}{|a''|^p|Z/|Z|-a''|^p}$$

$$\leq \frac{1}{|Z|^{2(p-1)}} \int_{\mathbb{R}^2} \frac{da''}{|a''|^p|(1,0)-a''|^p}$$

which completes the proof of the case $p>1$ in view of the fact that

$$\int_{\mathbb{R}^2} \frac{da''}{|a''|^p|(1,0)-a''|^p} \leq c(\int_{B_{\mathbb{R}^2}(0,1/2)} \frac{da}{|a|^p} + \int_{B_{\mathbb{R}^2}(0,2)-B_{\mathbb{R}^2}(0,1/2)} \frac{da}{|a-(1,0)|^p}$$

$$+ \int_{B_{\mathbb{R}^2}(0,2)^c} \frac{da}{|a|^{2p}}) < \infty.$$

When $p=1$, the proof is the same apart from the last estimate:

$$\int_{B_{\mathbb{R}^2}(-\frac{X}{|Z|},\frac{1}{10|Z|})} \frac{da''}{|a''||Z/|Z|-a''|} \leq C + C\int_{B_{\mathbb{R}^2}(-\frac{X}{|Z|},\frac{1}{10|Z|})\backslash B_{\mathbb{R}^2}(0,2)} \frac{da''}{|a''|^2}$$

and $$\int_{B_{\mathbb{R}^2}(-\frac{X}{|Z|},\frac{1}{10|Z|})\backslash B_{\mathbb{R}^2}(0,2)} \frac{da''}{|a''|^2} \leq \int_{B_{\mathbb{R}^2}(0,\frac{2}{|Z|})\backslash B_{\mathbb{R}^2}(0,2)} \frac{da''}{|a''|^2}$$

$$\leq C(|\ln|Z||+1).$$

□

Using Lemma 7 in (14) for $X = v_\epsilon(x)$ and $Y = v_\epsilon(y)$, we get that $\displaystyle\int_{B_{\mathbb{R}^2}(0,1/10)} \zeta(a)\,da$ is not greater than

$$C\int_{|v_\epsilon(x)|<1/2}\int_{S^N} dx\,dy|dv_\epsilon(x)|^p \frac{|v_\epsilon(x)-v_\epsilon(y)|^{2-p}}{d(x,y)^{N+\sigma p}}$$

when $p>1$ and

$$C\int_{|v_\epsilon(x)|<1/2}\int_{S^N} dx\,dy|dv_\epsilon(x)|\frac{|v_\epsilon(x)-v_\epsilon(y)|}{d(x,y)^{N+\sigma}}(1+|\ln|v_\epsilon(x)-v_\epsilon(y)||)$$

when $p = 1$. In the latter case, the term

$$\int_{|v_\epsilon(x)|<1/2} \int_{S^N} dx\, dy |dv_\epsilon(x)| \frac{|v_\epsilon(x) - v_\epsilon(y)|}{d(x,y)^{N+\sigma}}$$

can be easily handled using Corollary 1a) while the term

$$\int_{|v_\epsilon(x)|<1/2} \int_{S^N} dx\, dy |dv_\epsilon(x)| \frac{|v_\epsilon(x) - v_\epsilon(y)|}{d(x,y)^{N+\sigma}} | \ln |v_\epsilon(x) - v_\epsilon(y)||$$

is not greater than

$$C \int_{|v_\epsilon(x)|<1/2} \int_{S^N} dx\, dy |dv_\epsilon(x)| \frac{|v_\epsilon(x) - v_\epsilon(y)|^{1-\alpha}}{d(x,y)^{N+\sigma}}$$

for any $\alpha \in]0, 1-\sigma[$ and some $C = C(\alpha)$.

