# On the Lavrentiev phenomenon for multiple integral scalar variational problems

Pierre Bousquet Carlo Mariconda Giulia Treu

**Abstract**

Let $\phi$ be a Lipschitz map on $\mathbb{R}^n$. We prove the non occurrence of the Lavrentiev gap between Lipschitz functions and Sobolev functions for functionals of the form

$$I(u) = \int_\Omega F(u, \nabla u) \quad u \in W^{1,p}_\phi(\Omega)$$

when $\Omega$ belongs to a wide class of open and bounded subsets of $\mathbb{R}^n$ containing Lipschitz ones, and either $F$ is convex in both variables or $F(s, \xi) = a(s)g(\xi) + b(s)$ with $g$ convex and $s \mapsto a(s)g(0) + b(s)$ satisfying a non oscillatory condition at infinity. We derive the non occurrence of the Lavrentiev phenomenon for unnecessarily convex functionals of the gradient. No growth conditions are assumed.

**Keywords.** Lavrentiev, Lavrentiev phenomenon, Lavrentiev gap, regularity, Lipschitz approximation, star-shaped

# 1 Introduction

In this article, we study the Lavrentiev phenomenon for a multidimensional scalar problem in the calculus of variations. Given a function $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and an open subset $\Omega$ of $\mathbb{R}^n$, we consider the functional $I$ defined for $u$ in the Sobolev space $W^{1,p}(\Omega)$, $p \geq 1$, by

$$I(u) := \int_\Omega F(u(x), \nabla u(x))\, dx.$$

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The admissible maps are subject to a Dirichlet boundary condition given by a Lipschitz function $\phi : \mathbb{R}^n \to \mathbb{R}$. We denote by $W^{1,p}_\phi(\Omega)$ the set of those maps in $W^{1,p}(\Omega)$ which agree with $\phi$ on $\partial\Omega$ in the following sense : $u \in W^{1,p}_\phi(\Omega)$ if and only if the extension of $u$ by $\phi$ outside $\Omega$ belongs to $W^{1,p}_{loc}(\mathbb{R}^n)$. Similarly, the set $\mathrm{Lip}_\phi(\Omega)$ is the set of the Lipschitz functions on $\Omega$ which coincide with $\phi$ (in a pointwise sense) on $\partial\Omega$.
We say that the functional $I$ has no *Lavrentiev gap* at $u \in W^{1,p}_\phi(\Omega)$ if there is a sequence $(u_k)_k$ in $\mathrm{Lip}_\phi(\Omega)$ converging to $u$ in $W^{1,p}(\Omega)$ and such that

$$\lim_{k\to+\infty} I(u_k) = I(u).$$

The non occurrence of a Lavrentiev gap for every $u \in W^{1,p}_\phi(\Omega)$ implies the non occurrence of the Lavrentiev phenomenon which is defined by the following identity:

$$\inf\{I(u) : \ u \in W^{1,p}_\phi(\Omega)\} = \inf\{I(u) : \ u \in \mathrm{Lip}_\phi(\Omega)\}.$$

In general, it is substantially more difficult to prove the non occurrence of the Lavrentiev gap than to establish the non occurrence of the Lavrentiev phenomenon. For instance, consider the case of a functional depending only on the gradient with null boundary datum: when $F$ is convex, $0$ itself is a minimizer thus preventing the occurrence of the Lavrentiev phenomenon. Instead, the fact that the Lavrentiev gap does not occur at $u \in \ W^{1,1}_0(\Omega)$ means that one can approximate $u$ via a sequence of Lipschitz functions that are equal to $0$ on the boundary, and such that the values $I(u_k)$ converge to $I(u)$, a much more difficult task. The knowledge of the a priori non occurrence of the Lavrentiev gap/phenomenon is particularly important in the context of Numerical Analysis; it allows to approximate the values of a functional by means of the finite elements method.
The first example of a variational problem whose infimum among Lipschitz mappings is strictly greater than the infimum among absolutely continuous functions with prescribed boundary data was presented in the one-dimensional case by M. A. Lavrentiev [13] in 1927. The occurrence of this phenomenon was quite surprising at that time since Lipschitz functions are dense $W^{1,1}$: this illustrated the fact that these functionals were not continuous on $W^{1,1}$. Ball and Mizel [2] subsequently built a smooth and coercive Lagrangian exhibiting the same phenomenon. The occurrence of the Lavrentiev phenomenon is now formulated in terms of the relaxation of the functional $I$[7]: for every $u \in W^{1,p}_\phi(\Omega)$, the relaxed functional $\overline{I}(u)$ is defined by

$$\overline{I}(u) := \inf\{\liminf_{n\to+\infty} I(u_k) : (u_k)_{k\in\mathbb{N}} \subset \mathrm{Lip}_\phi(\Omega), u_k \xrightarrow[W^{1,p}(\Omega)]{} u\}.$$

The one dimensional scalar case has been thoroughly studied: let us quote the results concerning the Lipschitz continuity of the minimizers [10] under the standard Tonelli's assumptions and the non occurrence of the Lavrentiev gap [1] in full generality for autonomous functionals.

In the scalar multidimensional case it is well known that the Lavrentiev phenomenon may occur if the Lagrangian depends on $x$ and $\nabla u$ [9], even for functionals of the form $F(x,\xi) = a(x)|\xi|^2$ [18].
For scalar multidimensional autonomous functionals, i.e. when the Lagrangian does not depend on $x$, there are neither examples of the occurrence of the Lavrentiev phenomenon nor a definite answer about its non occurrence, except in the obvious case when the Lagrangian satisfies the so called natural growth conditions (i.e. a $p$-growth from below and from above) [6]. A paradigm that is widely spread over the community of the Calculus of Variations asserts that Lavrentiev gaps should not occur for functionals (just) of the gradient. However there are just a few results corroborating this statement. To our knowledge, the first general statement appears in [12, Proposition X.2.6] : it requires that $\phi = 0$, that $F$ depend only on the gradient, and that $\Omega$ be Lipschitz. We think however that in the proof given in [12], the boundary datum is not preserved in the construction of the approximating sequence (see Remark 4.6 below). In [5], the non occurence of Lavrentiev gaps is established for a Lagrangian $F$ which is convex with respect to both variables and when the domain is star-shaped. Still, in this latter result, there is no boundary condition: the functions of the approximating sequences do not have to share the same boundary datum as the limit function. Finally, in a recent paper [3] (see also [4]), the Lavrentiev phenomenon is shown not to occur when $F$ is *radial* (i.e. depends only on the Euclidean norm of the gradient) and both the boundary datum and the domain are of class $\mathcal{C}^2$.

In this article, we prove the absence of Lavrentiev gaps in two situations which substantially extend the above results without assuming growth assumptions of any kind. Here, the open set $\Omega$ belongs to the large class of locally strongly star-shaped domains. This class is introduced in Section 2. Let us simply mention that it contains any Lipschitz domains. In our first result, $F$ is required to be convex with respect to both variables.

**Theorem 1.1** (Convex Lagrangians)**.** *Assume that $\Omega$ is locally strongly star-shaped and that $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is convex. Let $u \in W^{1,p}_\phi(\Omega)$ be such that $F(u, \nabla u) \in L^1(\Omega)$. Then the Lavrentiev gap for $I$ does not occur at $u$, i.e. there exists a sequence $(u_k)_k$ in $\mathrm{Lip}_\phi(\Omega)$ converging to $u$ in $W^{1,p}(\Omega)$ and such that*

$$\lim_{k\to+\infty} I(u_k) = I(u). \tag{1.1}$$

*Moreover, if $u$ is bounded in $L^\infty(\Omega)$, the sequence $(u_k)_k$ may be taken to be bounded in $L^\infty(\Omega)$.*

In the framework of Theorem 1.1, the quantity $\displaystyle\int_\Omega F(u(x), \nabla u(x))\, dx$ has a well-defined meaning in $\mathbb{R} \cup \{+\infty\}$ when $u \in W^{1,p}_\phi(\Omega)$. This is a consequence of the fact that the convex function $F$ is bounded from below by an affine function.

**Remark 1.2.** Theorem 1.1 extends [4, Theorem 3.4]. In the latter, the Lagrangian $F$ must be of the form $F(u, \nabla u) = a(u)+g(|\nabla u|)$ where both $a$ and $g$ are convex, and the domain as well as the boundary datum are required to be smooth. Moreover, [4] only considers the non occurence of the Lavrentiev phenomenon. In contrast, in Theorem 1.1 as well as in Theorem 1.3 below, we prove a deeper result : the non occurrence of the Lavrentiev gap.

It is very plausible that there is no Lavrentiev gap even when $F$ is not convex in $u$. However, this problem is open even when the boundary condition is ignored. In the second situation that we consider in this article, we thus restrict our attention to Lagrangians of the form

$$F(s, \xi) = a(s)g(\xi) + b(s), \tag{1.2}$$

where $a : \mathbb{R} \to [0, +\infty[$ is continuous, $g : \mathbb{R}^n \to \mathbb{R}$ is convex and $b : \mathbb{R} \to \mathbb{R}$ is continuous.

**Non oscillatory condition at infinity.** *Given $p \in [1, n]$, we say that the map $c(s) = a(s)g(0) + b(s)$ satisfies a* non oscillatory condition at infinity *if there exist two positive sequences $(\tau_k)_k$, $(\sigma_k)_k$ such that*

$$\lim_{k\to\infty} \tau_k = \lim_{k\to\infty} \sigma_k = +\infty$$

*and*

$$\forall s \geq \tau_k, \quad c(\tau_k) \leq C_1 c(s) + D|s|^{p^*}, \tag{1.3a}$$

$$\forall s \leq -\sigma_k, \quad c(-\sigma_k) \leq C_2 c(s) + D|s|^{p^*}, \tag{1.3b}$$

*for some $C_1, C_2, D > 0$. Here $p^* = \dfrac{np}{n-p}$ if $p < n$ while $p^*$ is any positive number if $p = n$ (this condition will not be considered when $p > n$).*

**Theorem 1.3** (Non convex Lagrangians)**.** *Assume that $\Omega$ is locally strongly star-shaped and that $F$ is given by* (1.2)*. Let $u$ in $W^{1,p}_\phi(\Omega)$ be such that $F(u, \nabla u) \in L^1(\Omega)$. If $0 \in a(\mathbb{R})$, we also require that $g(\nabla u) \in L^1(\Omega)$. When $1 \leq p \leq n$, we further assume that the map $c(s) = a(s)g(0) + b(s)$ satisfies the non oscillatory condition* (1.3)*. Then the Lavrentiev gap for $I$ does not occur at $u$, i.e. there exists a sequence $(u_k)_k$ in $\mathrm{Lip}_\phi(\Omega)$ converging to $u$ in $W^{1,p}(\Omega)$ and such that*

$$\lim_{k\to+\infty} I(u_k) = I(u).$$

**Remark 1.4.** 1. The Lagrangian $F(s, \xi) = a(s)|\xi|^p$ satisfies the assumptions of Theorem 1.3 when $a \geq 0$ is continuous. Observe in particular that if $u \in W^{1,p}(\Omega)$, then $g(\nabla u) = |\nabla u|^p \in L^1(\Omega)$. This example was considered in the unpublished preprint

[16] where Percivale proves the non occurrence of the Lavrentiev gap without taking care of the boundary datum, under the further assumption that the zero level of the function $a$ does not contain limit points.

2. The assumption $g(\nabla u) \in L^1(\Omega)$ may seem artificial. It is automatically satisfied if $g(\xi) \sim |\xi|^p$ at infinity.

3. Condition (1.3) is fulfilled if for instance $c$ can be written as the sum of two functions $c = c_1 + c_2$ such that for some $r > 0$ and $D > 0$, $c_1$ is decreasing on $]-\infty, -r]$ and increasing on $[r, +\infty[$ while $c_2$ is $\mathcal{C}^1$ and satisfies $|c_2'(s)| \leq D|s|^{p^*-1}$ for every $|s| \geq r$.

The following section is devoted to the study of different classes of star-shaped domains and locally star-shaped domains: these concepts turn out to be crucial in the proof of the main results. Theorem 1.1 is proved in Section 4 while Theorem 1.3 is proved in Section 5; a much less technical proof of Theorem 1.1 is provided in Section 4.1 under the assumption that $\Omega$ is strongly star-shaped and when $F$ depends only on the gradient variable. In Section 5.2 we get, by means of a relaxation argument, the non occurrence of the Lavrentiev phenomenon for Lagrangians of the form $F(s, \xi) = g(\xi)$ when $g$ is not necessarily convex.

# 2 Star-shaped domains

In this section, we present some properties of star-shaped domains that will be used in the proof of Theorem 1.1 and Theorem 1.3. We say that the open bounded set $\Omega \subset \mathbb{R}^n$ is star-shaped with respect to a point $z$ if the segments joining $z$ to any point of $\Omega$ are entirely contained in $\Omega$. We denote by $\Gamma$ the boundary of $\Omega$.

We are interested in star-shaped domains whose homothetic retractions w.r.t. to a given point $z$ are relatively compact subsets of the domain.

