Fractional Sobolev Spaces and Topology

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Abstract

Consider the Sobolev class $W^{s,p}(M,N)$ where M and N are compact manifolds, and $p \geq 1, s \in (0, 1+1/p)$. We present a necessary and sufficient condition for two maps u and v in $W^{s,p}(M,N)$ to be continuously connected in $W^{s,p}(M,N)$. We also discuss the problem of connecting a map $u \in W^{s,p}(M,N)$ to a smooth map $f \in C^\infty(M,N)$.

Keywords Fractional Sobolev spaces between manifolds, homotopy.

1 Introduction

Let M and N be compact connected oriented smooth boundaryless Riemannian manifolds. Throughout the paper we assume that M and N are isometrically embedded into $\mathbb{R}^a$ and $\mathbb{R}^l$ respectively and that $m := \dim M \geq 2$. Our functional framework is the Sobolev space

$$W^{s,p}(M,N) = \{u \in W^{s,p}(M,\mathbb{R}^l) : u(x) \in N \text{ a.e. }\},$$

with $1 \leq p < \infty, 0 < s$. The space $W^{s,p}(M,N)$ is equipped with the standard metric $\mathrm{d}(u,v) = ||u-v||_{W^{s,p}}$. The main purpose of this paper is to determine whether or not $W^{s,p}(M,N)$ is path-connected and if not, when two elements u and v in $W^{s,p}(M,N)$ can be continuously connected in $W^{s,p}(M,N)$; that is, when there exists $H \in C^0([0,1], W^{s,p}(M,N))$ such that $H(0) = u$ and $H(1) = v$. If this is the case, we say that 'u and v are $W^{s,p}$ connected' (or $W^{s,p}$ homotopic).

Homotopy theory in the framework of Sobolev spaces is essential when studying certain problems in the calculus of variations. This is the case when the admissible functions are defined on a manifold M into a manifold N. One may hope to find multiple minimizers to these problems, ideally one in each homotopy class (see [15], [16] and also [3]).

The topology of $W^{s,p}(M,N)$ depends on two features of the problem, namely the topology of M and N, and the value of s and p. When $s = 1$, the

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study of the topology of $W^{1,p}(M,N)$ was initiated in [4] . The analysis of homotopy classes (for $s=1$) was subsequently tackled in [9] (see also [15], [16] for related and earlier results). These results have been generalized to $W^{s,p}(M,N)$ for non integer values of s and $1<p<\infty$ when M is a smooth, bounded, connected open set in an Euclidean space and when $N=S^1$ (see [5]). In this case, the proofs exploit in an essential way the fact that the target manifold is S^1. In contrast, our main concern is to determine to what extent the methods of [9] and the tools of [4] can be adapted to the case $s\neq 1$. Throughout the paper, we assume that $0<s<1+1/p$ or $sp\geq \dim M$.

Our first result gives some conditions which imply that $W^{s,p}(M,N)$ is path-connected:

Theorem 1 *Let $0<s<1+1/p$. Then the space $W^{s,p}(M,N)$ is path-connected when $sp<2$.*

When $s=1$, this result was proved in [4], where the condition $p<2$ (for $s=1$) is seen to be sharp. For instance, $W^{1,2}(S^1\times\Lambda,S^1)$, where Λ is any open connected set, is not path connnected.

In the case $sp\geq 2$, we have:

Theorem 2 *Assume that $0<s<1+1/p, 2\leq sp<dim M$ and that there exists $k\in\mathbb{N}$ with $k\leq [sp]-1$ such that $\pi_i(M)=0$ for $1\leq i\leq k, \pi_i(N)=0$ for $k+1\leq i\leq [sp]-1$. Then the space $W^{s,p}(M,N)$ is path-connected.*

The case $s=1$ of the above theorem is Corollary 1.1 in [9].

More generally, it is natural to compare the connected components of $W^{s,p}(M,N)$ to those of $C^0(M,N)$. In certain cases, this is indeed possible:

Theorem 3 *a) If $sp\geq dim M$ then $W^{s,p}(M,N)$ is path connected if and only if $C^0(M,N)$ is path connected.*
b) The $W^{s,p}$ homotopy classes are in bijection with the C^0 homotopy classes when $0<s<1+1/p, 2\leq sp<dim M$ and $\pi_i(N)=0$ for $[sp]\leq i\leq dim M$.

The statement a) is well-known and can be proved as in the appendix of [4]. Part b) for $s=1$ was obtained in [9], Corollary 5.2.

When $s=1$, Theorem 2 and Theorem 3 are particular cases of a more general result in [9] which asserts that there is a one-to-one map from the connected components of $W^{1,p}(M,N)$ into the connected components of $C^0(M^{[p]-1},N)$. Here, $M^{[p]-1}$ denotes a $[p]-1$ skeleton of M. This may be re-expressed as follows: two maps u and v in $W^{1,p}(M,N)$ are $W^{1,p}$ homotopic if and only if u is $[p]-1$ homotopic to v. For an accurate definition of $[p]-1$ homotopy, one should refer to [9] or to section 6. Roughly speaking, this means that for a generic $[p]-1$ skeleton $M^{[p]-1}$ of $M, u|_{M^{[p]-1}}$ and $v|_{M^{[p]-1}}$ are homotopic. This makes sense because for a generic $[p]-1$ skeleton, u and

v are both $W^{1,p}$ on these skeletons and hence continuous, by the Sobolev embedding. There is a corresponding version of this result in which $W^{1,p}$ is replaced by $W^{s,p}$:

Theorem 4 *Assume that* $0 < s < 1 + 1/p, 2 \leq sp < dim M$. *Let* $u, v \in W^{s,p}(M, N)$. *Then* u *and* v *are* $W^{s,p}$ *connected if and only if* u *is* $[sp] - 1$ *homotopic to* v.

The techniques in [9] can be adapted in order to prove not only Theorem 4 but also the more general result where the condition $2 \leq sp < \dim M$ is replaced by: $0 < sp < \dim M$, and $sp \neq 1$. In turn, this last result implies Theorem 1 when $sp < 2, sp \neq 1$. However, the case $sp = 1$ seems delicate to handle via these techniques. This is the reason why we give a proof of Theorem 1 based on the tools of [4]. Besides its independent interest, it turns out that the technical core of the proof of Theorem 1 is also the technical core of the proof of Theorem 4. Furthermore, the techniques in [4] are more likely to allow some extensions to the case $s > 1 + 1/p$.

Another strategy to show that two elements in $W^{s,p}(M, N)$ are $W^{s,p}$ connected is based on the property $P(u)$ defined for any $u \in W^{s,p}(M, N)$ by:

$(P(u))$ The map u is $W^{s,p}$ homotopic to some $\tilde{u} \in C^\infty(M, N)$.

We proceed to explain the interest of this property. Assume that $P(u)$ and $P(v)$ are true, where $u, v \in W^{s,p}(M, N)$, and that $\tilde{u}$ and $\tilde{v}$ are C^0 homotopic. So, there exists $F \in C^\infty([0, 1] \times M, N)$ such that $F(0, \cdot) = \tilde{u}$ and $F(1, \cdot) = \tilde{v}$, which implies that $\tilde{u}$ and $\tilde{v}$ are $W^{s,p}$ homotopic. Finally, u and v are $W^{s,p}$ homotopic. This shows the importance of the property P.

Theorem 5 *Each* $u \in W^{s,p}(M, N)$ *satisfies* $P(u)$ *when*
a) $sp \geq dim M$,
b) $0 < sp < 2, 0 < s < 1 + 1/p$,
c) $dim M = 2, 0 < s < 1 + 1/p$,
d) $M = S^m, 0 < s < 1 + 1/p$,
e) $0 < s < 1 + 1/p, 2 \leq sp$ *and* M *satisfies the* $[sp] - 1$ *extension property with respect to* N,
f) $0 < s < 1 + 1/p, 2 \leq sp < dim M$ *and* $\pi_i(N) = 0$ *for* $[sp] \leq i \leq dim M - 1$.

The case $sp \geq \dim M$ can be handled as in the appendix of [4]. If $0 < sp < 2$, then Theorem 1 shows that u can be connected to a constant map. The case $\dim M = 2$ is a consequence of a) and b). When $M = S^m$, we can even show that $W^{s,p}(S^m, N)$ is path-connected if $sp < m$ (see section 5). The statement f) follows from e) (see [9], Remark 5.1). For the meaning of the "$[sp] - 1$ extension property with respect to N", one should refer to [9] or to section 9. Roughly speaking, this means that for any smooth triangulation of M, and any continuous map $f : M^{[sp]} \to N$, we may find a continuous extension of $f|_{M^{[sp]-1}}$ to the whole M. Unfortunately, it is not the case that for any M, N, s, p, each $u \in W^{s,p}(M, N)$ satisfies $P(u)$, (see [9], Corollary 1.5.).

Remark 1 *In the above results, we have often assumed that* $s < 1+1/p, 1 < sp$. *This is closely linked to the strategy of our proofs because we glue several maps in* $W^{s,p}(M, N)$ *together. Let* $u_1 \in W^{s,p}(\Omega_1)$ *and* $u_2 \in W^{s,p}(\Omega_2)$, *where* Ω_1, Ω_2 *are two Lipschitz open subsets of* $\mathbb{R}^d$ *such that*

$$\Gamma := \bar{\Omega}_1 \cap \bar{\Omega}_2 \subset \partial\Omega_1 \cap \partial\Omega_2,$$

and $\Omega := \Omega_1 \cup \Omega_2 \cup \Gamma$ *is a Lipschitz open set. Since* $1 < sp$, *we can define the traces of* u_1, u_2. *Assume that* $tr u_1|_\Gamma = tr u_2|_\Gamma$. *Then, the map* u *defined by*

$$u(x) = \begin{cases} u_1(x) & \text{when } x \in \Omega_1, \\ u_2(x) & \text{when } x \in \Omega_2 \end{cases}$$

belongs to $W^{s,p}(\Omega)$ *when* $s < 1 + 1/p$. *In contrast, nothing can be said when* $s \geq 1 + 1/p$.

Note that when $sp = 1$, *we cannot glue maps in* $W^{s,p}$ *any more, since traces are not defined. However, there is a way to overcome this difficulty (see [4], Appendix B and also section 2.2). Finally, when* $sp < 1$, *maps can be glued without any trace compatibility conditions.*

Remark 2 *To simplify the presentation, we have assumed that* M *is boundaryless. Nevertheless, all the results above can be generalized to the case when* M *has a boundary (see [4], Remark 2.1 and [8], section 4).*

Remark 3 *Lemma 21 below and Theorem 4 show that there exists* $\eta > 0$ *such that for any* $f, g \in W^{s,p}(M, N)$, *if* $||f - g||_{W^{s,p}(M,N)} < \eta$, *then* f *and* g *are* $W^{s,p}$ *homotopic. Hence connected components coincide with path-connected components.*

The following section is the technical core of the article: it enumerates some variations of the technique 'filling a hole', a phrase coined by Brezis and Li [4]. Sections 3 and 4 present some consequences of this technique which allow us to generalize in section 5 the results of [4] ; that is, Theorem 1 and Theorem 5 d). In section 6 and section 7 , we recall and adapt some results of [9] which prepare the proof of Theorem 4 in section 8. In the final section, the corollaries of this theorem, namely Theorem 2, Theorem 3 b) and Theorem 5 e) are proved.

We now introduce some notations: In $\mathbb{R}^d$, B^d (or B when no confusion may arise) denotes the unit ball centered at $0, S^d$ (or S) its boundary, $B^d_r(x) := rB + x, S^d_r(x) := rS + x$ and $B_r = rB$, $S_r = rS$. We will use the convention that all the constants are denoted by the same letter C.

When X is a topological space and $u, v \in X$, we write $u \sim_X v$ to signify the fact that there exists $H \in C^0([0, 1], X)$ such that $H(0) = u$ and $H(1) =$

v. We abbreviate this notation writing $u \sim_{s,p} v$ when u and v are $W^{s,p}$ homotopic; similarly, $u \sim v$ means that u and v are C^0 homotopic.

Whenever $s \in (1, 1 + 1/p)$, we denote $\sigma := s - 1$.

For any k dimensional Lipschitz manifold D embedded in $\mathbb{R}^n$ and any measurable function f, we denote

$$[f]_{W^{\sigma,p}(D)} := \left(\int_D d\mathcal{H}^k(x) \int_D d\mathcal{H}^k(y) \frac{|f(x) - f(y)|^p}{|x - y|^{n+\sigma p}} \right)^{1/p}.$$

The set $W^{s,p}(M)$ denotes either $W^{s,p}(M, \mathbb{R})$ or $W^{s,p}(M, \mathbb{R}^l)$. This will be clear from the context.

2 Filling a hole

The technique 'Filling a hole' appears in [4], Proposition 1.3. We will first generalize it to our context. This will be useful in adapting other tools from [4], such as 'Bridging a map' (see Section 3) and 'Opening a map' (see Section 4). This will allow us to avoid analytical proofs devised in [4] which elude us in the context of fractional Sobolev spaces.

In this section, the underlying Euclidean space is $\mathbb{R}^n$.

2.1 The main result

In this subsection, we prove the following generalization of Lemma D.1 in [5]:

Lemma 1 *Let $0 < s < 2, sp < n$ and $u \in W^{s,p}(S)$. Then, the map $\tilde{u}(x) := u(x/|x|)$ belongs to $W^{s,p}(B)$ and we have*

$$||\tilde{u}||_{W^{s,p}(B)} \leq C||u||_{W^{s,p}(S)}. \tag{1}$$

Proof: We first prove that $\tilde{u} \in L^p(S)$:

$$\int_B |\tilde{u}(x)|^p \, dx = \int_S |u(\theta)|^p d\theta \int_0^1 r^{n-1} \, dr = 1/n||u||^p_{L^p(S)}.$$

We consider three cases: $s = 1, s > 1$ and $s < 1$. When $s = 1$, we have:

$$\int_B |D\tilde{u}(x)|^p \, dx \leq C \int_S |Du(\theta)|^p d\theta \int_0^1 r^{n-1-p} \, dr \leq C||Du||^p_{L^p(S)},$$

since $p < n$.