In any case, a variation on Lemma 2 implies that for any $\alpha \in]0, 1-\sigma p[$,

$$\int_{S^N} \frac{|v_\epsilon(x) - v_\epsilon(y)|^{1-\alpha}}{d(x,y)^{N+\sigma p}} dy \quad \leq \quad c([\mathcal{M}(|dv_\epsilon|)(x)]^{\sigma p} + 1). \tag{15}$$

To prove (15), we adapt an idea of Hedberg (see [14], see also [17]). There exists $\delta_0 > 0$ (independent of x) such that the exponential map $\exp_x$ is a smooth diffeomorphism from $B_{T_x S^N}(0, \delta_0)$ onto $B_{S^N}(x, \delta_0)$. Fix $\delta \in (0, \delta_0)$. First,

$$\int_{S^N \setminus B_{S^N}(x,\delta)} \frac{|v_\epsilon(x) - v_\epsilon(y)|^{1-\alpha}}{d(x,y)^{N+\sigma p}} dy \leq \sum_{k=0}^{\infty} \int_{\delta \leq \frac{d(x,y)}{2^k} < 2\delta} \frac{|v_\epsilon(x) - v_\epsilon(y)|^{1-\alpha}}{(2^k \delta)^{N+\sigma p}} dy$$

$$\leq C \sum_{k=0}^{\infty} \frac{(2^{k+1}\delta)^N}{(2^k\delta)^{N+\sigma p}} \fint_{B_{S^N}(x, 2^{k+1}\delta)} |v_\epsilon - v_\epsilon(x)|^{1-\alpha}$$

$$\leq C\delta^{-\sigma p} (\sum_{k=0}^{\infty} 2^{-k\sigma p}) \mathcal{M}|v_\epsilon - v_\epsilon(x)|^{1-\alpha}(x).$$

Furthermore, using the change of variable $y \mapsto k = (\exp_x)^{-1}(y)$, we get:

$$\int_{B_{S^N}(x,\delta)} \frac{|v_\epsilon(x) - v_\epsilon(y)|^{1-\alpha}}{d(x,y)^{N+\sigma p}} dy \leq C \int_{B_{T_x S^N}(0,\delta)} \frac{|v_\epsilon(x) - v_\epsilon(\exp_x(k))|^{1-\alpha}}{||k||^{N+\sigma p}} dk$$

$$\leq C \int_{B_{T_x S^N}(0,\delta)} \frac{dk}{||k||^{N+\sigma p}} (\int_0^1 |dv_\epsilon(\exp(tk)|\, dt)^{1-\alpha} ||k||^{1-\alpha}$$

$$\leq C \sum_{k=0}^{\infty} (\delta 2^{-k})^{1-\alpha-N-\sigma p} (\delta 2^{-k})^N \fint_{B_{T_x S^N}(0, \delta 2^{-k})} dk (\int_0^1 |dv_\epsilon(\exp(tk))|\, dt)^{1-\alpha}$$

$$\leq C\delta^{1-\alpha-\sigma p}\sup_{r>0} ⨍_{B_{T_xS^N}(0,r)} dk(\int_0^1 |dv_\epsilon(\exp(tk))|\,dt)^{1-\alpha}$$

$$\leq C\delta^{1-\alpha-\sigma p}(\sup_{r>0}\int_0^1 dt ⨍_{B_{T_xS^N}(0,tr)} dk|dv_\epsilon(\exp k)|)^{1-\alpha}$$

$$\leq C\delta^{1-\alpha-\sigma p}(\mathcal{M}|dv_\epsilon(x)|)^{1-\alpha}$$

Thus,

$$\int_{S^N}\frac{|v_\epsilon(x)-v_\epsilon(y)|^{1-\alpha}}{d(x,y)^{N+\sigma p}}\,dy$$

$$\leq C(\delta^{1-\alpha-\sigma p}(\mathcal{M}|dv_\epsilon|(x))^{1-\alpha}+\delta^{-\sigma p}\mathcal{M}|v_\epsilon - v_\epsilon(x)|^{1-\alpha}(x)).$$

Minimizing on $\delta\leq\delta_0$, we get:

$$\int_{S^N}\frac{|v_\epsilon(x)-v_\epsilon(y)|^{1-\alpha}}{d(x,y)^{N+\sigma p}}\,dy$$

$$\leq C(\mathcal{M}|dv_\epsilon|(x))^{\sigma p}(\mathcal{M}|v_\epsilon - v_\epsilon(x)|^{1-\alpha}(x))^{(1-\alpha-\sigma p)/(1-\alpha)}$$
$$+C\delta_0^{-\sigma p}(\mathcal{M}|v_\epsilon - v_\epsilon(x)|^{1-\alpha}(x)).$$

Using the fact that v_ϵ is uniformly bounded by 1, we get the expected result (15).

