Basics of Probability and Statistics

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Content

- **Statistical Model**
- **Probability Background**
- **Law of Large Numbers, Central Limit Theorem**
- **Gaussian Vectors**

- **Conditioning**
- **Estimation**
- **Confidence Set**
- **Basic of Regression**
- **Component Principal Analysis: Introduction**

Statistical Model

Definition

Let Ω be a set

Definition

$\mathcal{A} \subset \mathcal{P}(\Omega)$ is a σ-algebra on Ω if the following conditions are satisfied

1. $\Omega \in \mathcal{A}$
2. $\mathcal{A}$ is stable by the complementary operation i.e if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$
3. $\mathcal{A}$ is stable by countable union i.e if $(A_n)_n$ is a countable family of elements of $\mathcal{A}$ i.e $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$ then $\bigcup_n A_n \in \mathcal{A}$

1. $\{\emptyset, \Omega\}$ is the smallest $\sigma - algebra$
2. $\mathcal{P}(\Omega)$ is called the trivial $\sigma - algebra$, usually considered when Ω is discrete
3. When Ω is a topologic space equipped with a family of open sets, the smallest $\sigma-$ algebra which contains all these open is called the **Borel $\sigma-$algebra**. We denote it by $\mathcal{B}(\Omega)$. Why does it always exists?

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A set Ω equipped with a $\sigma-$algebra $\mathcal{A}$ is called a measurable space and we denote it by $(\Omega, \mathcal{A})$

Definition

A measure μ on $(\Omega, \mathcal{A})$ is an application from $\mathcal{A} \to [0, +\infty]$ such that

1. $\mu(\emptyset) = 0$
2. If $(A_n)_n$ is a countable family of elements of $\mathcal{A}$ mutally disjoints i.e $A_i \cap A_j = \emptyset$ if $i \neq j$ then

$$\mu(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

- Dirac measure δ_a. Counting measure $\sum_{n\in\mathbb{N}} \delta_n$.
- Lebesgue measure
  $\lambda([a, b]) = \lambda(]a, b]) = \lambda([a, b[) = \lambda(]a, b[) = b - a$

Definition

1. The triplet $(\Omega, \mathcal{A}, \mu)$ is called a measured set.
2. **When μ is of mass** 1 that is $\mu(\Omega) = 1$ we speak about **probability measure**. In this case we denote μ by $\mathbb{P}$.
3. **A probability space** is then a measurable space $(\Omega, \mathcal{A})$ equipped with a probability measure $\mathbb{P}$: $(\Omega, \mathcal{A}, \mathbb{P})$
4. One important situation in statistics is when the probability measure $\mathbb{P}$ depends on a **unknown parameter** θ^*. We usually denote $\mathbb{P}_{\theta^*}$ this probability.
5. We shall assume that the probability $\mathbb{P}_{\theta^*}$ belongs to a class of probability measure that we shall denote $\mathcal{P}$.
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Let E and F be two sets equipped with $\sigma-$algebras $\mathcal{A}$ for E and $\mathcal{B}$ for F. An application $f : (E, \mathcal{A}) \to (E, \mathcal{B})$ is called measurable if

$$\forall B \in \mathcal{B}, f^{-1}(B) \in \mathcal{A}$$

- Recall that a random variable X is a measurable function from Ω to $\mathbb{R}$ or a discrete or countable space
- Let us throw two dices and compute the sum $S : \{1, \dots, 6\}^2 \to \{2, \dots, 12\} : S(i, j) = i + j$ is a r.v
- When is X is valued on $\mathbb{R}^k$, $k > 1$, we usually speak of random vectors

Statistical Model

Definition

A statistical model is a triplet $(\Omega, \mathcal{A}, \mathcal{P})$ where

1. Ω is called the space of realizations
2. $\mathcal{A}$ is a $\sigma-$algebra
3. $\mathcal{P}$ is a family of probability measure defined on $\mathcal{A}$

- Family of Gaussian laws:

$$\mathcal{P} = \{\mathcal{N}(m, \sigma^2), m \in \mathbb{R}, \sigma \in \mathbb{R}_+^*\}$$

Recall that the density of $\mathcal{N}(m, \sigma^2)$ is given by

$$f_{\mathcal{N}(m,\sigma^2)}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

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  are usually associated with a random variable X whose law is either Gaussian or Bernoulli.

- Assume you want to extract information on m, σ or θ (these are unknown parameters). You can easily guess that one realization (one observation) of the value of X is not enough.

- Usually we are faced to n independent realizations of the same random variable. This way we consider $X_1, \ldots, X_n$ n r.v independent and identically distributed such as $X_i \sim X$ for all $i \in \{1, \ldots, n\}$

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Definition

In the situation where you have n observations i.i.d $X_1, \ldots, X_n$, the statical models can be described by

- Gaussian: $\Omega = \mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$ (n times), $\mathcal{A} = \mathcal{B}(\mathbb{R}^n)$,

$$\mathcal{P} = \{\mathcal{N}^{\otimes n}(m, \sigma), m \in \mathbb{R}, \sigma \in \mathbb{R}_+^*\}$$

- Bernoulli: $\Omega = \{0, 1\}^n$, $\mathcal{A} = \mathcal{P}(\Omega)$

$$\mathcal{P} = \{\mathcal{B}^{\otimes n}(\theta), \theta \in [0, 1]\}$$

the notation $\otimes n$ means that we consider the product of measure on the cartesian product $\mathbb{R}^n$ or $\{0, 1\}^n$. This corresponds to the fact that we consider independent situation.

- Exercise: describe the statistical model where you throw 100 times 10 dices and you just look at the sum of each result.

Definition

1. Other situations. Assume you observe n realizations of random variables X_i valued in $\mathbb{R}$ such that

$$\mathbb{E}[X_i] = i\theta$$

   where θ is an unknown parameter and the law of X_i are unknown (you do not know the forme of the density for example). Your focus is on θ! only and not on the distribution of X_i
   - $\Omega = \mathbb{R}^n$
   - $\mathcal{P} = \left\{\mathbb{P}_{X_1} \otimes \ldots \otimes \mathbb{P}_{X_n}, \int_{\mathbb{R}} x dP_{X_i}(x) = i\theta, \theta \in \mathbb{R}\right\}$

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1. **Parametric Model**: the family law is parametrized by a subset of $\mathbf{R}^d$.
2. **Semi- parametric Model**: the family laws is not parametrized by a subset of $\mathbf{R}^d$ but the quantity of interest is.
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1. Estimation: Assume you want to estimate an unknown parameter θ or a function $g(\theta)$. This estimation has to be based only on the observations; this is done by the **notion of estimator**. We shall concentrate only the i.i.d situation

Definition

Let $X_1, \ldots, X_n$ be a sample that is the r.v are independent and identically distributed. An estimator is a measurable function of the observations.

