On the lower bounded slope condition

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Abstract

Let Ω be a bounded open convex set in $\mathbb{R}^n$ and let $\phi : \Gamma := \partial\Omega \to \mathbb{R}$ be a function defined on its boundary. The *lower bounded slope condition* (on ϕ) is a hypothesis recently introduced by Clarke [3], who has shown its relevance to regularity theory in the calculus of variations. It corresponds to a weaker version of the traditional *bounded slope condition*, which also appears in the theory of elliptic differential equations. In this paper, we study the regularity properties of these functions and give intrinsic characterizations of them. Semiconvexity turns out to be a central tool in the proofs.

1 Introduction

Hilbert-Haar theory is one of the classical approaches to regularity in the multiple integral calculus of variations. The classical version of the Hilbert-Haar theorem can be stated as follows. Let $n \geq 2, F : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and Ω a bounded open set in $\mathbb{R}^n$. Let $\phi : \Gamma = \partial\Omega \to \mathbb{R}$ be a function which satisfies a Bounded Slope Condition (BSC) of rank Q. The BSC of rank Q is the requirement that given any point γ on the boundary, there exist two affine functions

$$y \mapsto \langle \zeta_\gamma^-, y - \gamma \rangle + \phi(\gamma), \ \ y \mapsto \langle \zeta_\gamma^+, y - \gamma \rangle + \phi(\gamma)$$

agreeing with ϕ at γ whose slopes satisfy $|\zeta_\gamma^-|, |\zeta_\gamma^+| \leq Q$ and such that

$$\langle \zeta_\gamma^-, \gamma' - \gamma \rangle + \phi(\gamma) \ \leq \ \phi(\gamma') \ \leq \ \langle \zeta_\gamma^+, \gamma' - \gamma \rangle + \phi(\gamma) \quad \forall \gamma' \in \Gamma.$$

Then the functional $I : u \mapsto \int_\Omega F(\nabla u)$ has a minimum over all the Lipschitz functions which assume the boundary values ϕ on Γ.

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When Hilbert or Haar gave their versions of this theorem (with $F(p) = |p|^2$, see [14] ; with $n = 2$, see [9]), they used a *three point condition* which is equivalent to the BSC (when $n = 2$). Hartman and Nirenberg [13] formulated the BSC (after Rado had done it for $n = 2$, see [21] and [22]) and Stampacchia [23] coined the term BSC and gave the first proof of the Hilbert-Haar theorem in dimensions greater than 2. The BSC has also been used in the context of elliptic pde's (see [11] , [24] and [6]).

Miranda published in [19] (see also [8] and [20]) the proof of the Hilbert-Haar theorem as stated above. One drawback of this theorem is that the BSC hypothesis is quite restrictive. First, if ϕ is not the restriction of a linear function, it implies that Ω is convex. Indeed, the BSC hypothesis implies the existence of a supporting hyperplane at any point γ of Γ, namely :

$$\{\gamma' \in \mathbb{R}^n : \langle \zeta_\gamma^- - \zeta_\gamma^+, \gamma' - \gamma \rangle = 0\}.$$

Secondly, the BSC hypothesis forces ϕ to be a Lipschitz function and to be affine on any segment in Γ. Additionally, Hartman [10] has shown that if Γ is smooth, then any ϕ satisfying the BSC must be smooth. (A precise statement appears below.)

All this has led Clarke [3] to introduce a new property so as to generalize Hilbert-Haar theory to a wider class of boundary functions, namely those functions which satisfy the so-called Lower Bounded Slope Condition (LBSC). The aim of this article is to understand how wide this class is and to characterize it.

Definition 1 *The function $\phi : \Gamma \to \mathbb{R}$ is said to satisfy the LBSC of rank Q if given any $x \in \Gamma$, there exists an affine function*

$$y \mapsto \langle \zeta_x, y - x \rangle + \phi(x)$$

with $|\zeta_x| \leq Q$ such that

$$\langle \zeta_x, y - x \rangle + \phi(x) \leq \phi(y) \quad \forall y \in \Gamma.$$

The following proposition gives a first characterization of functions satisfying the LBSC.

Proposition 1 *The function $\phi : \Gamma \to \mathbb{R}$ satisfies the LBSC if and only if it is the restriction to Γ of a (finite) convex function.*

In contrast, it is known that functions satisfying the BSC are precisely those which coincide on Γ with a convex function and also with a concave function (see [10]). The proof of Proposition 1 is given in section 3.

Actually, the proof will show that ϕ satisfies the LBSC of rank Q if and only if it is the restriction to Γ of a convex function which is globally Lipschitz of rank Q. As an example, one can show that the functions satisfying the LBSC on a square are the Lipschitz functions which are convex on each side of the square (see [1] for a proof).

We can also define the Upper Bounded Slope Condition (UBSC) which is satisfied by $\phi : \Gamma \to \mathbb{R}$ exactly when $-\phi$ satisfies the LBSC. Note that ϕ satisfies the BSC if and only if ϕ satisfies the LBSC and the UBSC.

Though the BSC forces boundary functions to be affine on flat parts of the boundary, it becomes more interesting when Ω is sufficiently curved.

Definition 2 *A convex set Ω is said to be uniformly convex if, for some $\epsilon > 0$, for every point γ on the boundary, there exists a unit vector $b_\gamma \in \mathbb{R}^n$ such that*

$$\langle b_\gamma, \gamma' - \gamma \rangle \geq \epsilon |\gamma' - \gamma|^2, \quad \forall \gamma' \in \Gamma.$$

Miranda's Theorem [19] states that when Ω is uniformly convex, then any ϕ of class C^2 (and actually $C^{1,1}$ is enough) satisfies the BSC. We can prove an analogue of this for functions satisfying the LBSC. The LBSC requires only the minoration inequality of the two inequalities defining the BSC. In that sense, the LBSC is a one-sided BSC. It turns out that the one-sided $C^{1,1}$ regularity (that is regularity required only "from below") is exactly semiconvexity (or equivalently, up to sign, semiconcavity, a familiar and useful property in pde's, see [2]).

Definition 3 *Let S be a subset of $\mathbb{R}^m$. The function $u : S \subset \mathbb{R}^m \to \mathbb{R}$ is said to be semiconvex if there exists a lower semicontinuous function which is nonincreasing $\omega : \mathbb{R}^+ \to \mathbb{R}^-$ such that $\lim_{\rho \to 0^+} \omega(\rho) = 0$ and*

$$\lambda u(x) + (1-\lambda)u(y) - u(\lambda x + (1-\lambda y)) \geq \lambda(1-\lambda)|x-y|\omega(|x-y|)$$

for any $x, y \in S$ such that $[x, y] \subseteq S$ and for any $\lambda \in [0, 1]$. We call such an ω a modulus of semiconvexity for u on S.

We say that a function is locally semiconvex if it is semiconvex on every compact subset of its domain of definition. A function is said to be [locally] semiconcave if its negative is [locally] semiconvex.

Finally, if ω is of the form $-C|\cdot|$ where $C \geq 0$, we say that u is linearly semiconvex.

This definition implies that convex functions are semiconvex functions with a vanishing modulus of semiconvexity $\omega = 0$. Actually, u is linearly semiconvex on an open convex set S with modulus of convexity $-C|\cdot|$ if and only if $u + C/2|\cdot|^2$ is convex on S (see [2], Proposition 1.1.3). Semiconvex functions share with convex functions the property of being locally Lipschitz.

Then we have the following:

Proposition 2 *When Ω is uniformly convex, ϕ satisfies the LBSC if and only if it is the restriction to Γ of a function which is locally linearly semiconvex on $\mathbb{R}^n$.*

See section 3 for a proof of Proposition 2.

In 1966, Hartman [10] found a converse to Miranda's earlier result.