We now use (15) in the estimate of $\int_{B_{\mathbb{R}^2}(0,1/10)}\zeta(a)\,da$. When $p>1$, we take $\alpha := p-1$. The map $\mathcal{M}$ being bounded on L^{sp},

$$\begin{aligned}\int_{B_{\mathbb{R}^2}(0,1/10)}\zeta(a)\,da &\leq C||dv_\epsilon||^p_{L^{sp}(|v_\epsilon|<1/2)}(||\mathcal{M}|dv_\epsilon|||^{p(s-1)}_{L^{sp}(S^N)}+1)\\ &\leq C||dv_\epsilon||^p_{L^{sp}(|v_\epsilon|<1/2)}(||dv_\epsilon||^{p(s-1)}_{L^{sp}(\Lambda^1 S^N)}+1)\end{aligned}$$

which converges to 0 when ϵ goes to 0, thanks to the dominated convergence theorem. When $p=1$, a similar estimate holds for any $\alpha\in]0,1-\sigma[$. This completes the proof of Claim 3 and Theorem 2.

□

6 The Laplacian on S^N

In this final section, we describe and prove some results concerning the regularity of the solutions of:

$$\Delta v = T \tag{16}$$

to be solved in fractional Sobolev spaces $W^{s,p}(\Lambda^l S^N, S^1)$, with $s,p\geq 1, sp>1$.

We recall here the main results, following Scott [24]. We will also prove few results, presumably well-known to experts, but that we could not find in the literature.

First, we define the harmonic *l fields* by

$$\mathcal{H}(\Lambda^l S^N) := \{h \in C^\infty(\Lambda^l S^N) : dh = \delta h = 0\}.$$

This is a finite dimensional vector space, whose orthogonal space (with respect to the inner product on l forms) will be denoted by $\mathcal{H}(\Lambda^l S^N)^\perp$. Then, we denote by $H(\omega)$ the harmonic projection into $\mathcal{H}(\Lambda^l S^N)$ of an l form ω, that is:

$$\langle \omega - H(\omega), h \rangle = 0$$

for any $h \in \mathcal{H}(\Lambda^l S^N)$. (In fact, $\mathcal{H}(\Lambda^l S^N) = \{0\}$ if $0 < l < N$. We have introduced these notations for the sake of generality, since all the results of this article can be generalized to the case when S^N is replaced by more general manifolds).

Now, (Definition 5.23 and Proposition 6.1 in [24]) for any $\omega \in L^p(\Lambda^l S^N)$, where $1 < p < \infty$, there exists some $G(\omega) \in W^{2,p}(\Lambda^l S^N) \cap \mathcal{H}(\Lambda^l S^N)^\perp$ such that

$$\Delta G(\omega) = \omega - H(\omega)$$

and G is a bounded linear operator from $L^p(\Lambda^l S^N)$ into $W^{2,p}(\Lambda^l S^N)$. Moreover, G is selfadjoint and commutes with the Laplacian, the differential and the codifferential.

The Green operator G and the harmonic projection H can be extended to $\mathcal{D}'(\Lambda^l S^N)$, by duality, setting $\langle G(\omega), \alpha \rangle = \langle \omega, G(\alpha) \rangle$ and the same for H. We still have $\Delta G(\omega) = \omega - H(\omega)$ for any $\omega \in \mathcal{D}'(\Lambda^l S^N)$.

By duality, G is also continuous from $W^{-2,p}(\Lambda^l S^N)$ into $L^p(\Lambda^l S^N), 1 < p < \infty$. Furthermore, if $T \in W^{-1,p}(\Lambda^l S^N)$ and $v := G(T)$, we already know that v is in $L^p(\Lambda^l S^N)$, since $T \in W^{-2,p}(\Lambda^l S^N)$, and for any $\alpha \in L^{p'}(\Lambda^l S^N)$, we have $\delta\alpha = \delta\Delta G(\alpha) = \Delta\delta G(\alpha)$, so that

$$\begin{aligned}
\langle dv, \alpha \rangle &= -\langle v, \delta\alpha \rangle \\
&= -\langle v, \Delta(\delta G(\alpha)) \rangle \\
&= -\langle T, \delta G(\alpha) \rangle \\
&\leq ||T||_{W^{-1,p}} ||\delta G(\alpha)||_{W^{1,p'}} \\
&\leq C||T||_{W^{-1,p}} (||d\delta G(\alpha)||_{L^{p'}} + ||\delta G(\alpha)||_{L^{p'}}) \quad \text{(see [24], Cor 4.12)} \\
&\leq C||T||_{W^{-1,p}} ||\alpha||_{L^{p'}} \quad \text{(see [24], Prop 5.15, Prop 5.17)}.
\end{aligned}$$