2. An estimator can not be defined with unknown parameters
3. Usual estimator take the form $T = f(X_1, \ldots, X_n)$. An estimator is a r.v. When you have an observations $(x_1, \ldots, x_n)$, the quantity $t = f(x_1, \ldots, x_n)$ is a realization of T and is called an estimation
4. Examples:

$$T = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad T = max(X_1, \ldots, X_n)$$

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Definition

1. Hypothesis testing: Assume that your unknown parameter $\theta^* \in \Theta = \Theta_1 \cup \Theta_2$ where the union is disjoint.
2. Within the observations you want to take a decision: the parameter θ^* belongs either to Θ_1 or to Θ_2
3. Again this decision has to be made in a measurable way with respect to the observations. A test is a measurable function of $(X_1, \ldots, X_n)$
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Definition

1. Before going further: **Important point:** making statistic is assuming that you are going to make mistakes, errors.
2. Indeed you won't be able, in general, to be sure having founded the unknown parameter only with a finite number of observations
3. Statisticians are Mathematicians who are able to control the error they will make by establishing qualitative analysis of their estimators or tests.
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Probability background

First concentration inequality

- This part will be a glossary of notions of probability we shall need in the sequel
- Let us start with two useful concentration inequalities. Let us consider a random variable X on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.
- If $X \in L^1$, the mean, average, expectation is denoted by $\mathbb{E}[X]$
- If $X \in L^2$, the variance is denoted by $Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$
- If X is L^1: **Markov inequality**

$$\mathbb{P}(|X| \geqslant t) \leqslant \frac{\mathbb{E}(|X|)}{t}$$

- If X is L^2: **Bienaymé-Tchebychev inequality**

$$\mathbb{P}(|X - \mathbb{E}(X)| \geqslant t) \leqslant \frac{Var(X)}{t^2}$$

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Characteristic function

Definition

The characteristic function of a r.v X is defined by

$$\phi_X(t) = \mathbb{E}[e^{itX}], \forall t \in \mathbb{R}$$

The characeristic function of a random vector is

$$\phi_X(u) = \mathbb{E}[e^{i<u,X>}], \forall u \in \mathbb{R}^d,$$

where $<,>$ denote the scalar product on $\mathbb{R}^d$.

characteristic function

- $X \sim \mathcal{B}(p)$ then $\phi_X(t) = 1 - p + pe^{it}$
- $X \sim \mathcal{B}(n,p)$ then $\phi_X(t) = (1 - p + pe^{it})^n$
- $X \sim \mathcal{P}(\lambda)$ then $\phi_X(t) = exp(\lambda(e^{it} - 1))$

- $X \sim \mathcal{U}([a,b])$ then $\phi_X(t) = \frac{e^{ibt} - e^{iat}}{(b-a)it}$
- $X \sim \mathcal{E}(\lambda)$ then $\phi_X(t) = \frac{\lambda}{\lambda - it}$
- $X \sim C(a)$ then $\phi_X(t) = exp(-a|t|)$
- $X \sim \mathcal{N}(m, \sigma^2)$ then $\phi_X(t) = exp(imt - \frac{\sigma^2 t^2}{2})$

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Proposition

Let X be a r.v which admits a moment of order p then its characteristic function is p times differentiable and we have

$$\phi_X^{(p)}(0) = i^p \mathbb{E}[X^p]$$

Other transformation

- The moment generator function of a r.v X with values in $S(X) \subset \mathbb{N}$ and $p_k = \mathbb{P}(X = k)$ is

$$G_X(t) = \mathbb{E}[t^X] = \sum_k p_k t^k$$

This function is C^∞ on $[0, 1[$ and p times differentiable on 1 if $\mathbb{E}[X^p] < +\infty$

$$G_X^{(k)}(0) = k!p_k, k \in \mathbb{N}$$

If the mean exists, we have $G'_X(1) = \mathbb{E}(X)$

- Laplace transform. For a r.v X, we call its Laplace transform

$$\phi_X(t) = \mathbb{E}[e^{tX}]$$

Other transformation

1. As we shall see in the sequel, we shall be interested in limits of estimator when the number of observations n goes to infinity.

2. This asks for convergence of random variables.

Definition

Definition

Let (X_n) be a sequence of r.v and X be a r.v. We say that (X_n) converge towards X

- **Almost surely a.s** if $\mathbb{P}(\lim X_n = X) = 1$ we note $X_n \xrightarrow{a.s} X$
- **In L^p norm** if $\lim\limits_{n\to+\infty} \mathbb{E}[|X_n - X|^p] = 0$ we note $X_n \xrightarrow{L^p} X$
- **In probability** if $\forall \epsilon > 0, \lim\limits_{n\to+\infty} \mathbb{P}[|X_n - X| > \epsilon] = 0$ we note $X_n \xrightarrow{\mathbb{P}} X$
- **In law** if for all continuous and bounded functions f we have $\lim\limits_{n\to+\infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X)]$ we note $X_n \xrightarrow{\mathcal{L}} X$

When the law of X depends on a unknown parameter θ we make appear this dependency.

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Convergence en loi

For a r.v we denote its partition function F_X and recall that ϕ_X denotes its characteristic function

Theorem

(X_n) converge in law towards X if and only if

$$F_{X_n}(t) \to F_X(t)$$

in all points where F_X is continuous i.e in all points t such that $\mathbb{P}(X = t) = 0$

Theorem

(X_n) converges in law towards X if and only if

$$\phi_{X_n}(t) \to \phi_X(t)$$

for all $t \in \mathbb{R}$.