Theorem 1 *Let Ω be a bounded open convex set and ϕ a function on $\Gamma = \partial\Omega$ satisfying a BSC. If Γ is C^1, then ϕ is C^1. If Γ is $C^{1,\lambda}$ for some $\lambda \in]0,1]$, then ϕ is $C^{1,\lambda}$.*

A function on an open set $A \subset \mathbb{R}^n$ is said to be of class $C^{1,\lambda}$ if it has continuous first order partial derivatives which are uniformly Hölder [or Lipschitz] continuous of order $\lambda, 0 < \lambda < 1$ [or $\lambda = 1$] on closed balls in A. A hypersurface $\Gamma \subset \mathbb{R}^n$ is said to be of class $C^{1,\lambda}$ if for any $x \in \Gamma$, there exists a parametrization $\rho : V \to \Gamma \cap U \ni x$ (that is V is an open set in $\mathbb{R}^{n-1}, U$ is an open set in $\mathbb{R}^n$ containing x and ρ is an immersion and a homeomorphism onto its image) which is of class $C^{1,\lambda}$. Finally, $\phi : \Gamma \to \mathbb{R}$ is said to be $C^{1,\lambda}$ if for any such parametrization $\rho : V \to U, \phi \circ \rho$ is of class $C^{1,\lambda}$. We will give in Section 3 a (new) short proof of Theorem 1, based on the natural link between the LBSC and semiconvexity. But it is a natural question to ask whether such a result still holds if one replaces (for ϕ) BSC by LBSC and $C^{1,\lambda}$ by semiconvexity. $\phi : \Gamma \to \mathbb{R}$ is said to be [linearly] *semiconvex* if $\phi \circ \rho$ is locally [linearly] semiconvex on the open set $V \subset \mathbb{R}^{n-1}$ (in the sense of Definition 3) for any parametrization $\rho : V \to U \cap \Gamma$. We will prove in Section 3 the following :

Theorem 2 *Let Ω be a bounded open convex set, $\Gamma := \partial\Omega$ being $C^{1,1}$ and ϕ a function on Γ. If ϕ satisfies the LBSC, then ϕ is linearly semiconvex. If moreover Ω is uniformly convex, then the converse is true; that is, if ϕ is linearly semiconvex, then ϕ satisfies the LBSC.*

The first part of the theorem is the counterpart for LBSC functions of Theorem 1. The last assertion does not coincide with Proposition 2 as it is of a local nature. It is a general principle in convexity theory that local properties are simultaneously global (see for instance Claim 3 in the proof of Theorem 2). This is not the case for semiconvexity on hypersurfaces in $\mathbb{R}^n$.

Even if the two articles [10], [12] deal with the BSC, most of the proofs stated there are valid for the LBSC. We enumerate some of these results (for the LBSC) in section 2 as well as a new result concerning continuity of minimizers in the multiple integral calculus of variations, see Theorem 5 below.

In Section 3, we prove Proposition 1 and 2 as well as Theorem 1 and Theorem 2, and underline the local nature of the LBSC. In Section 4, we will provide intrinsic characterizations of the LBSC in terms of subgradients of ϕ.

2 Some further results

We state now some further results about the LBSC, which can be deduced from the proofs appearing in [10] and [12]. In section 3, Theorem 3 will be used to show that a particular example of a function ϕ does not satisfy the LBSC.

The proof by Hartman of Theorem 1 has a geometrical flavour, and one of the main results in [10] states an equivalence between the BSC and the $n+1$ points condition. (Actually, the case $n = 2$ had been known for a long time, see [7] for a proof in this case.) We say that ϕ satisfies an $n+1$ points condition [with constant K] if for every set of $n+1$ points $x_0, ...x_n$ of Γ, there is a hyperplane $z = \langle a, x\rangle + c = \sum_{h=1}^{n} a^h x^h + c$ in $\mathbb{R}^{n+1}$ which passes through the points $(x, z) = (x_j, \phi(x_j))$ for $j = 0, ..., n$ and satisfies

$$|a| := (\sum_{k=1}^{n} |a^k|^2)^{1/2} \leq K.$$

It is easy to see that the same proof as in [12], Theorem 3.1 yields a similar result for functions satisfying the LBSC:

Proposition 3 *The function ϕ satisfies the LBSC if and only if there exists $S \in \mathbb{R}$ such that for every set of $n+1$ linearly independent points $x_0, ...x_n$ in Γ, the couple $(a, c) \in \mathbb{R}^n \times \mathbb{R}$ defined by*

$$\phi(x_i) = \langle a, x_i\rangle + c \quad \forall i = 0, ..., n$$

satisfies $\langle a, x\rangle + c \geq S, \forall x \in \Omega$.

In other words, any hyperplane in $\mathbb{R}^{n+1}$ passing through the points $(x_i, \phi(x_i))$ lies above the *horizontal* hyperplane $z = S$ on Ω. The proof of this Lemma is based on a quite effective characterization of functions satisfying the LBSC, which we now describe (see [10], Corollary 2.1 which holds for BSC functions but whose proof is also valid for LBSC functions).

Let $x_* \in \Omega$ be fixed and x_0, x_1 be distinct points in Γ. By a point x_{01} of Γ *between* x_0 and x_1 is meant a point of the form $x_{01} = x_* + \lambda(x_0 - x_*) + \mu(x_1 - x_*)$ with $\lambda, \mu > 0$. In the 2 dimensional plane π defined by the three points x_*, x_0, x_1, introduce rectangular coordinates (ξ, η) with x_* as origin such that if $(\xi_0, \eta_0), (\xi_1, \eta_1)$ denote the coordinates of x_0, x_1 respectively, then

$$\begin{vmatrix} \xi_0 & \eta_0 \\ \xi_1 & \eta_1 \end{vmatrix} > 0$$

(in other words, the basis defining the coordinates (ξ, η) has the same orientation in π as the basis $(x_0 - x_*, x_1 - x_*)$). Then we have

Theorem 3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open convex set, $\phi : \Gamma = \partial\Omega \to \mathbb{R}$, and $x_* \in \Omega$. Then ϕ satisfies a LBSC if and only if there exists a number S such that for $z_* \leq S$, the inequality*

$$\begin{vmatrix} \xi_0 & \eta_0 & \phi(x_0) - z^* \\ \xi_{01} & \eta_{01} & \phi(x_{01}) - z^* \\ \xi_1 & \eta_1 & \phi(x_1) - z^* \end{vmatrix} \geq 0 \qquad (1)$$

holds for all points $x_0, x_1 \in \Gamma$ and points x_{01} between them,$(\xi_0, \eta_0), (\xi_1, \eta_1)$ and (ξ_{01}, η_{01}) being the coordinates of x_0, x_1, x_{01} respectively.

Two years later, Hartman [12] made another significant contribution to the understanding of the BSC. Let Ω be a bounded open convex set. Let $\Lambda(\Gamma)$ be the set of all those $\phi : \Gamma \to \mathbb{R}$ such that ϕ is continuous on Γ and on every line segment $l \subset \Gamma, \phi_{|l}$ is the restriction of an affine function. Then

Theorem 4 *The set of all those ϕ satisfying the BSC is dense in $\Lambda(\Gamma)$ for the uniform norm.*

This result enabled Hartman to generalize Miranda's Theorem [19] concerning generalized solutions of the Dirichlet boundary value problem for the minimal surface equation and a continuous boundary function on a uniformly convex set. Actually, as seen by Hartman, this theorem and its proof still hold when Ω is an arbitrary bounded open convex set and ϕ is in the closure in $C^0(\Gamma)$ of the set of those functions satisfying the BSC. Indeed, the proof of Miranda's theorem is based on an approximation procedure and the Hilbert-Haar theorem. It is a striking feature of the Hilbert-Haar theory, that applying it to a sequence of problems can give useful information for a limit problem, associated with a boundary function which does not satisfy the BSC or the LBSC (see [19],[18] and [15]). We give here a regularity result of this kind in the multiple integral Calculus of Variations. To our knowledge, this result is new.

Theorem 5 *Let* Ω *be a bounded open convex set in* $\mathbb{R}^n, \phi \in \Lambda(\Gamma)$ *and* $I(u) := \int_\Omega F(\nabla u)$. *Here,* F *is a strictly convex function on* $\mathbb{R}^n$. *We consider the problem of minimizing* I *over the functions* $u \in W^{1,1}(\Omega)$ *that assume boundary values* ϕ. *If* u *is a solution, then it is continuous on* $\bar{\Omega}$.