When $s \in (1, 2)$, we claim that

$$I := \int_B dx \int_B dy \frac{|D\tilde{u}(x) - D\tilde{u}(y)|^p}{|x - y|^{n+\sigma p}} < +\infty.$$

We denote $f(x) := x/|x|$. We have

$$Df(x) = \frac{1}{|x|}Id - \frac{x \otimes x}{|x|^3} \quad \text{, where } x \otimes x = (x_i x_j)_{(i,j)\in[|1,n|]^2},$$

so that $|Df(x)| \leq C/|x|$ and

$$|Df(x) - Df(y)| \leq C\frac{|x-y|}{|x||y|}. \tag{2}$$

(Indeed, note that $Df(\lambda x) = x/\lambda$ and $Df(Rx) = R(Df(x))R^{-1}$ for any $\lambda > 0, R \in O(n)$. Hence, we can assume that $x = (1, 0, ..0)$ and $y = (r\cos\theta, r\sin\theta, 0, .., 0)$. Then, (2) can be easily shown).

Writing

$$|D\tilde{u}(x) - D\tilde{u}(y)| \leq |Du(x/|x|) - Du(y/|y|)||Df(x)|$$
$$+|Du(y/|y|)||Df(x) - Df(y)|, \tag{3}$$

we find $I \leq C(I_1 + I_2)$ with

$$I_1 := \int_S d\theta \int_S d\tau |Du(\theta) - Du(\tau)|^p \int_{r=0}^1 dr \int_{t=0}^1 \frac{r^{n-1-p}t^{n-1}}{|r\theta - t\tau|^{n+\sigma p}}\, dt,$$

$$I_2 := \int_B dx \int_B dy |Du(y/|y|)|^p \frac{|x-y|^p}{|x|^p|y|^p|x-y|^{n+\sigma p}}.$$

We claim that whenever $\theta \neq \tau$,

$$J := \int_{r=0}^1 dr \int_{t=0}^1 \frac{r^{n-1-p}t^{n-1}}{|r\theta - t\tau|^{n+\sigma p}}\, dt \leq \frac{C}{|\theta - \tau|^{n-1+\sigma p}}. \tag{4}$$

Indeed, after making the change of variable $t \to \lambda := t/r$, we get

$$J \leq \int_{r=0}^1 r^{n-1-sp}\, dr \int_{\lambda=0}^{\infty} \frac{\lambda^{n-1}}{|\theta - \lambda\tau|^{n+\sigma p}}\, d\lambda$$

$$\leq C \int_{\lambda=0}^{\infty} \frac{\lambda^{n-1}}{|\theta - \lambda\tau|^{n+\sigma p}}\, d\lambda \qquad \text{(since } sp < n\text{)}$$

$$\leq C(\int_{\lambda=0}^{2} \frac{d\lambda}{|\theta - \lambda\tau|^{n+\sigma p}} + \int_2^{\infty} \frac{\lambda^{n-1}}{\lambda^{n+\sigma p}}) \leq C(\int_{\lambda=0}^{2} \frac{d\lambda}{|\theta - \lambda\tau|^{n+\sigma p}} + 1).$$

Now, consider the 2 plane generated by θ and τ. In this plane, θ and τ belong to S^1, so that they can be written $\theta = e^{i\alpha}, \tau = e^{i\beta}, \alpha, \beta \in (-\pi, \pi]$. Hence, with $\gamma := \beta - \alpha$,

$$|\theta - \lambda\tau|^2 = |\lambda - e^{i\gamma}|^2 = (\lambda - \cos\gamma)^2 + \sin^2\gamma.$$

The change of variable $\mu := (\lambda - \cos\gamma)/\sin\gamma$, (when $\sin\gamma \neq 0$) yields

$$\int_{\lambda=0}^{2} \frac{d\lambda}{|\theta - \lambda\tau|^{n+\sigma p}} \leq \frac{1}{(\sin\gamma)^{n-1+\sigma p}} \int_{\mathbb{R}} \frac{d\mu}{(1+\mu^2)^{(n+\sigma p)/2}} \leq \frac{C}{(\sin\gamma)^{n-1+\sigma p}}.$$

Moreover,

$$|\theta - \tau|^2 = 2(1 - \cos\gamma) = 4\sin^2(\gamma/2)$$

and the map $\gamma \to \dfrac{\sin(\gamma/2)}{\sin\gamma}$ is bounded near 0, say for $|\gamma| \leq \pi/4$. This shows that

$$\int_{\lambda=0}^{2} \frac{d\lambda}{|\theta - \lambda\tau|^{n+\sigma p}} \leq \frac{C}{|\theta - \tau|^{n-1+\sigma p}}$$

when $|\beta - \alpha| \leq \pi/4$. On the other hand, this inequality is trivially true when $|\beta - \alpha| \geq \pi/4$ (by increasing C if necessary). This proves (4) and implies that

$$I_1 \leq C \int_S d\theta \int_S d\tau \frac{|Du(\theta) - Du(\tau)|^p}{|\theta - \tau|^{n-1+\sigma p}} = C[Du]^p_{W^{\sigma,p}(S)}.$$

We proceed to estimate I_2. We have

$$I_2 \leq \int_B |Du(y/|y|)|^p \, dy \int_{\mathbb{R}^n} \frac{dx}{|x|^p |y|^p |y - x|^{n+(\sigma-1)p}}$$

$$=: \int_B |Du(y/|y|)|^p K(y) \, dy.$$

Clearly, for any $y \neq 0$, $K(y) < \infty$ (since $p < n$), $K(y)$ depends only on $|y|$ and $K(\lambda y) = K(y)/\lambda^{sp}$. Thus, $K(y) = C/|y|^{sp}$. This shows that $I_2 \leq C||Du||^p_{L^p(S)}$. Moreover, we have established (1) when $s \in (1, 2)$.

When $s \in (0, 1)$, the calculation is easier, and is very similar to the treatment of I_1. The lemma is proved.

□

The same proof yields:

Corollary 1 *Let $0 < s < 2, sp < n$ and $u \in W^{s,p}(S)$. Then, $\tilde{u}(x) := u(x/|x|)$ belongs to $W^{s,p}_{loc}(\mathbb{R}^n)$.*

2.2 Filling a hole continuously

Consider a smooth bounded open set Ω in $\mathbb{R}^n$ and denote by Γ its boundary. There exists $\epsilon > 0$ such that the ϵ tubular neighborhood of Γ :

$$U_\epsilon := \{x \in \Omega : \text{dist}\,(x, \Gamma) < \epsilon\}$$

can be parametrized by:

$$\Phi : (x', r) \in \Gamma \times (0, \epsilon) \mapsto x' + r\nu(x'),$$

where $\nu(x')$ denotes the inner unit normal to Γ at x'. We also introduce the nearest point projection $\pi : U_\epsilon \to \Gamma$. Hence, for any $x \in U_\epsilon$, we have $\Phi^{-1}(x) = (\pi(x), \mathrm{dist}\,(x, \Gamma))$. Finally, we denote $\Gamma_r := \Phi(\Gamma \times \{r\})$.

Note that for any measurable function $u : \mathbb{R}^n \to \mathbb{R}$, defined almost everywhere, it makes sense to define its restriction $u|_{\Gamma_r}$ to Γ_r, for almost every $r \in (0, \epsilon)$. When $u \in W^{s,p}(\mathbb{R}^n)$ with $sp > 1$, this restriction is equal to the trace of u : $\mathrm{tr}\, u|_{\Gamma_r}$ for a.e. r. In the special case $sp = 1$, we need a substitute for the trace theory: the *good restrictions*, introduced in [5]. We proceed to present the definition of good restrictions for a map $u \in W^{s,p}(\Omega)$, when $s \in (0,1), sp = 1$. For a proof of the statements below, see [5].

For each $r \in (0, \epsilon)$, there is at most one function v defined on Γ_r such that the map

$$w_1^r(x) = \begin{cases} u(x) & \text{in } \Omega \setminus U_r, \\ v(\Phi(\pi(x), r)) & \text{in } \Omega \cap U_r \end{cases}$$

or equivalently, the map

$$w_2^r(x) = \begin{cases} u(x) - v(\Phi(\pi(x), r)) & \text{in } \Omega \setminus U_r, \\ 0 & \text{in } \Omega \cap U_r \end{cases}$$

belongs to $W^{s,p}(\Omega)$. Moreover, for a.e. $r \in (0, \epsilon)$, the function $v := u|_{\Gamma_r}$ has the property that $w_1^r, w_2^r \in W^{s,p}(\Omega)$. In fact, a necessary and sufficient condition for this property to hold is that $v \in W^{s,p}(\Gamma_r)$ and

$$\int_\Gamma d\mathcal{H}^{n-1}(x') \int_r^\epsilon dt \frac{|v(\Phi(x', r)) - u(\Phi(x', t))|^p}{(t-r)} < \infty.$$

For these values of r, we say that v is the inner good restriction of u to Γ_r. Similarly, we may define an outer good restriction. If v is both an inner and an outer good restriction, we call it a good restriction.

In particular, $u|_{\Gamma_r}$ is a good restriction if and only if

i) $u|_{\Gamma_r} \in W^{s,p}(\Gamma_r)$,

ii) $$\int_\Gamma d\mathcal{H}^{n-1}(x') \int_0^\epsilon dt \frac{|u(\Phi(x', r)) - u(\Phi(x', t))|^p}{|t-r|} < \infty.$$

Assume that Γ can be written as a finite union of subsets Γ^i which are open in Γ and such that i), ii) are true for each Γ^i instead of Γ. Then i), ii) are true for Γ. This shows that 'being a good restriction' is a *local* condition.

We will often use the following well-known consequence of the Fubini's Theorem:

Lemma 2 *Let $s \in (0, 2)$ and $u \in W^{s,p}(\Omega)$. Then for a.e. $r \in (0, \epsilon)$,*
i) when $sp > 1$, the trace $\mathrm{tr}\, u|_{\Gamma_r}$ coincides with $u|_{\Gamma_r}$ and belongs to $W^{s,p}(\Gamma_r)$,
ii) when $sp = 1$, $u|_{\Gamma_r}$ is a good restriction of u to Γ_r, (in particular, $u|_{\Gamma_r} \in W^{s,p}(\Gamma_r)$),
iii) when $sp < 1$, the restriction of u to Γ_r belongs to $W^{s,p}(\Gamma_r)$.

Such an r will be called '*good*'. We will also say that Γ_r is '*good for u*'.

In the following lemma, the set Ω is B_2, so that Γ_r is the sphere of radius $2-r$.

Lemma 3 *Let $0 < s < 1 + 1/p, 0 < sp < n$. Let $u \in W^{s,p}(B_2, N)$ and assume that S is good for u. For any $t \in [0,1)$, let*

$$u^t(x) = \begin{cases} u(x/(1-t)) & \textit{when } |x| \le 1-t, \\ u(x/|x|) & \textit{when } 1-t \le |x| \le 1, \\ u(x) & \textit{when } 1 \le |x| \le 2 \end{cases}$$

and

$$u^1(x) = \begin{cases} u(x/|x|) & \textit{when } |x| \le 1, \\ u(x) & \textit{when } 1 \le |x| \le 2. \end{cases}$$

Then,

$$t \in [0,1] \to u^t \in W^{s,p}(B_2, N)$$

is continuous and $u^t(x) = u(x)$ for any $t \in [0,1]$ and any $1 \le |x| \le 2$.

Proof: Consider the maps

$$v^t(x) = \begin{cases} u(x/(1-t)) & \text{when } |x| \le 1-t, \\ u(x/|x|) & \text{when } 1-t \le |x| \le 2 \end{cases}$$

and $v^1(x) = u(x/|x|)$. To prove Lemma 3, it is enough to show that $v^t \in C^0([0,1], W^{s,p}(B_2, N))$ since $u^t = v^t + z$ where z is defined by:

$$z(x) = \begin{cases} 0 & \text{when } |x| \le 1, \\ u(x) - u(x/|x|) & \text{when } 1 \le |x| \le 2. \end{cases}$$

(The map z belongs to $W^{s,p}$ since S is good for u.)

Consider first the case $sp > 1$. Then, Lemma 3 is essentially Lemma D.2 in [5] : condition $s < 1$ is replaced by $s < 1 + 1/p$ in our case.

Let

$$\tilde{v}(x) := \begin{cases} u(x) & \text{when } |x| \le 1, \\ u(x/|x|) & \text{when } 1 \le |x|. \end{cases}$$

Then $\tilde{v}$ belongs to $W^{s,p}_{\text{loc}}(\mathbb{R}^n)$. We have $v^t(x) = \tilde{v}(x/(1-t))$. This shows that $t \in [0,1) \mapsto v^t \in W^{s,p}(B_2, N)$ is continuous. Thus, there remains to show that v^t converges to v^1 when $t \to 1^-$. By Corollary 1, $v^1 \in W^{s,p}_{\text{loc}}(\mathbb{R}^n)$. Let $g := \tilde{v} - v^1$. Then, $g \in W^{s,p}(\mathbb{R}^n)$ because $g(x) = 0$ when $|x| \ge 1$. Moreover, $v^t(x) - v^1(x) = g(x/(1-t))$. We easily have

$$[g(\cdot/(1-t))]_{W^{s,p}(\mathbb{R}^n)} = (1-t)^{(n-sp)/p}[g]_{W^{s,p}(\mathbb{R}^n)}.$$

This shows the continuity at $t = 1$.

It remains to consider the case $sp \le 1$. Though we cannot define the trace anymore, the fact that $r = 1$ is good implies that $\tilde{v} \in W^{s,p}_{\text{loc}}(\mathbb{R}^n), g \in W^{s,p}(\mathbb{R}^n)$. As above, we find that $v^t \to v^1$ in $W^{s,p}(B_2)$.

This completes the proof of the lemma.

□

2.3 Filling an annulus continuously

As a corollary of Lemma 3, we get the following:

Lemma 4 *Let $s \in (0, 1 + 1/p)$ and $u \in W^{s,p}(B_2)$ such that S is good for u. Then, the map u^t defined by*

$$u^t(x) = \begin{cases} u(x/(1-t/2)) & \text{when } |x| \leq 1 - t/2, \\ u(x/|x|) & \text{when } 1 - t/2 \leq |x| \leq 1, \\ u(x) & \text{when } 1 \leq |x| \leq 2 \end{cases}$$

belongs to $C^0([0,1], W^{s,p}(B_2))$.

Lemma 4 can be immediately generalized to the case when B_2 is replaced by a smooth bounded open convex set Ω containing the origin, with the Euclidean norm replaced by the norm

$$j(x) := \inf\{t > 0 : x/t \in \Omega\}.$$

2.4 Filling a cylinder

In this subsection, we pick some $2 \leq k \leq n-1$ and we decompose $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$. We also denote $x \in \mathbb{R}^n$ as $(x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$.

Let T be the open set in $\mathbb{R}^n$ defined by:

$$T := \{(x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k} : |x'| < 1\}$$

and $2T := \{2x : x \in T\}$. Then we have:

Lemma 5 *Let $0 < s < 2, sp < k$ and $u \in W^{s,p}(\partial T)$. Then, the map $\tilde{u}$ defined by:*

$$\tilde{u}(x', x'') := u(x'/|x'|, x'')$$

belongs to $W^{s,p}(T)$.