Usual Convergence mode

In order to finish let us recall the usual convergence mode

Theorem

- ***Beppo Levy Theorem:*** *let* (X_n) *be a non decreasing sequence of non negative numbers then if* $\lim_n X_n = X$ *we have*

$$\lim_n \mathbb{E}[X_n] = \mathbb{E}[X]$$

- ***Fatou Lemma:*** *let* (X_n) *be a sequence of non negative numbers then*

$$\mathbb{E}[\liminf_n X_n] \leqslant \liminf_n \mathbb{E}[X_n]$$

- ***Lebesgue dominated convergence Theorem:*** *let* (X_n) *be a sequence such that* X_n *converges a.s to* X. *Let* Y *such that* $\mathbb{E}[|Y|] < \infty$ *and* $|X_n| < Y|$ *then*

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- ***Beppo Levy Theorem:*** *let* (X_n) *be a non decreasing sequence of non negative numbers then if* $\lim_n X_n = X$ *we have*

$$\lim_n \mathbb{E}[X_n] = \mathbb{E}[X]$$

- ***Fatou Lemma:*** *let* (X_n) *be a sequence of non negative numbers then*

$$\mathbb{E}[\liminf_n X_n] \leqslant \liminf_n \mathbb{E}[X_n]$$

- ***Lebesgue dominated convergence Theorem:*** *let* (X_n) *be a sequence such that* X_n *converges a.s to* X*. Let* Y *such that* $\mathbb{E}[|Y|] < \infty$ *and* $|X_n| < Y|$ *then*

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Links between convergence modes

Recall the usual links

- Almost sure convergence $\Longrightarrow$ Convergence in probability
- L^p Convergence $p \geqslant 1 \Longrightarrow L^1$ Convergence $\Longrightarrow$ Convergence in probability
- All convergence modes $\Longrightarrow$ Convergence in law
- Almost surely convergence + domination $\Longrightarrow L^1$ convergence
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- We say that $X_n = \circ_{\mathbb{P}}(Y_n)$ if

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Law of Large Numbers (LLN) and Central Limit Theorem (CLT)

Objectif

- The objective of this section is to understand the convergence of

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\sqrt{n}(\bar{X}_n - m) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - m)$$

  when (X_n) is a sequence of i.i.d random variables where $m = \mathbb{E}[X_1]$.
- As we shall see the first quantity is a good estimator of m and the second quantity allows to control the error we make when making estimation

Weak Law of Large Numbers L^2 and L^1

Theorem

Let (X_n) be a sequence of i.i.d r.v which are L^2 then

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- Let (X_n) be a sequence of i.i.d r.v $\mathcal{B}(p)$ then $M_n := \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\mathbb{P}} p$
- First step towards estimation of an unknown proportion

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$$\frac{1}{n}\sum_{i=1}^{n} X_i \overset{a.s}{\to} \mathbb{E}[X_1]$$

- Application: Monte Carlo Method. Let f be a measurable function such that $f(X_1)$ Let L^1

$$\frac{1}{n}\sum_{i=1}^{n} f(X_i) \overset{a.s}{\to} \mathbb{E}[f(X_1)]$$

  Rq: note that the advantage of this method is that we do not require any regularity property of f.
- $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}_n^2$ are estimators of the mean and of the variance

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Theorem

Central Limit Theorem: *Let (X_n) be a sequence of i.i.d r.v which are L^2. Let m be the common mean and σ^2 the common variance. We put*

$$S_n = \sum_{i=1}^{n} X_i = n\bar{X}_n$$

then

$$\frac{1}{\sqrt{n\sigma^2}} \sum_{i=1}^{n} (X_i - m) = \frac{S_n - nm}{\sqrt{n\sigma^2}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$$

Central Limit Theorem

- This is a strong refinement of the LLN: somehow it gives the rate of convergence of the empirical mean towards the mean.
- As we shall see later, this allows to construct confidence interval
- Sometimes we need to consider $f(\bar{X}_n)$ for f sufficiently smooth. It is easy to see that

$$f(\bar{X}_n) \stackrel{a.s}{\to} f(\mathbb{E}[X_1])$$

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- Concerning extension of CLT one is interested in convergence in law of

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Gaussian Vectors

Definition

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A random vector $X = (X_1, \dots, X_d)^t$ is called Gaussian vector if all linear combination of its coordinates are Gaussian, that is for all $a \in \mathbb{R}^d$ the r.v

$$< a, X >= \sum_{i=1}^{d} a_i X_i$$

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matrix de covaroiance

Definition

Let $X = (X_1, \ldots, X_d)^t$ be a Gaussian vector we note K its covariance matrix defined by

$$K_{i,j} = Cov(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j],$$

for all $i, j = 1, \ldots, d$. We shall also note

$$m = \mathbb{E}[X] = (\mathbb{E}[X_1], \ldots, \mathbb{E}[X_d])^t$$

the vector of mean. We shall note $X \sim \mathcal{N}_d(m, K)$

- The matrix K is semi-definite positive
- $\mathbb{E}[< a, X >] =< a, \mathbb{E}[X] >$
- $Var(< a, X >) = Var\left(\sum_{i=1}^{d} a_i X_i\right) = \sum_{i,j=1}^{d} a_i a_j Cov(X_i, X_j) = a^t K a =< a, Ka >$

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- One can check that

$$\phi_{<a,X>}(t) = \exp\left(i < a, m > t - \frac{1}{2} a^t K a \, t^2\right)$$

- $\phi_X(x) = \mathbb{E}[e^{i<x,X>}] = \phi_{<x,X>}(1)$

Proposition

The characteristic function of a Gaussian vector is given by

$$\phi_X(x) = \exp\left(i < x, m > -\frac{1}{2} x^t K x\right)$$

- The coordinates of a Gaussian vector are independent if and only if its covariance matrix is diagonal

Transformation linéaire

Proposition

Let $X \sim \mathcal{N}_d(m, K)$ *then for all matrices* $A \in \mathbb{M}_{p,d}(\mathbb{R})$ *then*

$$AX \sim \mathcal{N}_p(AX, AKA^t)$$

- If $X \sim \mathcal{N}_d(0, I_d)$ then the law of X is invariant by all rotation.