This shows that $dv \in L^p(\Lambda^{l+1} S^N)$ and $||dv||_{L^p(\Lambda^{l+1} S^N)} \leq C||T||_{W^{-1,p}(\Lambda^l S^N)}$. We have a similar estimate for $||\delta v||_{L^p(\Lambda^{l-1} S^N)}$. Hence (see [24], Cor 4.12), G is a bounded linear operator from $W^{-1,p}(\Lambda^l S^N)$ into $W^{1,p}(\Lambda^l S^N)$.

When $s \notin \mathbb{Z}, 1 < p < \infty$, the fractional Sobolev spaces $W^{s,p}$ can be defined by interpolation (see [22]). If we combine this with the previous remarks, we have:

Proposition 3 *The Green operator G is a bounded linear operator from $W^{s-2,p}(\Lambda^l S^N)$ into $W^{s,p}(\Lambda^l S^N)$, when $0 \leq s \leq 2, 1 < p < \infty$.*

The case $p = 1, 1 < s < 2$ is also needed and not covered by the previous proposition. This is the object of the remaining part of this section:

Theorem 4 *Fix $l \in [|0, N|]$ and $1 < s < 2$. There exists $C > 0$ such that for any $T \in W^{s-2,1}(\Lambda^l S^N)$ satisfying $H(T) = 0$, there is an $\omega \in W^{s,1}(\Lambda^l S^N)$ such that $\Delta\omega = T$ and*

$$||\omega||_{W^{s,1}(\Lambda^l S^N)} \leq C||T||_{W^{s-2,1}(\Lambda^l S^N)}.$$

It is well-known that this statement is false for $s = 1$. To prove the theorem, we use the Besov's spaces and the fact that they coincide with Sobolev's spaces for noninteger values of s. Actually, the proof of Theorem 4 is true when $W^{s,1}$ is replaced by $W^{s,p}$ for any $1 \leq p < \infty, s \geq 1$ and $(s,p) \notin \mathbb{N} \times \{1\}$. This fact was used in the proof of Theorem 3.

The proof of Theorem 4 rests on the following lemma:

Lemma 8 *There exists $C > 0$ such that for any $\omega \in C^\infty(\Lambda^l S^N)$, with $H(\omega) = 0$, we have:*

$$||\omega||_{W^{s,1}(\Lambda^l S^N)} \leq C||\Delta\omega||_{W^{s-2,1}(\Lambda^l S^N)}.$$

Indeed, if this lemma is true, let $T \in W^{s-2,1}(\Lambda^l S^N)$ satisfying $H(T) = 0$. Then, there is a sequence of smooth $T_n \in C^\infty(\Lambda^l S^N)$ converging to T in $W^{s-2,1}(\Lambda^l S^N)$. Since H is continuous on $W^{s-2,1}$ (into a finite dimensional space), the sequence $H(T_n)$ converges to 0. Hence, we can assume that $H(T_n) = 0$ (by replacing T_n with $T_n - H(T_n)$).

For each n, there exists $\omega_n \in C^\infty(\Lambda^l S^N)$ such that $\Delta\omega_n = T_n$ and $H(\omega_n) = 0$ for every n. From Lemma 8 and the fact that $\Delta(\omega_p - \omega_q) = T_p - T_q$, it follows that

$$||\omega_p - \omega_q||_{W^{s,1}(\Lambda^l S^N)} \leq C||T_p - T_q||_{W^{s-2,1}(\Lambda^l S^N)}.$$

This shows that (ω_n) is a Cauchy sequence in $W^{s,1}(\Lambda^l S^N)$. So, it converges to some $\omega \in W^{s,1}(\Lambda^l S^N)$ which satisfies $\Delta\omega = T$ and the estimate $||\omega||_{W^{s,1}(\Lambda^l S^N)} \leq C||T||_{W^{s-2,1}(\Lambda^l S^N)}$ follows.