Proof: An easy computation shows that

$$||\tilde{u}||_{W^{1,p}(T)} \leq C||u||_{W^{1,p}(\partial T)} \quad ;$$

this settles the case $s = 1$. When $s \in (1,2)$, it remains to show that

$$I := \int_T dx \int_T dy \frac{|D\tilde{u}(x) - D\tilde{u}(y)|^p}{|x-y|^{n+\sigma p}} < +\infty.$$

We have $I \leq C(I' + I'')$, where

$$I' := \int_{\mathbb{R}^{n-k}} dx'' \int_{x' \in \mathbb{R}^k, |x'|<1} dx' \int_{y' \in \mathbb{R}^k, |y'|<1} dy' \frac{|D\tilde{u}(x', x'') - D\tilde{u}(y', x'')|^p}{|x'-y'|^{k+\sigma p}},$$

$$I'' := \int_{\mathbb{R}^k, |y'|<1} dy' \int_{x'' \in \mathbb{R}^{n-k}} dx'' \int_{y'' \in \mathbb{R}^{n-k}} dy'' \frac{|D\tilde{u}(y', x'') - D\tilde{u}(y', y'')|^p}{|x'' - y''|^{n-k+\sigma p}}.$$

This is a Besov's type inequality (see [1] or [2]).

We first prove that $I'' \le C||Du||^p_{W^{\sigma,p}(\partial T)}$. Using the fact that $p < n$, we have

$$I'' \le \int_{|y'|<1} dy' \frac{1}{|y'|^p} \int_{\mathbb{R}^{n-k}} dx'' \int_{\mathbb{R}^{n-k}} dy'' \frac{|Du(y'/|y'|, x'') - Du(y'/|y'|, y'')|^p}{|x'' - y''|^{n-k+\sigma p}}$$

$$\le C \int_{S^{k-1}} d\theta \int_{\mathbb{R}^{n-k}} dx'' \int_{\mathbb{R}^{n-k}} dy'' \frac{|Du(\theta, x'') - Du(\theta, y'')|^p}{|x'' - y''|^{n-k+\sigma p}},$$

which implies that $I'' \le C||u||^p_{W^{s,p}(\partial T)}$.

We denote $f(x', x'') := (x'/|x'|, x'')$. We proceed to estimate I' by writing $I' \le C(I'_1 + I'_2)$ with

$$I'_1 := \int_{\mathbb{R}^{n-k}} dx'' \int_{|x'|<1} dx' \int_{|y'|<1} \frac{|Du(x'/|x'|, x'') - Du(y'/|y'|, x'')|^p}{|x'|^p |x' - y'|^{k+\sigma p}} dy'$$

$$I'_2 := \int_{\mathbb{R}^{n-k}} dx'' \int_{|x'|,|y'|<1} dx'\, dy' \frac{|Du(y'/|y'|, x'')|^p |Df(x', x'') - Df(y', x'')|^p}{|x' - y'|^{k+\sigma p}};$$

this follows from (3).

We can prove that $I'_2 \le C||Du||^p_{L^p(\partial T)}$ exactly as we estimated I_2 in the proof of Lemma 1.

On the other hand, we find that

$$I'_1 = \int_{\mathbb{R}^{n-k}} dx'' \int_{|x'|<1} dx' \int_{|y'|<1} dy' \frac{|Du(x'/|x'|, x'') - Du(y'/|y'|, x'')|^p}{|x'|^p |x' - y'|^{k+\sigma p}}$$

$$= \int_{\mathbb{R}^{n-k}} dx'' \int_{S^{k-1}} d\theta \int_{S^{k-1}} d\tau |Du(\theta, x'') - Du(\tau, x'')|^p \int_0^1 \int_0^1 \frac{r^{n-1} t^{n-1}}{r^p |r\theta - t\tau|^{k+\sigma p}}$$

$$\le C \int_{\mathbb{R}^{n-k}} dx'' \int_{S^{k-1}} d\theta \int_{S^{k-1}} d\tau \frac{|Du(\theta, x'') - Du(\tau, x'')|^p}{|\theta - \tau|^{k-1+\sigma p}},$$

(here, we use $\int_{r=0}^1 dr \int_{t=0}^1 dt \frac{r^{n-1-p} t^{n-1}}{|r\theta - t\tau|^{k+\sigma p}} \le \frac{C}{|\theta - \tau|^{k-1+\sigma p}}$, see the proof of (4)).

From the last inequality, we easily obtain $I'_1 \le C||u||^p_{W^{s,p}(\partial T)}$, which gives the required result when $s \in (1, 2)$. When $s \in (0, 1)$, the calculation is easier and we omit it. Lemma 5 is proved.

□

Lemma 5 implies the following (exactly as Lemma 1 implied Lemma 3):

Lemma 6 *Let $0 < s < 1 + 1/p$, $sp < k$ and $u \in W^{s,p}(2T)$ such that ∂T is good for u. Then the map u^t defined by*

$$u^t(x) := \begin{cases} u(x'/(1-t), x'') & \text{when } |x'| \le 1-t, \\ u(x'/|x'|, x'') & \text{when } 1-t \le |x'| \le 1, \\ u(x', x'') & \text{when } 1 \le |x'| \le 2 \end{cases}$$

belongs to $C^0([0,1], W^{s,p}(2T))$.

3 'Bridging' of maps

3.1 The case $n = 2$

Consider the square

$$\Omega := \{x = (x_1, x_2) : |x_1| < 20, \quad |x_2| < 20\}$$

and let $u \in W^{s,p}(\Omega, N)$.

We assume that u is constant, say Y_0, in the region $Q^+ \cup Q^-$ where

$$Q^+ = \{x = (x_1, x_2) : |x_1| < 20, \quad 1 < x_2 < 20\}$$

and

$$Q^- = \{x = (x_1, x_2) : |x_1| < 20, \quad -20 < x_2 < -1\}.$$

The following lemma corresponds to [4], Proposition 1.2.

Lemma 7 *If $0 < s < 1 + 1/p, sp < 2$, then there exists $u^t \in C^0([0,1], W^{s,p}(\Omega, N))$ such that*

$$u^0 = u,$$
$$u^t(x) = u(x) \ \forall t \in [0,1], \quad \forall x \notin (-5,5) \times (-1,1),$$
$$u^1(x) = Y_0 \ \forall x \in (1, 3/2) \times (-20, 20).$$

Proof: First, choose two circles C_1, C_2 with the same radius larger than $2/\sqrt{3}$, centered on the line $\{x = (x_1, x_2) : x_2 = 0\}$ such that the center of C_1 belongs to C_2. This implies that C_1 and C_2 intersects at two points which belongs to Q^+ and Q^-. Moreover, we require that C_1 and C_2 are good for u. Without loss of generality, we may assume that C_1 is centered at $(0,0)$ and that C_2 is centered at $(2,0)$, their common radius being 2. Now, by filling the hole inside C_1 (see Lemma 3), we can link u to some u_1 which is equal to u outside C_1 and which is equal to Y_0 on the set $\{(x_1, x_2) : |x_2| \ge |x_1|/\sqrt{3}\}$.

We claim that C_2 is still good for u_1. In fact, in the subset of C_2 where u has been changed, u_1 is equal to Y_0 and when $sp > 1$, the trace of u on $C_2 \cap \{x : x_1 \le 2\}$ is equal to Y_0. This settles the cases $sp > 1$. The case $sp < 1$ is obvious. When $sp = 1$, it remains to prove that the constant map equal to Y_0 is a good restriction for u to $C_2 \cap \{x : x_1 \le 2\}$ (since the concept of good restrictions is local). But this is a mere consequence of Lemma 8 below. The claim is proved.

Finally, by filling the hole inside C_2, we can connect u_1 to some u_2 which is equal to u_1 outside C_2 while inside C_2, u_2 is equal to Y_0 except on the domain $\{(x_1,x_2) : x_1 > 2+\sqrt{3}|x_2|\}$. In particular, u_2 is equal to u on $\{(x_1,x_2) : |x_1| > 4\}$ and is equal to Y_0 on

$$Q^+ \cup Q^- \cup \{(x_1,x_2) : 0 < x_1 < 2\}.$$

This completes the proof of the lemma.

□

Lemma 8 *Let $sp = 1$ and $u \in W^{s,p}((-1,1)^2)$ such that $u = Y_0$ on $\{x : |x_1| < |x_2|\}$. Then the constant map equal to Y_0 on the line $D := \{x_1 = 0\}$ is a good restriction of u to D.*

Proof: It is sufficient to prove that

$$I := \int_{-1}^{1} dx_2 \int_{-1}^{1} \frac{|u(x_1,x_2) - Y_0|^p}{|x_1|}\, dx_1 < \infty.$$

Since N is compact, there exists $C > 0$ such that $|u(x_1,x_2) - Y_0|^p \leq C$ for any (x_1,x_2). Then the lemma follows from the fact that:

$$\begin{aligned} I &= \int_{-1}^{1} dx_2 \int_{|x_2|\leq|x_1|\leq 1} \frac{|u(x_1,x_2) - Y_0|^p}{|x_1|}\, dx_1 \\ &\leq C \int_{-1}^{1} dx_2 \int_{|x_2|}^{1} \frac{dx_1}{|x_1|} \leq C. \end{aligned}$$

□

3.2 The case $n \geq 2$

We work in $\mathbb{R}^n, n \geq 2$ and we distinguish some special variables. For $0 \leq l \leq n-2$, we write

$$x = (x_1', x'', x_2')$$

where $x_1' = x_1, x_2' = (x_{n-l}, .., x_n)$ and $x'' = (x_2, .., x_{n-l-1})$ (when $l = n-2$, we omit x''). We also write $x' = (x_1', x_2')$. Let

$$\Omega := \{(x_1', x'', x_2') : |x_1'| < 20, |x''| < 20, |x_2'| < 20\}.$$

Set $k := l + 2$.

Lemma 9 *Assume that $0 < s < 1 + 1/p, sp < k$ and $u \in W^{s,p}(\Omega, N)$ with $u(x) = Y_0$ for any $x \in \Omega$ such that $1 < |x_2'|$, for some $Y_0 \in N$. Then there exists $u^t \in C^0([0,1], W^{s,p}(\Omega, N))$ such that $u^0 = u, u^t(x) = u(x)$ for any $0 \leq t \leq 1$ and any x outside $\{x : |x| < 15\}$ and $u^1(x) = Y_0$ for any $x, |(x_1', x'')| < 1/8$.*

Proof: If $k = n$, then the proof is exactly the same as in the previous subsection (except that circles are replaced by n dimensional balls). Hence, we may assume that $k < n$. Let $\delta : \mathbb{R}^{n-k} \to \mathbb{R}^k$ be a smooth function to be chosen later. We define the cylinder C_1 by

$$C_1 := \{x = (x'_1, x'', x'_2) : |x' - \delta(x'')| = a\}$$

and the tube T_1 by

$$T_1 := \{x = (x'_1, x'', x'_2) : |x' - \delta(x'')| < a\},$$

for some $a > 1$ to be determined below. We may choose a and δ such that:
i) when $|x''| < 2$, we have $\delta(x'') = 0$,
ii) when $|x''| \geq 4$, we have $x \in T_1 \Rightarrow |x'_2| > 1$,
iii) C_1 is good for u.

Note that C_1 can be chosen as a smooth deformation of a *straight* cylinder as defined in subsection 2.4. Note also that even if $C_1 \cap \Omega$ is a *finite* cylinder (contrary to those of subsection 2.4), the *ends* of this cylinder are contained in a domain where u is equal to the constant Y_0, where 'nothing happens'. Hence, we can apply Lemma 6 to C_1 : u can be connected to some $\bar{u}$ which equals Y_0 on $\{x \in \Omega : |x''| < 2, |x'_2| \geq |x'_1|/\sqrt{a^2 - 1}\}$.

The computation in the proof of Lemma 8 yields easily that $\bar{u}$ has a good restriction (equal to Y_0) on the set $\{|x''| < 2, x'_1 = 0\}$. This implies that the map:

$$w(x'_1, x'', x'_2) := \begin{cases} 0 \quad \text{when } x'_1 \leq 0, \\ \bar{u}(x'_1, x'', x'_2) - Y_0 \quad \text{when } x'_1 \geq 0 \end{cases}$$

belongs to $W^{s,p}(\Omega_0)$, where $\Omega_0 := \{x \in \Omega : |x''| < 2\}$.

Let $\rho : \mathbb{R} \to [0, 1]$ be a smooth function which vanishes on $\{t : |t| \geq 2\}$, which is equal to 1 on $\{t : |t| \leq 1\}$ and such that $|\rho'| \leq 2$. Then we define

$$\Xi_t(x'_1, x'', x'_2) := (x'_1 - \frac{t\rho(2|x''|^2)\rho(2x'_1)}{8}, x'', x'_2).$$

The map Ξ_t is a smooth diffeomorphism of $\mathbb{R}^n$ which maps Ω_0 onto Ω_0.

By the *diffeomorphism property* in $W^{s,p}$ (see [14]), there exists $C > 0$ such that for any $t \in [0, 1]$, and any $g \in W^{s,p}(\Omega_0)$, we have

$$||g \circ \Xi_t||_{W^{s,p}(\Omega_0)} \leq C||g||_{W^{s,p}(\Omega_0)}.$$

Let $\epsilon > 0$. Then there exists $z \in C^\infty(\bar{\Omega}_0)$ such that $||z - w||_{W^{s,p}(\Omega_0)} < \epsilon$. Hence, for any $t, s \in [0, 1]$,

$$||w \circ \Xi_t - w \circ \Xi_s||_{W^{s,p}(\Omega_0)} \leq ||w \circ \Xi_t - z \circ \Xi_t||_{W^{s,p}(\Omega_0)} + ||z \circ \Xi_t - z \circ \Xi_s||_{W^{s,p}(\Omega_0)}$$

$$+||z \circ \Xi_s - w \circ \Xi_s||_{W^{s,p}(\Omega_0)} \leq C||z - w||_{W^{s,p}(\Omega_0)} + ||z \circ \Xi_t - z \circ \Xi_s||_{W^{s,p}(\Omega_0)}$$

$$\leq C\epsilon + ||z \circ \Xi_t - z \circ \Xi_s||_{W^{s,p}(\Omega_0)}.$$

Since the last term goes to 0 when $|s-t| \to 0$, the map $t \to w \circ \Xi_t$ belongs to $C^0([0,1], W^{s,p}(\Omega_0))$.