Centrer and réduire un Gaussian vector

- We shall say that a Gaussian vector X is degenerate if its covariance matrix K is non invertible
- In the degenerate case, there exists a such that $Ka = 0$ which implies that

$$Var(< a, X >) = 0$$

  and then $< a, X >= b$ a.s. Then X leaves in the affine space

$$\{< a, x >= b, x \in \mathbb{R}^d\}$$

- If K is invertible then $\sqrt{K}^{-1}(X - m) \sim \mathcal{N}(0, I_d)$
- If $N \sim \mathcal{N}(0, I_d)$ then $X = \sqrt{K}N + m \in \mathcal{N}(m, K)$

Density

- If $X \sim \mathcal{N}_d(0, I_d)$ then the coordinates $(X_i)_{i=1,\ldots,d}$ are i.i.d and $X_1 \sim \mathcal{N}(0, 1)$. Then the density of X is given by the product of densities i.e

$$f_X(x_1, \ldots, x_d) = \frac{1}{\sqrt{2\pi}^d} \exp\left(-\frac{1}{2}\sum_{i=1}^{d} x_i^2\right)$$

- In the case where K is invertible we have

$$f_X(x_1, \ldots, x_d) = \frac{1}{\sqrt{(2\pi)^d \det K}} \exp(-\frac{1}{2} < (x-m), K^{-1}(x-m) >)$$

- Rq: if X is Gaussian all its coordinates are Gaussian, the converse is not true in general.

CLT multidimensional

Theorem

Let $X^{(n)}$ *be a sequence of random vectors of* $\mathbb{R}^d$ *which are i.i.d and* L^2 *of mean vector* m *and of covariance matrix* K. *We put* $S^{(n)} = \sum_{i=1}^{n} X^{(i)}$ *then we have*

$$n^{-1/2}\sqrt{K}^{-1}(S^{(n)} - nm) \xrightarrow{\mathcal{L}} \mathcal{N}_d(0, I_d)$$

or

$$n^{-1/2}(S^{(n)} - nm) \xrightarrow{\mathcal{L}} \mathcal{N}_d(0, K)$$

Transformation of Gaussian law

- Let $X \sim \mathcal{N}(0,1)$ and consider $Z = X^2$. Let f be a continuous and bounded function

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\begin{aligned}
\mathbb{E}[f(Z)] &= \mathbb{E}[f(X^2)] \\
&= \int_{\mathbb{R}} f(x^2) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= 2 \int_0^{+\infty} f(x^2) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \int_0^{+\infty} f(z) \frac{1}{\sqrt{2\pi}} e^{-\frac{z}{2}} (\sqrt{z})^{-1} dz
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- Then $Z \sim \chi^2(1)$ where $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z}{2}} (\sqrt{z})^{-1} \mathbf{1}_{z \geqslant 0}$

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Transformation of Gaussian law

- Let $X = (X_1, \ldots, X_d)$ a Gaussian random vector where (X_i) are i.i.d of law $\mathcal{N}(0,1)$ then
$$Z = \sum_{i=1}^{d} X_i^2$$
is a random variable whose law is $\chi^2(d)$ where d is called the degree of freedom
- The density of this r.v is
$$f_Z(z) = \frac{1}{2\Gamma(k/2)} z^{\frac{k}{2}-1} e^{-\frac{z}{2}} \mathbf{1}_{z \geqslant 0}$$
where Γ is the Gamma function

Transformation of Gaussian law

- Let $X \sim \mathcal{N}(0,1)$ and $Z \sim \chi^2(k)$ then the r.v

$$T = \frac{X}{\sqrt{Z/k}}$$

is said to be distributed as the Student law of degree k

- The density is given by

$$f_T(t) = \frac{1}{\sqrt{k\pi}} \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})} (1 + \frac{t^2}{2})^{-\frac{k+1}{2}}$$

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Cochran Theorem

Proposition

Let $X \sim \mathcal{N}_d(0, I_d)$ and let $\mathbb{R}^d = F_1 \oplus \ldots \oplus F_k$ a decomposition in orthogonal space with $\dim(F_i) = d_i$. We note $P_{F_i}, i = 1, \ldots, k$ the orthogonal projectors associated with space $F_i, i = 1 \ldots, k$. In this case the vectors $P_{F_1}(X), \ldots, P_{F_k}(X)$ are independent Gaussian vectors. We have also

$$\|P_{F_i}(X)\|^2 \sim \chi^2(d_i), i = 1, \ldots, k$$

- This is linear algebra
- We can express a more general result $X \sim \mathcal{N}(0, K)$ with non degenerate K by introducing a scalar product with respect to K i.e $< a, b >_K = < a, Kb >$.

chi^2 Test

Test of adequation χ^2:

- We observe a random variable X where the set of values $S(X) = \{a_1, \ldots, a_r\}$ and $p_j = \mathbb{P}(X = a_j) = Q(\{a_j\}), j = 1, \ldots, r$ unknown. We note $p = (p_1, \ldots, p_r)$ the corresponding vector of probability.
- We consider a reference probability $Q_0 = \sum_i \pi_i \delta_i$ with same support but with a known vector $\pi = (\pi_1, \ldots, \pi_r)$ where $\pi_i > 0$
- The Hypothesis testing is $H_0 : Q = Q_0$ against $H_1 : Q \neq Q_0$.
- Let (X_n) be a sequence of i.i.d.r.v of law Q. For $n \in \mathbb{N}$, we put

$$N_j = \sum_{i=1}^{n} \mathbf{1}_{X_i = a_j}$$

- The random vector $N = (N_1, N_2, \ldots, N_r)^t$ follows a multinomial law $\mathcal{M}(n, p_1, \ldots, p_r)$ i.e

$$\mathbb{P}(N_1 = n_1, \ldots, N_r = n_r) = \frac{n!}{n_1! \ldots n_r!} p_1^{n_1} \ldots p_r^{n_r}, \quad n_1 + \ldots + n_r = n$$

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chi^2 Test

Test of adequation χ^2:

- We observe a random variable X where the set of values $S(X) = \{a_1, \ldots, a_r\}$ and $p_j = \mathbb{P}(X = a_j) = Q(\{a_j\}), j = 1, \ldots, r$ unknown. We note $p = (p_1, \ldots, p_r)$ the corresponding vector of probability.
- We consider a reference probability $Q_0 = \sum_i \pi_i \delta_i$ with same support but with a known vector $\pi = (\pi_1, \ldots, \pi_r)$ where $\pi_i > 0$
- The Hypothesis testing is $H_0 : Q = Q_0$ against $H_1 : Q \neq Q_0$.
- Let (X_n) be a sequence of i.i.d.r.v of law Q. For $n \in \mathbb{N}$, we put

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Test of *chi*^{2}

- We put

$$T_n = \sum_{j=1}^{r} \frac{(N_j - n\pi_j)^2}{n\pi_j}$$

- Under H_0 this quantity is close to 0 whereas under H_1 this quantity is big.