**Definition 2.1.** An open set $\Omega$ which is star-shaped with respect to a point $z$ is called *strongly star-shaped*[17] if the relative interior of each segment from $z$ to a point of $\Gamma$ is entirely contained in $\Omega$ or, equivalently, the intersection of $\Gamma$ with any half line originating at $z$ is a point.

Thus, $\Omega$ is strongly star-shaped if and only if, for every $h \in [0, 1[$,

$$z + h(\Omega - z) \subset\subset \Omega,$$

i.e. $z + h(\Omega - z)$ is relatively compact in $\Omega$. One sometimes needs to quantify more precisely the distance from $z + h(\Omega - z)$ to $\Gamma$.

**Definition 2.2.** An open set $\Omega$ which is star-shaped with respect to a point $z$ is called *uniformly star-shaped* if there exist $C > 0$ such that for any $0 \leq h \leq 1$,

$$\operatorname{dist}(h(\Gamma - z) + z, \Gamma) \geq C(1 - h). \tag{2.1}$$

Notice that if (2.1) holds then the open ball $B_C(z)$ of center $z$ and radius $C$ is entirely contained in $\Omega$, due to the fact that for $h = 0$ the condition turns out to be equivalent to $\operatorname{dist}(z, \Gamma) \geq C$.
In particular a uniformly star-shaped open set with respect to a point is strongly star-shaped with respect to the same point; the converse is not true, as it will be shown in Example 2.8.

**Remark 2.3.** Condition (2.1) is equivalent to

$$\forall h \in [0, 1[, \qquad \inf_{x,y \in \Gamma} \frac{|hx - y + (1 - h)z|}{1 - h} \geq C. \tag{2.2}$$

In order to give some characterizations of uniformly star-shaped domains, we introduce two definitions. Given a point $z \in \Omega$, a *radial direction* at a point $\alpha$ of the boundary $\Gamma$ of $\Omega$ is a vector that is parallel to $\alpha - z$. Let us also recall the definition of the *paratingent tangent cone* to a set.

**Definition 2.4** (Paratingent tangent cones)**.** The *Bouligand paratingent cone* $P_K(\alpha)$ to a set $K \subseteq \mathbb{R}^n$ at $\alpha \in K$ is the set of vectors $v \in \mathbb{R}^n$ such that

$$\lim_{k \to +\infty} t_k(x_k - y_k) = v$$

for some sequences $(t_k)_k$ of positive numbers with $\lim\limits_{k \to +\infty} t_k = +\infty$ and $(x_k)_k, (y_k)_k \in K$ both converging to $\alpha$. The set $P_K(\alpha)$ is a symmetric cone with respect to the origin.

Here is a characterization of the sets that are uniformly star-shaped: it turns out in particular that these sets are star-shaped with respect to a ball, which is the most common way under which this class of star-shaped sets appears in analysis. A further characterization for domains that are strongly or uniformly star-shaped in terms of the jauge and radii function will be settled in Proposition 6.1.

**Example 2.5.** Assume that $\Omega$ is star-shaped with respect to $z$. The following conditions are equivalent:

1. the set $\Omega$ is uniformly star-shaped w.r. to $z$;

2. For all $\alpha \in \Gamma$
$$\liminf_{\substack{x,y \in \Gamma \to \alpha \\ h \uparrow 1}} \frac{|hx - y + (1 - h)z|}{1 - h} > 0; \tag{2.3}$$

3. the paratingent tangent cone at any points of $\Gamma$ does not contain radial directions, i.e.
$$\forall \alpha \in \Gamma \qquad \alpha - z \notin P_\Gamma(\alpha); \tag{2.4}$$

4. the set $\Omega$ is star-shaped with respect to every point of an open ball contained in $\Omega$ of center $z$; actually (2.1) holds if and only if $\Omega$ is star-shaped with respect to the points of the ball $B_C(z)$.

We postpone the proof of Proposition 2.5 after the following lemmata; just the equivalence $(1) \Leftrightarrow (4)$ will be used in the sequel.

**Lemma 2.6.** *Let $K$ be a non empty subset of $\mathbb{R}^n$. Then $\alpha \in P_K(\alpha)$ if and only if there exist sequences $(x_k)_k, (y_k)_k$ in $K$ both converging to $\alpha$ and a sequence $(h_k)_k$ in $[0,1[$ converging to 1 such that*
$$\lim_{k\to+\infty} \frac{h_k x_k - y_k}{1-h_k} = 0. \tag{2.5}$$

*Proof.* Assume that $\alpha \in P_K(\alpha)$. Then there exist two sequences $(x_k)_k, (y_k)_k$ in $K$ both converging to $\alpha$ and a sequence $(t_k)_k$ of positive terms with $t_k \to +\infty$ such that
$$\lim_{k\to+\infty} t_k(x_k - y_k) = \alpha.$$

Set
$$h_k := \frac{t_k}{1+t_k} \qquad \forall k \in \mathbb{N}.$$

Clearly $h_k \in [0,1[$ for all $k$ and $(h_k)_k$ converges to 1 as $k \to +\infty$. Since
$$\begin{aligned} t_k(x_k - y_k) - \alpha &= t_k(x_k - y_k) - y_k + (y_k - \alpha) \\ &= \frac{h_k}{1-h_k}(x_k - y_k) - y_k + (y_k - \alpha) \\ &= \frac{h_k x_k - y_k}{1-h_k} + (y_k - \alpha) \end{aligned} \tag{2.6}$$

and $y_k - \alpha \to 0$ as $k \to +\infty$, it turns out that $\lim\limits_{k\to+\infty} \dfrac{h_k x_k - y_k}{1-h_k} = 0$.

Conversely, assume that (2.5) holds for some sequence $(h_k)_k$ converging to $1$ and sequences $(x_k)_k, (y_k)_k$ both in $\Gamma$ and converging to $\alpha$. Then if we set $t_k = \dfrac{h_k}{1-h_k}$, we have $h_k = \dfrac{t_k}{1+t_k}$ and (2.6) yields the conclusion. □

**Lemma 2.7.** *Assume that $\Omega$ is star-shaped w.r. to the points of an open ball $B \subset \Omega$. Then $\Omega$ is strongly star-shaped w.r. to the points of $B$, i.e. it contains the relative interior of each segment joining a point of $B$ to a point of $\Gamma$.*

*Proof.* Let $q \in B$, $\gamma \in \Gamma$ and $p \in ]q, \gamma[$. By a change of coordinates it is not restrictive to assume that

$$q = 0, \quad \gamma = (0, \gamma_n), \quad p = (0, p_n), \quad B = B_C,$$

for some $\gamma_n > 0, p_n > 0, C > 0$. Fix $h, \delta > 0$ satisfying $h < \gamma_n - p_n$ and $\delta < Ch/p_n$; since the segment joining $p$ to $\gamma$ is in $\overline{\Omega}$, there is $(v, v_n) \in \Omega$ with $v \in \mathbb{R}^{n-1}$ such that $|v| < \delta$ and $v_n > p_n + h$. We claim that $p$ belongs to the relative interior of the segment $[(v, v_n), (\xi, 0)]$, joining $(v, v_n)$ to a point $(\xi, 0)$ for some $\xi \in B_C^{n-1}$, the open ball in $\mathbb{R}^{n-1}$ centered at the origin. Indeed the points of the segment $[(v, v_n), (\xi, 0)]$ are of the form

$$\psi(\xi, \lambda) := (v, v_n) + \lambda((\xi, 0) - (v, v_n)) \quad \lambda \in [0, 1]$$

and $\psi(\xi, \lambda) = p = (0, p_n)$ if and only if

$$\lambda = \overline{\lambda} := 1 - \frac{p_n}{v_n}, \qquad \xi = \overline{\xi} = -\frac{v p_n}{v_n - p_n}.$$

Now $\overline{\lambda} \in ]0, 1[$ and

$$|\overline{\xi}| < \frac{|v| p_n}{h} < \frac{\delta p_n}{h} < C,$$

proving the claim. Hence, $p \in \Omega$ and Lemma 2.7 is proved. □

*Proof of Proposition 2.5.* It is not restrictive to assume that $z = 0$.
$(1) \Leftrightarrow (2)$. If $\Omega$ is uniformly star-shaped w.r. to $0$ there is $C > 0$ satisfying

$$\forall h \in [0, 1[, \qquad \inf_{x, y \in \Gamma} \frac{|hx - y|}{1 - h} \geq C,$$

and thus for all $\alpha \in \Gamma$ we have

$$\liminf_{\substack{x, y \in \Gamma \to \alpha \\ h \uparrow 1}} \frac{|hx - y|}{1 - h} \geq C > 0,$$

proving the validity of (2.3). Conversely, if $\Omega$ is not uniformly star-shaped w.r. to $0$ then for all $k = 1, 2, ...$ there are $h_k \in [0, 1[$ and $x_k, y_k \in \Gamma$ such that

$$\frac{|h_k x_k - y_k|}{1 - h_k} \leq \frac{1}{k}.$$

Modulo a subsequence we may assume that

$$\lim_k x_k = \alpha \in \Gamma, \quad \lim_k y_k = \beta \in \Gamma, \quad \lim_k h_k = h \in [0, 1].$$

Two cases may occur. If $h = 1$ then $\alpha = \beta$ and

$$\liminf_{\substack{x, y \in \Gamma \to \alpha \\ h \uparrow 1}} \frac{|hx - y|}{1 - h} = 0. \tag{2.7}$$

Otherwise, if $h < 1$, then $h\alpha = \beta \in \Gamma$ and the segment joining $h\alpha$ to $\alpha$ is contained in $\Gamma$. Indeed, since $[0, \alpha] \subset \overline{\Omega}$, any $x \in ]h\alpha, \alpha[$ belongs to $\overline{\Omega}$. Assume by contradiction that $x \in \Omega$. The set $\Omega$ being star-shaped with respect to $0$, this implies $[0, x] \subset \Omega$, hence $h\alpha \in \Omega$ : a contradiction. This proves that $[h\alpha, \alpha] \subset \Gamma$. Then, if for all $k$ we set

$$\widetilde{x}_k = \alpha, \quad \widetilde{y}_k = (1 - \frac{1}{k})\alpha$$

we have that $\widetilde{x}_k, \widetilde{y}_k \in \Gamma$ for $k$ large enough and $(1 - \frac{1}{k})\widetilde{x}_k - \widetilde{y}_k = 0$ so that (2.7) holds. In both cases $h = 1$ and $h < 1$ condition (2.3) is violated.
$(2) \Leftrightarrow (3)$ directly follows from Lemma 2.6.
We now prove the equivalence between $(1)$ and $(4)$, more precisely that (2.1) holds if and only if $\Omega$ is star-shaped with respect to the points of the ball $B_C(z)$. First assume that (2.1) holds. If $\Omega$ fails to be star-shaped w.r. to the points of $B_C$, there are $0 < \varepsilon < C$, a unit vector $u$ and $\gamma \in \Gamma$ such that the relative interior of the segment joining $\varepsilon u$ with $\gamma$ is not entirely contained in $\Omega$. Hence, there is $0 < h < 1$ such that

$$h(\varepsilon u) + (1 - h)\gamma := \gamma' \in \Gamma.$$

Thus

$$\varepsilon h = |\gamma' - (1 - h)\gamma| \geq \mathrm{dist}(\Gamma, (1 - h)\Gamma) \geq Ch,$$

implying that $\varepsilon \geq C$, a contradiction. Conversely, assume that $\Omega$ is star-shaped w.r. to the points of $B_C$. If (2.1) fails to be true, there is $h$ in $[0, 1]$ such that $\mathrm{dist}(h\Gamma, \Gamma) < C(1-h)$; notice that since $B_C \subset \Omega$, $h \neq 0$, and of course $h \neq 1$. Therefore there are $\gamma, \gamma' \in \Gamma$ satisfying

$$|\gamma' - h\gamma| < C(1 - h)$$

so that

$$\gamma' - h\gamma = \varepsilon(1 - h)u,$$

for some $0 \leq \varepsilon < C$ and a unit vector $u$. Thus

$$h\gamma + (1 - h)(\varepsilon u) = \gamma' \in \Gamma, \qquad 0 < h < 1$$

so that the relative interior of the segment joining $\varepsilon u \in B_C$ to $\gamma \in \Gamma$ is not contained in $\Omega$, contradicting Lemma 2.7. It follows that (2.1) is true and $\Omega$ is uniformly star-shaped w.r. to $0$. □

**Example 2.8.** The open bounded region $\Omega$ whose boundary is the cardioid described in Figure 1 is strongly star-shaped w.r.t. $z = 0$ but not uniformly star-shaped w.r.t. $0$ since the tangent cone to $\Gamma$ at $(0, 1)$ is vertical and thus contains a radial direction; here, we use (3) of Proposition 2.5.

