THE EULER EQUATION IN THE MULTIPLE INTEGRALS CALCULUS OF VARIATIONS

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Abstract. For a multiple integrals problem in the calculus of variations, we establish the validity of the Euler equation when the Lagrangian L satisfies a mild growth assumption *from below* at infinity. We do not assume that the map L is differentiable or convex.

1. Introduction

We consider the following problem (P) in the multiple integrals calculus of variations :

$$\text{To minimize} \quad I : u \mapsto \int_\Omega L(x, u(x), \nabla u(x))\, dx \tag{1.1}$$

over the set of those $u \in W_0^{1,1}(\Omega) + \varphi$. Here, Ω is a bounded open set in $\mathbb{R}^n$ and $\varphi \in W^{1,1}(\Omega)$. The map $L : (x, p, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^+$ is measurable with respect to x and locally Lipschitz continuous with respect to (p, ξ). In particular, for any $u \in W^{1,1}(\Omega)$, the map $x \mapsto L(x, u(x), \nabla u(x))$ is measurable and nonnegative on Ω, so that the integral in (1.1) is well defined.

We assume that there exists a solution u_* to (P) : $u_* \in W_0^{1,1}(\Omega) + \varphi$, $I(u_*) < \infty$ and u_* minimizes I over $W_0^{1,1}(\Omega)+\varphi$. The existence of a solution can be established with the direct method in the calculus of variations. It generally requires convexity and coercivity with respect to ξ (see e.g. [14] Theorem 3.4.1). However, it is sometimes possible to prove the existence of a solution when these properties are not satisfied (for nonconvex variational problems, see [6, 17] and the references therein).

When L is sufficiently smooth, we say that u_* satisfies the Euler equation if for every $\theta \in C_c^\infty(\Omega)$, we have

$$\int_\Omega \langle (\theta(x), \nabla\theta(x)), \nabla_{p,\xi} L(x, u_*(x), \nabla u_*(x))\rangle\, dx = 0. \tag{1.2}$$

In writing this, we implicitly require that $\nabla_{p,\xi} L(x, u_*(x), \nabla u_*(x))$ belongs to $L^1_{loc}(\Omega)$. We have denoted by $\langle \cdot, \cdot \rangle$ the standard inner product in $\mathbb{R} \times \mathbb{R}^n$.

Even in the one dimensional case $n = 1$ and when L is smooth and strictly convex with respect to ξ, it may happen that a minimum does not satisfy the Euler equation. Several examples are presented in [1]. However, when $n = 1$, general conditions are now available to ensure the validity of the Euler equation, even when L is neither smooth nor convex, see e.g. [12], chapter 4.

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In the multidimensional setting $n > 1$, the Euler equation is satisfied by any minimum of (P) when L satisfies growth conditions of polynomial type (see e.g. [14] Theorem 3.4.4). Clarke [10, 11] has established the Euler equation when the growth of L is at most exponential. Since L is merely locally Lipschitz continuous, the Euler equation stated in [10] is expressed in terms of the generalized subdifferential ∂L of L with respect to (p, ξ) (the definition of ∂L is detailed in the following section). More precisely, assume that L does not depend on x (in order to simplify the presentation) and that L satisfies the following growth condition: there exists $k > 0$ such that for any $(p, \xi) \in \mathbb{R} \times \mathbb{R}^n$,

$$|\partial L(p, \xi)| \leq k(1 + |L(p, \xi)| + |(p, \xi)|). \tag{1.3}$$

Then there exists $p \in L^1(\Omega, \mathbb{R}^n)$ such that $\operatorname{div} p \in L^1(\Omega)$ and

$$(\operatorname{div} p(x), p(x)) \in \partial L(u_*(x), \nabla u_*(x)) \quad \text{a.e. } x \in \Omega.$$

The divergence has to be understood in the distributional sense. When L is C^1, then $\partial L(u_*(x), \nabla u_*(x))$ only contains $\nabla L(u_*(x), \nabla u_*(x))$ and the above inclusion coincides with the standard Euler equation (1.2).

In [7], Cellina considers the case when L has the form $L(x, p, \xi) = F(|\xi|) + G(x, p)$ where $F(|\cdot|)$ is convex and differentiable and G is a Caratheodory function which satisfies some growth assumptions of polynomial type with respect to u. Euler equation is then established under a further growth assumption on F, which is more general than the exponential growth. For related results, see also [2, 8, 9, 18].

Very recently, Degiovanni and Marzocchi [15] have obtained the validity of the Euler equation when $L(x, p, \xi) = L(x, \xi)$ does not depend on p, and is C^1 and convex with respect to ξ. Moreover, φ is required to be in $L^\infty_{loc}(\Omega)$. We emphasize the fact that no growth assumption is needed on F. This result was later generalized in [4] to Lagrangians of the form $F(\xi) + G(x, p)$ with F C^1 and convex. Here, G must be concave with respect to p and satisfy some growth assumptions of polynomial type.

In this paper, we establish the Euler equation when L is not necessarily convex or C^1. Our main assumption requires that L does not decrease too fast at infinity, with respect to (p, ξ). When L is C^1, this is implied by the following condition:

$$\liminf_{\substack{|(p,\xi)| \to +\infty \\ x \in \Omega}} \langle (p, \xi), \frac{\nabla L_x(p, \xi)}{|\nabla L_x(p, \xi)|} \rangle = +\infty.$$

In contrast to [10] or [7], we do not require any growth assumptions *from above* on L. In the particular case when $L(x, p, \xi) = F(\xi) + G(x, p)$ and the minimum u_* is locally bounded, we prove that the convexity of F alone is a sufficient condition for the validity of the Euler equation (here, G is merely assumed to be locally Lipschitz in p, uniformly with respect to x). Detailed statements of our results are given in the following section.

2. Statement of the main results

Throughout the paper, we assume that $L : (x, p, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \mapsto L(x, p, \xi) \in \mathbb{R}^+$ is measurable in x and locally Lipschitz in (p, ξ) uniformly with respect to $x \in \Omega$. More precisely, for any $(p, \xi) \in \mathbb{R} \times \mathbb{R}^n$, there exist

$\varepsilon > 0, T > 0$ such that for any $(p_1, \xi_1), (p_2, \xi_2) \in B^{n+1}((p, \xi), \varepsilon)$, for a.e. $x \in \Omega$, we have

$$(H0) \quad |L(x, p_1, \xi_1) - L(x, p_2, \xi_2)| \leq T|(p_1, \xi_1) - (p_2, \xi_2)|.$$

We often write $L_x(p, \xi) := L(x, p, \xi)$.

We next define the generalized subdifferential of a locally Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}$. For any $a, v \in \mathbb{R}^m$, the generalized directional derivative of f at a in the direction v is

$$f^0(a, v) := \limsup_{\substack{b \to a \\ \lambda \downarrow 0}} \frac{f(b + \lambda v) - f(b)}{\lambda},$$

where $b \in \mathbb{R}^m$ and $\lambda \in (0, \infty)$. It is a consequence of the Hahn-Banach Theorem (see e.g. [13], Chapter 2 for details) that there exists a uniquely defined compact convex subset $\partial f(a) \subset \mathbb{R}^m$ such that for any $v \in \mathbb{R}^m$,

$$f^0(a, v) = \max_{\zeta \in \partial f(a)} \langle \zeta, v \rangle.$$

The set $\partial f(a)$ is called the generalized subdifferential of f at a.