Theorem

- *Under H_0 we have*

$$T_n \xrightarrow{\mathcal{L}} \chi^2(r-1)$$

- *Under H_1 we have*

$$T_n \xrightarrow{a.s} +\infty$$

- Homogeneity Test, Independency Test

Delta method

- Let (X_n) be a sequence of i.i.d r.v L^2. Denote $\theta = \mathbb{E}[X_1]$ and $\sigma^2 = Var(X_1)$. Recall
$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$
- Recall that the CLT says
$$\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2)$$
- As already announced, for a particular class of f we would like to understand the convergence of
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Delta method

- Keep in mind the CLT

$$\sqrt{n}(\bar{X}_n - \theta) \stackrel{\mathcal{L}}{\to} \mathcal{N}(0, \sigma^2)$$

- First let us consider $f(x) = ax + b$ then we have

$$\sqrt{n}\left(a\bar{X}_n - a\theta\right) \stackrel{\mathcal{L}}{\to} a\mathcal{N}(0, \sigma^2) = \mathcal{N}(0, a^2\sigma^2)$$

- Now suppose that f is differentiable in θ you can write $f(x) = f(\theta) + f'(\theta)(x - \theta) + \circ(|x - \theta|)$. Since $\bar{X}_n - \theta$ converges to 0 almost surely it converges to 0 in probability which allows to write

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- Now the term $1 + o_{\mathbb{P}}(1)$ converges towards 1 in probability and then in Law (since the limit is a constant). Using the Slutsky Lemma allows to conclude that

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$$r_n(T_n - \theta) \xrightarrow{\mathcal{L}} T$$

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Delta method: General version

Theorem

Let θ in $\mathbb{R}^k$. Let ϕ be an application from $\mathbb{R}^k$ to $\mathbb{R}^m$ differentiable in θ. We denote $D_\theta\phi(.)$ the corresponding differential application. Let (T_n) be a sequence of random vectors of $\mathbb{R}^k$ such that there exists a sequence (r_n) and a random vector T such that

$$r_n(T_n - \theta) \xrightarrow{\mathcal{L}} T$$

then we have

$$r_n(\phi(T_n) - \phi(\theta)) \xrightarrow{\mathcal{L}} D_\theta\phi(T)$$

- In the Gaussian case if $Z \sim \mathcal{N}(0, K)$ where K is the covariance matrix and Z a Gaussian vector, then we have

$$D_\theta\phi(Z) \sim \mathcal{N}(0, J_\theta\phi K J_\theta\phi^t),$$

where $J_\theta\phi$ is the Jacobian matrix of ϕ.

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Conditioning

Definition

Let B be a event of non zero probability i.e $\mathbb{P}(B) \neq 0$. For all events A we define the conditional probability A knowing B by

$$\mathbb{P}_B(A) = \mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

- $\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B)$
- The application $\mathbb{P}(\cdot|B)$ defines a measure on $(\Omega, \mathcal{A})$
- If $A \perp\!\!\!\perp B$ then $\mathbb{P}(A|B) = \mathbb{P}(A)$

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Total probability law formula and Bayes formula

- Total probability law:

$$\mathbb{P}(A) = \mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|B^c)\mathbb{P}(B^c)$$

- Two players A and B owns respectively a and b euros. They throw a dice where a odd number apear with probability p. The player B gives 1 euro to A if a odd number appear and the converse if a even number appears. We define u_a the probability that A bankrupt. We have

$$u_a = pu_{a+1} + (1-p)u_{a-1}$$

- Bayes law:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A)}$$

In the practice, the total probability law is used to compute $\mathbb{P}(A)$.

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Total probability law formula and Bayes formula

Proposition

Let $A_1, \ldots, A_N$ a partition of Ω then

$$\mathbb{P}(A) = \sum_{i=1}^{N} \mathbb{P}(A|A_i)\mathbb{P}(A_i)$$

$$\mathbb{P}(A_i|A) = \frac{\mathbb{P}(A|A_i)\mathbb{P}(A_i)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|A_i)\mathbb{P}(A_i)}{\sum_{i=1}^{N} \mathbb{P}(A|A_i)\mathbb{P}(A_i)}$$

Conditional Law

- Let X and Y two random variables. One can write

$$\mathbb{P}(Y \in A, X \in B) = \int \mathbb{P}(Y \in A|X = x)\mathbb{P}_X(dx) = \mathbb{E}[\mathbf{1}_B \mathbb{P}(Y \in A|X)]$$

- The quantity $\mathbb{P}(Y \in A|X = x)$ is a notation which corresponds to the Radon Nykodym derivative
- The family $(\mathbb{P}(Y \in \cdot|X = x)_{x \in \mathbb{R}}$ is called conditional probability law family of Y knowing X.
- The conditional law of Y knowing X is denoted by $\mathbb{P}(Y \in \cdot|X)$

Conditional Law

- In the discrete case, the conditional probability law family is easy to obtain. In particular

$$\mathbb{P}(Y = y|X = x) = \frac{\mathbb{P}(Y = y, X = x)}{\mathbb{P}(X = x)}$$

  we have then

$$\mathbb{P}(Y \in \cdot|X) = \sum_{x \in S(X)} \mathbb{P}(Y = y|X = x)\mathbf{1}_{X=x}$$

- In the continuous case, we speak about conditional density. To this end, we put

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x, y)}{f_X(x)}\mathbf{1}_{f_X(x)>0}$$

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Conditional expectation

- So far we have addressed conditional probability. We want to construct a notion of conditional expectation. Let us consider the following

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \mathbb{E}[\mathbf{1}_A|B]$$

- Then one is tempting to define the conditional expectation of a r.v knowing an event by

$$\mathbb{E}[X|B] = \frac{\mathbb{E}[X\mathbf{1}_B]}{\mathbb{P}[B]}$$

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- Now we aim to extend this notation to the conditional expectation to a r.v knowing a σ-algebra $\mathcal{B}$:

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Conditional expectation

- Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let B such that $0 < \mathbb{P}[B] < 1$. Consider $\mathcal{B} = \sigma(B)$ the σ-algebra generated by B.

$$\mathcal{B} = \{\emptyset, B, B^c, \Omega\},$$

- We put for X a L^1 r.v

$$\mathbb{E}[X|\mathcal{B}] = \mathbb{E}[X|B]\mathbf{1}_B + \mathbb{E}[X|B^c]\mathbf{1}_{B^c}$$