Proof : Let ϕ_i be a sequence of functions satisfying the BSC and uniformly converging to ϕ on Γ (Theorem 4 provides the existence of this sequence). The Hilbert-Haar theorem yields the existence of a Lipschitz function u_i which minimizes I relative to all Lipschitz functions having value ϕ_i on Γ. Mariconda and Treu [16] have shown that no Lavrentiev phenomenon can occur; that is, u_i minimizes I over all $v \in W_0^{1,1}(\Omega) + \phi$. To see this, apply the main theorem in [4] which yields the existence of a bounded function k which is a measurable selection of the convex subgradient of F along $\nabla u_i(x)$, that is, $k(x) \in \partial F(\nabla u_i(x))$ a.e. on Ω, such that

$$\int_\Omega \langle k(x), \nabla \eta(x) \rangle \, dx = 0 \quad \forall \eta \in C_c^\infty(\Omega)$$

and then (as k is bounded) this remains true for any $\eta \in W_0^{1,1}(\Omega)$. So

$$I(u_i + \eta) \geq I(u_i) + \int_\Omega \langle k(x), \nabla \eta(x) \rangle \, dx = I(u_i)$$

in view of the definition of the convex subgradient.

Now, u and u_i being minimisers of I, we can apply the comparison principle stated in [17] to deduce

$$|u(x) - u_i(x)| \leq ||\phi - \phi_i||_{L^\infty(\Gamma)} \quad \forall x \in \Omega.$$

So, the sequence u_i is a Cauchy sequence in $C^0(\bar{\Omega})$ which converges to a continuous representative of u. This completes the proof.

□

The proof of Theorem 4 by Hartman (see Proposition 3.5 in [12]) shows in particular that:

Theorem 6 *The set of those functions* $\phi : \Gamma \to \mathbb{R}$ *satisfying the LBSC is dense (for the uniform norm) in the subset of those continuous functions which are convex on any line segment* $l \subset \Gamma$.

3 The Lower Bounded Slope Condition and Semiconvexity

First, we show the characterization of the LBSC given in the Introduction which makes a link between the LBSC and convexity.

Proof of Proposition 1 : If ϕ is the restriction to Γ of a convex function $\tilde{\phi}$ defined on $\mathbb{R}^n$, then for every $x \in \Gamma$, there exists ζ in the convex subgradient of $\tilde{\phi}$ at $x, \zeta \in \partial\phi(x)$, which means

$$\tilde{\phi}(y) \geq \tilde{\phi}(x) + \langle \zeta, y - x \rangle \quad \forall y \in \mathbb{R}^n.$$

Since a convex function is locally Lipschitz, there exists some $Q \geq 0$ such that $\tilde{\phi}$ is Q Lipschitz on a neighborhood of $\bar{\Omega}$, which implies $\partial\tilde{\phi}(x) \subset \bar{B}(0, Q)$. Hence, there exists $Q \geq 0$ such that for every $x \in \Gamma$, there exists $\zeta \in \mathbb{R}^n$ such that

$$\phi(y) \geq \phi(x) + \langle \zeta, y - x \rangle \quad \forall y \in \Gamma,$$

which is the LBSC of rank Q. Conversely, if ϕ satisfies the LBSC of rank Q, then let us define

$$\Phi(y) := \sup_{x \in \Gamma} (\phi(x) + \langle \zeta_x, y - x \rangle),$$

where $\zeta_x \in \mathbb{R}^n$ is such that

$$\phi(y) \geq \phi(x) + \langle \zeta_x, y - x \rangle, \forall y \in \Gamma$$

and $|\zeta_x| \leq Q$. Then, the supremum is finite and no greater than $\phi(y)$. Moreover, for any $y \in \Gamma, \phi(y) = \phi(y) + \langle \zeta_y, y - y \rangle$, so that $\Phi(y) \geq \phi(y)$. So ϕ is the restriction to Γ of Φ, which is a convex function as the supremum of affine functions.

□

Proposition 2 improves this result when Ω is uniformly convex.

Proof of Proposition 2 : The *only if* part is obvious in view of the fact that convex functions are semiconvex, and in view of Proposition 1 . It does not require the uniform convexity of Ω. Conversely, if ϕ is the restriction to Γ of a locally linearly semiconvex function $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}$, then there exists $C, Q \geq 0$ such that for all $x \in \Gamma$, there exists $\zeta_x \in \bar{B}(0, Q)$ satisfying

$$\phi(y) \geq \phi(x) + \langle \zeta_x, y - x \rangle - C|y - x|^2 \quad \forall y \in \Gamma.$$

(see [2], Propositions 3.3.1 and 3.3.4). Here, ζ_x is a Frechet subgradient to $\tilde{\phi}$ at x, $-C|\cdot|$ is a modulus of semiconvexity on some neighborhood of $\bar{\Omega}$ and Q is a Lipschitz constant for $\tilde{\phi}$ on this neighborhood. Furthermore by uniform convexity, there exists $\epsilon > 0$ and a unit vector $b_x \in \mathbb{R}^n$ such that

$$\langle b_x, y - x\rangle \geq \epsilon |y - x|^2 \quad \forall y \in \Gamma.$$

When put together, these inequalities imply

$$\phi(y) \geq \phi(x) + \langle \zeta_x - \frac{C}{\epsilon} b_x, y - x\rangle \;\; \forall y \in \Gamma.$$

Therefore ϕ satisfies the LBSC of rank $Q + \frac{C}{\epsilon}$. This completes the proof.

□

As said before, semiconvexity is a useful tool to deal with the LBSC. The two following propositions will be crucial in the sequel (the first one is exactly Theorem 3.3.7 in [2] whereas the second one corresponds to Proposition 2.1.12 and its proof there).

Proposition 4 *If $u : V \to \mathbb{R}$, with V open, is both semiconvex and semiconcave in V, then $u \in C^1(V)$. Moreover, if the moduli of semiconvexity and semiconcavity of u both have the form $\omega(r) = Cr^\alpha$, for the same $\alpha \in]0, 1]$, then $u \in C^{1,\alpha}(V)$.*

The same is true when semiconvex [semiconcave] is replaced by locally semiconvex [locally semiconcave] and the moduli of semiconvexity [semiconcavity] depend on the compact subset $S \subset V$. Recall that $C^{1,\alpha}$ in this article means the derivative is Hölderian on any closed ball in V (and not necessarily globally on V).

Proposition 5 *Let $u : A \to \mathbb{R}$ be a locally semiconvex function on an open set A and $\rho : V \to A$ a function of class C^1 on an open set V of $\mathbb{R}^{n-1}$. Then $u \circ \rho$ is locally semiconvex on V. More precisely, if S is a compact subset of V, such that $\mathrm{co}[\rho(S)] \subset A$, then $L_1\omega_2(\cdot) + \omega_1(L_2\cdot)$ is a modulus of semiconvexity of $u \circ \rho$ on S where ω_2 [resp. ω_1] is the modulus of continuity of $D\rho$ on S [resp. the modulus of semiconvexity of u on $\mathrm{co}[\rho(S)]$ and L_1 [resp. L_2] a Lipschitz constant for u on $\rho(S)$ [resp. of ρ on S].*

Theorem 1 is an easy consequence of the properties satisfied by semiconvex functions.

Proof of Theorem 1 : Let $\rho : V \to U$ be a parametrization of class $C^{1,\lambda}$. We must show that $\phi \circ \rho$ is of class $C^{1,\lambda}$. Since ϕ satisfies the $LBSC$, it is the restriction of a convex function $\hat{\phi} : \mathbb{R}^n \to \mathbb{R}$. Let S be any compact subset of V and let $\hat{L} \in \mathbb{R}$ be a Lipschitz constant for $\hat{\phi}$ on a neighborhood of $\mathrm{co}[\rho(S)]$. Then, thanks to Proposition 5, $\phi \circ \rho = \hat{\phi} \circ \rho$ is semiconvex on S, a modulus of semiconvexity being $\hat{L}\omega_S$ where ω_S is the modulus of continuity of $D\rho$ on S. Using now the fact that ϕ is UBSC, we can find in a similar way that there exists $\check{L} \in \mathbb{R}$ such that $\phi \circ \rho$ is semiconcave on S with modulus of semiconcavity $\check{L}\omega_S$.

Since ρ is $C^{1,\lambda}$, the modulus of continuity of $D\rho$ on S is of the form $\omega_S(r) = C_S|r|^\lambda$ with some $C_S \geq 0$. Proposition 4 then implies that $\phi \circ \rho$ is of class $C^{1,\lambda}$ on $\mathrm{int}\, S$. This shows that $\phi \circ \rho$ is $C^{1,\lambda}$ on V and completes the proof.