We require that L *does not decrease too fast at infinity.* More precisely, we assume that for every $R > 0$ there exist a nonnegative summable map $K_R^0 \in L^1(\Omega)$ and a constant $K_R^1 > 0$ such that for every $(p, \xi) \in \mathbb{R} \times \mathbb{R}^n$,

$$(H1) \max_{|(p', \xi')| \leq R} L_x^0((p, \xi), (p' - p, \xi' - \xi)) \leq K_R^0(x) + K_R^1(L_x(p, \xi) + |(p, \xi)|).$$

In case when L_x is C^1, $L_x^0((p, \xi), (p' - p, \xi' - \xi)) = \langle \nabla L_x(p, \xi), (p' - p, \xi' - \xi) \rangle$ and $(H1)$ is equivalent to

$$R|\nabla L_x(p, \xi)| - \langle \nabla L_x(p, \xi), (p, \xi) \rangle \leq K_R^0(x) + K_R^1(L_x(p, \xi) + |(p, \xi)|).$$

Property $(H1)$ only depends on the behavior of L when $|(p, \xi)| \to \infty$. It is substantially weaker than the assumptions needed to establish the Euler equation in the papers quoted in the introduction. In order to clarify this fact, here is a list of sufficient conditions that imply $(H1)$ (for the sake of clarity, we consider the case of a C^1 map L that depends only on ξ; a more general statement is given in the last section).

Proposition 2.1. *The map $L : \mathbb{R}^n \to \mathbb{R}^+$ satisfies $(H1)$ if one of the following assumptions is satisfied:*

i) *There exists $S > 0$ such that L coincides with a convex map $\widetilde{L} : \mathbb{R}^n \to \mathbb{R}^+$ outside $B^n(0, S)$.*

ii) *There exists $C > 0$ such that for any $|\xi| \leq |\xi'|$, we have*

$$L(\xi) \leq L(\xi') + C|\xi - \xi'|.$$

iii) *There exist $\alpha > 0, \beta \in \mathbb{R}$ such that $L(\xi) \geq \alpha|\xi|^2 + \beta$, and L is semiconvex: there exists $C > 0$ such that for every $\xi, \xi' \in \mathbb{R}^n$, for every $\theta \in (0, 1)$, we have*

$$L(\theta\xi + (1 - \theta)\xi') \leq \theta L(\xi) + (1 - \theta)L(\xi') + C\theta(1 - \theta)|\xi - \xi'|^2.$$

iv) *The map L satisfies the following radial growth condition from below:*

$$\liminf_{|\xi| \to +\infty} \langle \xi, \frac{\nabla L(\xi)}{|\nabla L(\xi)|} \rangle = +\infty.$$

v) *The map L is non decreasing in the following sense $\langle \nabla L(\xi), \xi \rangle \geq 0$ and the growth of L is at most exponential: there exists $K > 0$ such that*

$$|\nabla L(\xi)| \leq K(1 + L(\xi) + |\xi|) \quad , \quad \xi \in \mathbb{R}^n.$$

Roughly speaking, a C^1 map $L : \mathbb{R}^n \to \mathbb{R}^+$ fails to satisfy $(H1)$ when the quantity $\langle \xi, \frac{\nabla L(\xi)}{|\nabla L(\xi)|} \rangle$ becomes 'too negative' for arbitrarily large values of $|\xi|$. This is for instance the case of $L(\xi) = 1 + \sin(|\xi|^2)$.

Given a map $u \in W^{1,1}(\Omega)$, we say that $u|_{\partial\Omega}$ is bounded if there exists $M > 0$ such that the map $u^M := \max(-M, \min(u, M))$ belongs to $u + W_0^{1,1}(\Omega)$. Observe that

$$u^M(x) = \begin{cases} M \text{ if } u(x) > M, \\ u(x) \text{ if } |u(x)| \leq M, \\ -M \text{ if } u(x) < -M. \end{cases} \tag{2.1}$$

When Ω is smooth, $u|_{\partial\Omega}$ is bounded if and only if the trace of u belongs to $L^\infty(\partial\Omega)$.

We now state our main result :

Theorem 1. *If $L : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ satisfies $(H1)$ and $u_*|_{\partial\Omega}$ is bounded, then there exists $(q, \zeta) \in L^1_{loc}(\Omega) \times L^1_{loc}(\Omega)$ such that*

1) *for a.e. $x \in \Omega$, $(q(x), \zeta(x)) \in \partial L_x(u_*(x), \nabla u_*(x))$,*
2) *$qu_* + \langle \zeta, \nabla u_* \rangle \in L^1_{loc}(\Omega)$,*
3) *for any $\theta \in C_c^\infty(\Omega)$,*

$$\int_\Omega q(x)\theta(x) + \langle \zeta(x), \nabla\theta(x) \rangle \, dx = 0. \tag{2.2}$$

It is often possible to prove *a priori* that any minimum of (P) is bounded. This is the case when L is a convex function of ξ, and does not depend either on x or on p. Then any minimum is bounded on Ω provided that $\varphi|_{\partial\Omega}$ is bounded. When L has the form $(x, p, \xi) \mapsto F(\xi) + G(x, p)$, certain growth assumptions on G together with the uniform convexity of F guarantee the boundedness of any minimum (see e.g. [19]).

When we know that the minimum u_* is (locally) bounded, it is natural to provide separate assumptions regarding the dependence of L with respect to p and ξ. Roughly speaking, we assume in the following statement that $\xi \mapsto L_{(x,p)}(\xi) := L(x, p, \xi)$ does not decrease too fast at infinity, uniformly with respect to (x, p), and that $p \mapsto L_{(x,\xi)}(p) := L(x, p, \xi)$ is locally Lipschitz. More precisely,

Theorem 2. *We assume that for every $M > 0$, there exists a nonnegative summable map $C_M^0 \in L^1(\Omega)$ and a constant $C_M^1 > 0$ such that for a.e. $x \in \Omega$, for every $p \in (-M, M)$, for every $\xi \in \mathbb{R}^n$,*

$$(H2) \qquad |\partial L_{x,\xi}(p)| \leq C_M^0(x) + C_M^1(L_x(p, \xi) + |\xi|).$$

We also assume that for every $R > 0$ there exist a nonnegative summable map $K_R^0 \in L^1(\Omega)$ and a constant $K_R^1 > 0$ such that for every $p \in (-R, R)$, for every $\xi \in \mathbb{R}^n$,

$$(H3) \qquad \max_{|\xi'| \leq R} L^0_{x,p}(\xi, \xi' - \xi) \leq K_R^0(x) + K_R^1(L_x(p, \xi) + |\xi|).$$

If $u_* \in L^\infty_{loc}(\Omega)$*, then there exists* $(q,\zeta) \in L^1_{loc}(\Omega) \times L^1_{loc}(\Omega)$ *such that*

1) *for a.e.* $x \in \Omega$, $(q(x), \zeta(x)) \in \partial L_x(u_*(x), \nabla u_*(x))$,
2) $qu_* + \langle \zeta, \nabla u_* \rangle \in L^1_{loc}(\Omega)$,
3) *for any* $\theta \in C^\infty_c(\Omega)$,

$$\int_\Omega q(x)\theta(x) + \langle \zeta(x), \nabla\theta(x)\rangle \, dx = 0. \tag{2.3}$$

In particular, if L has the form $L(x,p,\xi) = F(\xi) + G(x,p)$ with F convex and G_x locally Lipschitz (uniformly with respect to x), L satisfies the assumptions of Theorem 2.