- This is a random variable called conditional expectation of X knowing $\mathcal{B}$.
- Note that this r.v is measurable with respect to $\mathcal{B}$

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Conditional expectation

- Let us investigate the property of this random variable

$$Y = \mathbb{E}[X|\mathcal{B}] = \mathbb{E}[X|B]\mathbf{1}_B + \mathbb{E}[X|B^c]\mathbf{1}_{B^c}$$

- First note that

$$\begin{aligned}
\mathbb{E}[Y\mathbf{1}_B] &= \mathbb{E}[(\mathbb{E}[X|B]\mathbf{1}_B + \mathbb{E}[X|B^c]\mathbf{1}_{B^c})\,\mathbf{1}_B] \\
&= \mathbb{E}[(\mathbb{E}[X|B])\,\mathbf{1}_B] \\
&= \mathbb{E}[X|B]\mathbb{E}[\mathbf{1}_B] \\
&= \frac{\mathbb{E}[X\mathbf{1}_B]}{\mathbb{P}[B]}\mathbb{P}[B] \\
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&= \mathbb{E}[X|B]\mathbb{E}[\mathbf{1}_B] \\
&= \frac{\mathbb{E}[X\mathbf{1}_B]}{\mathbb{P}[B]}\mathbb{P}[B] \\
&= \mathbb{E}[X\mathbf{1}_B] \\
\mathbb{E}[Y\mathbf{1}_{B^c}] &= \mathbb{E}[X\mathbf{1}_{B^c}]
\end{aligned}$$

- We easy see also that $\mathbb{E}[Y\mathbf{1}_\emptyset] = \mathbb{E}[X\mathbf{1}_\emptyset]$ and $\mathbb{E}[Y] = \mathbb{E}[Y\mathbf{1}_\Omega] = \mathbb{E}[X\mathbf{1}_\Omega] = \mathbb{E}[X]$

Conditional expectation

- Let us investigate the property of this random variable

$$Y = \mathbb{E}[X|\mathcal{B}] = \mathbb{E}[X|B]\mathbf{1}_B + \mathbb{E}[X|B^c]\mathbf{1}_{B^c}$$

- First note that

$$\begin{aligned}
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Conditional expectation

- As a conclusion we can see that for all event $G \in \mathcal{B} = \{\emptyset, B, B^c, \Omega\}$ we have
$$\mathbb{E}[Y\mathbf{1}_G] = \mathbb{E}[X\mathbf{1}_G] \tag{1}$$

- The r.v $Y = \mathbb{E}[X|\mathcal{B}]$ is the only r.v $\mathcal{B}$ mesurable satisfying the above property.
- Indeed a $\mathcal{B}$ mesurable r.v Z can be written in form of
$$Z = \alpha\mathbf{1}_B + \beta\mathbf{1}_{B^c}$$
then asking (1) implies $\alpha = \mathbb{E}[X|B]$ and $\beta = \mathbb{E}[X|B^c]$

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Conditional expectation

- Let us go further and consider $\mathcal{B} = \sigma\{B_i, i = 1, \ldots, N\}$, where B_i is a partition of Ω, that is

$$\Omega = \bigcup_{i=1}^{N} B_i, \quad B_i \cap B_j = \emptyset, i \neq j$$

- We define

$$\mathbb{E}[X|\mathcal{B}] = \sum_{i=1}^{N} \mathbb{E}[X|B_i]\mathbf{1}_{B_i}$$

- One can verify that for all $G \in \mathcal{B}$

$$\mathbb{E}\left[\mathbb{E}[X|\mathcal{B}]\mathbf{1}_G\right] = \mathbb{E}[X\mathbf{1}_G]$$

and this is the only $\mathcal{B}$ mesurable r.v satisfying such a property.

Conditional expectation

- We have the following theorem

Theorem

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $\mathcal{B} \subset \mathcal{A}$. Let X be a L^1 r.v. There exists a unique r.v Y with is $\mathcal{B}$ mesurable such that

$$\mathbb{E}[Y\mathbf{1}_G] = \mathbb{E}[X\mathbf{1}_G],$$

for all $G \in \mathcal{B}$. We denote this r.v

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- The conditioning calls for partial information and as we shall see the r.v $\mathbb{E}[X|\mathcal{B}]$ is somehow best "approximation" of X knowing only the information included in $\mathcal{B}$.
- Come back to $\mathcal{B} = \{\emptyset, B, B^c, \Omega\}$ we recall that

$$\begin{aligned}
\mathbb{E}[X|\mathcal{B}] &= \mathbb{E}[X|B]\mathbf{1}_B + \mathbb{E}[X|B^c]\mathbf{1}_{B^c} && (2)\\
&= \sqrt{\mathbb{P}[B]}\mathbb{E}[X|B]\frac{\mathbf{1}_B}{\sqrt{\mathbb{P}[B]}} + \sqrt{\mathbb{P}[B^c]}\mathbb{E}[X|B^c]\frac{\mathbf{1}_{B^c}}{\sqrt{\mathbb{P}[B^c]}} && (3)\\
&= \mathbb{E}\left[X\frac{\mathbf{1}_B}{\sqrt{\mathbb{P}[B]}}\right]\frac{\mathbf{1}_B}{\sqrt{\mathbb{P}[B]}} + \mathbb{E}\left[X\frac{\mathbf{1}_{B^c}}{\sqrt{\mathbb{P}[B^c]}}\right]\frac{\mathbf{1}_{B^c}}{\sqrt{\mathbb{P}[B^c]}} && (4)
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Conditional expectation