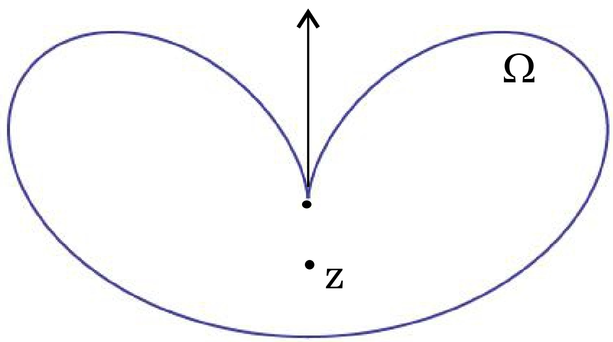


Figure 1: A set $\Omega$ that is strongly but not uniformly star-shaped w.r.t. $z$.

We now introduce the domains that will play a crucial role in our main result Theorem 1.1.

**Definition 2.9.** An open and bounded set $\Omega$ is called *locally strongly star-shaped* if for every $p \in \partial\Omega$, there exists an open set $H \subset \mathbb{R}^n$ such that $p \in H$ and $H \cap \Omega$ is strongly star-shaped.

We now present a large class of sets which are locally strongly star-shaped: Lipschitz domains belong to this class, but we may even allow cusps at some boundary points. First, we introduce some notation and assumption. Given $p = (p_1, \dots, p_n) \in \mathbb{R}^n$, let us write $p = (p', p_n)$ for $p' = (p_1, \dots, p_{n-1}) \in \mathbb{R}^{n-1}$, $p_n \in \mathbb{R}$. We assume that for each $p := (p', p_n) \in \partial\Omega$, there exist, upon rotating and relabeling the coordinate axes if necessary, $R > 0, a < 0 < p_n < b$ and a continuous function $\theta : \overline{B_R^{n-1}(p')} \to ]0, b[$ such that

$$\Omega \cap \left(\overline{B_R^{n-1}(p')} \times ]a, b[\right) = \{(y', y_n) : |y' - p'| \le R,\ a < y_n < \theta(y')\};$$

$$\Gamma \cap \left(\overline{B_R^{n-1}(p')} \times ]a, b[\right) = \{(y', y_n) :\ |y' - p'| \le R,\ y_n = \theta(y')\}.$$

**Example 2.10** (Locally strongly star-shaped domains)**.** Under the above assumptions on $\Omega$ and $p = (p', p_n)$, the following properties hold.

a) The set $\Omega \cap \left(B_R^{n-1}(p') \times ]a, b[\right)$ is strongly star-shaped with respect to $(p', 0) \in \Omega$ if and only if for all $\lambda \in ]0, 1[$ and $y' \in \overline{B_R^{n-1}(p')}$

$$\theta(\lambda y' + (1 - \lambda)p') > \lambda\theta(y'); \tag{2.8}$$

in particular, condition (2.8) is satisfied if either $\theta$ is concave or $\theta$ is non increasing on the segment $[p', y']$.

b) If $\theta$ is Lipschitz then there is an open neighborhood $H$ of $p$ which is contained in $B_R^{n-1}(p') \times ]a, b[$ and such that $\Omega \cap H$ is uniformly (whence strongly) star-shaped with respect to $(p', 0) \in \Omega$.

*Proof.* *a)* The set $H := \Omega \cap \left(B_R^{n-1}(p') \times ]a,b[\right)$ is strongly star-shaped w.r. to $(p',0)$ if and only if the relative interior of the segments joining $(p',0)$ to the points of

$$\partial H = \left(\partial\Omega \cap \left(\overline{B_R^{n-1}(p')} \times ]a,b[\right)\right) \bigcup \left(\Omega \cap \left(\partial B_R^{n-1}(p') \times ]a,b[\right)\right) \\ \bigcup \left(\Omega \cap \left(\overline{B_R^{n-1}(p')} \times \{a\}\right)\right)$$

is contained in $\Omega$. This occurs if and only if (2.8) holds for every $\lambda \in ]0,1[$ and $y' \in \overline{B_R^{n-1}(p')}$. If $\theta$ is concave on the segment $[p',y']$ for $\lambda \in ]0,1[$ we have

$$\theta(\lambda y' + (1-\lambda)p') \geq \lambda\theta(y') + (1-\lambda)\theta(p') > \lambda\theta(y');$$

whereas if $\theta$ is decreasing on the segment $[p',y']$ for $\lambda \in ]0,1[$ we have

$$\theta(\lambda y' + (1-\lambda)p') \geq \theta(y') > \lambda\theta(y'),$$

due to the fact that $\theta > 0$. In both cases (2.8) is fulfilled.

*b)* We now assume that $\theta$ is Lipschitz on $B_R^{n-1}$. It is not restrictive to take $p' = 0$. We claim that if $0 < r \leq R$ is small enough then the set $(B_r^{n-1} \times ]a,b[) \cap \Omega$ is star-shaped w.r. to to the points of a suitable ball that is centered at the origin. It is enough to prove that the relative interior of the segment joining each point of $\left(\overline{B_r^{n-1}} \times ]a,b[\right) \cap \Gamma$ to an open neighborhood of $0$ is contained in $(B_r^{n-1} \times ]a,b[) \cap \Omega$. Given $r \leq R$, let $(x',x_n)$ in $(B_r^{n-1} \times ]a,b[) \cap \Omega$ and $(y',\theta(y'))$ in $\left(\overline{B_r^{n-1}} \times ]a,b[\right) \cap \Gamma$. The open segment joining $(x',x_n)$ to $(y',\theta(y'))$ is contained in $(B_r^{n-1} \times ]a,b[) \cap \Omega$ if and only if

$$\forall \lambda \in ]0,1[ \qquad x_n + \lambda(\theta(y') - x_n) < \theta(x' + \lambda(y' - x')). \tag{2.9}$$

Let $k$ be the Lipschitz rank of $\theta$ on $B_R^{n-1}$. Since

$$\theta(x' + \lambda(y'-x')) \geq \theta(y') - k|y' - x' - \lambda(y'-x')| = \theta(y') - k|y'-x'|(1-\lambda),$$

it follows that (2.9) is satisfied whenever

$$x_n(1-\lambda) + k|y'-x'|(1-\lambda) < (1-\lambda)\theta(y'),$$

or equivalently,

$$x_n + k|y'-x'| < \theta(y') \quad \forall \lambda \in ]0,1[. \tag{2.10}$$

Since $\theta(0) = p_n > 0$, we can choose $r$ in order to satisfy the following condition:

$$\min\{\theta(y') : y' \in \overline{B_r^{n-1}}\} > p_n/2.$$

Then (2.10) is fulfilled if just $x_n + k|x'| + k|y'| < p_n/2$ and this occurs for instance if

$$x_n + k|x'| < p_n/4, \qquad |y'| < p_n/4k.$$

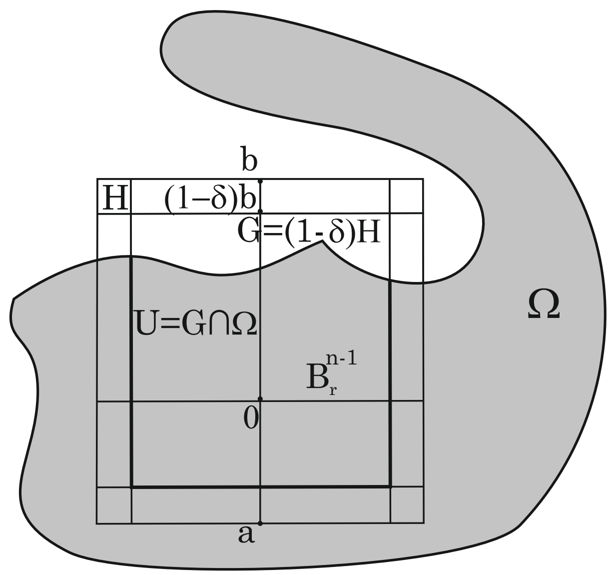


Figure 2: Proof of Proposition 2.11 iii).

Hence, if we further require that $r < p_n/4k$, then it follows that $(B_r^{n-1}\times]a,b[) \cap \Omega$ is star-shaped with respect to a ball centered at the origin; Proposition 2.5 implies that $(B_r^{n-1}\times]a,b[) \cap \Omega$ is uniformly star-shaped, henceforth strongly star-shaped, with respect to the origin. □

In the next result we prove that the boundary of a locally strongly star-shaped domain can be covered by a finite union of sets whose intersection with the domain is strongly star-shaped.

**Example 2.11.** Let $\Omega$ be a locally strongly star-shaped subset of $\mathbb{R}^n$. Then there exists a finite number of open sets $G_1, ..., G_N$ such that

i) $\Gamma \subset \bigcup_{j=1}^{N} G_j$;

ii) each of the sets $U_j := \Omega \cap G_j$, $(j = 1, ..., N)$ is strongly star-shaped with respect to a point $z_j \in U_j$;

iii) there exists $\delta > 0$ such that for every $h \in ]1-\delta, 1[$,

$$(z_j + h(-z_j + \Omega)) \cap U_j \subset\subset \Omega \quad j = 1, ..., N. \tag{2.11}$$

*Proof.* By a standard covering argument, we only need to prove that for every $p \in \Gamma$, there exists an open neighborhood $G$ of $p$ in $\mathbb{R}^n$ such that $U := \Omega \cap G$ is strongly star-shaped with respect to a point $z \in U$ and there is $\delta > 0$ such that for every $h \in ]1-\delta, 1[$,

$$(z + h(-z + \Omega)) \cap U \subset\subset \Omega. \tag{2.12}$$

By assumption, there exists an open neighborhood $H$ of $p$ such that $H \cap \Omega$ is strongly star-shaped with respect to a point $z$. The set $H$ contains a convex neighborhood $H_0$ of the segment $[z, p]$ and $H_0 \cap \Omega$ is strongly star-shaped with respect to $z$. We can thus assume from the beginning that $H$ itself is convex. Without loss of generality, we can also assume that $z = 0$. Hence

$$h(H \cap \Omega) \subset\subset H \cap \Omega \quad \forall h \in [0, 1[$$

so that

$$\operatorname{dist}(h(H \cap \Omega), \partial(H \cap \Omega)) > 0 \quad \forall h \in [0, 1[. \tag{2.13}$$

Let $\delta \in ]0, 1[$ be such that $p$ belongs to the set $G := (1-\delta)H \subset H$ and define $U := G \cap \Omega$. Since for every $h \in [0, 1[$, $hU \subset hG \subset\subset G$ and $hU \subset h(H \cap \Omega) \subset\subset H \cap \Omega$, we get

$$hU \subset\subset G \cap \Omega = U.$$

Hence, $U$ is strongly star-shaped with respect to $0$. We proceed to prove that

$$\operatorname{dist}((h\Omega) \cap U, \partial\Omega) > 0. \tag{2.14}$$

Since $U \subset \Omega \cap H$, $(h\Omega) \cap U \subset H \cap \Omega \subset \Omega$, for every $h \in [0, 1]$. This implies

$$\operatorname{dist}((h\Omega) \cap U, \partial\Omega) \geq \operatorname{dist}((h\Omega) \cap U, \partial(H \cap \Omega)). \tag{2.15}$$

Now $U$ is star-shaped with respect to the origin, so that for $h \in ]1-\delta, 1[$ we get

$$\frac{1}{h}U \subset \frac{1}{1-\delta}U \subset \frac{1}{1-\delta}G = H.$$

Hence $\Omega \cap \frac{1}{h}U \subset \Omega \cap H$, or equivalently

$$(h\Omega) \cap U \subset h(H \cap \Omega) \subset H \cap \Omega.$$

It then follows from (2.13) and (2.15) that

$$\operatorname{dist}((h\Omega) \cap U, \partial\Omega) \geq \operatorname{dist}((h\Omega) \cap U, \partial(H \cap \Omega)) \geq \operatorname{dist}(h(H \cap \Omega), \partial(H \cap \Omega)) > 0,$$

proving (2.14). $\square$

**Remark 2.12.** Claims i) and ii) of Proposition 2.11 extend [8, Proposition 2.5.4] where the authors prove the result for Lipschitz domains: Proposition 2.10 shows that we are able to deal with domains that are not Lipschitz. It is important to underline how claims

i) and ii) differ from other similar ones concerning covering of open sets with strongly star-shaped domains. It is well known [15, Lemma I.1] that an open set that has the cone property is a finite union of open subsets that are star-shaped with respect to a ball, and thus uniformly star-shaped thanks to Proposition 2.5. Here, for any point $p$ of the boundary of $\Omega$, we need a neighborhood of $p$ whose intersection with $\Omega$ is strongly star-shaped w.r.t one of its points: this forces $\Omega$ to be just on "one side" of the boundary.

Consider for instance the set

$$\Omega := B_1^2 \setminus \{(x_1, x_2) \in \mathbb{R}^2 : \, x_1 \geq 0, x_2 = 0\} :$$

it has the cone property and it is not locally strongly star-shaped around the points of the segment $L := \{(x, 0) : \, x \in [0, 1[\} \subset \partial\Omega$.