When $n = 1$, any minimum is bounded and Theorem 2 applies (Theorem 1 is still valid but less interesting in that setting). In all the counterexamples presented in [1], the map $x \mapsto \nabla_p L(x, u_*(x), u_*'(x))$ is *not* summable on Ω (an interval in that case). In our framework, such a phenomenon is impossible in view of $(H2)$. Actually, in the one-dimensional case, one can establish a generalized form of the Euler equation without $(H0)$ and $(H3)$ and under a weaker version of $(H2)$, the so-called 'generalized Tonelli-Morrey growth condition'. It requires that for every $M > 0$, there exist a summable function K^0 and a constant K^1 such that for a.e. x, for every $p \in (-M, M)$, for every $\xi \in \mathbb{R}$, for every $(\zeta, \psi) \in \partial L_x(p, \xi)$, one has

$$\frac{|\zeta|}{1 + |\psi|} \le K^0(x) + K^1(L_x(p,\xi) + |\xi|).$$

Then by [12] Theorem 4.3.2, a generalized form of the Euler equation holds true.

In view of the one dimensional case, it is thus very plausible that the conclusion of Theorem 2 remains true under a weaker version of $(H2)$ alone, without any assumption on $\partial L_{x,p}$. Moreover, $(H0)$ is a quite restrictive assumption regarding the dependence with respect to x. In [15], the dependence with respect to x was controlled by a very mild assumption, but only for lagrangians not depending on p, and which were C^1 and convex with respect to ξ.

Theorem 1 is proved in the next section while Theorem 2 is proved in section 3. The last section is devoted to the proof of (a more general version of) Proposition 2.1.

3. Proof of Theorem 1

We denote by $\mathcal{G}$ the set of those measurable maps $(q,\zeta) : \Omega \to \mathbb{R} \times \mathbb{R}^n$ such that for a.e. $x \in \Omega, (q(x), \zeta(x)) \in \partial L_x(u_*(x), \nabla u_*(x))$. By the measurable selection theorem ([5], see also [11] Theorem 3.1.1), the set $\mathcal{G}$ is not empty.

We also consider for $R > 0$ the set $\mathcal{A}_R$ of those $\eta \in W^{1,1}_0(\Omega) + u_*$ such that for a.e. $x \in \Omega$, $(\eta(x), \nabla\eta(x))$ belongs to the convex hull

$$\text{co}\left(\{(u_*(x), \nabla u_*(x))\} \cup \overline{B}^{n+1}(0,R) \right).$$

For $k \ge 0$, we introduce the measurable set

$$E_k := \{x \in \Omega : k \le |(u_*(x), \nabla u_*(x))| < k+1\}.$$

As a consequence of $(H0)$, for any $K \geq 0$, there exists $M_K \geq 0$ such that for $x \in \cup_{k\leq K} E_k$, for any $(q,\zeta) \in \partial L_x(u_*(x), \nabla u_*(x))$, we have

$$|(q,\zeta)| \leq M_K. \tag{3.1}$$

We also consider for $K > 0$ the set $\mathcal{G}_K$ of those measurable maps $(q,\zeta) : \Omega \to \mathbb{R}\times\mathbb{R}^n$ such that

$$\begin{cases} (q(x),\zeta(x)) \in \partial L_x(u_*(x), \nabla u_*(x)) \quad , \quad \text{a.e. } x \in \cup_{k\leq K} E_k, \\ (q(x),\zeta(x)) = (0,0) \quad , \quad \text{a.e. } x \in \cup_{k>K} E_k. \end{cases}$$

By the above remark, $\mathcal{G}_K$ is weakly* compact in $L^\infty(\Omega)\times L^\infty(\Omega)^n$. The convexity of $\partial L_x(u_*(x), \nabla u_*(x))$ implies that $\mathcal{G}_K$ is convex as well. Moreover,

Lemma 3.1. i) *If $\{(q_K,\zeta_K)\}$ is a sequence of measurable maps such that for every $K \geq 0$, $(q_K,\zeta_K) \in \mathcal{G}_K$, then there exist $(q,\zeta) \in \mathcal{G}$ and a subsequence $\{(q_{K_i},\zeta_{K_i})\}_{i\geq 0}$ such that for any $k \geq 0$,*

$(q_{K_i}|_{E_k}, \zeta_{K_i}|_{E_k})$ weakly converges in $L^\infty(E_k)$ to $(q|_{E_k}, \zeta|_{E_k})$.*

ii) *If $\{(q^R,\zeta^R)\}$ is a sequence in $\mathcal{G}$, then there exist $(q,\zeta) \in \mathcal{G}$ and a subsequence $\{(q^{R_i},\zeta^{R_i})\}_{i\geq 0}$ such that for any $k \geq 0$,*

$(q^{R_i}|_{E_k}, \zeta^{R_i}|_{E_k})$ weakly converges in $L^\infty(E_k)$ to $(q|_{E_k}, \zeta|_{E_k})$.*

Proof. For any $k \geq 0$, the sequence $\{(q_K|_{E_k}, \zeta_K|_{E_k})\}_{K\geq 0}$ is bounded in $L^\infty(E_k)$. By a diagonal process, we can thus extract a subsequence $\{(q_{K_i},\zeta_{K_i})\}_{i\geq 0}$ such that for any $k \geq 0$, the sequence $\{(q_{K_i}|_{E_k}, \zeta_{K_i}|_{E_k})\}_{i\geq 0}$ weakly* converges in $L^\infty(E_k)$ to some limit that we denote by (q^k,ζ^k). We then define the measurable map $(q,\zeta) : \Omega \to \mathbb{R}\times\mathbb{R}^n$ by $(q|_{E_k}, \zeta|_{E_k}) = (q^k,\zeta^k)$ (observe that $\{E_k\}_{k\geq 0}$ is a partition of Ω up to a negligeable set).

We now prove that $(q,\zeta) \in \mathcal{G}$. We introduce the map $H : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$

$$H(x,r,\gamma) := \max_{(q,\zeta)\in\partial L_x(u_*(x),\nabla u_*(x))} qr + \langle\zeta,\gamma\rangle \quad , \quad (r,\gamma) \in \mathbb{R}\times\mathbb{R}^n, x \in \Omega.$$

We write $H_x(r,\gamma) = H(x,r,\gamma)$. The Hahn-Banach theorem implies that for a.e. $x \in \Omega$, we have $(q,\zeta) \in \partial L_x(u_*(x), \nabla u_*(x))$ if and only if $qr + \langle\zeta,\gamma\rangle \leq H_x(r,\gamma)$ for every $(r,\gamma) \in \mathbb{R}\times\mathbb{R}^n$.

Fix $(r,\gamma) \in \mathbb{R}\times\mathbb{R}^n$ and $K \geq 0$. For any $K_i \geq K$, $(q_{K_i},\zeta_{K_i}) \in \mathcal{G}_{K_i}$ so that

$$q_{K_i}(x)r + \langle\zeta_{K_i}(x),\gamma\rangle \leq H_x(r,\gamma) \quad , \text{ a.e. } x \in \cup_{k\leq K} E_k.$$

Hence, for any measurable subset $A \subset \Omega$, we have

$$\int_{A\cap(\cup_{k\leq K} E_k)} q_{K_i}(x)r + \langle\zeta_{K_i}(x),\gamma\rangle\, dx \leq \int_{A\cap(\cup_{k\leq K} E_k)} H_x(r,\gamma)\, dx.$$

By letting $i \to \infty$, we get

$$\int_{A\cap(\cup_{k\leq K} E_k)} q(x)r + \langle\zeta(x),\gamma\rangle\, dx \leq \int_{A\cap(\cup_{k\leq K} E_k)} H_x(r,\gamma)\, dx.$$

Since A is arbitrary, it then follows that for a.e. $x \in \cup_{k\leq K} E_k$,

$$q(x)r + \langle\zeta(x)),\gamma\rangle \leq H_x(r,\gamma).$$

This implies $(q(x),\zeta(x)) \in \partial L_x(u_*(x), \nabla u_*(x))$, which completes the proof of i). The proof of ii) is very similar and we omit it. ☐

We proceed to state two consequences of $(H1)$.