- If X is L^2 one can write

$$\mathbb{E}[X|\mathcal{B}] = \mathbb{E}\left[X\frac{\mathbf{1}_B}{\sqrt{\mathbb{P}[B]}}\right]\frac{\mathbf{1}_B}{\sqrt{\mathbb{P}[B]}} + \mathbb{E}\left[X\frac{\mathbf{1}_{B^c}}{\sqrt{\mathbb{P}[B^c]}}\right]\frac{\mathbf{1}_{B^c}}{\sqrt{\mathbb{P}[B^c]}} \tag{5}$$

  in the form

$$\mathbb{E}[X|\mathcal{B}] = \left\langle X, \frac{\mathbf{1}_B}{\sqrt{\mathbb{P}[B]}}\right\rangle\frac{\mathbf{1}_B}{\sqrt{\mathbb{P}[B]}} + \left\langle X, \frac{\mathbf{1}_{B^c}}{\sqrt{\mathbb{P}[B^c]}}\right\rangle\frac{\mathbf{1}_{B^c}}{\sqrt{\mathbb{P}[B^c]}} \tag{6}$$

  where

$$\langle X, Y\rangle = \mathbb{E}[XY],$$

  is the scalar product in L^2

- Note that one can easily check that $\left\{\frac{\mathbf{1}_B}{\sqrt{\mathbb{P}[B]}}, \frac{\mathbf{1}_{B^c}}{\sqrt{\mathbb{P}[B^c]}}\right\}$ is an orthonormal basis of $L^2((\Omega, \mathcal{B}, \mathbb{P}))$
- $\mathbb{E}[X|\mathcal{B}]$ is then just the L^2 orthonormal projection of X onto $L^2((\Omega, \mathcal{B}, \mathbb{P}))$.

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Conditional expectation

- In fact, in the case where X is L^2, the property

$$\mathbb{E}[\mathbb{E}[X|\mathcal{B}]\mathbf{1}_G] = \mathbb{E}[X\mathbf{1}_G]$$

  for all $G \in \mathcal{B}$ means that $\mathbb{E}[X|\mathcal{B}]$ is the orthogonal projection of X onto $L^2((\Omega, \mathcal{B}, \mathbb{P}))$
- We can then express the following result which is useful in some situation (for example in the Gaussian context)

Theorem

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $\mathcal{B} \subset \mathcal{A}$. Let X be a L^2 r.v.

The conditional expectation of X knowing $\mathcal{B}$ is the orthogonal projection of X onto $L^2((\Omega, \mathcal{B}, \mathbb{P}))$

Conditional expectation

- Recall that the conditional law of Y knowing X was given by

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} \mathbf{1}_{f_X(x)>0}, \quad f_{Y|X}(y) = \frac{f_{X,Y}(X,y)}{f_X(X)} \mathbf{1}_{f_X(x)>0}$$

with

$$f_X(x) = \int f_{X,Y}(x,y)dy$$

- Let denote $\mathbb{E}[h(Y)|X] = \mathbb{E}[h(Y)|\sigma(X)]$, where $\sigma(X)$ is the σ-algebra generated by X
- We have

$$\mathbb{E}[h(Y)|X] = \int h(y) f_{Y|X}(X,y)dy$$

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Conditional expectation

- Some useful properties

$$\mathbb{E}[\mathbb{E}[X|\mathcal{B}]] = \mathbb{E}[X]$$

- if X is independent of $\mathcal{B}$

$$\mathbb{E}[X|\mathcal{B}] = \mathbb{E}[X]$$

- If X is $\mathcal{B}$ mesurable

$$\mathbb{E}[X|\mathcal{B}] = X$$

- If Z is $\mathcal{B}$ mesurable

$$\mathbb{E}[X\,Z|\mathcal{B}] = \mathbb{E}[X|\mathcal{B}]Z$$

Estimation

Generality

- Let us consider a parametric model where θ is an unknown parameter valued in $\Theta \subset \mathbb{R}^d$
- Recall that an estimator of θ is a r.v which is measurable with respect to a n sample $X_1, \ldots, X_n$

Definition

- An estimator T is said to be unbiased if for all $\theta \in \Theta$

$$\mathbb{E}_\theta[T] = \theta$$

- T is said to be consistent if for all $\theta \in \Theta$

$$T(X_1, \ldots, X_n) \to_{n\to\infty} \theta$$

in probability or almost surely (with respect to $\mathbb{P}_\theta$)
- T is said asymptotically normal if there exists a sequence (a_n) converging to ∞ such that

$$a_n\left(T(X_1, \ldots, X_n) - \theta\right) \to \mathcal{N}(0, 1)$$

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Moment estimation

- Let $(X_1, \ldots, X_n)$ a sample
- Recall that the moment of order k for a r.v is

$$\mathbb{E}[X_1^k] = \mathbb{E}[X_i^k], i = 1, \ldots, k$$

- We can replace these moments by their empirical version that is

$$\bar{X_n^k} = \frac{1}{n}\sum_{i=1}^{n} X_i^k$$

- The centered version

$$\mathbb{E}\left[(X_1 - E[X_1])^k\right]$$

$$\bar{X_n^k} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^k$$

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Moment estimation

- Method principle
- Assuming that you can apply the Law of large numbers we have

$$\bar{X}_n^k = \frac{1}{n}\sum_{i=1}^{n} X_i^k \xrightarrow{a.s} \mathbb{E}X_1^k$$

- Assume that $X = (X_1, \ldots, X_n)$ is distributed along $\mathbb{P}_\theta$ where $\theta \in \Theta$ is unknown.
- Hope: extract information on θ by knowing the moment

Moment estimation

- Example
- Bernoulli of parameter θ: $\mathcal{B}(\theta)$

$$\mathbb{E}[X_1] = \theta$$

we can use the first moment

$$T = \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{a.s} \theta$$

- We also have

$$\mathbb{E}[X_1^2] = \theta$$

we can use the second moment

$$\bar{X}_n \xrightarrow{a.s} \theta, \quad T = \bar{X}_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{a.s} \theta$$

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Moment estimation

- Example
- Binomial of parameter (k, θ). Assume you know k and just want to estimate θ

$$\mathbb{E}[X_1] = k\theta$$

we can use the first moment

$$T = \frac{1}{k}\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{a.s} \theta$$

- Assume you do not know k and need to estimate k and θ you should use also the second moment

$$Var(X_1) = \mathbb{E}[(X_1 - \mathbb{E}(X_1))^2] = k\theta(1-\theta) = \mathbb{E}[X_1](1-\theta)$$

- Then

$$\theta = 1 - \frac{Var(X_1)}{\mathbb{E}[X_1]}, \quad k = \frac{\mathbb{E}[X_1]}{1 - \frac{Var(X_1)}{\mathbb{E}[X_1]}}$$

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- Then we can estimate k and θ by putting

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$$

and defining

$$\hat{\theta}(X_1, \ldots, X_n) = 1 - \frac{\hat{\sigma}_n^2}{\bar{X}_n}, \quad \hat{k}(X_1, \ldots, X_n) = \frac{\bar{X}_n}{1 - \frac{\hat{\sigma}_n^2}{\bar{X}_n}}$$