□

Proof of Theorem 2 : If Γ is $C^{1,1}$ and ϕ satisfies the LBSC, then ϕ is the restriction to Γ of a convex function $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}$, so that $\phi \circ \rho = \tilde{\phi} \circ \rho$ is locally linearly semiconvex for any parametrisation $\rho : V \subset \mathbb{R}^n \to U \cap \Gamma$ (thanks to Proposition 5). This means that ϕ is linearly semiconvex.

The converse is not so straightforward, as the semiconvexity of ϕ is a local property and the LBSC appears (as far as its definition is concerned) as a global property (involving all of Γ).

Definition 4 *If U is an open set in $\mathbb{R}^n$, we say that $\phi_{|U}$ satisfies the LBSC if there exists $Q \geq 0$ such that for any $x \in U \cap \Gamma$, there exists $\zeta_x \in \bar{B}(0, Q)$ such that $\phi(y) \geq \phi(x) + \langle \zeta_x, y - x \rangle, \forall y \in \Gamma \cap U$.*

Fix some x_* in Ω. For each $x \in \Gamma$, there exists a parametrization $\rho : V \subset \mathbb{R}^{n-1} \to U \cap \Gamma \ni x$ which is $C^{1,1}$. Moreover,

Claim 1 *For any $x \in \rho(V)$, there exists $U_1 \subset U, V_1 \subset \bar{V}_1 \subset V$, with V_1 an open convex set and U_1 an open set, $\psi : U_1 \to V_1$ of class $C^{1,1}$ such that*

$$\rho \circ \psi(x') = x' \ \forall x' \in \Gamma \cap U_1, \quad \psi \circ \rho(v') = v', \ \forall v' \in V_1. \tag{2}$$

This Claim is an easy consequence of the Inverse Function Theorem applied to the function $\tilde{\rho} : (v, t) \in V \times \mathbb{R} \mapsto \rho(v) + tn$ where n is any vector in $\mathbb{R}^n$ not belonging to $D\rho(\rho^{-1}(x))\mathbb{R}^{n-1}$. There exist $\epsilon > 0$, an open convex set V_1 in $\mathbb{R}^{n-1}$ containing $\rho^{-1}(x)$, and an open set U_0 in $\mathbb{R}^n$ containing x such that $\tilde{\rho}$ is a $C^{1,1}$ diffeomorphism from $V_1 \times]-\epsilon, \epsilon[$ onto U_0. Define an open set $U_1 \subset U_0$ such that $U_1 \cap \Gamma = \rho(V_1)$, and $\psi := \Pi \circ \tilde{\rho}^{-1} : U_1 \to V_1$, where

$$\Pi : (v_1, ... v_{n-1}, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \to (v_1, ..., v_{n-1}) \in \mathbb{R}^{n-1}.$$

Then (2) holds.

For any $x \in \rho(V)$, and U_1, V_1, ψ as in the Claim, $\phi \circ \rho$ is linearly semiconvex and Lipschitz on V_1. Hence, $\phi \circ \rho \circ \psi$ is locally linearly semiconvex on U_1. As $\phi \circ \rho \circ \psi = \phi$ on $\Gamma \cap U_1$, we see that ϕ is the restriction to $\Gamma \cap U_1$ of a locally linearly semiconvex function defined on U_1. Therefore, using the fact that Ω is uniformly convex (exactly as in Proposition 2), $\phi_{|U_x}$ satisfies the LBSC for any open set U_x satisfying $x \in U_x \subset \bar{U}_x \subset U_1$.

When x runs through Γ, the corresponding sets U_x constitute a covering of the compact set Γ. We extract from this covering a finite one which we will denote for ease of notation $U_1, .., U_m$, and correspondingly $\Xi_1, ..., \Xi_m$, will denote the following open subsets of $\mathbb{R}^n$:

$$\Xi_i := \{x_* + t(x - x_*) : t > 0, x \in U_i\}.$$

Finally, Γ_i will denote $U_i \cap \Gamma$. Theorem 2 will be then a direct consequence of the following Lemma, which is of independent interest.

Lemma 1 *If for any $i = 1, ..., m, \phi_{|U_i}$ satisfies the LBSC of rank Q_i, then ϕ satisfies the LBSC.*

We now prove this Lemma. For each $z_* \in \mathbb{R}^-$, we consider the function (as in [10])

$$\tau_{z_*} : x \in \mathbb{R}^n \to z = z_* + t[\phi(x_0) - z_*] \tag{3}$$

where (t, x_0) is defined as :

if $x \neq x_*, x_0$ is the unique point of Γ of the form $x_* + s(x - x_*)$, with $s > 0$ and t is defined by $x = x_* + t(x_0 - x_*)$;

if $x = x_*$, we set $t = 0$ (and x_0 any point in Γ). In the notation of [3], $x_0 = \pi_\Gamma(x_*|x)$ and $t = |x - x_*|/d_\Gamma(x_*|x)$.

Claim 2 *For any $i = 1, ..., m$, if $\phi_{|U_i}$ satisfies the LBSC of rank Q_i, then there exists $N_i \in \mathbb{R}$ such that for any $z_* \leq N_i, \tau_{z_*|\Xi_i}$ is convex.*

This claim is a local one-sided version of Theorem 2.1 in [10].

Proof of Claim 2: For every $x \in \Gamma_i$, there exists $\zeta_x \in \bar{B}(0, Q_i)$ such that $\phi(y) \geq \phi(x) + < \zeta_x, y - x >=: v_x(y)$, $\forall y \in \Gamma_i$. There exists a number $N_i \leq -||\phi||_\infty$ (depending only on $||\phi||_{L^\infty}, Q_i, \operatorname{diam}\Omega$, where the latter denotes $\sup_{x,y\in\Omega} |x - y|$) such that $v_x(x_*) \geq N_i$ for any $x \in \Gamma_i$. Let $z_* \leq N_i$. Let $a_x \neq 0$ be in the convex cone to Ω at x and $\mu_x \geq 0$ such that

$$v_x(x_*) + \langle \mu_x a_x, x_* - x \rangle = z_*$$

(μ_x certainly exists because $< a_x, x_* - x >< 0$). Then we claim that

$$\tau_{z_*}(y) = \sup_{x\in\Gamma_i}(v_x(y) + \langle \mu_x a_x, y - x\rangle) \quad \forall y \in \Xi_i.$$

Indeed, let $x_0 \in \Gamma_i, t \geq 0$ and $y = x_* + t(x_0 - x_*)$. Then

$$\begin{aligned} \tau_{z_*}(y) = z_* + t(\phi(x_0) - z_*) &= v_{x_0}(y) + \langle \mu_{x_0} a_{x_0}, y - x_0\rangle \\ &\leq \sup_{x\in\Gamma_i}(v_x(y) + \langle \mu_x a_x, y - x\rangle). \end{aligned}$$

And for any $x \in \Gamma_i$,

$$\begin{aligned} v_x(y) + \langle \mu_x a_x, y - x\rangle &= (1-t)(v_x(x_*) + \langle \mu_x a_x, x_* - x\rangle) \\ &\quad + t(\underbrace{v_x(x_0)}_{\leq\phi(x_0)} + \underbrace{\langle \mu_x a_x, x_0 - x\rangle}_{\leq 0}) \\ &\leq (1-t)z_* + t\phi(x_0). \end{aligned}$$

Hence, τ_{z^*} is a convex function on Ξ_i as the supremum of affine functions (though Ξ_i might not be convex). The conclusion of the Claim follows from that.
We can now finish the proof of Lemma 1. Indeed, setting $N := \min_{1\leq i\leq m} N_i$, we have for any $z_* \leq N$, that $\tau_{z_*|\Xi_i}$ is convex.

The following remark is useful.