So it remains to prove Lemma 8. The proof relies on the following three lemmas:

Lemma 9 *There exists $C_0 > 0$ such that for any $w \in C_c^\infty(\mathbb{R}^N)$, we have:*

$$||w||_{W^{s,1}(\mathbb{R}^N)} \leq C_0(||w||_{W^{s-2,1}(\mathbb{R}^N)} + ||\Delta w||_{W^{s-2,1}(\mathbb{R}^N)}).$$

Proof: Thanks to the lifting property (see [22], Proposition 2.1.4.1), we have:

$$\begin{aligned}||w||_{W^{s,1}(\mathbb{R}^N)} &\leq C||\mathcal{F}^{-1}(1+|y|^2)\mathcal{F}w||_{W^{s-2,1}(\mathbb{R}^N)}\\ &= C||(-\Delta+I)w||_{W^{s-2,1}(\mathbb{R}^N)}\\ &\leq C(||\Delta w||_{W^{s-2,1}(\mathbb{R}^N)}+||w||_{W^{s-2,1}(\mathbb{R}^N)}).\end{aligned}$$

□

We proceed with the slightly more elaborate lemma, where we use the notation $I(l,N) := \{(i_1 < .. < i_l) : 1 \leq i_1 < .. < i_n \leq N\}$.

Lemma 10 *Let V be an open neighborhood of $0 \in \mathbb{R}^N$. Let $a^{IJ\alpha\beta} \in C^\infty(\bar{V})$ for any $I \in I(l,N), J \in I(l,N)$ and any $\alpha \in [|1,N|], \beta \in [|1,N|]$. We assume that $a^{IJ\alpha\beta}(0) = \delta_{IJ}\delta_{\alpha\beta}$. Then, there exists $\rho > 0, C > 0$ such that for any $\omega_J \in C_c^\infty(B(0,\rho)), J \in I(l,N)$, we have:*

$$||(\omega_J)||_{W^{s,1}(\Lambda^l\mathbb{R}^N)} \leq C(||(T_J)||_{W^{s-2,1}(\Lambda^l\mathbb{R}^N)} + ||(\omega_J)||_{W^{s-1,1}(\Lambda^l\mathbb{R}^N)})$$

where T_I denotes:

$$T_I := \sum_J \sum_{\alpha,\beta} a^{IJ\alpha\beta}\frac{\partial^2\omega_J}{\partial x_\alpha \partial x_\beta}\ ,\ I \in I(l,N).$$

Here, the norm $||(\omega_J)||_{W^{s,1}(\mathbb{R}^N)}$ means (for instance)

$$||(\omega_J)||_{W^{s,1}(\Lambda^l\mathbb{R}^N)} := \sum_J ||\omega_J||_{W^{s,1}(\mathbb{R}^N)}.$$

Proof of Lemma 10: Let us pick some $\rho > 0$ which will be subsequently subject to some restrictions (independent from the ω_J's). Let $\omega_J \in C_c^\infty(B(0,\rho))$, $J \in I(l,N)$. For any I, we have:

$$||\sum_\alpha \partial_{x_\alpha}\partial_{x_\alpha}\omega_I||_{W^{s-2,1}(\mathbb{R}^N)} = ||\sum_{J,\alpha,\beta} a^{IJ\alpha\beta}(0)\partial_{x_\alpha}\partial_{x_\beta}\omega_J||_{W^{s-2,1}(\mathbb{R}^N)}$$

$$\leq ||\sum_{J,\alpha,\beta}\partial_{x_\alpha}\partial_{x_\beta}((a^{IJ\alpha\beta}(0)-a^{IJ\alpha\beta})\omega_J)||_{W^{s-2,1}(\mathbb{R}^N)}$$

$$+||\sum_{J,\alpha,\beta}\partial_{x_\alpha}\partial_{x_\beta}(a^{IJ\alpha\beta}\omega_J)||_{W^{s-2,1}(\mathbb{R}^N)}$$