**Remark 2.13.** The fact that open sets with a Lipschitz boundary can be covered by open sets $G_1, ..., G_N$ satisfying ii) of Proposition 2.10 is a well established fact that can be found for instance in [8, Proposition 2.5.4]. The new fact here is that not only the latter property holds for a class of domains more general than Lipschitz, but that also one can choose the sets $G_1, ..., G_N$ of the covering in such a way that (2.11) holds. This property is a fundamental tool in the proof of Proposition 2.15 and the subsequent main Theorem 1.1: we are unable to find it explicitly elsewhere in the current literature.

In Example 2.14 below, we give an open covering $\{G_1, G_2\}$ of an open Lipschitz set $\Omega$ such that each $G_i \cap \Omega, 1 \leq i \leq 2$, is convex (thus uniformly star-shaped) but for which (2.11) of Proposition 2.11 does not hold.

**Example 2.14.** Let $\Omega = (]0, 2[\times]0, 4[) \cup (]0, 4[\times]0, 2[)$,

$$G_1 := \{(x, y) \in \mathbb{R}^2 : x < 2\}, \quad G_2 := \{(x, y) \in \mathbb{R}^2 : y < 2\}.$$

Let $z_1 := (1, 3)$ and $z_2 := (3, 1)$. Then $U_1 := G_1 \cap \Omega$ and $U_2 := G_2 \cap \Omega$ form an open covering of $\Omega$, they are both convex and thus uniformly star-shaped with respect to $z_1, z_2$ respectively. However, for every $h \in ]0, 1[$ sufficiently close to 1, the point $p = (2, 2)$ of $\partial\Omega$ belongs to the closure of both $(z_i + h(\Omega - z_1)) \cap U_i$, $i = 1, 2$.

The next result, that will be used in the proof of Theorem 1.1 explains the role played by the locally strongly star-shaped domains. It is an important tool that has an interest in itself and that allows to build, given a positive function $v \in W_0^{1,p}(\Omega)$, a nearby function in the $W^{1,p}$ topology that is equal to zero in a neighborhood of $\partial\Omega$. The crucial fact is that we do not use a partition of unity, which would require, in the proof of Theorem 1.1, growth assumptions on the Lagrangian $F$.

As usual every function in $W_0^{1,p}(\Omega)$ is implicitly extended by 0 out of $\Omega$.

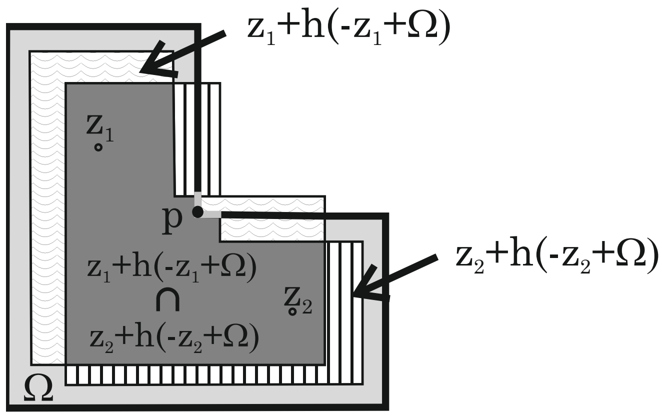


Figure 3: The set $\Omega$ and its dilations in Example 2.14

**Example 2.15.** Assume that $\Omega$ is locally strongly star-shaped and let $U_1, ..., U_N$, $z_1, ..., z_N$ be as in Proposition 2.11. Let $v \in W_0^{1,p}(\Omega)$, with $v \geq 0$ a.e. on $\Omega$. For every $h \in ]0, 1[$ and $j = 1, ..., N$ let

$$v_{h,j}(x) := v\left(z_j + \frac{x - z_j}{h}\right) \qquad v_h := \min\{v_{h,1}, ..., v_{h,N}\}.$$

Then $|v_h|_{L^\infty} \leq |v|_{L^\infty}$ and, for $h$ sufficiently close to 1, $v_h = 0$ in a neighborhood of $\partial\Omega$; moreover the functions $v_h$ tend to $v$ in $W^{1,p}(\Omega)$ as $h$ tends to 1.

*Proof.* For each $j = 1, ..., N$ the function $v_{h,j}$ converges to $v$ in $W^{1,p}$ as $h$ tends to 1, whence so does $v_h$. For $j = 1, ..., N$ the support of $v_{h,j}$ is contained in $z_j + h(-z_j + \overline{\Omega})$ and it follows from Proposition 2.11 that, for all $h \in ]1 - \delta, 1[$, there is $\varepsilon_h > 0$ satisfying

$$\text{dist}\left((z_j + h(-z_j + \Omega)) \cap U_j, \Gamma\right) > \varepsilon_h \quad j = 1, ..., N.$$

Thus,

$$\{x \in \Omega : \text{dist}(x, \Gamma) < \varepsilon_h\} \cap U_j \subset U_j \setminus \left(z_j + h(-z_j + \overline{\Omega})\right). \tag{2.16}$$

Now, since $\varepsilon_h \to 0$ as $h \to 1$, there is $\delta' \leq \delta$ such that

$$\{x \in \Omega : \text{dist}(x, \Gamma) < \varepsilon_h\} \subset U_1 \cup ... \cup U_N \quad \forall h > 1 - \delta'.$$

It follows from (2.16) that, for $h > 1 - \delta'$,

$$\{x \in \Omega : \text{dist}(x, \Gamma) < \varepsilon_h\} \subset \bigcup_{j=1}^{N} U_j \setminus \left(z_j + h_k(-z_j + \overline{\Omega})\right).$$

Since $v_h = 0$ whenever at least one of the $v_{h,j}$ equals 0, we deduce that, for $h > 1 - \delta'$, the function $v_h = 0$ a.e. on a neighborhood of $\partial\Omega$. $\square$

**Remark 2.16.** The construction of the function $v_h$ in Proposition 2.15 may fail to produce a function that equals 0 around $\partial\Omega$ if the covering $U_1, \dots, U_N$ of strongly star-shaped sets in Proposition 2.11 does not satisfy (2.11). For instance, consider again the domain

$$\Omega = (]0,2[\times]0,4[) \cup (]0,4[\times]0,2[,$$

the sets $U_i$ and the points $z_i$ of Example 2.14. Let $v \in \mathcal{C}^1(\overline{\Omega})$ with $v > 0$ on $\Omega$ and $v = 0$ on $\partial\Omega$. Then, for $h$ not too far from 1, both $v_{h,1}$ and $v_{h,2}$ are non zero around $p = (2,2)$, so that $v_h = \min\{v_{h,1}, v_{h,2}\}$ does not vanish on a neighborhood of $\partial\Omega$.

# 3 Two approximation lemma

As a first step, we prove that there is no Lavrentiev gap for a map $u \in W^{1,p}_\phi(\Omega)$ which is Lipschitz continuous on a neighborhood of $\Gamma$ and such that $F(u, \nabla u) \in L^1(\Omega)$.
We use the following notation: for every measurable subset $A$ of $\Omega$,

$$\forall u \in W^{1,p}_\phi(\Omega), \qquad I(A,u) := \int_A F(u(x), \nabla u(x))\,dx.$$

## 3.1 Preliminary results

We first prove that $I(W^{1,p}_\phi(\Omega) \cap L^\infty(\Omega))$ is dense in $I(W^{1,p}_\phi(\Omega))$. As a consequence, in order to deduce that there is no Lavrentiev gap for $I$ at $u \in W^{1,p}_\phi(\Omega)$, in what follows we will assume without restriction that $u$ is bounded. We have defined the space $W^{1,p}_\phi(\Omega)$ as the set of those functions $u \in W^{1,p}(\Omega)$ such that the extension of $u$ by $\phi$ on $\mathbb{R}^n \setminus \Omega$ belongs to $W^{1,p}_{loc}(\mathbb{R}^n)$. We still denote by $u$ this extension. In particular, $(u - \phi)$ belongs to $W^{1,p}(\mathbb{R}^n)$ and has compact support. We can apply the Sobolev embeddings on $\mathbb{R}^n$ to $u - \phi$ (no matter how regular $\Omega$ is). This implies that

- if $p > n$, then $u \in L^\infty(\Omega)$,
- if $p = n$, then $u \in L^q(\Omega)$ for every $q \in [1, \infty)$,
- if $p < n$, then $u \in L^{p^*}(\Omega)$ where $p^* = \frac{np}{n-p}$.

This explains why in the next lemma, we can assume without loss of generality that $p \leq n$.

**Lemma 3.1.** *Let $p \in [1,n]$ and $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy one of the following assumptions:*

*(A) $F$ is convex with respect to both variables,*

*(B) $F$ can be written in the following form $F(s,\xi) = a(s)g(\xi) + b(s)$ with $a : \mathbb{R} \to \mathbb{R}^+$, $b : \mathbb{R} \to \mathbb{R}$ continuous, $g : R^n \to \mathbb{R}$ convex and the function $c(s) = a(s)g(0) + b(s)$ satisfies the non oscillatory condition* (1.3).

*Then for every $u$ in $W^{1,p}_\phi(\Omega)$ such that $F(u,\nabla u)\in L^1(\Omega)$, there exists a sequence $(u_k)_k$ in $W^{1,p}_\phi(\Omega)\cap L^\infty(\Omega)$ such that $(u_k)_k$ converges to $u$ in $W^{1,p}_\phi(\Omega)$ and*

$$\lim_{k\to+\infty} I(u_k)=I(u).$$

*Proof.* Let $(\tau_k)_k$ and $(\sigma_k)_k$ be positive sequences diverging to $+\infty$. For $k$ large enough such that both $\tau_k>|\phi|_{L^\infty}$ and $\sigma_k>|\phi|_{L^\infty}$, we define $u_k$ by

$$u_k(x)=(u^+\wedge\tau_k)(x)-(u^-\wedge\sigma_k)(x)=\begin{cases} u(x) \text{ if } -\sigma_k\le u(x)\le\tau_k,\\ \tau_k \text{ if } u(x)\ge\tau_k,\\ -\sigma_k \text{ if } u(x)\le-\sigma_k.\end{cases}$$

It is clear that $u_k\in W^{1,p}_\phi(\Omega)\cap L^\infty(\Omega)$ and that $u_k$ converges to $u$ in $W^{1,p}_\phi(\Omega)$. We write that

$$I(u_k)=\int_{\{-\sigma_k\le u\le\tau_k\}}F(u,\nabla u)\,dx+\int_{\{u\ge\tau_k\}}F(\tau_k,0)\,dx+\int_{\{u\le-\sigma_k\}}F(-\sigma_k,0)\,dx. \quad (3.1)$$

We now prove that

$$\limsup_{k\to+\infty} I(u_k)\le I(u); \quad (3.2)$$

afterwards the weak lower semicontinuity of $I$ allows to conclude. We now consider two different cases, depending if $F$ satisfies (A) or (B).

*i) Assume that $F$ satisfies (A).* We set $\sigma_k=\tau_k=k$. Let $(q,\zeta)\in\partial F(k,0)$: since

$$F(u,\nabla u)\ge F(k,0)+q(u-k)+\zeta\cdot\nabla(u-k) \text{ a.e.},$$

we get

$$\begin{aligned}\int_{\{u\ge k\}}F(u,\nabla u)\,dx &\ge \int_{\{u\ge k\}}(F(k,0)-|q|u)\,dx+\int_{\{u\ge k\}}\zeta\cdot\nabla(u-k)\,dx\\ &\ge \int_{\{u\ge k\}}(F(k,0)-|q|u)\,dx+\int_\Omega\zeta\cdot\nabla(u-k)^+\,dx\\ &= \int_{\{u\ge k\}}(F(k,0)-|q|u)\,dx\end{aligned}$$

since $(u-k)^+\in W^{1,p}_0(\Omega)$. Analogously we get

$$\int_{\{u\le-k\}}F(u,\nabla u)\,dx\ge\int_{\{u\le-k\}}(F(-k,0)+|q|u)\,dx.$$

It follows from (3.1) that

$$\begin{aligned}I(u_k)\le&\int_{\{|u|\le k\}}F(u,\nabla u)\,dx+\int_{\{u\ge k\}}F(u,\nabla u)\,dx\\ &+\int_{\{u\le -k\}}F(u,\nabla u)\,dx+|q|\int_{\{|u|\ge k\}}|u|=I(u)+|q|\int_{\{|u|\ge k\}}|u|.\end{aligned}$$

Since $u \in L^1(\Omega)$, the dominated convergence theorem implies that

$$\limsup_{k\to+\infty} I(u_k) \leq I(u),$$

proving (3.2).