Claim 3 *Let I be a nontrivial interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ a locally convex function, in the following sense: for any $x \in I$, there exists $\epsilon > 0$ such that f restricted to $(x - \epsilon, x + \epsilon) \cap I$ is convex. Then f is convex on I.*

This is a well known fact which we admit. Now let x, x' be two points in $\mathbb{R}^n$. If $x_* \in (x, x')$, then $\tau_{z_*|(x_*,x)}$ is affine and so is $\tau_{z_*|(x_*,x')}$, with $\tau_{z_*}(x_*) \leq \tau_{z_*}(x), \tau_{z_*}(x')$. Then $\tau_{z_*|(x,x')}$ is convex. In the other case, $(x, x') \not\ni x_*$ and then $(x, x') = \bigcup_{i\in\{1,\dots,m\}}((x, x') \cap \Xi_i))$, so that $\tau_{z_*|(x,x')}$ is locally convex, hence convex. As the restriction of τ_{z_*} to any straight line is convex, τ_{z_*} itself is convex. This shows that ϕ is the restriction to Γ of a convex function, and thus satisfies the LBSC.

□

Let us give an application of these results. The following example is used in [3] to show that even for the Dirichlet Lagrangian on the open disk, minimizers are not necessarily globally Lipschitz when the boundary function satisfies only the LBSC (but local Lipschitz continuity in the interior is obtained). We now show that the function involved satisfies the LBSC but not the BSC.

Example 1 *Let $\Omega \subset \mathbb{R}^2$ be the unit disc and*

$$\phi : (x,y) \in \Gamma \mapsto -\frac{\pi^2}{6} + \frac{\pi}{2}\theta - \frac{\theta^2}{4}.$$

where $\theta \in [0, 2\pi[$ is such that $(x,y) = (\cos\theta, \sin\theta)$. Then ϕ is a Lipschitz function satisfying the LBSC but not the BSC.

We will use $\rho : \theta \in \mathbb{R} \mapsto (\cos\theta, \sin\theta)$, which is a parametrization when restricted to any interval of length less than 2π. We have then $\phi \circ \rho : \theta \mapsto -\frac{\pi^2}{6} + \frac{\pi}{2}\theta - \frac{\theta^2}{4}$ when the right hand side is extended by 2π periodicity all over $\mathbb{R}$. The derivative of $\phi \circ \rho$ has a discontinuity at each point of the form $2k\pi, k \in \mathbb{Z}$. On $\Gamma \backslash \{(1,0)\}, \phi$ is smooth so that ϕ restricted to $\Gamma \backslash [1/2, +\infty[\times \mathbb{R}$ satisfies the LBSC (see the proof of Proposition 2). To show that ϕ satisfies the LBSC, it is enough to check that there exists $\sigma \geq 0$ such that

$$\forall \theta_0 \in]-\pi, \pi[, \exists \zeta \in \bar{B}(0, \pi/2)$$

such that for any $\theta \in]-\pi, \pi[$

$$\phi \circ \rho(\theta) \geq \phi \circ \rho(\theta_0) + < \zeta, \theta - \theta_0 > -\sigma|\theta - \theta_0|^2$$

which is equivalent to verifying four cases

$$\underbrace{\frac{\pi}{2}(\theta - \theta_0) - \frac{\theta^2}{4} + \frac{\theta_0^2}{4}}_{(\frac{\pi}{2} - \frac{\theta_0}{2})(\theta - \theta_0) - \frac{1}{4}(\theta - \theta_0)^2} \geq \langle \zeta, \theta - \theta_0 \rangle - \sigma|\theta - \theta_0|^2 \quad \forall \theta, \theta_0 \in [0, \pi[,$$

$$\underbrace{\frac{\pi}{2}(-\theta + \theta_0) - \frac{\theta^2}{4} + \frac{\theta_0^2}{4}}_{(-\frac{\pi}{2} - \frac{\theta_0}{2})(\theta - \theta_0) - \frac{1}{4}(\theta - \theta_0)^2} \geq \langle \zeta, \theta - \theta_0 \rangle - \sigma|\theta - \theta_0|^2 \quad \forall \theta, \theta_0 \in]-\pi, 0],$$

$$\frac{\pi}{2}(\theta + \theta_0) - \frac{\theta^2}{4} + \frac{\theta_0^2}{4} \geq \langle \zeta, \theta - \theta_0 \rangle - \sigma|\theta - \theta_0|^2 \quad \forall \theta \in [0, \pi[, \theta_0 \in]-\pi, 0],$$

$$\frac{\pi}{2}(-\theta - \theta_0) - \frac{\theta^2}{4} + \frac{\theta_0^2}{4} \geq \langle \zeta, \theta - \theta_0 \rangle - \sigma|\theta - \theta_0|^2 \quad \forall \theta_0 \in [0, \pi[, \theta \in]-\pi, 0].$$

In the third case,

$$\begin{aligned}
\frac{\pi}{2}(\theta + \theta_0) - \frac{\theta^2}{4} + \frac{\theta_0^2}{4} &= \pi\theta_0 + (\theta - \theta_0)(\frac{\pi}{2} - \frac{\theta_0}{2}) - \frac{1}{4}(\theta - \theta_0)^2 \\
&\geq -\pi(\theta - \theta_0) + (\theta - \theta_0)(\frac{\pi}{2} - \frac{\theta_0}{2}) - \frac{1}{4}(\theta - \theta_0)^2 \\
&= (\theta - \theta_0)(-\frac{\pi}{2} - \frac{\theta_0}{2}) - \frac{1}{4}(\theta - \theta_0)^2.
\end{aligned}$$

In the last case, similarly,

$$\frac{\pi}{2}(-\theta-\theta_0)-\frac{\theta^2}{4}+\frac{\theta_0^2}{4}\geq(\frac{\pi}{2}-\frac{\theta_0}{2})(\theta-\theta_0)-\frac{(\theta-\theta_0)^2}{4}.$$

When $\theta_0 \geq 0$, (that is, in the first and the last case), we can take $\zeta = \frac{\pi}{2} - \frac{\theta_0}{2}$ and when $\theta_0 \leq 0$, (the second and third case), we can take $\zeta = -\frac{\pi}{2} - \frac{\theta_0}{2}$. When $\theta_0 = 0$, any of these two values of θ_0 will do.

Let us show now that $-\phi$ does not satisfy the LBSC, by contradicting Theorem 3. For any $\epsilon > 0$, fix $x_0 = (\cos\epsilon, -\sin\epsilon), x_{01} = (1,0)$ and $x_1 = (\cos\epsilon, \sin\epsilon)$. Then with $x_* = (0,0)$,

$$\frac{\begin{vmatrix} \xi_0 & \eta_0 & -\phi(x_0) \\ \xi_{01} & \eta_{01} & -\phi(x_{01}) \\ \xi_1 & \eta_1 & -\phi(x_1) \end{vmatrix}}{\begin{vmatrix} \xi_0 & \eta_0 & 1 \\ \xi_{01} & \eta_{01} & 1 \\ \xi_1 & \eta_1 & 1 \end{vmatrix}} = \frac{(\frac{\pi^2}{6}-\frac{\pi}{2}\epsilon+\frac{\epsilon^2}{4})\sin\epsilon-\frac{\pi^2}{6}\sin\epsilon\cos\epsilon}{\sin\epsilon(1-\cos\epsilon)}$$

$$= \frac{\frac{\pi^2}{6}\sin\epsilon(1-\cos\epsilon)+(-\frac{\pi}{2}\epsilon+\frac{\epsilon^2}{4})\sin\epsilon}{\sin\epsilon(1-\cos\epsilon)}$$

$$= \frac{\pi^2}{6}+\frac{-\frac{\pi\epsilon}{2}+\frac{\epsilon^2}{4}}{1-\cos\epsilon}$$

As $1-\cos\epsilon \sim \frac{\epsilon^2}{2}, \epsilon \to 0$, the last term tends to $-\infty$ when $\epsilon \to 0$. This prevents Hartman's criterion (1) from holding. So $-\phi$ does not satisfy the LBSC, and ϕ fails to satisfy the BSC.

4 Subgradients

The aim of this section is to give intrinsic characterizations (i.e. without any parametrization) for a function $\phi : \Gamma \to \mathbb{R}$ to satisfy the LBSC. This characterization is in term of subgradients. Using the same ideas will enable us to improve Lemma 1 to give a pointwise (rather than local) condition for a function to satisfy the LBSC.

If $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function, we define $\operatorname{dom} f := \{x : f(x) < +\infty\}$. We say that ζ is a proximal subgradient of f at $x \in \operatorname{dom} f$, and we note $\zeta \in \partial_P f(x)$, if there exists $\eta > 0$ and $\sigma \geq 0$ such that

$$f(y) \geq f(x) + \langle \zeta, y - x\rangle - \sigma|y-x|^2 \quad \forall y \in B(x,\eta).$$

When f is convex, proximal subgradients coincide with convex subgradients.