$$\leq ||\sum_{J,\alpha,\beta}(a^{IJ\alpha\beta}(0)-a^{IJ\alpha\beta})\omega_J||_{W^{s,1}(\mathbb{R}^N)} + ||\sum_{J,\alpha,\beta} a^{IJ\alpha\beta}\partial_{x_\alpha}\partial_{x_\beta}\omega_J||_{W^{s-2,1}(\mathbb{R}^N)}$$

$$+c||(\omega_J)||_{W^{s-1,1}(\Lambda^l\mathbb{R}^N)} =: a_1 + a_2 + a_3.$$

where c depends only on the $a^{IJ\alpha\beta}$'s.

To estimate the term a_1, we use Lemma 4.6.2.2 in [22] with ϕ being a function in $C_c^\infty(\mathbb{R}^N)$ equal to 1 on a neighborhood of $\bar{B}(0,1)$ and $\sigma := s-1$:

$$||[a^{IJ\alpha\beta}(.) - a^{IJ\alpha\beta}(0)]\omega_J||_{W^{s,1}(\mathbb{R}^N)} \leq c(\rho||\omega_J||_{W^{s,1}(\mathbb{R}^N)} + C_\rho||\omega_J||_{W^{\sigma,1}(\mathbb{R}^N)})$$

where c depends only on the $a^{IJ\alpha\beta}$'s. This implies that a_1 is not greater than

$$N^2 c\rho||(\omega_J)||_{W^{s,1}(\Lambda^l\mathbb{R}^N)} + N^2 cC_\rho||(\omega_J)||_{W^{\sigma,1}(\Lambda^l\mathbb{R}^N)}.$$

The term a_2 is exactly $||T_I||_{W^{s-2,1}(\mathbb{R}^N)}$. Finally, we have shown that:

$$||\Delta\omega_I||_{W^{s-2,1}(\mathbb{R}^N)} \leq C||(T_J)||_{W^{s-2,1}(\Lambda^l\mathbb{R}^N)} + C||(\omega_J)||_{W^{s-1,1}(\Lambda^l\mathbb{R}^N)}$$

$$+N^2 c\rho||(\omega_J)||_{W^{s,1}(\Lambda^l\mathbb{R}^N)}.$$

This implies (thanks to Lemma 9) that:

$$||(\omega_J)||_{W^{s,1}(\Lambda^l\mathbb{R}^N)} \leq C||(T_J)||_{W^{s-2,1}(\Lambda^l\mathbb{R}^N)} + C||(\omega_J)||_{W^{s-1,1}(\Lambda^l\mathbb{R}^N)}$$

$$+N^3 c\rho||(\omega_J)||_{W^{s,1}(\Lambda^l\mathbb{R}^N)}$$

and finally if we choose $\rho < 1/(2N^3c)$ (which depends only on the $a^{IJ\alpha\beta}$'s),

$$||(\omega_J)||_{W^{s,1}(\Lambda^l\mathbb{R}^N)} \leq C||(T_J)||_{W^{s-2,1}(\Lambda^l\mathbb{R}^N)} + C||(\omega_J)||_{W^{s-1,1}(\Lambda^l\mathbb{R}^N)}.$$

Lemma 10 is proved.

□

Lemma 11 *Let $x_0 \in S^N$. Then, there exists an open neighborhood U of x_0 and some constant $C > 0$ such that for any $\omega \in C^\infty(\Lambda^l S^N)$ compactly supported in U we have*

$$||\omega||_{W^{s,1}(\Lambda^l S^N)} \leq C(||\Delta\omega||_{W^{s-2,1}(\Lambda^l S^N)} + ||\omega||_{W^{s-1,1}(\Lambda^l S^N)}).$$

Proof of Lemma 11: The point x_0 belongs to the domain U_0 of a chart ϕ_0 such that $\phi_0(x_0) = 0$ and $g_{ij}(x_0) = \delta_{ij}$. Let $V_0 := \phi(U_0)$. Let $\omega \in C_c^\infty(\Lambda^l U_0)$ and $T := \Delta\omega$. Then, for any $\eta \in C_c^\infty(\Lambda^l U_0)$, we have:

$$\langle d\omega, d\eta\rangle + \langle\delta\omega, \delta\eta\rangle = -\langle T, \eta\rangle$$