*ii) Assume that $F(s,\xi) = a(s)g(\xi) + b(s)$ satisfies (B).* We choose here $\tau_k$ and $\sigma_k$ as in (1.3). If $\zeta \in \partial g(0)$ then

$$g(\nabla u) \geq g(0) + \zeta \cdot \nabla u$$

so that, since $a \geq 0$,

$$F(u,\nabla u) = a(u)g(\nabla u) + b(u) \geq a(u)g(0) + b(u) + a(u)\zeta \cdot \nabla u = c(u) + a(u)\zeta \cdot \nabla u.$$

If $u(x) \geq \tau_k$, (1.3) implies

$$F(u(x),\nabla u(x)) \geq \frac{1}{C_1} c(\tau_k) - \frac{D}{C_1}|u(x)|^{p^*} + a(u(x))\zeta \cdot \nabla u(x) \tag{3.3}$$

or equivalently

$$F(\tau_k,0) = c(\tau_k) \leq C_1 \left( F(u(x),\nabla u(x)) + \frac{D}{C_1}|u(x)|^{p^*} - a(u(x))\zeta \cdot \nabla u(x) \right). \tag{3.4}$$

By the Sobolev embeddings, $u \in L^{p^*}(\Omega)$. By assumption, $F(u,\nabla u) \in L^1(\Omega)$. We thus have

$$\limsup_{k\to+\infty} \int_{\{u\geq\tau_k\}} F(u,\nabla u) + \frac{D}{C_1}|u|^{p^*} = 0.$$

Notice that, if $v \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, then

$$a(v)\zeta \cdot \nabla v = \nabla A(v),$$

where $A(s) = \displaystyle\int_0^s a(t)\,dt$. Now, for every $k \in \mathbb{N}$, for every $v \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ such that $v \leq \tau_k$ on $\Gamma$, i.e. $(v-\tau_k)^+ \in W_0^{1,1}(\Omega)$, we have $A(\max(v,\tau_k)) - A(\tau_k) \in W_0^{1,1}(\Omega)$ and thus

$$\begin{aligned}\int_{\{v\geq\tau_k\}} a(v)\zeta \cdot \nabla v &= \int_{\{v\geq\tau_k\}} \zeta \cdot \nabla[A(v)] \\ &= \int_\Omega \zeta \cdot \nabla[A(\max(v,\tau_k)) - A(\tau_k)] = 0.\end{aligned} \tag{3.5}$$

For a fixed $k$, by applying (3.5) to the maps $v_i = \min(u^+, i)$ for every $i$ in $\mathbb{N}$ with $i \geq \tau_k$, we get

$$\begin{aligned}0 &= \int_{\{v_i\geq\tau_k\}} a(v_i)\zeta \cdot \nabla v_i \\ &= \int_{\{i\geq u^+\geq\tau_k\}} a(u^+)\zeta \cdot \nabla u^+ = \int_{\{i\geq u\geq\tau_k\}} a(u)\zeta \cdot \nabla u.\end{aligned}$$

By (3.3), for every $i \geq \tau_k$:

$$\chi_{\{i \geq u \geq \tau_k\}} a(u)\zeta \cdot \nabla u \leq \max\left(0, F(u, \nabla u) + \frac{D}{C_1}|u|^{p^*} + \frac{|c(\tau_k)|}{C_1}\right).$$

Since the map in the right hand side is summable, we can use Fatou Lemma to get that

$$0 = \limsup_{i \to +\infty} \int_{\{i \geq u \geq \tau_k\}} a(u)\zeta \cdot \nabla u \leq \int_{\{u \geq \tau_k\}} a(u)\zeta \cdot \nabla u.$$

In view of (3.4), this gives

$$\int_{\{u \geq \tau_k\}} F(\tau_k, 0) \leq C_1 \int_{\{u \geq \tau_k\}} F(u, \nabla u) + \frac{D}{C_1}|u|^{p^*}.$$

It follows that

$$\limsup_{k \to +\infty} \int_{\{u \geq \tau_k\}} F(\tau_k, 0) \leq 0.$$

Similarly,

$$\limsup_{k \to +\infty} \int_{\{u \leq -\sigma_k\}} F(-\sigma_k, 0) \leq 0$$

and (3.2) now follows from (3.1).

□

We now prove that there is no Lavrentiev gap at $u \in W^{1,p}_\phi(\Omega)$ if $u$ is Lipschitz continuous on a neighborhood of $\Gamma$.

**Lemma 3.2.** *Assume that $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is convex with respect to both variables. Let $u$ in $W^{1,p}_\phi(\Omega)$ be such that $F(u, \nabla u) \in L^1(\Omega)$. If $u$ is Lipschitz continuous on a neighborhood of $\Gamma$, then there exists a sequence $(u_k)_k$ in $\mathrm{Lip}_\phi(\Omega)$ such that $(u_k)_k$ converges to $u$ in $W^{1,p}_\phi(\Omega)$ and*

$$\lim_{k \to +\infty} I(u_k) = I(u).$$

*Moreover, if $u$ is bounded in $L^\infty(\Omega)$, then the sequence $(u_k)_k$ may be taken to be bounded in $L^\infty(\Omega)$.*

*Proof.* From Lemma 3.1 it is not restrictive to assume that $u$ is bounded. We may consider $u$ as extended by $\phi$ out of $\Omega$. By assumption, there exists an open set $V \subset \mathbb{R}^n$ such that $\Gamma \subset V$ and $u$ is Lipschitz continuous on $V \cap \Omega$. In particular $u$ and $\nabla u$ are in $L^\infty(V \cap \Omega)$. Let $\rho \in C^\infty_c(B_1, \mathbb{R}^+)$, $\int_{\mathbb{R}^n} \rho \, dx = 1$ and for $k = 1, 2, ...$, $(\rho_k)_k$ be the sequence of mollifiers defined by $\rho_k(x) := k^n \rho(kx)$. Let also $\theta \in C^\infty_c(\Omega, [0, 1])$ be such that $\theta = 1$ on a neighborhood of $\Omega \setminus V$. We then define

$$u_k = \theta(u * \rho_k) + (1 - \theta)u.$$

Clearly, $u_k \in \mathrm{Lip}_\phi(\Omega)$ and $(u_k)_k$ converges to $u$ in $W^{1,p}_\phi(\Omega)$. This implies

$$\liminf_{k\to+\infty} I(u_k) \geq I(u).$$

It remains to show that

$$\limsup_{k\to+\infty} I(u_k) \leq I(u). \tag{3.6}$$

For this purpose, we decompose $I(u_k) = \displaystyle\int_\Omega F(u_k, \nabla u_k)\, dx$ as the sum

$$I(u_k) = \int_{\{\theta=1\}} F(u_k, \nabla u_k)\, dx + \int_{\{\theta=0\}} F(u, \nabla u)\, dx + \int_{\{0<\theta<1\}} F(u_k, \nabla u_k)\, dx. \tag{3.7}$$

On the set $\{0<\theta<1\} \subset V\cap\Omega$, $\nabla u \in L^\infty(V\cap\Omega)$ and

$$u_k = \theta(u * \rho_k) + (1-\theta)u,$$

$$\nabla u_k = \theta(\nabla u * \rho_k) + (1-\theta)\nabla u + \nabla\theta(u*\rho_k - u).$$

We have $\nabla\theta(u*\rho_k) = 0$ out of $V$. Let $\overline{k}$ be such that

$$\forall k \geq \overline{k} \qquad \{0<\theta<1\} + B_{1/k} \subset V\cap\Omega.$$

Then, for $k \geq \overline{k}$ and $x \in \{0<\theta<1\}$ we have

$$|u_k(x)| \leq 2|u|_{L^\infty(V\cap\Omega)};$$

$$|\nabla u_k(x)| \leq 2|\nabla u|_{L^\infty(V\cap\Omega)} + 2|\nabla\theta|_{L^\infty(V\cap\Omega)}|u|_{L^\infty(V\cap\Omega)}$$

which in turn means that under the above assumptions both $u_k$ and $\nabla u_k$ are bounded by a constant that does not depend on $k$. Since $(u_k)_k$ converges to $u$ in $W^{1,p}(\Omega)$ we may assume, by taking a subsequence, that $(u_k, \nabla u_k)_k$ converges a.e. to $(u, \nabla u)$. Now, since $F$ is bounded on bounded sets, by Lebesgue's Theorem we have

$$\lim_{k\to+\infty} \int_{\{0<\theta<1\}} F(u_k, \nabla u_k)\, dx = \int_{\{0<\theta<1\}} F(u, \nabla u)\, dx. \tag{3.8}$$

On the set $\{\theta = 1\}$ we have

$$u_k = u * \rho_k, \quad \nabla u_k = \nabla u * \rho_k.$$

It remains to show that

$$\limsup_{k\to 0} \int_{\{\theta=1\}} F(u*\rho_k, \nabla u * \rho_k) \leq \int_{\{\theta=1\}} F(u, \nabla u) : \tag{3.9}$$

afterwards, in view of (3.7) and (3.8), we get (3.6).

By Jensen's inequality,

$$F(u*\rho_k, \nabla u*\rho_k) \leq F(u, \nabla u) * \rho_k.$$

Whence

$$\int_{\{\theta=1\}} F(u*\rho_k, \nabla u*\rho_k) \leq \int_{\{\theta=1\}} F(u,\nabla u) * \rho_k.$$

Since $F(u, \nabla u) \in L^1(\Omega)$, we get (3.9). □

# 4 the convex case

In this section, we prove Theorem 1.1. We first present a proof of a simplified version of Theorem 1.1 where $\Omega$ is assumed to be uniformly star-shaped and $F$ depends only on the gradient. We think that it enlightens one of the main ideas of the proof of Theorem 1.1, which is given in the last part of this section.

## 4.1 The case of a uniformly star-shaped domain

Here, we establish a version of Theorem 1.1 under restrictive assumptions : $F$ depends only on the gradient and $\Omega$ is uniformly star-shaped. Our aim is both to provide an elementary proof of the main results in a special case, and to give a flavour of some ideas that are developed more thoroughly in the sequel. The good approximating sequence to a given function $u \in W_{\phi}^{1,1}(\Omega)$ is obtained here by first retracting $u$ inside $\Omega$ and then by "glueing" it with a function that equals $\phi$ on the boundary and that is Lipschitz in a neighborhood of $\partial\Omega$.

Actually, the main tool is Jensen's inequality in connection with an argument of approximation by convolution. This idea was already used in [5, Remark 3.6] to prove the non occurence of the Lavrentiev phenomenon for a Lagrangian $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ which is convex with respect to both variables, on a star-shaped domain, and when the boundary datum is ignored.

**Theorem 4.1** (Convex Lagrangians)**.** *Assume that $\Omega$ is uniformly star-shaped and that $F : \mathbb{R}^n \to \mathbb{R}$ is convex. Let $u \in W_{\phi}^{1,p}(\Omega)$ be such that $F(\nabla u) \in L^1(\Omega)$. Then the Lavrentiev gap for $I$ does not occur at $u$, i.e. there exists a sequence $(u_k)_k$ in $\mathrm{Lip}_{\phi}(\Omega)$ converging to $u$ in $W^{1,p}(\Omega)$ and such that*

$$\lim_{k \to +\infty} I(u_k) = I(u). \tag{4.1}$$

*Moreover, if $u$ is bounded in $L^{\infty}(\Omega)$, the sequence $(u_k)_k$ may be taken to be bounded in $L^{\infty}(\Omega)$.*

**Remark 4.2.** The proof of Theorem 4.1 could be easily adapted to the case of a Lagrangian $F$ which is convex in both variables. Still, if one simply repeats the same arguments as in the proof below for a Lagrangian $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ convex in $(s, \xi)$, one will require a further assumption, namely $F(0, \nabla u) \in L^1(\Omega)$. It is interesting to note that in [5] instead, the authors subsume that $F(u, 0) \in L^1(\Omega)$, a fact that is not a consequence of their assumptions as it is shown in Example 4.3. A new idea will be required in the proof of Theorem 1.1 to avoid any technical assumption of this type.

The following example shows that the condition $F(u, 0) \in L^1(\Omega)$ does not follow in general from the convexity of $F$ and the property that $F(u, \nabla u)$ is in $L^1(\Omega)$.

**Example 4.3.** Let $g:\mathbb{R}\to\mathbb{R}$ be the function defined by

$$g(t)=\begin{cases} t^2 & \text{if } t\geq 0,\\ 0 & \text{if } t<0. \end{cases} \tag{4.2}$$

We consider the function defined in $\mathbb{R}\times\mathbb{R}^2$ by $F(u,(\xi_1,\xi_2)) = g\left(\xi_1+\frac{1}{2}|u|^3\right)$. $F$ is convex in both variables $(u,\xi)$, where $\xi=(\xi_1,\xi_2)$. We assume that

$$\Omega=\{x=(x_1,x_2)\in\mathbb{R}^2 : 0<|x_2|<x_1<1\}.$$

The function $u(x)=|x_1|^{-\frac{1}{2}}$ belongs to the space $W^{1,1}(\Omega)$. Moreover, $F(u(x),\nabla u(x))\equiv 0$ in $\Omega$ and $F(u(x),0)=\frac{1}{4}|x_1|^{-3}$, so that $F(u(x),0)$ is not summable. The example can be easily modified in order to get a coercive Lagrangian by adding the term $|t|^p$ to $g$, for any $1<p<\frac{4}{3}$: indeed $u\in W^{1,p}$ for every such $p$.