Similarly we may define

$$\tilde{w}(x_1', x'', x_2') := \begin{cases} \bar{u}(x_1', x'', x_2') - Y_0 & \text{when } x_1' \le 0, \\ 0 \text{ when } x_1' \ge 0 \end{cases}$$

and

$$\tilde{\Xi}_t(x_1', x'', x_2') := (x_1' + \frac{t\rho(2|x''|^2)\rho(2x_1')}{8}, x'', x_2').$$

As above, $\tilde{w} \circ \tilde{\Xi}_t \in C^0([0,1], W^{s,p}(\Omega_0))$. This yields

$$w \circ \Xi_t + \tilde{w} \circ \tilde{\Xi}_t \in C^0([0,1], W^{s,p}(\Omega_0)).$$

If we denote by v^t the map $w \circ \Xi_t + \tilde{w} \circ \tilde{\Xi}_t + Y_0$, we have $v^t =$

$$\begin{cases} \bar{u}(x_1' + t\rho(2|x''|^2)\rho(2x_1')/8, x'', x_2') & \text{when } x_1' \le -t\rho(2|x''|^2)\rho(2x_1')/8, \\ \\ Y_0 \text{ when } -t\rho(2|x''|^2)\rho(2x_1')/8 \le x_1' \le t\rho(2|x''|^2)\rho(2x_1')/8, \\ \\ \bar{u}(x_1' - t\rho(2|x''|^2)\rho(2x_1')/8, x'', x_2') & \text{when } t\rho(2|x''|^2)\rho(2x_1')/8 \le x_1'. \end{cases}$$

Note in particular that $v^t = \bar{u}$ when $|x''| > 1$ or $|x_1'| > 1$. Hence we can extend v^t by $\bar{u}$ on Ω and we still have $v^t \in C^0([0,1], W^{s,p}(\Omega))$. Finally, $v^t = Y_0$ when $|x''| < 1/\sqrt{2}$ and $|x_1'| \le t/8$. This completes the proof of the lemma.

□

4 Opening of Maps

Lemma 10 *Let $0 < s < 1 + 1/p$ and $u \in W^{s,p}(B_{10}, N)$. Then, there exists $u^t \in C^0([0,1], W^{s,p}(B_{10}, N))$ such that $u^0 = u, u^1 = Y_0$ on an open subset of B_5 for some $Y_0 \in N$ and $u^t = u$ on $B_{10} \setminus B_9, 0 \le t \le 1$.*

Proof: We first introduce the concept of *smooth cubes*. A smooth cube is simply a cube with smooth corners, or equivalently, a sphere with faces. Formally, a smooth open set G of $\mathbb{R}^n$ will be called a smooth cube of side R if it is a smooth convex set G which satisfies:

$$\cup_{i=1}^n \{(x_1, .., x_n) : |x_i| < R, |x_j| < 4R/5 \quad \forall j \ne i\} \subset G \subset (-R, R)^n.$$

For such a set G, we define the i^{th} face:

$$F_i := \{(x_1, ..x_n) : x_i = R, |x_j| < 4R/5\}.$$

For any $i = 1, ..n$, let

$$G_i := \{tx : x \in F_i, t \in (1/5, 1)\}.$$

The set G is a smooth convex set, so that the technique of 'filling an annulus' (see Lemma 4) applies. More precisely, consider some $v \in W^{s,p}(\mathbb{R}^n)$ such that ∂G is good for v. Then v can be connected to a map $w \in W^{s,p}(\mathbb{R}^n)$ which is equal to v on $\mathbb{R}^n \setminus G$ and which satisfies

$$w(tx) = v(x) \ \ \forall tx \in G_i.$$

Returning to the proof of Lemma 10, let $v \in W^{s,p}(B_{10})$ and G be a smooth cube of side R such that $G \subset B_5$ and ∂G is good for v. Assume that $v|_{F_i}(x_1,..,x_n)$ does not depend on $x_1,..,x_{i-1}$. By this, we mean that for $\mathcal{H}^{n-i+1}$ a.e. $x_i,..,x_n \in \mathbb{R}^{n-i+1}$, the map $(x_1,..x_{i-1}) \in \mathbb{R}^{i-1} \to \chi_{F_i}(x)v(x)$ is $\mathcal{H}^{i-1}$ a.e. constant. Then on $G_i, w(tx) = v(x)$ (with $x \in F_i, t \in (1/5,1)$), does not depend neither on $x_1,..,x_{i-1}$ nor on t.

Consider the map

$$\phi_i : tx \in G_i \mapsto \sum_{j \neq i} \frac{5x_j}{4R} e_j + \frac{5t-3}{2} e_i \in (-1,1)^n.$$

Here (e_k) denotes the canonical basis of $\mathbb{R}^n$. Observe that ϕ_i^{-1} is a smooth diffeomorphism from $[-1,1]^n$ onto $\bar{G}_i$. Then, $w \circ \phi_i^{-1} \in W^{s,p}((-1,1)^n)$ and does not depend on $x_1,..,x_i$.

We now prove the lemma by induction: We claim that for each $1 \leq k \leq n, u$ can be connected to some $u_k \in W^{s,p}(B_{10})$ such that $u_k = u$ outside B_9 and such that there exists a smooth diffeomorphism ψ_k from $[-1,1]^n$ into B_5 such that $u_k \circ \psi_k$ does not depend on $x_1,..,x_k$ on $(-1,1)^n$.

For $k = 1$, select a smooth cube $G \subset B_5$ such that ∂G is good for u. Then as explained above, we can connect u to some u_1 which is equal to u on $B_{10} \setminus G$ and such that $u_1(tx) = u(x)$ for any $x \in F_1, t \in (1/5,1)$. Then $u_1 \circ \phi_1^{-1}$ belongs to $W^{s,p}((-1,1)^n)$ and does not depend on x_1. We can choose $\psi_1 = \phi_1^{-1}$.

Assume the claim is true up to k. We can select a smooth cube G inside $(-1,1)^n$, such that ∂G is good for $u_k \circ \psi_k$ and $u_k \circ \psi_k$ does not depend on $x_1,..,x_k$ on G. Then, as explained previously, we can connect $u_k \circ \psi_k$ to some $w \in W^{s,p}((-1,1)^n)$ such that $w = u_k \circ \psi_k$ on $(-1,1)^n \setminus G$ and $w(tx) = u_k \circ \psi_k(x)$ for any $x \in F_{k+1}, F_{k+1}$ being the $(k+1)^{th}$ face relative to G. Then $w \circ \phi_{k+1}^{-1}$ (ϕ_{k+1} being defined for G) belongs to $W^{s,p}((-1,1)^n)$ and does not depend on $x_1,..,x_{k+1}$. We can choose $\psi_{k+1} = \psi_k \circ \phi_{k+1}^{-1}$ and define

$$u_{k+1}(x) := \begin{cases} u_k(x) & \text{when } \ x \in B_{10} \setminus \psi_k(G), \\ w \circ \psi_k^{-1}(x) & \text{when } \ x \in \psi_k(G). \end{cases}$$

The claim is proved for $k+1$. Finally, we have connected u to a map $u_n \in W^{s,p}(B_{10})$ which is a.e. constant on $\psi_n((-1,1)^n)$, namely an open subset of B_5.

□

5 Proof of Theorem 1 and Theorem 5 c)

The tools 'Connecting constants' and 'Propagation of constants' in [4] can be readily generalized to the case $W^{s,p}$.

Then, the same proof as in [4], Theorem 0.2 shows that $W^{s,p}(M,N)$ is path connected when $sp < 2$; that is, Theorem 1. The fact that $W^{s,p}(S^m,N)$ is path-connected when $s \in (0, 1+1/p)$ can be proved as in [4], Proposition 0.1. This shows Theorem 5 c).

In the sections below, we assume that $s \in (0, 1+1/p), p \in [1,\infty), 1 < sp$.

We denote by Π_M the nearest point projection onto M, which is defined and smooth on an ϵ_M tubular neighborhood of M :

$$M_{\epsilon_M} := \{x \in \mathbb{R}^a : \operatorname{dist}(x, M) < \epsilon_M\}.$$

Similarly, we introduce $\Pi_N : N_{\epsilon_N} \subset \mathbb{R}^l \to N$.

6 Definition of $[sp-1]$ homotopy

6.1 Triangulations and homotopy

We define a rectilinear cell, its dimension, its faces and a rectilinear cell complex as in [12], Chapter 7. In particular, the p skeleton of a rectilinear cell complex K, denoted by K^p, is the collection of all cells having dimension at most p. Any complex considered below is finite. The *polytope* $|K|$ of a complex K is the union of the cells of K. We will use the fact that the boundary $\partial\Delta$ of a simplex Δ can be identified with a complex in an obvious way.

We also introduce some notation. Let Δ be a rectilinear cell, $y \in \operatorname{Int}\Delta$. Then, for any $x \in \Delta$, we set

$$|x|_{y,\Delta} := \inf\{t > 0 : x \in y + t(\Delta - y)\}.$$

This is the usual Minkowski functional of Δ with respect to y. When it is clear what y and Δ are, we simply write $|x|$ instead of $|x|_{y,\Delta}$.

The concepts of smooth maps and immersions on a complex K are defined as in [12], Chapter 8. A smooth immersion which is a homeomorphism onto M is called a triangulation of M. Actually, the word 'triangulation' is mostly used for the case when K is simplicial. In the general case, we will also use the phrase 'rectilinear cell decomposition'. Each smooth boundaryless manifold M has a triangulation ([12], Theorem 10.6). The proof of this result shows that we can choose a simplicial m dimensional complex K (where m is the dimension of M) such that the polytope $|K|$ is the union of its m simplices. Consider such a simplicial complex and denote by $f : K \to M$ a triangulation. The set $f(\Delta)$ is a Lipschitz domain in M for each cell Δ.

Assume that $u \in W^{s,p}(M)$. Then $u \circ f|_\Delta$ belongs to $W^{s,p}(\Delta)$ for each m cell $\Delta \in K$, because $f|_\Delta$ is a smooth diffeomorphism onto $f(\Delta) \subset M$. Conversely, assume that $u \in L^p(M)$ is such that u belongs to $W^{s,p}(f(\Delta))$ for each m cell $\Delta \in K$. Since $sp > 1$, we can define the trace of u on $\partial f(\Delta)$. Assume that for any m cells $\Delta_1, \Delta_2 \in K$ satisfying $\Delta_1 \cap \Delta_2 \neq \emptyset$, the maps $u|_{f(\Delta_1)}$ and $u|_{f(\Delta_2)}$ have the same trace on $f(\Delta_1 \cap \Delta_2)$. This certainly implies that u belongs to $W^{s,p}(f(\Delta_1 \cup \Delta_2))$ when $s \leq 1$. But this holds true even when $s \in (1, 1 + 1/p)$, because in that case the derivatives of $u|_{f(\Delta_1)}$ and $u|_{f(\Delta_2)}$ belong to $W^{\sigma,p}(f(\Delta_1))$ and $W^{\sigma,p}(f(\Delta_2))$ respectively, with now $\sigma p = (s-1)p < 1$. This implies that the derivatives of u belong to $W^{\sigma,p}(f(\Delta_1 \cup \Delta_2))$. Hence, $u \in W^{s,p}(f(\Delta_1 \cup \Delta_2))$.

The following lemma shows that we can glue homotopies together:

Lemma 11 *Let $f : K \to M$ be a smooth triangulation, with m being the common dimension of K and M. Assume that Δ_1 and Δ_2 are two m simplices in K such that $\Delta_1 \cap \Delta_2 = \Sigma$, where Σ is $m-1$ dimensional. Let $F_1 \in C^0([0,1], W^{s,p}(f(\Delta_1)))$, $F_2 \in C^0([0,1], W^{s,p}(f(\Delta_2)))$ and $\forall t \in [0,1]$,*

$$tr\, F_1(t)|_{f(\Sigma)} = tr\, F_2(t)|_{f(\Sigma)}.$$

Then $F \in C^0([0,1], W^{s,p}(f(\Delta_1 \cup \Delta_2)))$ where

$$F(t)(x) = \begin{cases} F_1(t)(x) & \textit{when } x \in \Delta_1, \\ F_2(t)(x) & \textit{when } x \in \Delta_2. \end{cases}$$

Proof: Let us define the closed subset of $W^{s,p}(f(\Delta_1)) \times W^{s,p}(f(\Delta_2))$:

$$\mathcal{F} := \{(u_1, u_2) \in W^{s,p}(f(\Delta_1)) \times W^{s,p}(f(\Delta_2)) : \mathrm{tr}\, u_1|_{f(\Sigma)} = \mathrm{tr}\, u_2|_{f(\Sigma)}\}.$$

Then the remarks above show that the map: $(u_1, u_2) \in \mathcal{F} \to u \in W^{s,p}(f(\Delta_1 \cup \Delta_2))$ where

$$u(x) = \begin{cases} u_1(x) & \text{when } x \in f(\Delta_1), \\ u_2(x) & \text{when } x \in f(\Delta_2) \end{cases}$$

is well defined.

The Closed Graph Theorem shows that this map is continuous into $W^{s,p}(f(\Delta_1 \cup \Delta_2))$. In particular, there exists $C > 0$ such that for any $(u_1, u_2) \in \mathcal{F}$,

$$||u||_{W^{s,p}(f(\Delta_1 \cup \Delta_2))} \leq C[||u_1||_{W^{s,p}(f(\Delta_1))} + ||u_2||_{W^{s,p}(f(\Delta_2))}]. \quad (5)$$

Whence

$$||F(t) - F(t')||_{W^{s,p}(f(\Delta_1 \cup \Delta_2))} \leq C[||F_1(t) - F_1(t')||_{W^{s,p}(f(\Delta_1))} + ||F_2(t) - F_2(t')||_{W^{s,p}(f(\Delta_2))}].$$

The lemma follows.

□

6.2 Definition of $\mathcal{W}^{s,p}(K)$

Let K be a finite rectilinear cell complex. Recall that N is smoothly embedded in $\mathbb{R}^l$. Let $f, g : |K| \to \mathbb{R}^l$ be two everywhere defined Borel measurable functions. We say that f and g are equivalent if for any $\Delta \in K$, $f|_\Delta = g|_\Delta$ $\mathcal{H}^d$ a.e. on Δ, where $d = \dim \Delta$. From now on, we identify two such functions and an equivalence class is called a *Borel function.*

Following [9], we introduce the space $\mathcal{W}^{s,p}(K)$ of those Borel functions $f : |K| \to \mathbb{R}^l$ such that for any cell Δ, the restriction $f|_\Delta$ belongs to $W^{s,p}(\Delta)$ and its trace $\mathrm{tr}\, f|_{\partial\Delta}$ is equal to $f|_{\partial\Delta}, \mathcal{H}^{d-1} a.e.\ x \in \partial\Delta$.