We mention here some properties of proximal subgradients that will be used in the sequel (see [5] for proofs of these properties; the hypotheses stated here are far from being optimal).

First, consider the indicator function I_Γ of a $C^{1,1}$ hypersurface $\Gamma \subset \mathbb{R}^n$, that is $I_\Gamma(x) = 0$ if $x \in \Gamma$ and is $+\infty$ elsewhere. Then, for any $x \in \Gamma$, the set of proximal subgradients of I_Γ at x is the normal to the hypersurface Γ at $x, N_\Gamma(x)$.

If $f : U \to \mathbb{R}$ is a Lipschitz function on an open convex U, then f is convex if and only if for any $x, x' \in U$,

$$\langle \zeta - \zeta', x - x' \rangle \geq 0 \ \ , \ \forall \zeta \in \partial_P f(x) \ , \ \forall \zeta' \in \partial_P f(x').$$

When f is a Lipschitz function of Lipschitz rank K, then its proximal subgradients are bounded by K.

Nonsmooth analysis also provides several sum rules. Suppose that $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function and $\theta : \mathbb{R}^n \to \mathbb{R}$ is a $C^{1,1}$ function. If $\zeta \in \partial_P(f + \theta)(x)$, then $\zeta - \nabla\theta(x) \in \partial_P f(x)$.

If $\phi : \Gamma \to \mathbb{R}$ is a lower semicontinuous function, then we can extend it into a lower semicontinuous function $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by setting $\phi(x) = +\infty$ whenever $x \notin \Gamma$. Then a proximal subgradient ζ of ϕ at $x \in \Gamma$ (we still note $\zeta \in \partial_P \phi(x)$) will be defined as any proximal subgradient of $\tilde{\phi}$ at x, that is: there exist $\sigma \geq 0, \eta > 0$ such that

$$\phi(y) \geq \phi(x) + \langle \zeta, y - x \rangle - \sigma |y - x|^2 \quad \forall y \in B(x, \eta) \cap \Gamma.$$

We consider a bounded open set Ω which is supposed to be of class $C^{1,1}$ and uniformly convex. Then, the tangent plane to Ω at any $x \in \Gamma$ is well-defined. For any $\zeta \in \partial_P \phi(x)$, we will note $\tilde{\zeta}$ the tangential component of ζ, that is the orthogonal projection of ζ on the tangent plane to Ω at x.

The main result of this section is

Theorem 7 *Suppose Ω is a bounded open uniformly convex set of class $C^{1,1}$ and consider $\phi : \Gamma \to \mathbb{R}$ a Lipschitz function of rank K. Then ϕ satisfies the LBSC if and only if for any $x \in \Gamma$, there is an open set U_x in $\mathbb{R}^n$ and some $Q \geq 0$ such that*

$$\langle \tilde{\zeta} - \tilde{\zeta}', y - y' \rangle \geq -Q|y - y'|^2, \tag{4}$$

for any $y, y' \in U_x \cap \Gamma$ and any $\zeta \in \partial_P \phi(y), \zeta' \in \partial_P \phi(y')$.

To prove Theorem 7, we are going to apply Theorem 2 for one implication and Proposition 2 for the other one. Suppose first that for any $x_0 \in \Gamma$, there is an open set U in $\mathbb{R}^n$ and some $Q \geq 0$ such that

$$\langle \tilde{\zeta} - \tilde{\zeta}', y - y' \rangle \geq -Q|y - y'|^2 \quad \forall y, y' \in U \cap \Gamma, \forall \zeta \in \partial_P \phi(y), \zeta' \in \partial_P \phi(y').$$

In view of the regularity of Γ, there exist open sets $U_1 \subset U, V \subset \mathbb{R}^{n-1}$ and $\rho : V \to \Gamma \cap U_1$ of class $C^{1,1}$ such that ρ is an immersion and a homeomorphism onto $\Gamma \cap U_1$. We can also suppose (see Claim 1) that there exists $\psi : U_1 \to V$ which is Lispchitz, $C^{1,1}$ and satisfies

$$\psi \circ \rho(v) = v \quad \forall v \in V, \quad \rho \circ \psi(x) = x \quad \forall x \in \Gamma \cap U_1.$$

Finally, shrinking V and U_1 if necessary, we can suppose that $\rho, D\rho$ are Lipschitz on $\operatorname{co} V$ and similarly $\psi, D\psi$ are Lipschitz on $\operatorname{co} U_1$. We will denote by R a Lipschitz constant for all these functions on these sets. Then to show that ϕ satisfies the LBSC, it is enough to prove that $\phi \circ \rho$ is linearly semiconvex. We need first to link the subgradients of ϕ to those of $\phi \circ \rho$, thanks to the following chain rule:

Lemma 2 *For any $v \in V, \xi \in \partial_P(\phi \circ \rho)(v)$, there exists $\zeta \in \partial_P \phi(\rho(v))$ such that $\xi = D\rho(v)^* \zeta = D\rho(v)^* \tilde{\zeta}$.*

Proof of Lemma 2: Let $v \in V, \xi \in \partial_P(\phi \circ \rho)(v)$. There exist $\eta > 0, \sigma \geq 0$ such that

$$\phi(\rho(v')) - \phi(\rho(v)) - \langle \xi, v' - v \rangle \geq -\sigma |v' - v|^2$$

for any $v' \in B(v, \eta)$. Denote $x' = \rho(v')$ and $x = \rho(v)$ so that $\psi(x') = v'$ and $\psi(x) = v$. There exists $F : U_1 \times U_1 \to \mathbb{R}^{n-1}$ uniformly bounded by R such that

$$\psi(y') - \psi(y) = D\psi(y)(y' - y) + F(y', y)|y' - y|^2$$

for any $y, y' \in U_1$. This implies

$$\begin{aligned}
\phi(x') - \phi(x) - \langle D\psi(x)^* \xi, x' - x \rangle &= \phi(x') - \phi(x) - \langle \xi, \psi(x') - \psi(x) \rangle \\
&\quad - |x' - x|^2 \langle \xi, F(x', x) \rangle \\
&= \phi(\rho(v')) - \phi(\rho(v)) - \langle \xi, v' - v \rangle \\
&\quad - |x' - x|^2 \langle \xi, F(x', x) \rangle \\
&\geq -\sigma |v' - v|^2 - |\xi||F(x', x)||x' - x|^2 \\
&\geq -R(\sigma + |\xi|)|x' - x|^2
\end{aligned}$$

because ψ is Lipschitz of rank R and F bounded by R. This inequality holds for any $x' \in \rho(B(v,\eta))$ which is a neighborhood of x in Γ. It follows that $D\psi(x)^*\xi \in \partial_P\phi(x)$. Let $\zeta := D\psi(x)^*\xi$. Then

$$D\rho(v)^*\zeta = D\rho(v)^* \circ D\psi(x)^*\xi = D(\psi\circ\rho)(v)^*\xi.$$

Moreover, for any $w \in V, \psi\circ\rho(w) = w$, hence $D(\psi\circ\rho)(v) = Id$ and so $D(\psi\circ\rho)(v)^* = Id$. We can conclude

$$D\rho(v)^*\zeta = \xi.$$

Finally, $D\rho(v)^*\zeta = D\rho(v)^*\tilde{\zeta}$, since the kernel of $D\rho(v)^*$ is exactly the normal $N_\Gamma(\rho(v))$ to the tangent plane to Ω at $\rho(v)$.

The lemma is proved.

□

We will also use the following well known result for semiconvex functions on open sets of $\mathbb{R}^{n-1}$.

Lemma 3 *Let $\theta : V \subseteq \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function on an open set V. We suppose that there exists $Q \geq 0$ such that for any $v, v' \in V$ satisfying $[v,v'] \subset V$, we have*

$$\langle \xi' - \xi, v' - v\rangle \geq -Q|v'-v|^2 \quad \forall \xi \in \partial_P\theta(v), \xi' \in \partial_P\theta(v'). \tag{5}$$

Then θ is linearly semiconvex on V.