*Proof of Theorem 4.1.* By Lemma 3.1 it is not restrictive to assume that $u\in L^\infty(\Omega)$. In view of Lemma 3.2 it is enough to show that there exists a sequence $(u_j)_{j\geq 1}$ in $W^{1,p}_\phi(\Omega)$ such that each $u_j$ is Lipschitz continuous on a neighborhood of $\Gamma$ and $(F(\nabla u_j))_{j\geq1}$ converges to $F(\nabla u)$ in $L^1(\Omega)$. Without loss of generality, we can assume that $\Omega$ is uniformly star-shaped with respect to 0. For $h\in]\frac{1}{2},1[$, we consider the map

$$\phi_h(x):=\begin{cases} \phi(x) \text{ when } x\in\Gamma,\\ h\phi\left(\dfrac{x}{h}\right) \text{ when } x\in h\Gamma. \end{cases}$$

We claim that $\phi_h$ is Lipschitz continuous, and that its rank does not depend on $h$. Indeed, it follows from Definition 2.2 that there exists $C>0$ such that

$$\text{dist }(\Gamma,h\Gamma)\geq C(1-h).$$

Now, let $x\in h\Gamma$, $y\in\Gamma$. Hence, $|1-h|\leq\frac{1}{C}|x-y|$. Denoting by $L$ the Lipschitz rank of $\phi$, we get

$$\begin{aligned}
|\phi_h(x)-\phi_h(y)| = \left|h\phi\left(\frac{x}{h}\right)-\phi(y)\right| &\leq \left|h\phi\left(\frac{x}{h}\right)-\phi\left(\frac{x}{h}\right)\right| + \left|\phi\left(\frac{x}{h}\right)-\phi(y)\right|\\
&\leq |\phi|_{L^\infty(\Gamma)}\,|1-h| + L\left|\frac{x}{h}-y\right|\\
&\leq |\phi|_{L^\infty(\Gamma)}\,|1-h| + 2L\,|x-hy|\\
&\leq |\phi|_{L^\infty(\Gamma)}\,|1-h| + 2L|x||1-h| + 2Lh|x-y| \leq K|x-y|
\end{aligned}$$

where $K$ depends only on $\phi$ and $\Omega$. Hence, $\phi_h$ has a Lipschitz continuous extension of rank $K$ on $\mathbb{R}^n$ that we still denote by $\phi_h$.
We now introduce the map

$$u_h(x):=\begin{cases} hu\,(x/h) \text{ when } x\in h\Omega,\\ \phi_h(x) \text{ when } x\in\Omega\setminus h\Omega. \end{cases}$$

Clearly $u_h$ tends to $u$ in $W^{1,p}(\Omega)$ as $h \to 1$. Moreover $u_h(x) = \phi_h(x)$ in a neighborhood of $\Gamma$. This implies that $u_h$ is Lipschitz continuous on a neighborhood of $\Gamma$ and that $u_h = \phi$ on $\Gamma$. Moreover,

$$\int_\Omega F(\nabla u_h(x))\, dx = X_h + Y_h,$$

where

$$X_h := \int_{\Omega \setminus h\Omega} F\left(\nabla \phi_h(x)\right)\, dx \quad , \quad Y_h := \int_{h\Omega} F\left(\nabla u\left(\frac{x}{h}\right)\right)\, dx.$$

We have

$$|X_h| \leq |\Omega \setminus h\Omega| \sup_{|\xi| \leq K} |F(\xi)|,$$

which goes to $0$ when $h \to 1$. Moreover

$$Y_h = \int_{h\Omega} F\left(\nabla u\left(\frac{x}{h}\right)\right)\, dx = h^n \int_\Omega F(\nabla u(x))\, dx.$$

Hence $Y_h \to I(u)$ as $h \to 1$. This completes the proof of Theorem 4.1. □

## 4.2 Proof of Theorem 1.1

The following technical result is one of the tools that allows us to avoid growth conditions in Theorem 1.1 and Theorem 1.3.

**Lemma 4.4.** *Let $g \in L^1(\mathbb{R}^n)$ and $N \geq 1$ in $\mathbb{N}$. Assume that for every $k$ in $\mathbb{N}$ there are:*

1. *$N$ points $x_{k,1}, ..., x_{k,N}$ in $\mathbb{R}^n$ with $\lim_{k \to +\infty} x_{k,j} = 0$, $j = 1, ..., N$;*
2. *a measurable partition $E_{k,1}, ..., E_{k,N}$ of $\mathbb{R}^n$.*

*Then, for every increasing sequence of positive numbers $(h_k)_k$ converging to 1 we have*

$$\lim_{k \to +\infty} \sum_{j=1}^N \int_{E_{k,j}} g\left(\frac{x - x_{k,j}}{h_k}\right)\, dx = \int_{\mathbb{R}^n} g(x)\, dx.$$

*Proof.* Assume first that $g$ is of class $\mathcal{C}^1$ and with compact support, say contained in $B_R$. Now

$$\sum_{j=1}^N \int_{E_{k,j}} g\left(\frac{x - x_{k,j}}{h_k}\right)\, dx - \int_{\mathbb{R}^n} g(x)\, dx = \sum_{j=1}^N \int_{E_{k,j}} g\left(\frac{x - x_{k,j}}{h_k}\right) - g(x)\, dx.$$

Let $k$ be large enough in such a way that $|x_{j,k}| \leq 1$ for $j = 1, ..., N$. If $x \notin B_{R+1}$ the integrands in the above formula are equal to 0; otherwise for each $j = 1, ..., N$ we have

$$\left|\frac{x - x_{k,j}}{h_k} - x\right| = \frac{1}{h_k}|(1 - h_k)x - x_{k,j}| \leq \frac{1}{h_k}((R + 1)(1 - h_k) + |x_{k,j}|),$$

so that $g\left(\dfrac{x-x_{k,j}}{h_k}\right)$ converges uniformly to $g(x)$ as $k\to+\infty$: the conclusion follows immediately.
In the general case fix $\varepsilon>0$; let $h$ be $\mathcal{C}^1$ with compact support and such that

$$\int_{\mathbb{R}^n}|g-h|(x)\,dx\le\varepsilon.$$

We have

$$\left|\sum_{j=1}^N\int_{E_{k,j}}g\left(\frac{x-x_{k,j}}{h_k}\right)\,dx-\int_{\mathbb{R}^n}g(x)\,dx\right|\le X_k+Y_k+Z_k$$

where

$$\begin{aligned}X_k=\sum_{j=1}^N\int_{E_{k,j}}|g-h|\left(\frac{x-x_{k,j}}{h_k}\right)\,dx&=\sum_{j=1}^N h_k^n\int_{\frac{E_{k,j}-x_{k,j}}{h_k}}|g-h|(y)\,dy\\&\le\sum_{j=1}^N\int_{\mathbb{R}^n}|g-h|(y)\,dy;\end{aligned}$$

$$Y_k=\left|\sum_{j=1}^N\int_{E_{k,j}}h\left(\frac{x-x_{k,j}}{h_k}\right)\,dx-\int_{\mathbb{R}^n}h(x)\,dx\right|;$$

$$Z_k=\int_{\mathbb{R}^n}|h-g|(x)\,dx.$$

Now $X_k\le N\varepsilon$ and $Z_k\le\varepsilon$; moreover due to the fact that $h$ is continuous with compact support we get that $Y_k\to 0$ as $k\to+\infty$. Thus

$$\left|\sum_{j=1}^N\int_{E_{k,j}}g\left(\frac{x-x_{k,j}}{h_k}\right)\,dx-\int_{\mathbb{R}^n}g(x)\,dx\right|\le(N+1)\varepsilon+o(1),\quad k\to+\infty.$$

The conclusion follows, $\varepsilon$ being arbitrary. $\square$

*Proof of Theorem 1.1.* Without loss of generality, we can assume that $F\ge 0$. Indeed, since $F$ is convex with respect to both variables, if $(\overline{q},\overline{\zeta})\in\partial F(0,0)$ then

$$G(s,\xi):=F(s,\xi)-\overline{q}s-\overline{\zeta}\cdot\xi-F(0,0)\ge 0.$$

Moreover additive affine terms do not perturb our convergence results: if a sequence $(u_k)_k$ converges to $u$ in $W^{1,p}(\Omega)$ then $(I(u_k))_k$ converges to $I(u)$ if and only if $\int_\Omega G(u_k,\nabla u_k)\,dx$ converges to $\int_\Omega G(u,\nabla u)\,dx$.

In view of Lemma 3.2 it is enough to provide a sequence $(u_k)_k$ in $W^{1,p}_\phi(\Omega)$ satisfying the conditions of the claim with the exception that it is just Lipschitz continuous in a neighborhood of $\Gamma$ (instead of Lipschitz on $\Omega$). We may consider $u$ to be extended by $\phi$ out of $\Omega$. Also, in view of Lemma 3.1, it is not restrictive to assume that $u$ is bounded. Let $U_1, ..., U_N$ and $z_1, ..., z_N$ be as in Proposition 2.11. Let $(u-\phi)^+$ and $(u-\phi)^-$ be, respectively, the positive and negative part of $u-\phi$: clearly $u-\phi = (u-\phi)^+ - (u-\phi)^-$. For every $k \in \mathbb{N}$ let $h_k = 1 - 1/k$; for $j = 1, ..., N$ we define

$$T_{k,j}(x) := z_j + \frac{x - z_j}{h_k} \tag{4.3}$$

$$v_{k,j}(x) := (u-\phi)^+ (T_{k,j}(x)) \quad j = 1, ..., N;$$
$$v_k := \min\{v_{k,1}, ..., v_{k,N}\}. \tag{4.4}$$

Analogously we define the sequence $w_k \in W_0^{1,p}(\Omega)$ by

$$w_{k,j}(x) := (u-\phi)^- (T_{k,j}(x)) \quad j = 1, ..., N;$$

$$w_k := \min\{w_{k,1}, ..., w_{k,N}\}. \tag{4.5}$$

It follows from Proposition 2.15 that, for $k$ large enough, $v_k = w_k = 0$ a.e. in a neighborhood of $\Gamma$; moreover $(v_k)_k$ converges to $(u-\phi)^+$ and $(w_k)_k$ converges to $(u-\phi)^-$ in $W^{1,p}(\Omega)$.
We define, for every $\lambda \in (0,1)$ and every $k \in \mathbb{N}$, $u_k^\lambda$ by

$$u_k^\lambda := \phi + \lambda h_k (v_k - w_k). \tag{4.6}$$

Then, for $k$ large enough, $u_k^\lambda \in W_\phi^{1,p}(\Omega)$ and $u_k^\lambda = \phi$ in a neighborhood of $\Gamma$. Moreover $u_k^\lambda$ converges to $\phi + \lambda(u-\phi)^+ - \lambda(u-\phi)^- = \lambda u + (1-\lambda)\phi$ in $W^{1,p}(\Omega)$.
For each $k$ let $A_k$ and $B_k$ be the two essentially disjoint subsets of $\Omega$ defined by

$$A_k = \{x : v_k(x) > 0\}, B_k = \{x \in \Omega : w_k(x) > 0\}.$$

We now compute the integral

$$I(u_k^\lambda) = \int_{\Omega \setminus A_k \cup B_k} F(\phi, \nabla\phi)\, dx + I(A_k, \phi + \lambda h_k v_k) + I(B_k, \phi - \lambda h_k w_k).$$

Now, since $v_k = 0$ out of $A_k$ and $w_k = 0$ out of $B_k$ we have

$$I(A_k, \phi + \lambda h_k v_k) = I(\phi + \lambda h_k v_k) - \int_{\Omega \setminus A_k} F(\phi, \nabla\phi)\, dx,$$

$$I(B_k, \phi - \lambda h_k w_k) = I(\phi - \lambda h_k w_k) - \int_{\Omega \setminus B_k} F(\phi, \nabla\phi)\, dx,$$

so that

$$I(u_k^\lambda) = I_k + J_k - \int_\Omega F(\phi, \nabla\phi)\, dx \tag{4.7}$$

where we set

$$I_k := \int_\Omega F(\phi + \lambda h_k v_k, \nabla\phi + \lambda h_k \nabla v_k)\, dx, \tag{4.8}$$

$$J_k := \int_\Omega F(\phi - \lambda h_k w_k, \nabla\phi - \lambda h_k \nabla w_k)\, dx. \tag{4.9}$$

By the subsequent Lemma 4.5 we get

$$\limsup_{k\to+\infty} I_k \le \lambda I\left(\max(u,\phi)\right) + N(1-\lambda)I(\phi),$$

$$\limsup_{k\to+\infty} J_k \le \lambda I\left(\min(u,\phi)\right)\, dx + N(1-\lambda)I(\phi).$$

Thus (4.7) yields

$$\begin{aligned}
\limsup_{k\to+\infty} I(u_k^\lambda) &\le \limsup_{k\to+\infty} I_k + \limsup_{k\to+\infty} J_k - I(\phi) \\
&\le \lambda\left(I\left(\max(u,\phi)\right) + I\left(\min(u,\phi)\right)\right) + 2N(1-\lambda)I(\phi) - I(\phi) \\
&\le \lambda\left(I(u) + I(\phi)\right) + 2N(1-\lambda)I(\phi) - I(\phi) \\
&\le \lambda I(u) + (1-\lambda)(2N-1)I(\phi).
\end{aligned}$$