Let $\mu := \phi_{0\sharp}\omega =: \sum_I \mu_I e_I^*$ (where $e_I^* = e_{i_1}^* \wedge .. \wedge e_{i_l}^*$ and (e_i^*) is the dual basis of the canonical basis (e_i) of $\mathbb{R}^N$). Then, for each I, the μ_J's satisfy an equation of the form (see [19], chapter 7):

$$\sum_{J,\alpha,\beta} a^{IJ\alpha\beta}\partial_{x_\alpha}\partial_{x_\beta}\mu_J = T_I$$

on V_0, where T_I is a sum of terms involving $\phi_{0\sharp}T, \mu_J$ and the first derivatives of the μ_J's. Hence, the following estimate holds:

$$||T_I||_{W^{s-2,1}(\mathbb{R}^N)} \leq C(||T||_{W^{s-2,1}(\Lambda^l S^N)} + ||\omega||_{W^{s-1,1}(\Lambda^l S^N)}).$$

Thanks to Lemma 10 for these $a^{IJ\alpha\beta}$, (which satisfy $a^{IJ\alpha\beta}(0) = \delta_{IJ}\delta_{\alpha\beta}$, see [19], page 296), there exists $\rho > 0$ such that

$$||(\mu_I)||_{W^{s,1}(\Lambda^l\mathbb{R}^N)} \leq C(||(T_I)||_{W^{s-2,1}(\Lambda^l\mathbb{R}^N)} + ||(\mu_I)||_{W^{s-1,1}(\Lambda^l\mathbb{R}^N)})$$

if ω is compactly supported in $U := \phi_0^{-1}(B(0,\rho))$. This shows that

$$||\omega||_{W^{s,1}(\Lambda^l S^N)} \leq C(||\Delta\omega||_{W^{s-2,1}(\Lambda^l S^N)} + ||\omega||_{W^{s-1,1}(\Lambda^l S^N)}),$$

as required. Lemma 11 is proved.

□

We now complete the proof of Lemma 8. There exists a finite covering $U_1, .., U_r$ around some points $x_1, ..., x_r$ such that the previous lemma is true on each of these U_i. We introduce a partition of unity (ζ_i) corresponding to this covering. Now, let $\omega \in C^\infty(\Lambda^l S^N)$ and $\omega^j := \zeta_j\omega$. Thanks to Lemma 11, we have for every j :

$$\begin{aligned} ||\omega^j||_{W^{s,1}(\Lambda^l S^N)} &\leq C(||\Delta\omega^j||_{W^{s-2,1}(\Lambda^l S^N)} + ||\omega^j||_{W^{s-1,1}(\Lambda^l S^N)}) \\ &\leq C(||\Delta\omega||_{W^{s-2,1}(\Lambda^l S^N)} + ||\omega||_{W^{s-1,1}(\Lambda^l S^N)}), \end{aligned} \tag{17}$$

thanks to the multiplication property. Furthermore, the Green operator is continuous from $W^{s-2,1}(\Lambda^l S^N)$ into $W^{s-1,1}(\Lambda^l S^N)$. Indeed, the space $W^{s-2,1}(\Lambda^l S^N)$ is continuously embedded into $W^{-1,1+\epsilon}(\Lambda^l S^N)$ (say for $\epsilon := (s-1)/(N+1-s)$, see [22], Theorem 2.2.3). The Green operator is continuous from $W^{-1,1+\epsilon}(\Lambda^l S^N)$ into $W^{1,1+\epsilon}(\Lambda^l S^N)$ (thanks to Proposition 3), which is continuously embedded in $W^{s-1,1}(\Lambda^l S^N)$. This implies that for some constant C, we have:

$$||\omega||_{W^{s-1,1}(\Lambda^l S^N)} \leq C||\Delta\omega||_{W^{s-2,1}(\Lambda^l S^N)}$$

(since, by hypothesis, $H(\omega) = 0$). Then, (17) implies

$$||\omega^j||_{W^{s,1}(\Lambda^l S^N)} \leq C||\Delta\omega||_{W^{s-2,1}(\Lambda^l S^N)}.$$

This completes the proof of Lemma 8.

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