Proof : Consider the function $\theta_Q := \theta + Q/2|\cdot|^2$. Then on any convex subset of V, θ is linearly semiconvex if θ_Q is convex. Note also that $\zeta \in \partial_P\theta_Q(x)$ if and only if $\zeta - Qx \in \partial_P\theta(x)$. Hence, inequality 5 means

$$\langle \xi' - \xi, v' - v\rangle \geq 0 \ , \quad \forall \xi \in \partial_P\theta_Q(v), \xi' \in \partial_P\theta_Q(v'),$$

which implies the convexity of θ_Q on any convex subset of V. Hence θ is semiconvex on any convex subset of V, with the same modulus of semiconvexity $-Q|\cdot|$. This implies the semiconvexity of θ on V. Lemma 3 is proved.

□

We can now show that $\phi\circ\rho$ is linearly semiconvex. Let $v, v' \in V$ such that $[v,v'] \subset V$, $\xi' \in \partial_P\phi\circ\rho(v'), \xi \in \partial_P\phi\circ\rho(v)$. Let us estimate $< \xi' - \xi, v' - v >$. Thanks to Lemma 2, there exist $\zeta \in \partial_P\phi(\rho(v)), \zeta' \in \partial_P\phi(\rho(v'))$ such that

$$\xi = D\rho(v)^*\tilde{\zeta}, \quad \xi' = D\rho(v')^*\tilde{\zeta}'.$$

There exists a function $E : V \times V \to \mathbb{R}^n$ bounded by R such that

$$\rho(w) - \rho(w') = D\rho(w')(w - w') + E(w, w')|w - w'|^2 \qquad (6)$$

for any $w, w' \in V$. With $w' = v'$ and $w = v$, we get

$$\rho(v') - \rho(v) + E(v, v')|v - v'|^2 = D\rho(v')(v' - v).$$

With $w' = v$ and $w = v'$, we get

$$\rho(v') - \rho(v) - E(v', v)|v - v'|^2 = D\rho(v)(v' - v).$$

Thus, using (4),

$$\begin{aligned}
\langle \xi' - \xi, v' - v\rangle &= \langle D\rho(v')^* \tilde{\zeta}' - D\rho(v)^* \tilde{\zeta}, v' - v\rangle \\
&= \langle \tilde{\zeta}', D\rho(v')(v' - v)\rangle - \langle \tilde{\zeta}, D\rho(v)(v' - v)\rangle \\
&= \langle \tilde{\zeta}', \rho(v') - \rho(v)\rangle - \langle \tilde{\zeta}, \rho(v') - \rho(v)\rangle \\
&\quad + \langle \tilde{\zeta}', E(v, v')|v - v'|^2\rangle + \langle \tilde{\zeta}, E(v', v)|v - v'|^2\rangle \\
&\geq -Q|\rho(v') - \rho(v)|^2 - 2RK|v - v'|^2 \\
&\geq -(QR^2 - 2RK)|v' - v|^2
\end{aligned}$$

(Q is given by (4), R is a Lipschitz constant for ρ on V, K is a Lipschitz constant for ϕ on Γ. Finally, E is bounded by R on $V \times V$.)

Apply now Lemma 3 to conclude that $\phi \circ \rho$ is linearly semiconvex on V, and so ϕ restricted to $\rho(V)$ satisfies the LBSC. As in the proof of Theorem 2, we infer from this fact that ϕ satisfies the LBSC.

Let us now prove the converse. We could reverse the arguments of the first part of the proof but here is a different strategy. Suppose that ϕ satisfies the LBSC. Then ϕ is the restriction to Γ of a convex function $\bar{\phi} : \mathbb{R}^n \to \mathbb{R}$. Let I_Γ be the indicator function of Γ. Let $\tilde{\phi} := \bar{\phi} + I_\Gamma$. Then for any $x \in \Gamma, \partial_P \phi(x) = \partial_P \tilde{\phi}(x)$ (by definition of $\partial_P \phi(x)$).

The *limiting sum rule* (see [5], Proposition 10.1) shows that for any $x \in \Gamma, \zeta \in \partial_P \phi(x)$, there exist $\nu \in N_\Gamma(x), \lambda \in \partial \bar{\phi}(x)$ (recall that for a convex function, proximal subgradients are convex subgradients) such that

$$\zeta = \nu + \lambda.$$

Considering the orthogonal projection of this equality on the tangent hyperplane to Γ at x, we have:

$$\tilde{\zeta} = \tilde{\lambda}.$$

Hence, to show that inequality (4) holds, it is enough to show that for any $x \in \Gamma$, there exist an open set U_x in $\mathbb{R}^n$ and some $Q \geq 0$ such that

$$\langle \tilde{\lambda} - \tilde{\lambda}', y - y' \rangle \geq -Q|y - y'|^2 \tag{7}$$

for any $y, y' \in \Gamma \cap U_x$, and $\lambda \in \partial\bar{\phi}(y), \lambda' \in \partial\bar{\phi}(y')$. For any $\lambda \in \partial\bar{\phi}(x)$, note that $\lambda - \tilde{\lambda} \in N_\Gamma(x)$. Then, inequality (7) is an easy consequence of the following:

Lemma 4 *For any $x \in \Gamma$, there is an open set U_x in $\mathbb{R}^n$ and some $Q_0 \geq 0$ such that for any $y, y' \in \Gamma \cap U_x$, and any $\nu \in N_\Gamma(y), \nu' \in N_\Gamma(y')$, we have*

$$\langle \nu - \nu', y - y' \rangle \leq Q_0(|\nu| + |\nu'|)|y - y'|^2.$$

Suppose that Lemma 4 is true. Then, let $y \in \Gamma$ and U_x, Q_0 as in the lemma. For any $y, y' \in \Gamma \cap U_x$, and $\lambda \in \partial\bar{\phi}(y), \lambda' \in \partial\bar{\phi}(y')$, we have (with $\nu = \lambda - \tilde{\lambda}$, $\nu' = \lambda' - \tilde{\lambda}'$)

$$\begin{aligned}
\langle \tilde{\lambda} - \tilde{\lambda}', y - y' \rangle &\geq \langle \lambda - \lambda', y - y' \rangle - \langle \nu - \nu', y - y' \rangle \\
&\geq 0 - Q_0(|\nu| + |\nu'|)|y - y'|^2 \text{ (because } \bar{\phi} \text{ is convex)} \\
&\geq -Q|y - y'|^2
\end{aligned}$$

The last line follows from the fact that $\bar{\phi}$ is Lipschitz on a neighborhood of Γ, which implies that its convex subgradients are locally bounded and so are the normal components of these. Then inequality (7) holds provided that we show Lemma 4.

Let $x \in \Gamma$ and $\rho : V \to U_x$ a parametrization near x as in the first part of the proof of Theorem 7. Then for any $y = \rho(v), y' = \rho(v'), \nu \in N_\Gamma(y), \nu' \in N_\Gamma(y')$, we have:

$$\begin{aligned}
\langle \nu - \nu', y - y' \rangle &= \langle \nu - \nu', \rho(v) - \rho(v') \rangle \\
&= \langle \nu, \rho(v) - \rho(v') \rangle - \langle \nu', \rho(v) - \rho(v') \rangle \\
&\leq \langle \nu, D\rho(v)(v - v') \rangle - \langle \nu', D\rho(v')(v - v') \rangle \\
&\qquad + R(|\nu| + |\nu'|)|v - v'|^2 \quad \text{(thanks to (6))} \\
&\leq R(|\nu| + |\nu'|)|v - v'|^2
\end{aligned}$$

because ν is in the kernel of $D\rho(v)^*$ and the same is true for $\nu', D\rho(v')^*$.

Finally, using the fact that $\psi_{|\Gamma} = \rho^{-1}$ is R Lipschitz on U_x, we find

$$\langle \nu - \nu', y - y' \rangle \leq R^2(|\nu| + |\nu'|)|y - y'|^2,$$

which is the desired estimate with $Q_0 = R^2$.

□

The developments above lead to the following result, which significantly improves Lemma 1, as it shows that merely a pointwise condition guarantees the LBSC.

Proposition 6 *Let Ω be a uniformly convex bounded open set of class $C^{1,1}$. We suppose that $\phi : \Gamma \to \mathbb{R}$ is continuous and there exists $Q \geq 0$ such that for any $x \in \Gamma$, there exists $\zeta \in \bar{B}(0, Q)$ satisfying*

$$\phi(y) \geq \phi(x) + \langle \zeta, y - x \rangle$$

for any $y \in \Gamma$ near x. Then ϕ satisfies the LBSC.