Lemma 3.2. *For any $R > 0$, there exists $\ell_R \in L^1(\Omega)$ such that for every $\eta \in \mathcal{A}_R$, for a.e. $x \in \Omega$,*

i) *for every $(q, \zeta) \in \mathcal{G}$,*

$$q(x)(\eta(x) - u_*(x)) + \langle \zeta(x), \nabla\eta(x) - \nabla u_*(x)\rangle \leq \ell_R(x), \tag{3.2}$$

ii) *for every $\lambda \in (0, 1/2)$, we have*

$$\begin{aligned}\frac{1}{\lambda}\,(L_x(u_*(x) + \lambda(\eta(x) - u_*(x)), \nabla u_*(x) + \lambda(\nabla\eta(x) - \nabla u_*(x))) \\ -L_x(u_*(x), \nabla u_*(x))) \leq \ell_R(x).\end{aligned} \tag{3.3}$$

Proof. In order to prove (i), we first write

$$\begin{aligned}&q(x)(\eta(x) - u_*(x)) + \langle \zeta(x), \nabla\eta(x) - \nabla u_*(x)\rangle \\ &\qquad \leq L_x^0((u_*(x), \nabla u_*(x)), (\eta(x) - u_*(x), \nabla\eta(x) - \nabla u_*(x))).\end{aligned}$$

Next, we use an equivalent form of $(H1)$ where the maximum on $\overline{B}^{n+1}(0, R)$ in the right hand side of $(H1)$ is replaced by a maximum on $\mathrm{co}\,\left(\{(p, \xi)\} \cup \overline{B}^{n+1}(0, R)\right)$. This follows from the fact that $L_x^0((p, \xi), \cdot)$ is positively homogeneous. We thus get

$$\begin{aligned}&q(x)(\eta(x) - u_*(x)) + \langle \zeta(x), \nabla\eta(x) - \nabla u_*(x)\rangle \\ &\qquad \leq K_R^0(x) + K_R^1(L_x(u_*(x), \nabla u_*(x)) + |(u_*(x), \nabla u_*(x))|).\end{aligned}$$

The right hand side is summable. We only need to take $\ell_R(x) \geq K_R^0(x) + K_R^1(L_x(u_*(x), \nabla u_*(x)) + |(u_*(x), \nabla u_*(x))|)$ to obtain (3.2).

For (ii), we first prove that there exist a nonnegative summable map $C_R^0 \in L^1(\Omega)$ and a constant $C_R^1 > 0$ such that for every $(p, \xi) \in \mathbb{R} \times \mathbb{R}^n$, for every $(p', \xi') \in \mathrm{co}\,\left(\{(p, \xi)\} \cup \overline{B}^{n+1}(0, R)\right)$ and for every $\lambda \in (0, 1/2)$, we have

$$\frac{L_x((p, \xi) + \lambda(p' - p, \xi' - \xi)) - L_x(p, \xi)}{\lambda} \leq C_R^0(x) + C_R^1(L_x(p, \xi) + |(p, \xi)|). \tag{3.4}$$

We simplify the notation by writing $\alpha = (p, \xi)$ and $\alpha' = (p', \xi')$. Let $g(\lambda) = L_x(\alpha + \lambda(\alpha' - \alpha)) - L_x(\alpha)$. Then (see e.g. [13] Theorem 2.4)

$$\partial g(\lambda) \subset \langle \partial L_x(\alpha + \lambda(\alpha' - \alpha)), \alpha' - \alpha\rangle.$$

Since g is locally Lipschitz, it is differentiable a.e., and the derivative $g'(\lambda)$ then belongs to $\partial g(\lambda)$. This gives

$$g'(\lambda) \leq L_x^0(\alpha + \lambda(\alpha' - \alpha), \alpha' - \alpha) = \frac{1}{1-\lambda} L_x^0(\alpha + \lambda(\alpha' - \alpha), (1-\lambda)(\alpha' - \alpha)).$$

Since $(1-\lambda)(\alpha' - \alpha) = \alpha' - (\alpha + \lambda(\alpha' - \alpha))$ and $\alpha' \in \mathrm{co}\,\left(\{\alpha + \lambda(\alpha' - \alpha)\} \cup \overline{B}^{n+1}(0, R)\right)$, it follows from $(H1)$ that for a.e. $\lambda \in (0, 1/2)$

$$\begin{aligned}g'(\lambda) &\leq 2\left(K_R^0(x) + K_R^1(L_x(\alpha + \lambda(\alpha' - \alpha)) + |\alpha + \lambda(\alpha' - \alpha)|)\right) \\ &\leq 2K_R^1 g(\lambda) + 2K_R^0(x) + 2K_R^1(L_x(\alpha) + |\alpha| + R).\end{aligned}$$

By a Gronwall type argument, we get

$$\frac{L_x(\alpha+\lambda(\alpha'-\alpha))-L_x(\alpha)}{\lambda} \leq \left(\frac{K_R^0(x)}{K_R^1}+L_x(\alpha)+|\alpha|+R\right)\frac{e^{2K_R^1\lambda}-1}{\lambda}.$$

Since $\lambda \mapsto \frac{e^{2K_R^1\lambda}-1}{\lambda}$ is bounded on $(0,1/2)$, inequality (3.4) follows for suitable $C_R^0 \in L^1(\Omega)$ and $C_R^1 > 0$. Hence,

$$\begin{aligned}\frac{1}{\lambda}\,(L_x(u_*(x)+\lambda(\eta(x)-u_*(x)),\nabla u_*(x)+\lambda(\nabla\eta(x)-\nabla u_*(x)))\\ -L_x(u_*(x),\nabla u_*(x)))\\ \leq C_R^0(x)+C_R^1(L_x(u_*(x),\nabla u_*(x))+|(u_*(x),\nabla u_*(x))|).\end{aligned}$$

The right hand side is summable, which implies the existence of ℓ_R.

□

By (3.2), the integral $\int_\Omega q(x)(\eta(x)-u_*(x))+\langle\zeta(x),\nabla\eta(x)-\nabla u_*(x)\rangle\,dx$ is well defined in $[-\infty,\infty)$ for every $(q,\zeta)\in\mathcal{G}$ and for every $\eta\in\mathcal{A}_R$, $R>0$.

As in [10], in order to handle the fact that $\partial L_x(u_*(x),\nabla u_*(x))$ is not a singleton in general, we use a minimax theorem that we apply to the function

$$f:((q,\zeta),\eta)\mapsto\int_\Omega q(\eta-u_*)+\langle\zeta,\nabla\eta-\nabla u_*\rangle.$$

The map f is continuous on $L^\infty(\Omega)\times L^\infty(\Omega)^n\times W^{1,1}(\Omega)$. Moreover, it is linear (or affine) with respect to q,ζ and η.