We write $||f||_{\mathcal{W}^{s,p}(K)} := \sum_{\Delta\in K} ||f|_\Delta||_{W^{s,p}(\Delta)}$.

As in [9], we also define a similar function space as follows. Let K be a finite rectilinear cell complex of dimension m. Assume that

$$|K| = \cup_{\Delta\in K, \dim\Delta = m}\Delta.$$

We define $\tilde{\mathcal{W}}^{s,p}(K)$ as the set of those Borel functions $f : |K| \to \mathbb{R}^l$ such that

i) the map $f|_\Delta \in W^{s,p}(\Delta)$ for any $\Delta \in K$ with $\dim\Delta = m$,

ii) for any $\Sigma \in K$ with $\dim\Sigma = m - 1, \Sigma \subset \partial\Delta_i, \dim\Delta_i = m$ for $i = 1, 2$, we have

$$\mathrm{tr}\,(f|_{\Delta_1})|_\Sigma = \mathrm{tr}\,(f|_{\Delta_2})|_\Sigma.$$

We also write:

$$||f||_{\tilde{\mathcal{W}}^{s,p}(K)} = \sum_{\Delta\in K, \dim\Delta = m} ||f|_\Delta||_{W^{s,p}(\Delta)}.$$

Finally, we define

$$\mathcal{W}^{s,p}(K, N) := \{u \in \mathcal{W}^{s,p}(K) : \forall\Delta \in K, u(x) \in N \;\; \mathcal{H}^{\dim\Delta}\ a.e.\}$$

and similarly for $\tilde{\mathcal{W}}^{s,p}(K, N)$.

6.3 Interpolation

We consider X_0, X_1 two Banach spaces such that X_1 is continuously embedded in X_0. We denote by $|| \cdot ||_{X_i}$ the norm in $X_i, i = 0, 1$ and for each fixed $t > 0$, we define

$$K(t; u) := \inf\{||u_0||_{X_0} + t||u_1||_{X_1} : u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\}.$$

Let $1 \le q < \infty$ and $0 < \theta < 1$. Then we define:

$$(X_0, X_1)_{\theta,q} := \{u \in X_0 : (2^{-i\theta} K(2^i; u))_{i\in\mathbb{Z}} \in l^q(\mathbb{Z})\},$$

which is a Banach space with the norm

$$||u||_{(X_0,X_1)_{\theta,q}} := ||(2^{-i\theta} K(2^i; u))_{i\in\mathbb{Z}}||_{l^q(\mathbb{Z})}.$$

Theorem 6 *([1], Theorem 7.48) Let Ω be a rectilinear cell or a smooth bounded open set in $\mathbb{R}^n$. Then we have:*

$$\text{When } \ s \in (0,1), \quad W^{s,p}(\Omega) = (L^p(\Omega), W^{1,p}(\Omega))_{s,p}.$$

$$\text{When } \ s \in (1,2), \quad W^{s,p}(\Omega) = (W^{1,p}(\Omega), W^{2,p}(\Omega))_{s-1,p}.$$

6.4 Perturbation

In this section, we follow [9] to explain how we choose *generic* skeletons of a given triangulation of a manifold. Nevertheless, it seems difficult to rewrite exactly the proof of [9] for the case $W^{s,p}$. This is the reason why we use the interpolation method.

Recall that M is an m dimensional Riemannian manifold without boundary. Assume that the parameter space P is a k dimensional Riemannian manifold, Q is a d dimensional Riemannian manifold without boundary, $D \subset Q$ is a domain with compact closure and Lipschitz boundary, and the dimensions satisfy $d + k \geq m$.

In the following, we will need

Lemma 12 *Assume $s \in (0,1)$. Let $X_0 := L^p(P, L^p(D))$, $X_1 := L^p(P, W^{1,p}(D))$, and $Z_0 := L^p(D)$, $Z_1 := W^{1,p}(D)$. Then we have:*

$$(X_0, X_1)_{s,p} \subset L^p(P, (Z_0, Z_1)_{s,p}) = L^p(P, W^{s,p}(D)).$$

Proof: Let $u \in (X_0, X_1)_{s,p}$ and $\epsilon > 0$. Then, for each $i \in \mathbb{Z}$, there exists $u_0^i \in X_0, u_1^i \in X_1$ such that $u = u_0^i + u_1^i$ and

$$||u_0^i||_{X_0} + 2^i ||u_1^i||_{X_1} < K_i(u) + \epsilon/(1+|i|)!$$

where

$$K_i(u) := \inf\{||u_0||_{X_0} + 2^i ||u_1||_{X_1} : u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\}.$$

Then, for $\mathcal{H}^k$ a.e. $\xi \in P, u(\xi) = u_0^i(\xi) + u_1^i(\xi), u_0^i(\xi) \in Z_0, u_1^i(\xi) \in Z_1$. Hence,

$$\inf\{||v_0||_{Z_0} + 2^i ||v_1||_{Z_1} : u(\xi) = v_0 + v_1, v_0 \in Z_0, v_1 \in Z_1\} \leq$$

$$||u_0^i(\xi)||_{Z_0} + 2^i ||u_1^i(\xi)||_{Z_1}$$

so that

$$||u(\xi)||_{(Z_0,Z_1)_{s,p}} \leq ||(2^{-is}(||u_0^i(\xi)||_{Z_0} + 2^i ||u_1^i(\xi)||_{Z_1}))_{i\in\mathbb{Z}}||_{l^p(\mathbb{Z})}.$$

Finally,

$$||u||_{L^p(P,(Z_0,Z_1)_{s,p})} \leq || \ ||(2^{-is}(||u_0^i(\cdot)||_{Z_0} + 2^i ||u_1^i(\cdot)||_{Z_1}))_{i\in\mathbb{Z}}||_{l^p(\mathbb{Z})}||_{L^p(P)}$$

$$= ||(2^{-is}||\ ||u_0^i(\cdot)||_{Z_0} + 2^i||u_1^i(\cdot)||_{Z_1}||_{L^p(P)})_{i\in\mathbb{Z}}||_{l^p(\mathbb{Z})}$$
$$\leq ||(2^{-is}(||u_0^i||_{X_0} + 2^i||u_1^i||_{X_1}))_{i\in\mathbb{Z}}||_{l^p(\mathbb{Z})}$$
$$\leq ||(2^{-is}(K_i(u) + \epsilon/(1+|i|!)))_{i\in\mathbb{Z}}||_{l^p(\mathbb{Z})}$$
$$\leq ||(2^{-is}K_i(u))_{i\in\mathbb{Z}}||_{l^p(\mathbb{Z})} + \epsilon||(2^{-is}/(1+|i|)!)_{i\in\mathbb{Z}}||_{l^p(\mathbb{Z})}$$
$$= ||u||_{(X_0,X_1)_{s,p}} + C\epsilon.$$

This shows the required inclusion when $\epsilon \to 0$.

□

Similarly, when $s \in (1,2)$, we have:

$$(L^p(P, W^{1,p}(D)), L^p(P, W^{2,p}(D)))_{s-1,p} \subset L^p(P, W^{s,p}(D)). \tag{6}$$

Given a map $H : \bar{D} \times P \to M$, we assume that H satisfies:
(H1) $H \in C^2(\bar{D} \times P)$ and $[H(\cdot,\xi)]_{\mathrm{Lip}(\bar{D})} \leq c_0$ for any $\xi \in P$.
(H2) There exists a positive number c_1 such that the m dimensional Jacobian $J_H(x,\xi) \geq c_1, \mathcal{H}^{d+k}$ a.e $(x,\xi) \in \bar{D} \times P$.
(H3) There exists a positive number c_2 such that $\mathcal{H}^{d+k-m}(H^{-1}(y)) \leq c_2$ for $\mathcal{H}^m$ a.e. $y \in M$.

We will denote $H(\cdot,\xi)$ by H_ξ or h_ξ. Then, we have:

Lemma 13 *([9], Lemma 3.3) For any Borel function $\chi : M \to \mathbb{R}^+ \cup \{+\infty\}$, we have:*

$$\int_P d\mathcal{H}^k(\xi) \int_D \chi(H_\xi(x))\, d\mathcal{H}^d(x) \leq c_1^{-1} c_2 \int_M \chi(y)\, d\mathcal{H}^m(y).$$

In particular, for any Borel subset $E \subset M$, we have

$$\int_P \mathcal{H}^d(H_\xi^{-1}(E))\, d\mathcal{H}^k(\xi) \leq c_1^{-1} c_2 \mathcal{H}^m(E).$$

If in addition $\mathcal{H}^m(E) = 0$, then $\mathcal{H}^d(H_\xi^{-1}(E)) = 0$ for $\mathcal{H}^k$ a.e. $\xi \in P$.

The following lemma will allow us to give the definition of $[sp]-1$ homotopy.

Lemma 14 *i) Let $f \in W^{s,p}(M)$. Then, there exists a Borel set $E \subset P$ such that $\mathcal{H}^k(E) = 0$ and for any $\xi \in P \setminus E, f \circ H_\xi \in W^{s,p}(D)$.*

ii) If we define $\tilde{f}$ by $\tilde{f}(\xi) = f \circ H_\xi$ for any $\xi \in P$, then $\tilde{f} \in L^p(P, W^{s,p}(D))$. In addition,

$$||\tilde{f}||_{L^p(P,W^{s,p}(D))} \leq c||f||_{W^{s,p}(M)},$$

where c depends only on p, c_0, c_1 and c_2.

iii) If $f_i \in C^2(M)$ converges to f in $W^{s,p}(M)$, then $\tilde{f}_i$ converges to $\tilde{f}$ in $L^p(P, W^{s,p}(D))$. Moreover, there exists a subsequence $f_{i'}$ and a Borel set $E \subset P$ such that $\mathcal{H}^k(E) = 0$, and for any $\xi \in P \setminus E, f_{i'} \circ H_\xi \to f \circ H_\xi$ in $W^{s,p}(D)$.

Proof: This lemma corresponds to Lemma 3.4 in [9], the proof of which shows that the map $f \to \tilde{f}$ is continuous from $L^p(M)$ into $L^p(P, L^p(D))$ and from $W^{1,p}(M)$ into $L^p(P, W^{1,p}(D))$. In light of Lemma 12, we deduce that this map is continuous from $W^{s,p}(M)$ into $L^p(P, W^{s,p}(D))$ in the case $s \in (0,1)$. This proves ii) when $s \leq 1$. To complete the proof of ii), it remains to consider the case $s \in (1, 1+1/p)$. To this end, we claim that the map $f \to \tilde{f}$ is continuous from $W^{2,p}(M)$ into $L^p(P, W^{2,p}(D))$. This will prove the required result by interpolation as before (using (6) instead of Lemma 12).

The proof of the claim is similar to the proof of [9] Lemma 3.4., except that $||f||_{W^{1,p}(M)} = ||f||_{L^p(M)} + ||df||_{L^p(M)}$ is replaced by (see [13]):

$$||f||_{W^{2,p}(M)} = ||f||_{L^p(M)} + ||df||_{L^p(M)} + ||d^*df||_{L^p(M)}$$

where d^* is the formal adjoint of the differential operator d on differential forms on M. (The notations df, d^*df have to be understood in a distributional sense).

The rest of the proof is the same and we omit it.

□

Lemma 14 implies the following corollary exactly as Lemma 3.4 implies Corollary 3.1 in [9].

Corollary 2 *Let $f \in W^{s,p}(M), K$ be a finite rectilinear cell complex, $H : |K| \times P \to M$ be a map such that $H|_{\Delta \times P}$ satisfies $(H1), (H2)$ and $(H3)$ for any $\Delta \in K$. Then, there exists a Borel set $E \subset P$ such that $\mathcal{H}^k(E) = 0$ and for any $\xi \in P \setminus E$, we have $f \circ H_\xi \in \mathcal{W}^{s,p}(K)$; in addition, the map $\tilde{f} = f \circ H_\xi$ for $\xi \in P$ belongs to $L^p(P, \mathcal{W}^{s,p}(K))$.*

6.5 Filling a hole (bis)

Lemma 3 is valid for any hole diffeomorphic to a ball. When $s \in (1, 1+1/p)$, we have a similar result when the 'hole' is a rectilinear cell.

Proposition 1 *Let Δ be a rectilinear cell and $y_\Delta \in Int\,\Delta$. Let $u \in W^{s,p}(\Delta)$ be such that $tr u|_{\partial\Delta} = f \in \tilde{\mathcal{W}}^{s,p}(\partial\Delta)$. Then the map u^t defined by*

$$u^t(x) := \begin{cases} u(x/(1-t)) & \text{when } |x|_\Delta \leq 1-t, \\ f(x/|x|_\Delta) & \text{when } |x|_\Delta \geq 1-t \end{cases}$$

belongs to $C^0([0,1), W^{s,p}(\Delta))$.

Moreover, when $sp < dim\,\Delta$, the map u^t is continuous on $[0,1]$.

We will say that u^1 is the homogeneous degree-zero extension of f.

Proof: We denote by d the dimension of Δ. Let $\Sigma_1, .., \Sigma_r$ be the $d-1$ faces of Δ and $\Delta_1, .., \Delta_r$ be the rectilinear cells defined by

$$\Delta_i := \{\lambda y_\Delta + (1-\lambda)x : x \in \Sigma_i, 0 \leq \lambda \leq 1\}.$$

Since

$$\mathrm{tr}\,(u^t|_{\Delta_i})|_{\Delta_i\cap\Delta_j} = \mathrm{tr}\,(u^t|_{\Delta_j})|_{\Delta_i\cap\Delta_j},$$

in light of Lemma 11, it suffices to show that $u^t|_{\Delta_i}$ is continuous into $W^{s,p}(\Delta_i)$.

There exists a C^2 diffeomorphism Φ_i between each Δ_i and a subset of B_1^d of the form $\{\lambda x : \lambda \in [0,1], x \in U_i\}$ where U_i is a connected compact subset of S_1^d, which is isometric in the sense that $|\Phi_i(x)| = |x|_{\Delta_i}, x \in \Delta_i$.

Hence, the continuity of $u^t|_{\Delta_i}$ is a mere consequence of Lemma 3. The proposition is proved.