Since the right-hand side term of the latter inequality tends to $I(u)$ as $\lambda$ tends to 1 then for every $i \in \mathbb{N}$, $i \ge 1$, there are $\lambda_i$ and $k_i \in \mathbb{N}$ with $k_i \ge i$ such that

$$I(u_{k_i}^{\lambda_i}) \le I(u) + \frac{1}{i};$$

in particular we get

$$\limsup_{i\to+\infty} I(u_{k_i}^{\lambda_i}) \le I(u). \tag{4.10}$$

On the other hand, since $\left(u_{k_i}^{\lambda_i}\right)_{i\ge1}$ converges to $u$ in $W^{1,p}(\Omega)$, the weak lower semicontinuity of $I$ yields

$$\liminf_{i\to+\infty} I(u_{k_i}^{\lambda_i})\, dx \ge I(u),$$

which, together with (4.10), gives

$$\lim_{i\to+\infty} I(u_{k_i}^{\lambda_i}) = I(u).$$

To prove the last part of the claim of Theorem 1.1 it is enough to remark that, thanks to Proposition 2.15,

$$\begin{aligned}
|u_k^\lambda|_{L^\infty} &\le |\phi|_{L^\infty} + |v_k|_{L^\infty} + |w_k|_{L^\infty} \\
&\le |\phi|_{L^\infty} + |(u-\phi)^+|_{L^\infty} + |(u-\phi)^-|_{L^\infty} \le 3|\phi|_{L^\infty} + 2|u|_{L^\infty},
\end{aligned}$$

This completes the proof of the theorem. □

**Lemma 4.5.** *Let, for $k \in \mathbb{N}$, $I_k$ and $J_k$ be defined as in (4.8) and (4.9). For every $\lambda \in ]0, 1[$ the following estimates hold:*

$$\limsup_{k\to+\infty} I_k \le \lambda I\left(\max(u,\phi)\right) + N(1-\lambda)I(\phi); \tag{4.11}$$

$$\limsup_{k\to+\infty} J_k \le \lambda I\left(\min(u,\phi)\right)\, dx + N(1-\lambda)I(\phi). \tag{4.12}$$

*Proof.* We only prove (4.11), the proof of (4.12) being similar. For every $k$ there is a measurable partition $\{A_{k,1}, ..., A_{k,N}\}$ of $\Omega$ such that

$$v_k = v_{k,j} \text{ on } A_{k,j}, \quad j = 1, ..., N,$$

so that $I_k$ can be rewritten as

$$I_k = \sum_{j=1}^{N} \int_{A_{k,j}} F(\phi + \lambda h_k v_{k,j}, \nabla\phi + \lambda h_k \nabla v_{k,j})\, dx.$$

By means of the equality

$$(u-\phi)^+ = -\phi + \max(u,\phi),$$

for each $j = 1, ..., N$ we may write

$$\begin{aligned} \phi + \lambda h_k v_{k,j} &= \phi + \lambda h_k (u-\phi)^+ \left(T_{k,j}\right) \\ &= \lambda h_k \max(u,\phi)\left(T_{k,j}\right) + \phi - \lambda h_k \phi\left(T_{k,j}\right); \end{aligned} \tag{4.13}$$

we recall that, for $k \in \mathbb{N}$ and $j = 1, ..., N$, the functions $T_{k,j}$ are the dilations defined in (4.3). For $k$ and $j$ fixed we now proceed to write both $\phi + \lambda h_k v_{k,j}$ and its gradient as a suitable convex convex combination, with coefficients $\lambda$ and $1-\lambda$:

$$\begin{aligned} \phi + \lambda h_k v_{k,j} &= \lambda h_k \max(u,\phi)\left(T_{k,j}\right) + \phi - \lambda h_k \phi\left(T_{k,j}\right) = \\ &= \lambda \max(u,\phi)\left(T_{k,j}\right) + \lambda(h_k - 1)\max(u,\phi)\left(T_{k,j}\right) + \phi - \lambda h_k \phi\left(T_{k,j}\right) \\ &= \lambda \max(u,\phi)\left(T_{k,j}\right) + (1-\lambda)\left(\frac{\lambda(h_k-1)\max(u,\phi)\left(T_{k,j}\right) + \phi - \lambda h_k \phi\left(T_{k,j}\right)}{1-\lambda}\right) \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \nabla\phi + \lambda h_k \nabla v_{k,j} &= \lambda \nabla \max(u,\phi)\left(T_{k,j}\right) + \nabla\phi - \lambda\nabla\phi\left(T_{k,j}\right) \\ &= \lambda \nabla \max(u,\phi)\left(T_{k,j}\right) + (1-\lambda)\left(\frac{\nabla\phi - \lambda\nabla\phi\left(T_{k,j}\right)}{1-\lambda}\right). \end{aligned} \tag{4.15}$$

Let $k \in \mathbb{N}$ and $j = 1, ..., N$. By means of (4.14) and (4.15) the convexity of $F$ yields the following estimate:

$$F(\phi + \lambda h_k v_{k,j}, \nabla\phi + \lambda h_k \nabla v_{k,j}) \leq \lambda F\left(\max(u,\phi)\left(T_{k,j}\right), \nabla \max(u,\phi)\left(T_{k,j}\right)\right) + (1-\lambda) F\left(s^\lambda_{k,j}, \xi^\lambda_{k,j}\right),$$

where we set

$$s^\lambda_{k,j} := \frac{\lambda(h_k - 1)\max(u,\phi)\left(T_{k,j}\right) + \phi - \lambda h_k \phi\left(T_{k,j}\right)}{1-\lambda},$$

$$\xi^\lambda_{k,j} := \frac{\nabla\phi - \lambda\nabla\phi\left(T_{k,j}\right)}{1-\lambda}.$$

It follows that

$$I_k \leq \lambda \sum_{j=1}^N \int_{A_{k,j}} F\left(\max(u,\phi)\left(T_{k,j}\right), \nabla\max(u,\phi)\left(T_{k,j}\right)\right)\, dx + \\ + (1-\lambda)\sum_{j=1}^N \int_{A_{k,j}} F\left(s^\lambda_{k,j}, \xi^\lambda_{k,j}\right)\, dx. \quad (4.16)$$

The sequences $(\max(u,\phi)(T_{k,j}))_k$ and $(\nabla\phi(T_{k,j}))_k$ converges in $L^1(\Omega)$ as $k \to +\infty$ to $\max(u,\phi)$ and $\nabla\phi$ respectively. Hence, modulo a subsequence, we may assume that $(\max(u,\phi)(T_{k,j}))_k$ and $(\nabla\phi(T_{k,j}))_k$ converges to $\max(u,\phi)$ and $\nabla\phi$ a.e. This implies that $s^\lambda_{k,j}$ converges to $\phi$ a.e. as $k \to +\infty$ while $\xi^\lambda_{k,j}$ converges to $\nabla\phi$ a.e.

Moreover, for all $k \in \mathbb{N}$ we have

$$|s^\lambda_{k,j}| \leq \frac{3}{1-\lambda}\max\{\|u\|_{L^\infty}, \|\phi\|_{L^\infty}\} \text{ a.e.}$$

and since $\phi$ is Lipschitz continuous, for every $k \in \mathbb{N}$, we have

$$|\xi^\lambda_{k,j}| \leq \frac{2}{1-\lambda}\|\nabla\phi\|_{L^\infty} \text{ a.e..}$$

By the dominated convergence theorem, we get

$$\lim_{k\to+\infty} \int_\Omega F\left(s^\lambda_{k,j}, \xi^\lambda_{k,j}\right)\, dx = I(\phi).$$

The fact that $F \geq 0$ yields

$$\limsup_{k\to+\infty} \sum_{j=1}^N \int_{A_{k,j}} F\left(s^\lambda_{k,j}, \xi^\lambda_{k,j}\right)\, dx \leq N I(\phi).$$

Moreover, Lemma 4.4 yields

$$\lim_{k\to+\infty} \sum_{j=1}^N \int_{A_{k,j}} F\left(\max(u,\phi)\left(T_{k,j}\right), \nabla\max(u,\phi)\left(T_{k,j}\right)\right)\, dx = I\left(\max(u,\phi)\right).$$

The validity of (4.11) then follows from (4.16). $\square$

**Remark 4.6.** The initial part of the proof of Theorem 1.1, which involves the covering of $\Gamma$ with a union of uniformly star-shaped sets, is inspired by that of [12, Proposition X.2.6] where the authors prove the result in the particular case where $F$ does not depend on $u$ and the boundary datum $\phi$ equals 0. We point out however that in the proof of [12, Proposition X.2.6] it is not clear why the approximating functions should have compact support in $\Omega$, due to the fact that the covering of $\Gamma$ that is involved there consists of sets that are *just* star-shaped, and do not satisfy (2.11) , see also Remark 2.16.

**Remark 4.7.** One could have been tempted to introduce the approximating sequence $(u_k)_k$ as

$$u_k(x) = \phi + h_k(v_k - w_k), \tag{4.17}$$

by omitting the multiplicative term $\lambda$ in front of the right-hand term of (4.6). However in this case one has to deal with the problem of studying the convergence of

$$\int_\Omega F\left(\mathbf{h_k} u\left(\frac{x}{h_k}\right), \nabla u\left(\frac{x}{h_k}\right)\right)\, dx$$

as $k \to +\infty$, which is not obvious unless $F(0, \nabla u) \in L^1(\Omega)$. The "trick" of introducing a new parameter $\lambda$ was used in [11].

# 5 The non convex case

## 5.1 Proof of Theorem 1.3

*Proof of Theorem 1.3.* By Lemma 3.1 it is not restrictive to assume that $u$ is bounded. Assume first that $g(\nabla u) \in L^1(\Omega)$. By Theorem 1.1 applied to $F(s, \xi) = g(\xi)$ there is a sequence $(u_k)_k$ in $\mathrm{Lip}_\phi(\Omega)$ that is bounded in $L^\infty(\Omega)$, converging to $u$ in $W^{1,p}$ and such that

$$\lim_{k\to+\infty} \int_\Omega g(\nabla u_k) = \int_\Omega g(\nabla u).$$

Modulo a subsequence we may assume that $(u_k)_k$ converges a.e. to $u$ and that there exists a summable map dominating a.e. each $|g(\nabla u_k)|$. Since $a$ is continuous, the bounded sequence $(a(u_k))_k$ converges to $a(u)$ a.e. By the dominated convergence theorem, the sequence $(a(u_k)g(\nabla u_k))_k$ converges to $a(u)g(\nabla u)$ in $L^1(\Omega)$. The continuity of $b$ also implies that (up to a subsequence), $(b(u_k))_k$ converges to $b(u)$. Finally,

$$\lim_{k\to+\infty} I(u_k) = I(u) :$$

proving the claim under the assumption that $g(\nabla u) \in L^1(\Omega)$.
Assume now that $a > 0$. Since $u \in L^\infty(\Omega)$, there exists $m > 0$ such that $a(u(x)) > m$ a.e. Moreover, $b(u) \in L^\infty(\Omega)$. Hence, the summability of $F(u, \nabla u) = a(u)g(\nabla u) + b(u)$ automatically implies the summability of $g(u)$, so that the result follows from the first case considered above. $\square$

## 5.2 The non occurrence of the Lavrentiev phenomenon for non convex Lagrangians of the gradient

As a consequence of Theorem 1.1 we get the non occurrence of the Lavrentiev phenomenon for non convex variational problems with Lagrangians of the form $F(s,\xi) = g(\xi)$, without assuming convexity in $\xi$.

**Corollary 5.1.** *Let $F(s,\xi) = g(\xi)$ where $g : \mathbb{R}^n \to \mathbb{R}$ is continuous and bounded below by an affine function; assume moreover that the epigraph of $g^{**}$ has no unbounded faces. If $\Omega$ is locally strongly star-shaped, then the Lavrentiev phenomenon for $I$ does not occur, i.e.*

$$\inf\{I(u) : \, u \in W^{1,p}_{\phi}(\Omega)\} = \inf\{I(u) : \, u \in \mathrm{Lip}_{\phi}(\Omega)\}.$$

**Remark 5.2.** The assumption that there are no unbounded faces on the epigraph of $g^{**}$ is a kind of growth condition from below; it is fulfilled if, for instance, $g$ has a superlinear growth.