Lemma 3.3. *For any $R>0, K\geq 0$ and any $\eta\in\mathcal{A}_R$, we have*

$$\sup_{(q,\zeta)\in\mathcal{G}_K} f((q,\zeta),\eta)\geq\alpha_K(R),$$

where $\alpha_K(R):=-\int_{\cup_{k>K}E_k}\ell_R(x)\,dx$ and ℓ_R is given by Lemma 3.2.

Proof. By the measurable selection theorem,

$$\sup_{(q,\zeta)\in\mathcal{G}_K} f((q,\zeta),\eta)=\int_{\cup_{k\leq K}E_k}\max_{(q,\zeta)\in\partial L_x(u_*(x),\nabla u_*(x))} q(\eta-u_*)+\langle\zeta,\nabla\eta-\nabla u_*\rangle.$$

We introduce the notation

$$M(x,\lambda)=L_x(u_*+\lambda(\eta-u_*),\nabla u_*+\lambda(\nabla\eta-\nabla u_*))-L_x(u_*,\nabla u_*).$$

We thus have

$$\sup_{(q,\zeta)\in\mathcal{G}_K} f((q,\zeta),\eta)\geq\int_{\cup_{k\leq K}E_k}\limsup_{\lambda\downarrow 0}\frac{1}{\lambda}M(x,\lambda)\,dx. \tag{3.5}$$

In view of (3.3), we can apply Fatou lemma in (3.5):

$$\sup_{(q,\zeta)\in\mathcal{G}_K} f((q,\zeta),\eta)\geq\limsup_{\lambda\downarrow 0}\frac{1}{\lambda}\int_{\cup_{k\leq K}E_k}M(x,\lambda)\,dx. \tag{3.6}$$

By minimality of u_*, we have

$$0\leq\int_\Omega M(x,\lambda)\,dx.$$

We now write

$$0 \le \limsup_{\lambda \downarrow 0} \frac{1}{\lambda} \int_{\Omega} \cdots \le \limsup_{\lambda \downarrow 0} \frac{1}{\lambda} \int_{\cup_{k \le K} E_k} \cdots + \limsup_{\lambda \downarrow 0} \frac{1}{\lambda} \int_{\cup_{k > K} E_k} \cdots .$$

By applying (3.6) and (3.3) successively, we get

$$\sup_{(q,\zeta) \in \mathcal{G}_K} f((q,\zeta),\eta) \ge - \limsup_{\lambda \downarrow 0} \frac{1}{\lambda} \int_{\cup_{k > K} E_k} M(x,\lambda)\, dx$$

$$\ge - \limsup_{\lambda \downarrow 0} \int_{\cup_{k > K} E_k} \ell_R(x)\, dx = \int_{\cup_{k > K} E_k} -\ell_R(x)\, dx,$$

which is the required result. □

We next state a version of the Sion-Ky Fan minimax theorem that is convenient for our purpose (see e.g. [16]):

Theorem 3. *Let A and B be nonempty convex subsets of two locally convex topological vector spaces, and let A be compact. Suppose that $f : A \times B \to \mathbb{R}$ is such that for each $a \in A$, $f(a,\cdot)$ is convex, and for each $b \in B$, $f(\cdot, b)$ is upper semicontinuous and concave. Then, if the quantity*

$$\beta = \inf_{b \in B} \sup_{a \in A} f(a,b)$$

is finite, we have $\beta = \sup_{a \in A} \inf_{b \in B} f(a,b)$ and there exists an element $a \in A$ such that $\inf_{b \in B} f(a,b) = \beta$.

We shall apply this result with the map f on $A = \mathcal{G}_K$ (which is a nonempty compact convex subset of $L^\infty(\Omega)^{n+1}$ endowed with the weak * topology), $B = \mathcal{A}_R$ (which is a nonempty convex subset of $W^{1,1}(\Omega)$).

Lemma 3.4. *There exists $(q,\zeta) \in \mathcal{G}$ such that for every $\eta \in \cup_{R>0} \mathcal{A}_R$, we have*

$$\int_\Omega q(\eta - u_*) + \langle \zeta, \nabla \eta - \nabla u_* \rangle \ge 0.$$

Proof. Fix $R > 0$. In view of Lemma 3.3 and by the Sion-Ky Fan minimax theorem, for any $K \ge 1$, there exists $(q_K, \zeta_K) \in \mathcal{G}_K$ such that for every $\eta \in \mathcal{A}_R$, we have

$$\int_\Omega q_K(\eta - u_*) + \langle \zeta_K, \nabla \eta - \nabla u_* \rangle \ge \alpha_K(R). \tag{3.7}$$

By Lemma 3.1 i), there exist $(q,\zeta) \in \mathcal{G}$ (which depends on R) and a subsequence (we do not relabel) such that for any $k \ge 0$, $\{(q_K|_{E_k}, \zeta_K|_{E_k})\}_{K \ge 0}$ weakly* converges to $(q|_{E_k}, \zeta|_{E_k})$ in $L^\infty(E_k)$. We claim that for every $K \ge 0$, for every $\overline{R} \le R$ and for every $\eta \in \mathcal{A}_{\overline{R}}$, we have

$$\int_{\cup_{k \le K} E_k} q(\eta - u_*) + \langle \zeta, \nabla \eta - \nabla u_* \rangle \ge \alpha_K(\overline{R}). \tag{3.8}$$

Indeed,

$$\int_{\cup_{k \le K} E_k} q(\eta - u_*) + \langle \zeta, \nabla \eta - \nabla u_* \rangle = \lim_{L \to +\infty} \int_{\cup_{k \le K} E_k} q_L(\eta - u_*) + \langle \zeta_L, \nabla \eta - \nabla u_* \rangle.$$

For any $L \geq K$,

$$\int_{\cup_{k\leq K}E_k} q_L(\eta - u_*) + \langle \zeta_L, \nabla\eta - \nabla u_*\rangle = \int_\Omega \cdots - \int_{\cup_{k>K}E_k} \cdots .$$

By (3.7), the first term in the right hand side is not lower than $\alpha_L(R)$. By using (3.2) with (q_L, ζ_L) in the second term, we get

$$\int_{\cup_{k\leq K}E_k} q_L(\eta - u_*) + \langle \zeta_L, \nabla\eta - \nabla u_*\rangle$$
$$\geq \alpha_L(R) - \int_{\cup_{k>K}E_k} \ell_{\overline{R}} = \alpha_L(R) + \alpha_K(\overline{R}).$$

Since $\lim_{L\to+\infty} \alpha_L(R) = 0$, (3.8) follows at once.

In order to emphasize the dependence of (q, ζ) with respect to R, we denote it by (q^R, ζ^R). By Lemma 3.1 ii), there exist $(q, \zeta) \in \mathcal{G}$ and a subsequence (we do not relabel) such that for any $k \geq 0$, $\{(q^R|_{E_k}, \zeta^R|_{E_k})\}_R$ weakly* converges to $(q|_{E_k}, \zeta|_{E_k})$ in $L^\infty(E_k)$. As a consequence of (3.8), for every $K \geq 0$, $\overline{R} > 0$ and $R \geq \overline{R} > 0$, we have

$$\int_{\cup_{k\leq K}E_k} q^R(\eta - u_*) + \langle \zeta^R, \nabla\eta - \nabla u_*\rangle \geq \alpha_K(\overline{R}) \quad , \eta \in \mathcal{A}_{\overline{R}}.$$

We then let $R \to \infty$ to get

$$\int_{\cup_{k\leq K}E_k} q(\eta - u_*) + \langle \zeta, \nabla\eta - \nabla u_*\rangle \geq \alpha_K(\overline{R}). \tag{3.9}$$

Since by (3.2), $q(\eta - u_*) + \langle \zeta, \nabla\eta - \nabla u_*\rangle \leq \ell_{\overline{R}}$, we can apply Fatou Lemma when $K \to +\infty$. This gives

$$\int_\Omega q(\eta - u_*) + \langle \zeta, \nabla\eta - \nabla u_*\rangle \geq \limsup_{K\to\infty} \alpha_K(\overline{R}) = 0. \tag{3.10}$$

This completes the proof of Lemma 3.4.