□

6.6 The final step for the definition of $[sp] - 1$ homotopy

Let X, Y be topological spaces. Then $[X, Y]$ denotes the set of all homotopy classes of continuous maps from X to Y. Given any $f \in C^0(X, Y)$, we use $[f]_{X,Y}$ (or simply $[f]$) to denote the homotopy class corresponding to f as a map from X to Y. If K is a complex, then for any $f \in \mathcal{W}^{s,p}(K, N)$ and $0 \le k < sp$, there exists a unique $g \in C^0(K^k, N)$ such that for any $\Delta \in K^k$, we have $f|_\Delta = g|_\Delta$ $\mathcal{H}^d$ a.e. on Δ with $d = \dim \Delta$. Hence, we may define the homotopy class $[f|_{K^k}]$ of f as the homotopy class $[g]$ of g (in $C^0(K^k, N)$).

Lemma 15 *(Lemma 4.4 in [9]) Assume that $d \in \mathbb{N}, 1 < d, sp = d, \Delta$ is a rectilinear cell of dimension d and $u \in W^{s,p}(\Delta, N)$ is such that the trace $tr\, u|_{\partial\Delta} = f \in \tilde{\mathcal{W}}^{s,p}(\partial\Delta, N) \subset C^0(\partial\Delta, N)$. Then, there exists $v \in C^0(\Delta, N) \cap W^{s,p}(\Delta, N)$ such that $v|_{\partial\Delta} = f$ and $v \sim_{W^{s,p}(\Delta,N)} u$.*

Proof: For any $\delta \in (0,1)$, we define $u_\delta(x) = u(x/(1-\delta))$ for $|x|_\Delta \le 1-\delta$ and $u_\delta(x) = f(x/|x|_\Delta)$ for $1-\delta \le |x|_\Delta \le 1$. Then $u_\delta \in W^{s,p}(\Delta)$ and $u_\delta \to u$ in $W^{s,p}(\Delta)$ as $\delta \to 0^+$ (here, we use Proposition 1).

Choose an $\eta \in C_c^\infty(\Delta, \mathbb{R})$ such that $0 \le \eta \le 1, \eta|_{\Delta_{1-\delta/2}} = 1$ and $\eta|_{\Delta\setminus\Delta_{1-\delta/3}} = 0$. The notation Δ_r signifies the set $\{x \in \Delta : |x|_\Delta < r\}$. For $\epsilon > 0$ small enough, we set $v_\epsilon(x) = ⨍_{B_\epsilon(x)} u_\delta$ for $x \in \Delta_{1-\delta/4}$. Then, we define:

$$w_\epsilon(x) = (1-\eta(x))u_\delta(x) + \eta(x)v_\epsilon(x) \quad \forall x \in \Delta.$$

Clearly, $w_\epsilon \in C^0(\bar{\Delta})$. Since u_δ is VMO, we have $\mathrm{dist}\,(v_\epsilon(x), N) \to 0$ uniformly for $x \in \Delta_{1-\delta/2}$, when $\epsilon \to 0^+$ (see [7], section I.2, Example 2). This implies that the same is true for w_ϵ on $\Delta_{1-\delta/2}$ because $v_\epsilon|_{\Delta_{1-\delta/2}} = w_\epsilon|_{\Delta_{1-\delta/2}}$. Moreover, from the uniform continuity of f, we know that $w_\epsilon(x) - u_\delta(x) \to 0$ uniformly for $x \in \Delta \setminus \Delta_{1-\delta/2}$ as $\epsilon \to 0^+$. Hence, $\mathrm{dist}\,(w_\epsilon(x), N) \to 0$ uniformly for $x \in \Delta$ as $\epsilon \to 0^+$, from which we deduce that $\Pi_N \circ w_\epsilon$ is well defined for ϵ sufficiently small. We have $v_\epsilon \to u_\delta$ when $\epsilon \to 0^+$ in $W^{s,p}(\Delta)$ (this can be shown as in the case of a regularization by a smooth kernel, see [11], Proposition 4.1.). Then w_ϵ converges to u_δ in $W^{s,p}(\Delta)$ when $\epsilon \to 0^+$.

We extend Π_N to the whole $\mathbb{R}^l$ and we may assume that Π_N vanishes outside a large ball. Since Π_N is smooth and N is bounded, by the *composition property* (see [6] and [10]), the map

$$z \in W^{s,p}(\Delta, \mathbb{R}^l) \mapsto \Pi_N \circ z W^{s,p}(\Delta, \mathbb{R}^l)$$

is continuous. Hence $\Pi_N \circ w_\epsilon \to u_\delta$ in $W^{s,p}(\Delta, N)$ when $\epsilon \to 0^+$ and $\Pi_N \circ w_{t\epsilon} \in C^0([0,1], W^{s,p}(\Delta, N))$. Since $u_\delta \sim_{W^{s,p}(\Delta,N)} u$ (by Proposition 1), we have $\Pi_N \circ w_\epsilon \sim_{W^{s,p}(\Delta,N)} u$. The map $v := \Pi_N \circ w_\epsilon$ satisfies the requirements of Lemma 15.

□

Lemma 16 *(Lemma 4.7 in [9]) Let $u \in W^{s,p}(M,N), K$ be a rectilinear cell complex. Assume that the parameter space P is a k dimensional connected Riemannian manifold, and that $H : |K| \times P \to M$ is a map such that $H|_{\Delta\times P}$ satisfies $(H_1), (H_2)$ and (H_3) for any $\Delta \in K$. Then*
i) there exists a Borel set $E \subset P$ such that $\mathcal{H}^k(E) = 0$ and $u \circ H_\xi \in \mathcal{W}^{s,p}(K,N)$ for any $\xi \in P \setminus E$.
ii) Let $0 \le d \le [sp]-1$. We can define $\chi = \chi_{d,H,u} : P \to [|K^d|, N]$ by setting $\chi(\xi) = [u \circ H_\xi|_{|K^d|}]$. Then χ is a constant $\mathcal{H}^k$ a.e. on P.

Proof: From Corollary 2 we know that there exists a Borel set $E_0 \subset P$ such that $\mathcal{H}^k(E_0) = 0$ and $u \circ H_\xi \in \mathcal{W}^{s,p}(K, \mathbb{R}^l)$ for any $\xi \in P \setminus E_0$. Since $u(x) \in N$ for almost every $x \in M$, Lemma 13 shows that there exists a Borel set $E \subset P$ such that $\mathcal{H}^k(E) = 0$ and $u \circ H_\xi \in \mathcal{W}^{s,p}(K,N)$ for any $\xi \in P \setminus E$; that is, the first assertion of the lemma.

The second assertion can be proved exactly as in [9] Lemma 4.7 except that in the proof, [9] Lemma 4.3 has to be replaced by i) and [9] Lemma 4.4 has to be replaced by our Lemma 15.

□

Finally, we give the definition of $[sp]-1$ homotopy (when $s \ge 1$, this definition is the same as in [9]).

Let K be a finite rectilinear cell complex and $h : K \to M$ be a triangulation of M. We define $H : |K| \times B^a_{\epsilon_M} \to M$ as $H(x,\xi) = \Pi_M(h(x) + \xi)$. Then H satisfies $(H1), (H2)$ and $(H3)$ for each $\Delta \in K$ with $P := B^a_{\epsilon_M}$ (see [9], page 72) so that $\chi_{[sp-1],H,u}$ is a constant a.e. on $B^a_{\epsilon_M}$. We denote this constant by $u_{\sharp,s,p}(h)$. When $s \in (1, 1+1/p), W^{s,p}(M,N) \subset W^{1,sp}(M,N)$ (because N is a bounded subset of $\mathbb{R}^l$) and $u_{\sharp,s,p}(h)$ is exactly the constant $u_{\sharp,sp}(h)$ defined in [9] (for $s = 1$).

We also remark that for ϵ_M sufficiently small, $H(\cdot,\xi)$ is a triangulation of M (see [12]). We will denote $H(\cdot,\xi)$ by H_ξ or h_ξ.

Lemma 4.8 and Lemma 4.9 in [9] show that if $u, v \in W^{s,p}(M,N)$ and $h_i : K_i \to M$ are triangulations for $i = 1,2$ (K_i being a rectilinear cell complex) and $u_{\sharp,s,p}(h_1) = v_{\sharp,s,p}(h_1)$, then $u_{\sharp,s,p}(h_2) = v_{\sharp,s,p}(h_2)$. In fact, when $s \in (0,1)$, the same proof as in the case $s = 1$ is valid. When $s \in (1, 1+1/p)$, one can use the inclusion $W^{s,p}(M,N) \subset W^{1,sp}(M,N)$ and apply directly the results in [9] with sp instead of p. Hence, we can define:

Definition 1 *Let $u, v \in W^{s,p}(M,N)$. If for any Lipschitz rectilinear cell decomposition $h : K \to M$, we have $u_{\sharp,s,p}(h) = v_{\sharp,s,p}(h)$, then we say that u is $[sp]-1$ homotopic to v.*

Clearly, this is an equivalence relation on $W^{s,p}(M,N)$.

7 A preliminary to the proof of Theorem 4

In [9], the fact that $\mathrm{Lip}(\Delta) \subset W^{1,p}(\Delta)$ for any simplex Δ is widely used. In contrast, $\mathrm{Lip}(\Delta) \not\subset W^{s,p}(\Delta)$ when $s > 1$. To overcome this difficulty, we have to substantially modify some parts of the proofs of [9]. This is the aim of this section.

Throughout this section, X denotes a rectilinear cell complex of dimension $k+1$ with $0 \le k \le sp-1$ and X^k its subcomplex of dimension k. We also define $[0,1] \times X^k \cup \{0\} \times X$ as the complex:

$$\{[0,1] \times \Delta : \Delta \in X^k\} \cup \{\{0\} \times \Delta : \Delta \in X\} \cup \{\{1\} \times \Delta : \Delta \in X^k\}.$$

If X is embedded in some $\mathbb{R}^S$ and $\Delta \in X^k$, then $[0,1] \times \Delta$ is a rectilinear cell in $\mathbb{R} \times \mathbb{R}^S$ and its boundary is

$$\{0\} \times \Delta \cup \{1\} \times \Delta \cup [0,1] \times \partial\Delta \ \subset \ [0,1] \times X^k \cup \{0\} \times X.$$

The proof of [9], Lemma 3.2 (with obvious modifications) shows the following

Lemma 17 *The set $C^0(X) \cap \mathcal{W}^{s,p}(X)$ is dense in the set $C^0(X)$.*

A consequence of Lemma 17 is given by

Lemma 18 *Let $H_0 \in C^0([0,1] \times X^k, N)$ be such that $H_0(0,\cdot)$ and $H_0(1,\cdot)$ belong to $\mathcal{W}^{s,p}(X^k, N)$. Then there exists*

$$H_1 \in \mathcal{W}^{s,p}([0,1] \times X^k, N) \cap C^0([0,1] \times X^k, N)$$

such that $H_0(0,\cdot) = H_1(0,\cdot)$ and $H_0(1,\cdot) = H_1(1,\cdot)$.

Proof: First, we may assume that $H_0(t,\cdot) = H_0(0,\cdot)$, $t \in [0,\delta]$ and $H_0(t,\cdot) = H_0(1,\cdot)$, $t \in [1-\delta, 1]$, for some $\delta \in (0, 1/4)$. Moreover, using Lemma 17, there exists G in $\mathcal{W}^{s,p}([0,1] \times X^k) \cap C^0([0,1] \times X^k)$ such that $|G(t,x) - H_0(t,x)| \le \epsilon_N$ for $(t,x) \in [0,1] \times |X^k|$.

Finally, let $\theta \in C^\infty(\mathbb{R}, [0,1])$ such that $\theta \equiv 1$ on $[\delta/2, 1-\delta/2]$ and $\theta \equiv 0$ on $[0, \delta/4] \cup [1-\delta/4, 1]$. Then we define

$$H(t,x) := \theta(t) G(t,x) + (1-\theta(t)) H_0(t,x).$$

The map H belongs to $\mathcal{W}^{s,p}([0,1] \times X^k, \mathbb{R}^l) \cap C^0([0,1] \times X^k, \mathbb{R}^l)$ and

$$|H(t,x) - H_0(t,x)| \leq \epsilon_N.$$

Thus, we can define $H_1(t,x) := \Pi_N \circ H(t,x)$. By the composition property, $H_1 \in \mathcal{W}^{s,p}([0,1] \times X^k, N) \cap C^0([0,1] \times X^k, N)$. We have $H_1(0,\cdot) = H_0(0,\cdot)$ and $H_1(1,\cdot) = H_0(1,\cdot)$. This completes the proof of the lemma.

□

Lemma 19 *Let $H_1 \in \mathcal{W}^{s,p}([0,1] \times X^k \cup \{0\} \times X, N) \cap C^0([0,1] \times X^k \cup \{0\} \times X, N)$. Then H_1 may be extended to a map*

$$H_2 \in \mathcal{W}^{s,p}([0,1] \times X, N) \cap C^0([0,1] \times X, N).$$

Proof: For each $\Delta \in X \setminus X^k$, consider its barycenter y_Δ and define $\bar{y}_\Delta := (2, y_\Delta) \in \bar{\Delta} := [0,4] \times \Delta$. Let ρ be the map defined on $[0,1] \times \Delta$ by

$$x \mapsto \bar{y}_\Delta + (x - \bar{y}_\Delta)/|x|_{\bar{\Delta}}.$$

Then

$$\rho(x) \in [0,1] \times \partial\Delta \cup \{0\} \times \Delta, \quad x \in [0,1] \times \Delta$$

and $\rho(x) = x$ for any $x \in [0,1] \times \partial\Delta \cup \{0\} \times \Delta$. Define ρ on each such $[0,1] \times \Delta$ for $\Delta \in X \setminus X^k$ and extend it to $[0,1] \times |X|$ by setting $\rho(x) = x$ on $[0,1] \times |X^k|$. Then ρ is a Lipschitz map from $[0,1] \times |X|$ into $[0,1] \times |X^k| \cup \{0\} \times |X|$, so that the map $H_2 := H_1 \circ \rho$ belongs to $C^0([0,1] \times X, N)$. Moreover, H_2 is an extension of H_1. To see that $H_2 \in \mathcal{W}^{s,p}([0,1] \times X, N)$, remark that on each cell $[0,1] \times \Delta$, with $\Delta \in X \setminus X^k$, H_2 is defined as the homogeneous degree-zero extension of H_1 (except that the center of the homogeneous degree-zero extension $\bar{y}_\Delta$ does not belong to the cell, which makes no trouble as the proof of Proposition 1 shows). Hence, $H_2|_{[0,1] \times \Delta} \in W^{s,p}$. That $H_2|_{\{1\} \times \Delta} \in W^{s,p}$ is an easy consequence of the fact that $H_1 \in \mathcal{W}^{s,p}([0,1] \times \partial\Delta \cup \{0\} \times \Delta)$ and that ρ^{-1} defined on the complex $[0,1] \times \partial\Delta \cup \{0\} \times \Delta$ is a triangulation of $\{1\} \times \Delta$ (see the remarks before Lemma 11). The lemma is proved.