Proof: It is enough to show that if $\rho : V \to \Gamma$ is a parametrization as in the proof of Theorem 7, then $\phi \circ \rho$ is linearly semiconvex. For any $v \in V$, there exists $\zeta \in \bar{B}(0, Q)$ such that

$$\phi \circ \rho(v') - \phi \circ \rho(v) \geq \langle \zeta, \rho(v') - \rho(v) \rangle$$

for any v' near v. Set $\xi := D\rho(v)^* \zeta$. We have

$$\phi \circ \rho(v') - \phi \circ \rho(v) \geq \langle \xi, v' - v \rangle - \sigma |v' - v|^2$$

for any v' near v, (σ does depend only on Q and on the modulus of continuity of $D\rho$). Set $\theta : V \to \mathbb{R}, \theta = \phi \circ \rho$. Then θ satisfies the hypotheses of the following lemma.

Lemma 5 *Let $\theta : V \to \mathbb{R}$ be a continuous function. We suppose there exists $\sigma \geq 0$ such that for any $v \in V$, there exists $\xi \in \mathbb{R}^n$, satisfying*

$$\theta(v') \geq \theta(v) + \langle \xi, v' - v \rangle - \sigma |v' - v|^2$$

for any v' near v. Then, θ is linearly semiconvex on V.

This lemma concludes the proof of the proposition. Let us now prove it. Set $g := \theta + \sigma |\cdot|^2$. Then for any $v \in V$, there exists $\xi \in \mathbb{R}^n$, such that

$$g(v') \geq g(v) + \langle \xi + 2\sigma v, v' - v \rangle$$

for any v' near v, so that g is convex. Then $\theta = g - \sigma |\cdot|^2$ is linearly semiconvex, and the lemma is proved.

In Proposition 6, the continuity assumption is necessary in view of the following example: Ω is the unit disc in $\mathbb{R}^2$ and $\phi : (\cos\theta, \sin\theta) \mapsto \theta \in [0, 2\pi[$. Furthermore, the existence of some *a priori* rank Q is unavoidable, as shown by the following example; here, Ω is the unit disc in $\mathbb{R}^2$ and Γ the unit circle.

Example 2 *There exists $\phi \in C^1(\Gamma)$, such that for any $x \in \Gamma$, there exists some $\zeta \in \mathbb{R}^n$ satisfying*

$$\phi(y) \geq \phi(x) + \langle \zeta, y - x \rangle \quad \forall y \in \Gamma, \tag{8}$$

and yet ϕ does not satisfy the LBSC.

Proof : There exists $g \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} - 2\pi\mathbb{Z}), 2\pi$ periodic, nonnegative, which is equal to $g(\theta) := |\theta|^3(sin\frac{1}{\theta} + 1)$ on a neighbourhood of 0 when $\theta \neq 0$ and vanishes at 0. Set $\phi(cos\theta, sin\theta) := g(\theta)$. g is nonnegative on a neighbourhood of 0 hence $(1, 0)$ is a global minimum of ϕ. Therefore,$\phi(y) \geq \phi(1, 0) + < (0, 0), y - (1, 0) >$ for any $y \in \Gamma$. On $\Gamma - \{(1, 0)\}, \phi$ is C^2 so that ϕ restricted to $\Gamma \cap] - \infty, 1[\times\mathbb{R}$ satisfies the LBSC (Ω being uniformly convex). To sum up, ϕ satisfies (8). Let us show that ϕ does not satisfy the LBSC. Let $x = (\cos\theta, \sin\theta) \in \Gamma$ near $(1,0)$ with $\theta > 0$ and $\zeta \in \mathbb{R}^n$ such that (8) holds. Then the tangential component of ζ is

$$\tilde{\zeta} = g'(\theta) = 3\theta^2(\sin\frac{1}{\theta} + 1) - \theta\cos\frac{1}{\theta}$$

and the normal component must satisfy (as a direct consequence of (8))

$$\hat{\zeta} \geq \frac{g(\theta') - g(\theta) - g'(\theta)\sin(\theta' - \theta)}{\cos(\theta' - \theta) - 1}$$

for any $\theta' \in \mathbb{R}$. When $\theta > 0$, the right hand side tends to $-g''(\theta) = -6\theta(\sin 1/\theta + 1) + 1/\theta \sin 1/\theta + 4\cos 1/\theta$ when $\theta' \to \theta$. Since $-g''(\frac{2}{(4n+1)\pi}) \to +\infty$ when $n \to +\infty$, we infer that $\hat{\zeta}$ cannot be majorized, hence ϕ does not satisfy the LBSC.

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References

[1] P. Bousquet. PhD thesis, Université Claude Bernard Lyon1. *In Preparation.*

[2] P. Cannarsa and C. Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control.* Progress in Nonlinear Differential Equations and their Applications, 58. Birkhäuser Boston Inc., Boston, MA, 2004.

[3] F. Clarke. Continuity of solutions to a basic problem in the calculus of variations. *Ann. Scuola Norm. Sup. Pisa,* In press.

[4] F. Clarke. Multiple integrals of Lipschitz functions in the calculus of variations. *Proc. Amer. Math. Soc.,* 64(2):260-264, 1977.

[5] F. Clarke, Y. S. Ledyaev, R. J. Stern, and P. R. Wolenski. *Nonsmooth analysis and control theory,* volume 178 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1998.

[6] D. Gilbarg. Boundary value problems for nonlinear elliptic equations in n variables. In *Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962)*, pages 151-159. Univ. of Wisconsin Press, Madison, Wis., 1963.

[7] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[8] E. Giusti. *Direct methods in the calculus of variations.* World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[9] A. Haar. Über das Plateausche Problem. *Math.Ann*, 97:124-258,1927.

[10] P. Hartman. On the bounded slope condition. *Pacific J. Math.,* 18:495-511, 1966.

[11] P. Hartman. On quasilinear elliptic functional-differential equations. *Proc. international symposium on differential equations and dynamical systems, (Puerto Rico, 1965) , Academic Press*, pages 393-407, 1967.

[12] P. Hartman. Convex sets and the bounded slope condition. *Pacific J. Math.,* 25:511-522, 1968.

[13] P. Hartman. and L. Nirenberg. On spherical image maps whose Jacobians do not change sign. *Amer. J. Math.,* 81:901-920, 1959.

[14] D. Hilbert. Über das Dirichletsche Prinzip. *Jber.Deut.Math.Ver.,* 8:184-188, 1900.

[15] P. Marcellini. Regularity for elliptic equations with general growth conditions. *J. Differential Equations,* 105(2):296-333, 1993.

[16] C. Mariconda and G. Treu. Existence and Lipschitz regularity for minima. *Proc. Amer. Math. Soc.,* 130(2):395-404, 2002.

[17] C. Mariconda and G. Treu. Gradient maximum principle for minima. *J. Optim. Theory Appl.,* 112(1):167-186, 2002.

[18] , U. Massari and M. Miranda. *Minimal surfaces of codimension one,* volume 91 of *North-Holland Mathematics Studies.* North-Holland Publishing Co., Amsterdam, 1984.

[19] M. Miranda. Un teorema di esistenza e unicità per il problema dell'area minima in n variabili. *Ann. Scuola Norm. Sup. Pisa (3)*, 19:233-249, 1965.

[20] C. B. Morrey, Jr. *Multiple integrals in the calculus of variations.* Die Grundlehren der mathematischen Wissenschaften, Band 130. Springer-Verlag New York, Inc., New York,1966.

[21] T. Rado. Geometrische betrachtungen über zweidimensional reguläre variations-probleme. *Acta Litterarum ac Scientiarum (Szeged)*, 2:228-253, 1924.

[22] T. Rado. On the problem of Plateau. *Ergeb d. Math., Springer, Berlin*, 2, 1933.

[23] G. Stampacchia. On some regular multiple integral problems in the calculus of variations. *Comm. Pure Appl. Math.,* 16:383-421, 1963.

[24] N. S. Trudinger. Lipschitz continuous solutions of elliptic equations of the form $\mathcal{A}(Du)D^2u = 0$. *Math. Z.,* 109:211-216, 1969.