□

We now complete the proof of Theorem 1 with the following proposition

Proposition 3.5. *Let $u_* \in W^{1,1}(\Omega)$ such that $u_*|_{\partial\Omega}$ is bounded. Assume that there exists $(q, \zeta) \in \mathcal{G}$ such that for every $\eta \in \cup_{R>0}\mathcal{A}_R$,*

$$0 \leq \int_\Omega q(\eta - u_*) + \langle \zeta, \nabla\eta - \nabla u_*\rangle. \tag{3.11}$$

Then

i) *$q \in L^1_{loc}(\Omega)$ and $\zeta \in L^1_{loc}(\Omega)$,*
ii) *$qu_* + \langle \zeta, \nabla u_*\rangle \in L^1_{loc}(\Omega)$,*
iii) *for any $\theta \in C_c^\infty(\Omega)$, we have*

$$\int_\Omega q\theta + \langle \zeta, \nabla\theta\rangle = 0.$$

Proof. The proof is reminiscent of the proof of [15] Theorem 2.4. The key observation is that for every $R > 0$, for every $\eta \in \mathcal{A}_R$, (3.11) holds true as well as

$$q(x)(\eta(x) - u_*(x)) + \langle \zeta(x), \nabla\eta(x) - \nabla u_*(x)\rangle \leq \ell_R(x) \quad , \quad \text{a.e. } x \in \Omega.$$

It then follows that the map $q(\eta - u_*) + \langle \zeta, \nabla\eta - \nabla u_*\rangle$ belongs to $L^1(\Omega)$.

We fix $M > 0$ such that the map u_*^M defined by (2.1) belongs to $u_* + W_0^{1,1}(\Omega)$.

Let Ω_0 be an open subset of Ω such that $\overline{\Omega_0} \subset \Omega$. Let $\theta_0 \in C_c^\infty(\Omega)$ such that $\theta_0 = 1$ on Ω_0. For $t \geq 1$, we then define the map $\eta_t := \max(u_*^M, tM(2\theta_0 - 1))$. The map η_t belongs to $\mathcal{A}_R$ for some $R > 0$. Hence, $q(\eta_t - u_*) + \langle \zeta, \nabla\eta_t - \nabla u_*\rangle \in L^1(\Omega)$. Since $\eta_t = tM \geq u_*^M$ on Ω_0, this implies $q(tM - u_*) + \langle \zeta, -\nabla u_*\rangle \in L^1(\Omega_0)$. In particular, this property is true for $t = 1$ and $t = 2$. Hence, $q \in L^1(\Omega_0)$ and thus $q \in L^1_{loc}(\Omega)$. In turn, this implies that $qu_* + \langle \zeta, \nabla u_*\rangle \in L^1(\Omega_0)$. This completes the proof of ii). Moreover, by writing for any $\eta \in \mathcal{A}_R$,

$$q\eta + \langle \zeta, \nabla\eta\rangle = q(\eta - u_*) + \langle \zeta, \nabla\eta - \nabla u_*\rangle + qu_* + \langle \zeta, \nabla u_*\rangle,$$

we have proved that $q\eta + \langle \zeta, \nabla\eta\rangle \in L^1_{loc}(\Omega)$.

Let $c > 0$ be such that $\Omega \subset (-c, c)^n$. We write $\zeta = (\zeta_1, \ldots, \zeta_n)$. We define $\eta := \max(u_*^M, M(2\theta_0 - 1)(x_1 + c + 1))$. Then $\eta \in \mathcal{A}_R$ for some $R > 0$ and $\eta = M(x_1 + c + 1) \geq u_*^M$ on Ω_0. This implies $\nabla\eta = M(1, 0, \ldots, 0)$. We know by the previous step that $q\eta + \langle \zeta, \nabla\eta\rangle \in L^1(\Omega_0)$. Since $(q\eta)|_{\Omega_0} = qM(x_1 + c + 1) \in L^1(\Omega_0)$, we get $\zeta_1 \in L^1(\Omega_0)$. Similarly, $\zeta_i \in L^1(\Omega_0)$, $1 \leq i \leq n$. This completes the proof of i).

We next prove iii). Let $\theta \in C_c^\infty(\Omega)$. For any $t > 0$, we consider $\eta := \max(u_*^M, t\theta - M)$. By inserting η in (3.11) and dividing by t, we obtain

$$\int_{[\theta > \frac{u_*^M + M}{t}]} q(\theta - \frac{1}{t}(M + u_*)) + \langle \zeta, \nabla\theta - \frac{1}{t}\nabla u_*\rangle$$
$$\geq \frac{-1}{t} \int_{[\theta \leq \frac{u_*^M + M}{t}]} q(u_*^M - u_*) + \langle \zeta, \nabla u_*^M - \nabla u_*\rangle.$$

Since $u_*^M \in \cup_{R>0}\mathcal{A}_R$, the map $q(u_*^M - u_*) + \langle \zeta, \nabla u_*^M - \nabla u_*\rangle$ belongs to $L^1(\Omega)$. Hence the right hand side goes to 0 when $t \to +\infty$.

For any $t > 0$, $[\theta > \frac{u_*^M + M}{t}]$ is a subset of supp θ. Since the maps q, ζ and $qu_* + \langle \zeta, \nabla u_*\rangle$ belong to $L^1_{loc}(\Omega)$, we can apply the dominated convergence theorem in the left hand side to get

$$\int_{[\theta \geq 0]} q\theta + \langle \zeta, \nabla\theta\rangle \geq 0.$$

We now insert $\eta := \min(u_*^M, t\theta + M)$ to obtain

$$\int_{[\theta \leq 0]} q\theta + \langle \zeta, \nabla\theta\rangle \geq 0.$$

This gives

$$\int_\Omega q\theta + \langle \zeta, \nabla\theta\rangle \geq 0.$$

Since the same inequality is true with $-\theta$ instead of θ, this completes the proof of iii). □

4. Proof of Theorem 2

We only indicate the major changes with respect to the proof of Theorem 1.

Proof. Let $\{\Omega_i\}_{i\geq 0}$ be an increasing sequence of open subsets compactly contained in Ω, such that $\Omega := \cup_{i\geq 0}\Omega_i$. For each $i \geq 0$, $u_*|_{\Omega_i}$ minimizes $u \mapsto \int_\Omega L_x(u, \nabla u)$ on $u_*|_{\Omega_i} + W_0^{1,1}(\Omega_i)$. Moreover, $u_*|_{\Omega_i} \in L^\infty(\Omega_i)$.