□

Lemma 20 *Let $H_2 \in C^0([0,1] \times X, N)$ be such that $H_2(0,\cdot)$ and $H_2(1,\cdot)$ belong to $\mathcal{W}^{s,p}(X, N)$. Then there exists $H_3 \in C^0([0,1], \mathcal{W}^{s,p}(X, N))$ such that $H_3(0) = H_2(0,\cdot)$ and $H_3(1) = H_2(1,\cdot)$.*

Proof: There exists $\delta > 0$ such that $|H_2(t_1, x_1) - H_2(t_2, x_2)| \leq \epsilon_N/8$ for any $|x_1 - x_2| + |t_1 - t_2| \leq \delta$. Pick some $m \in \mathbb{N}$ such that $1/m < \delta$. For any $1 \leq k \leq m-1$, there exists $L_{k/m} \in C^0(X) \cap \mathcal{W}^{s,p}(X)$ such that $|L_{k/m}(x) - H_2(k/m, x)| \leq \epsilon_N/8$ for $x \in |X|$. (Here, we use Lemma 17). We also define $L_0 := H_2(0,\cdot)$ and $L_1 := H_2(1,\cdot)$. For any $0 \leq k \leq m-1$, $t \in [k/m, (k+1)/m]$ and $x \in X$, we define

$$L(t)(x) = (k+1-mt)L_{k/m}(x) + (mt-k)L_{(k+1)/m}(x).$$

It is easy to see that

$$L \in C^0([0,1], \mathcal{W}^{s,p}(X, \mathbb{R}^l)) \cap C^0([0,1] \times X, \mathbb{R}^l)$$

and dist $(L(t)(x), N) < \epsilon_N, t \in [0,1], x \in |X|$.

We define $H_3(t)(x) := \Pi_N(L(t)(x))$. The composition property shows that the map $t \in [0,1] \mapsto \Pi_N \circ L(t) \in W^{s,p}(\Delta, N)$ is continuous for each $\Delta \in X$. This implies that $H_3 \in C^0([0,1], \mathcal{W}^{s,p}(X, N))$.

□

The proof of Theorem 4 is mainly based on the following proposition:

Proposition 2 *Let $u, v \in \mathcal{W}^{s,p}(X, N)$. Then $u|_{|X^k|}$ and $v|_{|X^k|}$ can be identified to elements in $C^0(X^k, N)$. Assume that $u|_{|X^k|} \sim_{C^0(X^k,N)} v|_{|X^k|}$. Then there exists $f \in \mathcal{W}^{s,p}(X, N) \cap C^0(X, N)$ such that $u \sim_{\mathcal{W}^{s,p}(X,N)} f$ and $f|_{|X^k|} = v|_{|X^k|}$.*

Proof: First, we claim that we may assume that $u \in C^0(X, N)$. Indeed, if $sp > k+1$, then this is a consequence of Sobolev's embeddings. If $sp = k+1$, then Lemma 15 applied to each $\Delta \in X \setminus X^k$ shows that there exists $u_1 \in \mathcal{W}^{s,p}(X, N) \cap C^0(X, N)$ such that $u_1|_{|X^k|} = u|_{|X^k|}$ and $u_1 \sim_{\mathcal{W}^{s,p}(X,N)} u$.

There exists $H_0 \in C^0([0,1] \times X^k, N)$ such that $H_0(0, \cdot) = u|_{|X^k|}$ and $H_0(1, \cdot) = v|_{|X^k|}$. Using Lemma 18, there exists

$$H_1 \in \mathcal{W}^{s,p}([0,1] \times X^k, N) \cap C^0([0,1] \times X^k, N)$$

such that $H_1(0, \cdot) = H_0(0, \cdot)$ and $H_1(1, \cdot) = H_0(1, \cdot)$.

Then extend H_1 to a map still denoted by H_1, defined on $[0,1] \times X^k \cup \{0\} \times X$ by setting $H_1(0, x) = u(x)$ for $x \in X$. It is clear that H_1 now belongs to the space

$$\mathcal{W}^{s,p}([0,1] \times X^k \cup \{0\} \times X, N) \cap C^0([0,1] \times X^k \cup \{0\} \times X, N).$$

In light of Lemma 19, we may extend H_1 to a map

$$H_2 \in \mathcal{W}^{s,p}([0,1] \times X, N) \cap C^0([0,1] \times X, N).$$

Finally, using Lemma 20, there exists $H_3 \in C^0([0,1], \mathcal{W}^{s,p}(X, N))$ such that $H_3(0) = H_2(0, \cdot) = u$ and $H_3(1) = H_2(1, \cdot)$. We have $H_2(1, \cdot)|_{|X^k|} = v|_{|X^k|}$. We can set $f := H_3(1)$.

□

8 Proof of Theorem 4

Lemma 21 *There exists $\eta > 0$ such that for any $u, v \in W^{s,p}(M, N)$ satisfying $||u - v||_{W^{s,p}(M, \mathbb{R}^l)} < \eta$, we have*

$$u \textit{ is } [sp] - 1 \textit{ homotopic to } v.$$

Proof: Fix a smooth triangulation of M, say $h : K \to M$. We may find a Borel set $E_1 \subset B^a_{\epsilon_M}$ such that $\mathcal{H}^a(E_1) = 0$ and for any $\xi \in B^a_{\epsilon_M} \setminus E_1$, we have $u \circ h_\xi, v \circ h_\xi \in \mathcal{W}^{s,p}(K, N)$ and

$$[u \circ h_\xi|_{|K^{[sp]-1}|}] = u_{\sharp,s,p}(h) \ , \ [v \circ h_\xi|_{|K^{[sp]-1}|}] = v_{\sharp,s,p}(h).$$

For any $\Delta \in K$, we have (see Lemma 14)

$$\int_{B^a_{\epsilon_M}} d\mathcal{H}^a(\xi)||u \circ h_\xi - v \circ h_\xi||^p_{W^{s,p}(\Delta,\mathbb{R}^l)} \leq C||u - v||^p_{W^{s,p}(M,\mathbb{R}^l)}.$$

This implies:

$$\mathcal{H}^a(\{\xi \in B^a_{\epsilon_M} : ||u \circ h_\xi - v \circ h_\xi||^p_{W^{s,p}(\Delta,\mathbb{R}^l)} \geq r\}) \leq C\frac{\epsilon^a_M||u - v||^p_{W^{s,p}(M,\mathbb{R}^l)}}{r}.$$

Hence, we may find a Borel set $E_2 \subset B^a_{\epsilon_M}$ such that $\mathcal{H}^a(E_2) > 0$ and for any $\xi \in E_2$, we have:
(i) $u \circ h_\xi, v \circ h_\xi \in \mathcal{W}^{s,p}(K, N)$
(ii) For any $\Delta \in K$, we have

$$||u \circ h_\xi - v \circ h_\xi||^p_{W^{s,p}(\Delta,\mathbb{R}^l)} \leq C||u - v||^p_{W^{s,p}(M,\mathbb{R}^l)}.$$

Hence, for any $\Delta \in K^{[sp-1]}$, we have:

$$\begin{aligned} ||u \circ h_\xi - v \circ h_\xi||_{L^\infty(\Delta)} &\leq C||u \circ h_\xi - v \circ h_\xi||_{W^{s,p}(\Delta,\mathbb{R}^l)} \\ &\leq C||u - v||_{W^{s,p}(M,\mathbb{R}^l)}. \end{aligned}$$

If $||u - v||_{W^{s,p}(M,\mathbb{R}^l)} \leq \eta := \epsilon_N/C$, then the continuous map

$$H(t, x) := \Pi_N((1 - t)u \circ h_\xi(x) + tv \circ h_\xi(x))$$

is well defined. This shows that u is $[sp] - 1$ homotopic to v.

□

Lemma 21 will allow us to prove one implication of Theorem 2. For the converse of this implication, we will need the two following propositions.

Proposition 3 *Assume that* $1 < sp < d$ *and that* f *is a continuous path in* $\tilde{\mathcal{W}}^{s,p}(\partial\Delta, N)$*, where* Δ *is a* d *dimensional rectilinear cell containing* 0*. Define* $\tilde{f}(t)(x) = f(t)(x/|x|)$ *for* $0 \leq t \leq 1$ *and* $x \in \Delta$*. (Here,* $|\cdot|$ *denotes the Minkowski functional of* Δ *with respect to* 0*). Then* $\tilde{f}$ *is a continuous path in* $W^{s,p}(\Delta, N)$*.*

Proof: In light of the proof of Proposition 1, Lemma 1 and (5), the proposition follows from

$$||\tilde{f}(t) - \tilde{f}(s)||_{W^{s,p}(\Delta)} = ||f\widetilde{(t) - f}(s)||_{W^{s,p}(\Delta)} \leq C||f(t) - f(s)||_{\tilde{\mathcal{W}}^{s,p}(\partial\Delta)}.$$

□

Proposition 4 *Consider a d dimensional rectilinear cell Δ containing 0. Assume that $1 < sp < d$. Let $u, v \in W^{s,p}(\Delta, N)$ be such that $tr\, u|_{\partial\Delta}, tr\, v|_{\partial\Delta} \in \tilde{\mathcal{W}}^{s,p}(\partial\Delta, N)$ and $tr\, u|_{\partial\Delta} \sim_{\tilde{\mathcal{W}}^{s,p}(\partial\Delta,N)} tr\, v|_{\partial\Delta}$. Then $u \sim_{W^{s,p}(\Delta,N)} v$.*

Proof: There exists $f \in C^0([0,1], \tilde{\mathcal{W}}^{s,p}(\partial\Delta, N))$ such that $\mathrm{tr}\, u = f(0)$, $\mathrm{tr}\, v = f(1)$. Then, Proposition 3 implies the existence of some

$$\tilde{f} \in C^0([0,1], W^{s,p}(\Delta, N))$$

satisfying $\tilde{f}(0) = \tilde{u}, \tilde{f}(1) = \tilde{v}$ with $\tilde{u}(x) = \mathrm{tr}\, u|_{\partial\Delta}(x/|x|)$ and similarly for $\tilde{v}$. Moreover, Proposition 1 shows that $\tilde{u} \sim_{W^{s,p}(\Delta)} u$, $\tilde{v} \sim_{W^{s,p}(\Delta)} v$. Finally, $u \sim_{W^{s,p}(\Delta)} v$.

□

We proceed to prove Theorem 4; that is,

Theorem 7 *Let $u, v \in W^{s,p}(M, N)$. Then $u \sim_{s,p} v$ if and only if u is $[sp]-1$ homotopic to v in $W^{s,p}(M, N)$.*

Proof: Let $u, v \in W^{s,p}(M, N)$. Assume that $u \sim_{s,p} v$. Then there exists a continuous map $H \in C^0([0,1], W^{s,p}(M, N))$ such that $H(0, \cdot) = u$ and $H(1, \cdot) = v$.

Let η be the number in Lemma 21. There exists $m \in \mathbb{N}$ such that for any $s, t \in [0,1]$ satisfying $|s - t| \leq 1/m$, we have:

$$||H(s) - H(t)||_{W^{s,p}(M,\mathbb{R}^l)} < \eta.$$

Then, for $i = 0, .., m-1$, we have $H(i/m)$ is $[sp]-1$ homotopic to $H((i+1)/m)$. This proves that u is $[sp]-1$ homotopic to v.

The converse is very close to [9]. Suppose that we are given two maps $u, v \in W^{s,p}(M, N)$ which are $[sp]-1$ homotopic. For convenience, we note $k = [sp]-1$. Let $h : K \to M$ be a smooth triangulation of M.

By definition of $[sp]-1$ homotopy, we may find a $\xi \in B^a_{\epsilon_M}$ such that $u \circ h_\xi, v \circ h_\xi \in \mathcal{W}^{s,p}(K, N)$ and $u \circ h_\xi|_{|K^k|} \sim v \circ h_\xi|_{|K^k|}$ as maps from $|K^k|$ to N. We remark that it is enough to prove that $u \circ h_\xi$ and $v \circ h_\xi$ are $\tilde{W}^{s,p}(K, N)$ homotopic. Indeed, if this is the case, u and v will be $W^{s,p}(h_\xi(\Delta), N)$ homotopic for each $\Delta \in K$ of dimension m (recall that h_ξ is a smooth diffeomorphism from Δ onto $h_\xi(\Delta)$). Then, Lemma 11 implies that $u \sim_{W^{s,p}(M,N)} v$.

Step 1: a reduction. We claim that we can assume that $u \circ h_\xi|_{|K^k|} = v \circ h_\xi|_{|K^k|}$. Indeed, since $u \circ h_\xi|_{|K^k|} \sim v \circ h_\xi|_{|K^k|}$ as maps from $|K^k|$ to N, we may apply Proposition 2 which shows that $u \circ h_\xi|_{K^{k+1}}$ is $\mathcal{W}^{s,p}(K^{k+1}, N)$ homotopic to a map $f \in \mathcal{W}^{s,p}(K^{k+1}, N) \cap C^0(K^{k+1}, N)$ which coincides with v on $|K^k|$. For each $(k+2)$ simplex Δ, f and $\mathrm{tr}\, u \circ h_\xi|_{\partial\Delta} = u \circ h_\xi|_{\partial\Delta}$ belongs to $\mathcal{W}^{s,p}(\partial\Delta)$. We choose the barycenter of Δ as origin and do homogeneous degree-zero extension from f to get $f_\Delta \in W^{s,p}(\Delta, N)$ on Δ. Define f_Δ on each such Δ to get $f_{k+2} \in \mathcal{W}^{s,p}(K^{k+2}, N)$. Proposition 4 shows that $u \circ h_\xi|_{K^{k+2}}$ is homotopic to f_{k+2} in $\mathcal{W}^{s,p}(K^{k+2}, N)$. Simply by induction we finish after working with n simplices.

Then, $u \circ h_\xi$ is $\mathcal{W}^{s,p}(K, N)$ homotopic to f. This completes the proof of step 1.

Step 2: completion of the proof. We now show that f can be connected to $v \circ h_\xi$ by a continuous path in $\tilde{\mathcal{W}}^{s,p}(K, N)$.