*Proof.* Let $g^{**}$ be the bipolar of $g$ and $I^{**}$ be the functional

$$I^{**}(u) = \int_{\Omega} g^{**}(\nabla u(x))\, dx.$$

By Theorem 1.1 we know that

$$\inf\{I^{**}(u) : \, u \in W^{1,p}_{\phi}(\Omega)\} = \inf\{I^{**}(u) : \, u \in \mathrm{Lip}_{\phi}(\Omega)\}.$$

Let $(u_n)_n$ in $\mathrm{Lip}_{\phi}(\Omega)$ be such that $I^{**}(u_n) \to \inf I^{**}$. Calling $K_n$ the Lipschitz constant of $u_n$ of course we have

$$\inf\{I^{**}(u) : \, u \in \mathrm{Lip}^{K_n}_{\phi}(\Omega)\} \leq I^{**}(u_n),$$

where by $\mathrm{Lip}^{K_n}_{\phi}(\Omega)$ we denote the set of functions in $\mathrm{Lip}_{\phi}(\Omega)$ with rank less than $K_n$. It follows from [14, Theorem 6.1] that, for each $n$, there is $w_n$ in $\mathrm{Lip}(\Omega, \phi)$ with

$$I(w_n) \leq \inf\{I^{**}(u) : \, u \in \mathrm{Lip}^{K_n}_{\phi}(\Omega)\} + \frac{1}{n}.$$

Thus for each $n$ we get

$$\inf I^{**} \leq \inf\{I(u) : \, u \in \mathrm{Lip}_{\phi}(\Omega)\} \leq I(w_n) \leq I^{**}(u_n) + \frac{1}{n},$$

so that, by passing to the limit as $n$ tends to $+\infty$ we obtain

$$\lim_{n\to+\infty} I(w_n) = \inf I^{**}.$$

The conclusion follows from the fact that $\inf I^{**} \leq \inf\{I(u) : \, u \in W^{1,p}_{\phi}(\Omega)\}$. $\square$

# 6 Strongly or uniformly star-shaped domains and regularity of the jauge function

Assume now that $\Omega$ is star-shaped with respect to $z = 0$. We relate condition (2.4) with the Lipschitz continuity of the jauge function of $\Omega$, defined for $x \in \mathbb{R}^n$ by

$$j(x) = \inf\{\lambda > 0 : \, x \in \lambda\Omega\}.$$

For any $u \in S := \{u \in \mathbb{R}^n, \, |u| = 1\}$ we set $\rho(u)$ to be the radius of $\Omega$ in the direction $u$, i.e.

$$\forall u \in S \qquad \rho(u) = \sup\{\lambda : \lambda u \in \Omega\}.$$

For any $u \in S$ the point $\rho(u)u$ belongs to the closure of $\Omega$, and

$$\Omega = \cup\{\lambda u : \, u \in S, \, 0 \leq \lambda < \rho(u)\}.$$

The functions $\rho$ and $j$ are related to each other by the formula

$$j(x) = \begin{cases} 0 \text{ if } x = 0, \\ \dfrac{|x|}{\rho(x/|x|)} \text{ otherwise.} \end{cases} \tag{6.1}$$

It is useful to notice that if $r, R > 0$ are such that $B_r \subset \Omega \subset B_R$ then

$$\frac{|x|}{R} \leq j(x) \leq \frac{|x|}{r}$$

so that, in particular, $j$ is continuous at 0 and $j(x) \neq 0$ if $x \neq 0$.

**Example 6.1.** Assume that $\Omega$ is star-shaped w.r. to the origin.

i) The jauge of $\Omega$ is continuous on $\mathbb{R}^n$ if and only if $\rho$ is continuous.

ii) The jauge of $\Omega$ is Lipschitz on $\mathbb{R}^n$ if and only if $\rho$ is Lipschitz.

*Proof.* Claim i) follows immediately from (6.1) and the fact that $j$ is continuous at 0. Let us prove ii). Clearly if the jauge is Lipschitz on $\mathbb{R}^n$ then the map $\rho(u) = \dfrac{1}{j(u)}$ is thus Lipschitz on $S$.

Conversely, assume that the map $u \in S \mapsto \rho(u)$ is Lipschitz. For any $x, y \in \mathbb{R}^n \setminus \{0\}$,

$$|j(y) - j(x)| = \left| \frac{|y|}{\rho(y/|y|)} - \frac{|x|}{\rho(x/|x|)} \right| = \left| \frac{|x|\rho(y/|y|) - |y|\rho(x/|x|)}{\rho(y/|y|)\rho(x/|x|)} \right|.$$

By the initial remark we get

$$|j(y) - j(x)| \leq C_1 ||x|\rho(y/|y|) - |y|\rho(x/|x|)|$$

$$\leq C_1|x||\rho(x/|x|) - \rho(y/|y|)| + C_2|x-y|.$$

By the Lipschitz continuity of $\rho$ on $S$,

$$|\rho(x/|x|) - \rho(y/|y|)| \leq C_3|x/|x| - y/|y|| \leq 2C_3\frac{|x-y|}{|x|}.$$

It follows that $|j(y) - j(x)| \leq C_4|x-y|$ for some $C_4 > 0$.

When one of the points (say $x$) is $0$, then we simply write

$$|j(y) - j(x)| = |j(y)| \leq C|y|,$$

which completes the proof of the proposition. □

The next result characterizes the star-shaped sets w.r. to 0 whose jauge function is continuous or Lipschitz: continuity is equivalent to the fact that $\Omega$ is strongly star-shaped whereas Lipschitzeanity is equivalent to the fact that $\Omega$ is uniformly star-shaped. Actually, one of the implications in ii) was established in [15, Section 1.1.8] with a different proof and formulation by means of sets that are star-shaped w.r.t a ball; by Proposition 2.5, these sets are uniformly star-shaped.

**Example 6.2.** Let $\Omega$ be bounded and star-shaped with respect to the origin.

i) The radii function $\rho$ is continuous if and only if $\Omega$ is strongly star-shaped with respect to $0$; in this case the boundary $\Gamma$ of $\Omega$ is the set of *radial extreme points* of $\Omega$ given by $\Gamma = \{\rho(u)u : \ u \in S\}$.

ii) Assume that $\Omega$ is strongly star-shaped with respect to $0$. The radii function of $\Omega$ is Lipschitz on $\mathbb{R}^n$ if and only if $\Omega$ is uniformly star-shaped w.r. to 0.

*Proof.* We denote by $[p,q]$ the segment joining $p, q$ in $\mathbb{R}^n$. Observe that $\Omega$ is strongly star-shaped w.r. to $0$ if and only if $\Gamma$ does not contain any segment of the form $[aw, bw]$ with $0 < a < b$ and $w \in S$.

i) Assume that $\rho$ is not continuous at $w \in S$. Then there are $0 < a < b$ and sequences $u_k, v_k$ converging to $w$ with $\rho(u_k) < a$ and $\rho(v_k) > b$. It follows that the segment $[0, bv_k]$ is in $\Omega$ for all $k$ so that $[0, bw]$ is in the closure $\overline{\Omega}$ of $\Omega$. However $[aw, bw] \cap \Omega = \emptyset$, otherwise there exists $t \in [a, b]$ such that $tw \in \Omega$. Since $\Omega$ is star-shaped with respect to $0$, this implies $[0, tw] \subset \Omega$ so that $[0, tu] \in \Omega$ for $u$ in a neighborhood of $w$, contradicting the fact that $\rho(u_k) < a$ for all $k$. It follows that $[aw, bw] \subset \Gamma$. Conversely assume that a segment $[aw, bw] \subset \Gamma$ for some $0 < a < b$ and $|w| = 1$. Assume by contradiction that $\rho$ is continuous at $w$: since $bw \in \overline{\Omega}$ there is a sequence $x_k \in \Omega$ that converges to $bw$. For every $k$, $\rho(\frac{x_k}{|x_k|}) \geq |x_k|$. The continuity assumption thus implies implies that $\rho(w) \geq b$. But then for $a < c < b$, $\rho(w) > c$ so that $cw \in \Omega$ which implies $[0, cw] \subset \Omega$ and in particular $[aw, cw] \subset \Omega$, contradicting the fact that $[aw, bw] \subset \Gamma$: thus $\rho$ is not continuous at $w$.

Now for every $u \in S$ the point $\rho(u)u$ belongs to $\Gamma$; conversely assume that $\rho$ is continuous and let $x \in \Gamma$. Set $u = x/|x|$: if $\rho(u) \neq |x|$ then the segment $[\rho(u)u, x]$ is non trivial and is contained in $\Gamma$, a contradiction, proving that $|x| = \rho(u)$, whence $x = \rho(u)u$.

ii) It is enough, from Proposition 2.5, to prove that the radii function of $\Omega$ is Lipschitz if and only if the Bouligand paratingent cone at every point of the boundary of $\Omega$ does not contain radial directions. Assume that the radii function of $\Omega$ is not Lipschitz: there are sequences $u_k, v_k$ in $S$ such that

$$\lim_{k\to+\infty} \left| \frac{\rho(v_k) - \rho(u_k)}{v_k - u_k} \right| = +\infty.$$

Modulo a subsequence, we may assume that $u_k$ and $v_k$ converge to $u_* \in S$. Let $\alpha = \rho(u_*)u_*$ and set

$$x_k = \rho(u_k)u_k, \quad y_k = \rho(v_k)v_k, \quad t_k = 1/|\rho(v_k) - \rho(u_k)|.$$

The continuity of $\rho$ ensured by i) yields $t_k \to +\infty$ as $k \to +\infty$. Moreover

$$\begin{aligned} t_k(y_k - x_k) &= \frac{\rho(v_k)v_k - \rho(u_k)u_k}{|\rho(v_k) - \rho(u_k)|} \\ &= \frac{\rho(v_k)(v_k - u_k) + (\rho(v_k) - \rho(u_k))u_k}{|\rho(v_k) - \rho(u_k)|} \\ &= \rho(v_k)\frac{v_k - u_k}{|\rho(v_k) - \rho(u_k)|} + \eta_k u_k \qquad \eta_k = \pm 1. \end{aligned}$$

By taking a subsequence we may assume $\eta_k = 1$ for all $k$ or $\eta_k = -1$ for all $k$. In the first case we obtain that $t_k(y_k - x_k) \to u_*$ as $k \to +\infty$ whereas in the second case we get $t_k(y_k - x_k) \to -u_*$ as $k \to +\infty$. It follows that $u_*$ or $-u_*$ (and thus $\alpha = \rho(u_*)u_*$) belongs to $P_\Gamma(\alpha)$.
Conversely assume that $\rho$ is Lipschitz. Let $\alpha \in \Gamma$ and $u_* \in S$ be such that $\alpha = \rho(u_*)u_*$. Let $0 \neq v \in P_\Gamma(\alpha)$: then $\lim_{k\to+\infty} t_k(y_k - x_k) = v$ for some $t_k \to +\infty$ and $x_k, y_k$ in $\Gamma$, $x_k \neq y_k$, both converging to $\alpha$. By the continuity assumption on $\rho$ we may suppose that, for all $k$, $x_k = \rho(u_k)u_k$ and $y_k = \rho(v_k)v_k$ for some $u_k$, $v_k \in S$ both converging to $u_*$. Since $\rho$ is continuous, $[x_k, y_k] \not\subset \Gamma$, whence $u_k \neq v_k$.

Now, for all $k$,

$$\begin{aligned} t_k(\rho(v_k)v_k - \rho(u_k)u_k) &= t_k(\rho(v_k)v_k - \rho(u_k)v_k) + t_k(\rho(u_k)v_k - \rho(u_k)u_k) \\ &= t_k|v_k - u_k| \left( \frac{\rho(v_k) - \rho(u_k)}{|v_k - u_k|} v_k + \rho(u_k)\frac{v_k - u_k}{|v_k - u_k|} \right). \end{aligned} \tag{6.2}$$

Since $\rho$ is Lipschitz, we may assume, modulo a subsequence, that $\dfrac{\rho(v_k) - \rho(u_k)}{|v_k - u_k|}$ converges to $r \in \mathbb{R}$; by compactness of the unit sphere we may also assume that $\dfrac{v_k - u_k}{|v_k - u_k|}$

converges to $v_* \in S$ as $k \to +\infty$. Notice that $v_*$ is orthogonal to $u_*$; indeed for all $k$ we have

$$|v_k - u_k|^2 = |v_k|^2 - |u_k|^2 - 2u_k \cdot (v_k - u_k) = -2u_k \cdot (v_k - u_k)$$

so that

$$v_* \cdot u_* = \lim_{k \to +\infty} u_k \cdot \frac{v_k - u_k}{|v_k - u_k|} = -\frac{1}{2} \lim_{k \to +\infty} |v_k - u_k| = 0.$$

Moreover by continuity $\rho(u_k)$ converges to $\rho(u_*)$. It follows that

$$\frac{\rho(v_k) - \rho(u_k)}{|v_k - u_k|} v_k + \rho(u_k) \frac{v_k - u_k}{|v_k - u_k|}$$

converges to $w := ru_* + \rho(u_*)v_*$; thus $w \neq 0$ and, passing to the limit in (6.2), we get

$$v = s(ru_* + \rho(u_*)v_*), \quad u_* \perp v_*, \quad \rho(u_*) > 0, \quad s = \frac{|v|}{|w|} > 0;$$

it follows that $v \notin \mathbb{R}u_* = \mathbb{R}\alpha$ and $P_\Gamma(\alpha) \cap \mathbb{R}\alpha = \{0\}$, proving ii). $\square$

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