We keep the notation introduced in the proof of Theorem 1. Lemma 3.1 remains true with the same proof. Lemma 3.2 has the following analogue:

Lemma 4.1. *For any $R > 0$, there exists $\ell_R \in L^1(\Omega)$ such that for every $\eta \in \mathcal{A}_R$, for a.e. $x \in \Omega$ satisfying $|u_*(x)| \leq R$,* (3.2) *and* (3.3) *hold true.*

Proof. Let $(p, \xi) \in \mathbb{R} \times \mathbb{R}^n$, $(r, \gamma) \in \mathbb{R} \times \mathbb{R}^n$ and $\lambda \in (0, 1/2)$. Then for a.e. $x \in \Omega$,

$$\begin{aligned}(4.1)\quad &\frac{L_x((p,\xi) + \lambda(r-p, \gamma-\xi)) - L_x(p,\xi)}{\lambda} \\ &= \frac{L_x((p,\xi) + \lambda(r-p, \gamma-\xi)) - L_x(p+\lambda(r-p),\xi)}{\lambda} \\ &\qquad + \frac{L_x(p+\lambda(r-p),\xi) - L_x(p,\xi)}{\lambda}.\end{aligned}$$

Let $R > 0$. Let $|p|, |r| \leq R$ and $\gamma \in \mathrm{co}\left(\{\xi\} \cup \overline{B}^n(0,R)\right)$. The assumption $(H2)$ in conjunction with a Gronwall type argument applied to the map $g(\lambda) = L_{x,\xi}(p + \lambda(r-p)) - L_{x,\xi}(p)$ (as in the proof of Lemma 3.2) imply

$$(4.2)\qquad \frac{L_{x,\xi}(p+\lambda(r-p)) - L_{x,\xi}(p)}{\lambda} \leq \tilde{C}_R^0(x) + \tilde{C}_R^1(L_x(p,\xi) + |\xi|).$$

Here $\tilde{C}_R^0 \in L^1(\Omega)$, $\tilde{C}_R^1 > 0$.

We next estimate the first term in the right hand side of (4.1). For $\lambda, t \in (0, 1/2)$, we consider

$$h(t) = L_{x,p+\lambda(r-p)}(\xi + t(\gamma - \xi)) - L_{x,p+\lambda(r-p)}(\xi).$$

By a now routine technique, we obtain from $(H3)$

$$h(t) \leq \left(\frac{K_R^0(x)}{K_R^1} + L_{x,p+\lambda(r-p)}(\xi) + |\xi| + R\right)(e^{2K_R^1 t} - 1).$$

In view of (4.2), we thus get

$$\begin{aligned}(4.3)\quad &\frac{L_x(p+\lambda(r-p), \xi + \lambda(\gamma-\xi)) - L_x(p+\lambda(r-p),\xi)}{\lambda} \\ &\qquad \leq \tilde{K}_R^0(x) + \tilde{K}_R^1(L_x(p,\xi) + |\xi|),\end{aligned}$$

with $\tilde{K}_R^0 \in L^1(\Omega)$ and $\tilde{K}_R^1 > 0$. By (4.1)-(4.3), we have thus proved that for every $|p|, |r| \leq R$, $\gamma \in \mathrm{co}\left(\{\xi\} \cup \overline{B}^n(0,R)\right)$ and for every $\xi \in \mathbb{R}^n$,

$$(4.4)\quad \frac{L_x((p,\xi) + \lambda(r-p, \gamma-\xi)) - L_x(p,\xi)}{\lambda} \leq T_R^0(x) + T_R^1(L_x(p,\xi) + |\xi|),$$

where $T^j_R = \tilde{C}^j_R + \tilde{K}^j_R, j = 0, 1$. For a.e. $x \in \Omega$ such that $|u_*(x)| \le R$, (4.4) implies (3.3) as well as

$$L^0_x(u_*(x), \nabla u_*(x))(\eta(x) - u_*(x), \nabla\eta(x) - \nabla u_*(x)) \\ \le T^0_{R+1}(x) + T^1_{R+1} L_x(u_*(x), \nabla u_*(x)) + T^1_{R+1}|\nabla u_*(x)|,$$

from which (3.2) follows.

This proves Lemma 4.1. □

For any i, u_* is bounded on Ω_i. Hence, there exists $R_i(= |u_*|_{L^\infty(\Omega_i)})$ such that (3.3) and (3.2) hold on Ω_i for every $R \ge R_i$.

By using exactly the same arguments as in the proof of Theorem 1, there exists a measurable map $(q_i, \zeta_i) : \Omega_i \to \mathbb{R} \times \mathbb{R}^n$ such that for a.e. $x \in \Omega_i$, $(q_i(x), \zeta_i(x)) \in \partial L_x(u_*(x), \nabla u_*(x))$ and moreover, for every $K \ge 0$, for every $R \ge R_i$ and for every $\eta \in \mathcal{A}_R$, we have

$$\int_{\Omega_i \cap \cup_{k\le K} E_k} q_i(\eta - u_*) + \langle \zeta_i, \nabla\eta - \nabla u_*\rangle \ge -\int_{\Omega_i \cap \cup_{k>K} E_k} \ell_R \ge \alpha_K(R). \tag{4.5}$$

We recall that $\alpha_K(R) = -\int_{\cup_{k>K} E_k} \ell_R$ (here, we also use the fact that ℓ_R can be assumed nonnegative without loss of generality). We extend (q_i, ζ_i) by 0 on the whole Ω.

As in the proof of Lemma 3.1, there exists $(q, \zeta) \in \mathcal{G}$ and a subsequence of $\{(q_i, \zeta_i)\}_i$ (we do not relabel) such that for each $k \ge 0$, $\{(q_i|_{E_k}, \zeta_i|_{E_k})\}_i$ weakly* converges to $(q|_{E_k}, \zeta|_{E_k})$.

We introduce the set $\widetilde{\mathcal{A}}_R, R > 0$, of those maps in $\mathcal{A}_R$ which coincide with u_* on a neighborhood of $\partial\Omega$.

Let $R > 0$ and $\eta \in \widetilde{\mathcal{A}}_R$. For i sufficiently large, say $i \ge i_0$, $\eta = u_*$ on $\Omega \setminus \Omega_i$. Hence, for any $K \ge 0$ and $i \ge i_0$, we have

$$\int_{\cup_{k\le K} E_k} q_i(\eta - u_*) + \langle \zeta_i, \nabla\eta - \nabla u_*\rangle \ge \alpha_K(\max(R, R_{i_0})).$$

We now let $i \to +\infty$. This gives

$$\int_{\cup_{k\le K} E_k} q(\eta - u_*) + \langle \zeta, \nabla\eta - \nabla u_*\rangle \ge \alpha_K(\max(R, R_{i_0})).$$

By Lemma 4.1 on Ω_{i_0}, the map under the integral sign is not larger than $\ell_{\max(R,R_{i_0})}$. We can thus apply Fatou Lemma to obtain

$$\int_\Omega q(\eta - u_*) + \langle \zeta, \nabla\eta - \nabla u_*\rangle \ge 0.$$

We now complete the proof of Theorem 2 with the following analogue of Proposition 3.5

Proposition 4.2. *Let $u_* \in W^{1,1}(\Omega) \cap L^\infty_{loc}(\Omega)$. Assume that there exists $(q, \zeta) \in \mathcal{G}$ such that for every $\eta \in \cup_{R>0}\widetilde{\mathcal{A}}_R$,*

$$0 \le \int_\Omega q(\eta - u_*) + \langle \zeta, \nabla\eta - \nabla u_*\rangle. \tag{4.6}$$

Then

i) $q \in L^1_{loc}(\Omega)$ *and* $\zeta \in L^1_{loc}(\Omega)$,
ii) $qu_* + \langle \zeta, \nabla u_*\rangle \in L^1_{loc}(\Omega)$,

iii) *for any* $\theta \in C_c^\infty(\Omega)$, *we have*

$$\int_\Omega q\theta + \langle \zeta, \nabla\theta \rangle = 0.$$

The proof is very similar to the proof of Proposition 3.5. As a matter of fact, it is exactly the same as the proof of [15] Theorem 2.4. We omit it.