Applying Proposition 1 to each $k+1$ simplex $\Delta \in K$, we may assume that $f|_{\Delta \setminus B_\delta(c_\Delta)} = v \circ h_\xi|_{\Delta \setminus B_\delta(c_\Delta)}$. Here c_Δ is the barycenter of Δ and δ is a small number. Note that f is continuous on Δ and that v is continuous on $\Delta \setminus B_\delta(c_\Delta)$.

Doing homogeneous degree-zero extension from $v \circ h_\xi|_{K^{k+1}}$ and $f|_{K^{k+1}}$ as we have done above, we may assume that $v \circ h_\xi$ and f are homogeneous of degree zero on $\Sigma \in K$ with $\dim \Sigma \geq k+2$. Then, on any $k+2$ simplex $\Sigma \in K$, f is continuous on $\Sigma \setminus \{c_\Sigma\}$ and $v \circ h_\xi$ is continuous on $\Sigma \setminus \{tz + (1-t)c_\Sigma : z \in \bar{B}_\delta(c_\Delta), t \in [0,1]\}$ (here, c_Σ is the barycenter of Σ and the center of the homogeneous degree-zero extension on Σ).

Fix a $k+1$ simplex Δ. It must be the face of several $k+2$ simplices, say $\Sigma_1, .., \Sigma_r, r \geq 2$. Now, for two small numbers $\delta' > \delta$ and $\epsilon > 0$, consider $\Omega := \cup_{i=1}^r \Omega_i$ where $\Omega_i \subset \Sigma_i$ is formally equal to $(\bar{B}_{2\delta'}(c_\Delta) \cap \Delta) \times [0, \epsilon]$, for which the product means that we go in the Σ_i in the normal direction by length ϵ. Define

$$\Omega_i' := (\bar{B}_{2\delta'}(c_\Delta) \cap \Delta) \times [0, \frac{1}{2}\epsilon], \Omega_i'' := (\bar{B}_{2\delta'}(c_\Delta) \cap \Delta) \times [\epsilon/2, \epsilon],$$

$$\Omega' = \cup_{i=1}^r \Omega_i', \Omega'' = \cup_{i=1}^r \Omega_i''.$$

We may choose δ' and ϵ such that $f|_{\partial\Omega_i \cup \partial\Omega_i''} \in \tilde{\mathcal{W}}^{s,p}(\partial\Omega_i \cup \partial\Omega_i'')$ and $v \circ h_\xi \in \tilde{\mathcal{W}}^{s,p}(\partial\Omega_i')$ (this amounts to Lemma 2 i); note also that the trace compatibility conditions are automatically satisfied for $\delta' > \delta$ and $\epsilon > 0$ sufficiently small: this follows from the continuity properties of f and $v \circ h_\xi$ stated above). This implies that $f|_{\partial\Omega} \in \tilde{\mathcal{W}}^{s,p}(\partial\Omega)$ (once again, the trace compatibility conditions are satisfied). If ϵ is taken sufficiently small (this depends only on the geometry of the $k+2$ simplices), we can assume that $v \circ h_\xi = f$ on a neighborhood of $\partial\Omega' \cap \partial\Omega$ (recall that on K^{k+2}, f and $v \circ h_\xi$ are now homogeneous of degree zero).

Now consider a w defined on $|K^{k+2}|$ by setting

$$w|_{\Omega'} = v \circ h_\xi|_{\Omega'}, w|_{|K^{k+2}| \setminus \Omega} = f|_{|K^{k+2}| \setminus \Omega}.$$

On each Ω_i'', we simply do homogeneous degree-zero extension with respect to a point in $\operatorname{int} \Omega_i''$ (here, we use the fact that the map equal to f on

$\partial\Omega_i'' \setminus \partial\Omega_i'$ and equal to $v \circ h_\xi$ on $\partial\Omega_i'' \cap \partial\Omega_i' = (\bar{B}_{2\delta}(c_\Delta) \cap \Delta) \times \{\epsilon/2\}$ belongs to $\tilde{\mathcal{W}}^{s,p}(\partial\Omega_i''))$. Clearly, $w \in \tilde{\mathcal{W}}^{s,p}(K^{k+2}, N)$.

We may connect w to $f|_{|K^{k+2}|}$ by a continuous path in $\tilde{\mathcal{W}}^{s,p}(K^{k+2}, N)$ since for any $1 \le i \ne j \le r$, $\Omega_i \cup \Omega_j$ is star-shaped with respect to c_Δ and we may apply Proposition 1 to w on this set (here, we use the fact that $w|_{\partial(\Omega_i \cup \Omega_j)} = f|_{\partial(\Omega_i \cup \Omega_j)}$ belongs to $\tilde{\mathcal{W}}^{s,p}(\partial(\Omega_i \cup \Omega_j)))$.

Define $\tilde{w}$ inductively to be the homogeneous degree-zero extension of w on each higher-dimensional simplex Δ with $\dim \Delta \ge k+3$, from its value on $\partial\Delta$ as described above. Then, one has $\tilde{w} \sim_{\tilde{\mathcal{W}}^{s,p}(K,N)} f$.

Since $\tilde{w}|_{|K^{k+1}|} = v \circ h_\xi|_{|K^{k+1}|}$, we have $\tilde{w} \sim_{\tilde{\mathcal{W}}^{s,p}(K,N)} v \circ h_\xi$ (by Proposition 4 and Lemma 11). Finally, $v \circ h_\xi \sim_{\tilde{\mathcal{W}}^{s,p}(K,N)} u \circ h_\xi$. This completes the proof of the theorem.

□

9 Consequences of Theorem 4

As in [9], Theorem 4 reduces certain problems about Sobolev mappings, which are analytical problems, to pure topology problems. In this section, we enumerate some of these results, which correspond to similar results in [9] (for $W^{1,p}$). We omit their proofs when they are similar to those of [9].

Proposition 5 *([9], Proposition 5.1) Assume that* $1 \le p$, $s \in (0, 1+1/p)$, $1 < sp < m$. *For any triangulation of* M, *say* $h : K \to M$, *we set* $M^j = h(|K^j|)$ *for any* j. *There is a bijection between the sets* $W^{s,p}(M,N)/\sim_{s,p}$ *and* $C^0(M^{[sp]}, N)/\sim_{M^{[sp]-1}}$. *Here for* $f, g \in C^0(M^{[sp]}, N)$, $f \sim_{M^{[sp]-1}} g$ *means that* $f|_{M^{[sp]-1}}$ *and* $g|_{M^{[sp]-1}}$ *are homotopic in* $C^0(M^{[sp]-1}, N)$.

Proof: A way to show this proposition is to introduce the space

$$X := (C^0(M^{[sp]}, N) \cap \mathcal{W}^{s,p}(M^{[sp]}, N))/\sim_{M^{[sp]-1}} .$$

The definition of $\mathcal{W}^{s,p}(M^{[sp]}, N)$ follows exactly the definition of $\mathcal{W}^{s,p}(K, N)$.

The natural map $G : X \to C^0(M^{[sp]}, N)/\sim_{M^{[sp]-1}}$ is one-to-one. The surjectivity of G is an easy consequence of Lemma 17. Indeed, let $u \in C^0(M^{[sp]}, N)$. Then Lemma 17 shows that there exists $v \in C^0(M^{[sp]}) \cap \mathcal{W}^{s,p}(M^{[sp]})$ such that $||u-v||_{L^\infty(M^{[sp]})} < \epsilon_N$ and $||\Pi_N(v)-u||_{L^\infty(M^{[sp]})} < \epsilon_N$. Hence u is continuously connected to $\Pi_N(v) \in C^0(M^{[sp]}, N) \cap \mathcal{W}^{s,p}(M^{[sp]}, N)$ by the map $H(t) := \Pi_N(t\Pi_N(v) + (1-t)u)$, so that $G(\Pi_N(v)) = u$.

Thus, there is a bijection between $C^0(M^{[sp]}, N)/\sim_{M^{[sp]-1}}$ and X. It remains to show that there is a bijection between X and $W^{s,p}(M,N)/\sim_{s,p}$.

We define a map from X into $W^{s,p}(M,N)/\sim_{s,p}$ as follows: For any $w \in C^0(M^{[sp]}, N) \cap \mathcal{W}^{s,p}(M^{[sp]}, N)$, using h to pull w to $K^{[sp]}$, after doing homogeneous degree-zero extension on higher-dimensional cells, we pull it to M by h and get $\tilde{w}$. Then we send the equivalence class corresponding to

w to the equivalence class corresponding to $\tilde{w}$. This map is well defined by the proof of Theorem 4.

We proceed to prove that this map is one-to-one. Let $u, v \in C^0(M^{[sp]}, N) \cap \mathcal{W}^{s,p}(M^{[sp]}, N)$ and $\tilde{u}, \tilde{v}$ their homogeneous degree-zero extension. Assume that $\tilde{u} \sim_{s,p} \tilde{v}$. Then by Theorem 4, $\tilde{u}_{\sharp,s,p}(h) = \tilde{v}_{\sharp,s,p}(h)$. It is easy to see that $\tilde{u}_{\sharp,s,p}(h) = [u \circ h|_{K^{[sp]-1}}]$ and similarly for v. Hence $u \sim_{M^{[sp]-1}} v$; that is, the map is one-to-one.

To prove the surjectivity, let $u \in W^{s,p}(M, N)$. There exists $\xi \in B^a_{\epsilon_M}$ such that $u \circ h_\xi \in \mathcal{W}^{s,p}(K, N)$. By the Sobolev embeddings or Lemma 15, there exists $f \in C^0(K^{sp}, N) \cap \mathcal{W}^{s,p}(K^{[sp]}, N)$ such that $f|_{|K^{[sp]-1}|} = u \circ h_\xi|_{|K^{[sp]-1}|}$. We extend f by degree-zero homogeneity. We denote by $\tilde{f}$ this extension. The proof of Theorem 4 (in fact, this is exactly 'step 2') shows that $u \circ h_\xi \sim_{\tilde{\mathcal{W}}^{s,p}(K,N)} \tilde{f}$. Hence, $u \circ h_\xi \circ h^{-1} \sim_{W^{s,p}(M,N)} \tilde{f} \circ h^{-1}$. Since $u \circ h_\xi \circ h^{-1} \sim_{W^{s,p}(M,N)} u$, the equivalence class corresponding to $f \circ h^{-1}|_{M^{[sp]}}$ is mapped to the equivalence class corresponding to u. That is, the map is onto.

□

For any $0 < s_1, s_2 \leq 1, 1 \leq p_1, p_2$, such that $W^{s_2,p_2} \subset W^{s_1,p_1}$, we have a map:

$$i : W^{s_2,p_2}/\sim_{s_2,p_2} \to W^{s_1,p_1}/\sim_{s_1,p_1}$$

defined in an obvious way. An immediate consequence of the above proposition is the following

Corollary 3 *([9], Corollary 5.1) Assume that* $[s_1p_1] = [s_2p_2]$. *Then* i *is a bijection.*

The following corollary implies Theorem 3 b).

Corollary 4 *([9], Corollary 5.2) Assume that* $1 \leq p, s \in (0, 1 + 1/p), 1 < sp < dim M$, *and* $\pi_i(N) = 0$ *for* $[sp] \leq i \leq dim M$. *Then there is a bijection between* $C^0(M, N)/\sim$ *and* $W^{s,p}(M, N)/\sim_{s,p}$ *.*

Corollary 5 *([9], Corollary 5.3) Assume that* $1 \leq p, s \in (0, 1 + 1/p), 1 < sp < m$. *If there exists some* $k \in \mathbb{Z}, k \leq [sp] - 1$ *such that* $\pi_i(M) = 0$ *for* $1 \leq i \leq k$, *and* $\pi_i(N) = 0$ *for* $k + 1 \leq i \leq [sp] - 1$, *then* $W^{s,p}(M, N)$ *is path-connected.*

This is Theorem 2.

We now turn to the question whether a given Sobolev map in $W^{s,p}(M, N)$ can be connected to a smooth map by a continuous path in $W^{s,p}(M, N)$. It turns out that there is a necessary and sufficient topological condition for this to be true.

Proposition 6 *([9], Proposition 5.2) Assume that* $1 \leq p, s \in (0, 1 + 1/p)$, $1 < sp < m$, $u \in W^{s,p}(M, N)$, *and that* $h : K \to M$ *is a triangulation. Then,*

u can be connected to a smooth map by a continuous path in $W^{s,p}(M,N)$ if and only if $u_{\sharp,s,p}(h)$ is extendible to M with respect to N, that is: for any $f \in C^0(K^{[sp]-1}, N)$ such that $f \in u_{\sharp,s,p}(h), f$ is the restriction of a map in $C^0(K,N)$.

Corollary 6 *([9], Corollary 5.4) Assume that $1 \le p, s \in (0, 1+1/p), 1 < sp < m$. Then every map in $W^{s,p}(M,N)$ can be connected by a continuous path in $W^{s,p}(M,N)$ to a smooth map if and only if M satisfies the $[sp]-1$ extension property with respect to N, that is: there exists a CW complex structure $(M^j)_{j\in\mathbb{Z}}$ of M such that every $f \in C^0(M^{[sp]}, N), f|_{M^{[sp]-1}}$ has a continuous extension to M.*

This is Theorem 5 e).

Proof: Fix a smooth triangulation of M, say $h : K \to M$. Assume that every map in $W^{s,p}(M,N)$ can be connected continuously to a smooth map. Let $f \in C^0(M^{[sp]}, N)$. Then using Lemma 17, there exists $f_1 \in C^0(K^{[sp]}, N) \cap \mathcal{W}^{s,p}(K^{[sp]}, N)$ such that $f_1 \sim_{C^0(K^{[sp]},N)} f \circ h$. Let g be the homogeneous degree-zero extension of f_1 to K. Then $u = g \circ h^{-1} \in W^{s,p}(M,N)$ and $u_{\sharp,s,p}(h) = [g|_{K^{[sp]-1}}]$. Since u can be connected continuously to a smooth map, from Proposition 6 we know that $f_1|_{|K^{[sp]-1}|}$ has a continuous extension to K with respect to N. Hence, $f|_{M^{[sp]-1}}$ has a continuous extension to M.

Conversely, assume that M satisfies the $([sp]-1)$ extension property with respect to N. Given any $u \in W^{s,p}(M,N)$, there exists $\xi \in B^a_{\epsilon_M}$ such that $u \circ h_\xi \in \mathcal{W}^{s,p}(K,N)$ and $u_{\sharp,s,p}(h) = [u \circ h_\xi|_{|K^{[sp]-1}|}]$. Using the Sobolev embeddings or Lemma 15, we may assume that $u \circ h_\xi \in C^0(K^{[sp]}, N)$. Hence, by Proposition 6, u may be connected continuously to a smooth map.

□

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