This completes the proof of Theorem 2.

□

5. Proof of Proposition 2.1

We first state a more general version of Proposition 2.1.

Proposition 5.1. *The map* $L : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ *satisfies* $(H1)$ *if one of the following assumptions is satisfied:*

i) *There exists* $S > 0$ *and a map* $\widetilde{L} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ *which is locally bounded, convex with respect to* (p, ξ) *and such that for a.e.* $x \in \Omega$, $L_x|_{\mathbb{R}^{n+1}\setminus B^{n+1}(0,S)} = \widetilde{L}_x|_{\mathbb{R}^{n+1}\setminus B^{n+1}(0,S)}$.

ii) *There exists* $C > 0$ *such that for any* $|(p,\xi)| \le |(p',\xi')|$, *for a.e.* $x \in \Omega$, *we have*
$$L_x(p,\xi) \le L_x(p',\xi') + C|(p,\xi) - (p',\xi')|.$$

iii) *There exist* $\alpha > 0, \beta \in \mathbb{R}$ *such that* $L_x(p,\xi) \ge \alpha|(p,\xi)|^2 + \beta$, $(x,p,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, L *is bounded on bounded sets and* L_x *is semiconvex: there exists* $C > 0$ *such that for every* $(p,\xi), (p',\xi') \in \mathbb{R}^n$, *for every* $\theta \in (0,1)$, *for a.e.* $x \in \Omega$, *we have*

$$\begin{aligned} &L_x(\theta p + (1-\theta)p', \theta\xi + (1-\theta)\xi') \\ &\qquad \le \theta L_x(p,\xi) + (1-\theta)L_x(p',\xi') + C\theta(1-\theta)|(p,\xi) - (p',\xi')|^2. \end{aligned}$$

iv) *The map* L *satisfies*
$$\liminf_{\substack{|(p,\xi)| \to +\infty \\ x \in \Omega}} \min_{\zeta \in \partial L_x(p,\xi)} \langle (p,\xi), \frac{\zeta}{|\zeta|} \rangle = +\infty.$$

v) *The map* L_x *is non decreasing:* $L_x^0((p,\xi), -(p,\xi)) \le 0$ *and the growth of* L_x *is at most exponential: there exists* $K^0 \in L^1(\Omega)$ *and* $K^1 > 0$ *such that*

$$\max_{\zeta \in \partial L_x(p,\xi)} |\zeta| \le K^0(x) + K^1(L_x(p,\xi) + |(p,\xi)|) \quad , \quad (x,p,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n.$$

Proposition 2.1 is an easy consequence of the above proposition.

Proof. In order to simplify the notation, we fix $x \in \Omega$, and we introduce for any $a = (p,\xi) \in \mathbb{R} \times \mathbb{R}^n$, the map $f(a) = L_x(p,\xi)$. Each of the assumptions (i)-(v) will imply the following version of $(H1)$: for every $R > 0$, there exist $S_R > 0$ and $K_R^0, K_R^1 : \Omega \to (0,\infty)$ such that for every $a \in \mathbb{R}^{n+1}\setminus\overline{B}^{n+1}(0,S_R)$,
$$\max_{|a'| \le R} f^0(a, a' - a) \le K_R^0 + K_R^1(f(a) + |a|).$$
In each case, K_R^0 will be a summable function of x and K_R^1 will be (essentially) bounded. Since f is globally Lipschitz on $\overline{B}^{n+1}(0,S_R)$, this will imply $(H1)$.

Case (i). There exists $S > 0$ such that $f|_{\mathbb{R}^{n+1}\setminus B^{n+1}(0,S)} = \widetilde{f}|_{\mathbb{R}^{n+1}\setminus B^{n+1}(0,S)}$, where $\widetilde{f}$ is convex on $\mathbb{R}^{n+1}$. For every $a, a' \in \mathbb{R}^{n+1}$, $\widetilde{f}(a') - \widetilde{f}(a) \geq \langle \xi, a' - a \rangle$ for any ξ in the convex subdifferential of $\widetilde{f}$ at a (which coincides with the generalized subdifferential $\partial \widetilde{f}(a)$). Hence,

$$\widetilde{f}^0(a, a' - a) \leq \widetilde{f}(a') - \widetilde{f}(a) \leq \widetilde{f}(a').$$

This implies that for every $|a| > S$, for every $R > 0$,

$$\max_{|a'| \leq R} f^0(a, a' - a) \leq |\widetilde{f}|_{L^\infty(B^{n+1}(0,R))}.$$

In view of the above discussion, this completes the proof of Proposition 5.1 in Case (i).
Case (ii). We know that there exists $C > 0$ such that for any $|a| \leq |a'|$, we have $f(a) \leq f(a') + C|a' - a|$. Let $R > 0$ and $|a| > R$. For any $|a'| \leq R$, for any $(\lambda, b) \in (0, \infty) \times \mathbb{R}^n$ sufficiently close to $(0, a)$, one has $|b + \lambda(a' - a)| \leq |b|$. This implies $f(b + \lambda(a' - a)) \leq f(b) + C\lambda|a' - a|$ so that

$$f^0(a, a' - a) \leq C|a' - a| \leq CR + C|a|.$$

Case (ii) follows at once.
Case (iii). Since f is semiconvex, there exists $C > 0$ such that for every $a, a' \in \mathbb{R}^{n+1}$,

$$f^0(a, a' - a) \leq f(a') - f(a) + C|a' - a|^2 \leq f(a') + 2C|a'|^2 + 2C|a|^2.$$

Since f is coercive of order 2, we get

$$\max_{|a'| \leq R} f^0(a, a' - a) \leq K_R^0 + K_R^1 f(a).$$

This proves Case (iii).
Case (iv). We have

$$\max_{|a'| \leq R} f^0(a, a' - a) = \max_{|a'| \leq R} \max_{\zeta \in \partial f(a)} \langle \zeta, a' - a \rangle \leq \max_{\zeta \in \partial f(a)} |\zeta|(R - \langle \frac{\zeta}{|\zeta|}, a \rangle).$$

By assumption, for every $R > 0$, there exists $S_R > 0$ such that for every $|a| \geq S_R$, for every $\zeta \in \partial f(a)$, we have $\langle \frac{\zeta}{|\zeta|}, a \rangle \geq R$. This implies $\max_{|a'| \leq R} f^0(a, a' - a) \leq 0$; that is, Case (iv).
Case (v). By subadditivity of $f^0(a, \cdot)$ and the fact that f is non decreasing, we have

$$\max_{|a'| \leq R} f^0(a, a' - a) \leq \max_{|a'| \leq R} f^0(a, a') + f^0(a, -a) \leq \max_{|a'| \leq R} f^0(a, a').$$

Now, we use the fact that the growth of f is at most exponential to get

$$\max_{|a'| \leq R} f^0(a, a' - a) \leq K_R^0 + K_R^1(f(a) + |a|).$$

This proves Case (v).

□

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