Moment estimation

- Case of a sample $(X_1, \ldots, X_n)$ whose density is $f_\theta(x) = \theta e^{-\theta x} \mathbf{1}_{\mathbb{R}^+}(x)$
- simple computation shows that

$$\mathbb{E}[X_1] = \frac{1}{\theta}$$

- Then our estimator of θ can be chosen as

$$\hat{\theta} = \frac{1}{\bar{X}_n}$$

- Exercise: do the same job for $(X_1, \ldots, X_n)$ distributed along $\mathcal{N}(\mu, \sigma^2)$

Moment estimation

- In order to summarize. Assume you want to estimate $g(\theta)$. First you should find h such that

$$\mathbb{E}[h(X_1)] = g(\theta)$$

- Determine the number p of moments you shall need to recover $g(\theta)$
- Then compute the p moments you need and connect them to the quantity you aim to estimate
- Replace these p moments by their empirical version.
- Unbiaised, asymptotic normality, Delta method

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- Comme back to the initial question with the notion of bias and asymtptotic normality.
- If you have found h such that $\mathbb{E}[h(X_1)] = g(\theta)$ then using

$$T = \frac{1}{n} \sum_{i=1}^{n} h(X_i) \xrightarrow{a.s} g(\theta)$$

  T is an unbiaised estimator of $g(\theta)$
- Assume that $Var(h(X_1)) = \sigma^2(\theta)$ then we have

$$\sqrt{n}\left(\frac{T - g(\theta)}{\sigma(\theta)}\right) \xrightarrow{\mathcal{L}_\theta} \mathcal{N}(0, 1)$$

Moment estimation

- One can see that the moment method has weakness
- First you can see that in the study of asymptotically normality one see that it depends on $\sigma(\theta)$ which is also unknown.
- You can avoid this obstacle using Slutsky Lemma, you look at

$$\sqrt{n}\left(\frac{T - g(\theta)}{\hat{\sigma}_n^2}\right) \xrightarrow{\mathcal{L}_\theta} \mathcal{N}(0, 1)$$

Moment estimation

- It is not evident to find h such that $\mathbb{E}[h(X_1)] = g(\theta)$. For example the density case where $f_\theta(x) = \theta e^{-\theta x}\mathbf{1}_{\mathbb{R}^+}(x)$, the estimator of θ was

$$T = \frac{n}{X_1 + \ldots + X_n}$$

and it is not even easy to compute $\mathbb{E}[T]$ which makes the study of bias not straightforward.

- Concerning the asymptotically normality property you have to use delta method to get

$$\sqrt{n}\left(\bar{X}_n - \frac{1}{\theta}\right) \xrightarrow{\mathcal{L}_\theta} \mathcal{N}(0, 1/\theta^2), \text{then} \quad \sqrt{n}\,(T - \theta) \xrightarrow{\mathcal{L}_\theta} \mathcal{N}(0, \theta^2)$$

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Maximum likelihood

- The framework is the following, we consider a parametric model $\mathcal{P} = \{\mathbb{P}_\theta, \theta \in \Theta\}$ and we consider that the model is dominated in the sense that for all θ there exists f_θ such that for all $A \in \mathcal{A}$:

$$\mathbb{P}_\theta(A) = \int_A f_\theta(x) d\mu(x)$$

Definition (Vraissemblance)

Let $(X_1, \ldots, X_n)$ be a n-sample of probability $\mathbf{P}_\theta$, we call likelihood of this sample, the joint density of this sample with respect to μ. We denote it as

$$L\left(x_1, \ldots, x_n; \theta;\right).$$

In general this can be expressed as

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- In the discrete case it takes the form

$$L_n(x_1, \ldots, x_n, \theta) = \mathbb{P}_\theta(X_1 = x_1) \ldots \mathbb{P}_\theta(X_n = x_n)$$

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- Example
- Let $(X_1, \ldots, X_n)$ be a n-sample of law $\mathcal{N}(m, \sigma^2)$. Assume that the unknown parameters are $\theta = (m, \sigma^2) \in \mathbf{R} \times \mathbf{R}_+$.

$$L\left(x_1, \ldots, x_n; \theta\right) = \prod_{i=1}^{n} \frac{1}{\sqrt{2 \prod \sigma^2}} e^{-\frac{(x_i - m)^2}{2\sigma^2}} = \frac{1}{\left(2 \prod \sigma^2\right)^{n/2}} e^{-\frac{\sum_{i=1}^{n}(x_i - m)^2}{2\sigma^2}}.$$

- Let $(X_1, \ldots, X_n)$ be a n-sample of law $\mathcal{P}(\theta)$. Assume that the unknown parameter $\theta \in \mathbf{R}$.

$$L\left(x_1, \ldots, x_n; \theta\right) = \prod_{i=1}^{n} e^{-\theta} \frac{\theta^{x_i}}{x_i!} = e^{-n\theta} \frac{\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}.$$

MLE

Definition

Let consider a statistical model dominated by a measure μ and let $L(X,\theta)$ be its likelihood function. All statistic $\hat{\theta}_n^{MV} = \hat{\theta}_n^{MV}(X_1,\dots,X_n)$ such that

$$L(X_1,\dots,X_n,\hat{\theta}_n^{MV}) = \max_{\theta} L(X_1,\dots,X_n,\theta)$$

is called estimator of the maximum likelihood. We shall denote

$$\hat{\theta}_n^{MV} = argmax\ L(X_1,\dots,X_n,\theta)$$

if there are several point where the maximum is reached, we can replace $=$ by $\in$

In the sequel, we shall denote the so-called log likelihood

$$l_n(\theta) = -\frac{1}{n}\sum_{i=1}^{n} \ln L(X_i,\theta).$$

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- Example
- Laplace model $f(x, \theta) = \frac{1}{2\sigma} \exp\left(-\frac{|x-\theta|}{\sigma}\right)$, $\theta \in \mathbf{R}$, unknown and σ known.

$$l_n(\theta) = \ln(2\sigma) + \frac{1}{n\sigma} \sum_{i=1}^{n} |X_i - \theta|.$$

- We shall need to find the minium of $\sum |X_i - \theta|$. Note that this function is almost surely differentiable and its differential h is given by

$$-\sum_{i=1}^{n} sign(X_i - \theta) = h(\theta).$$

if n is even the differential vanishes on every point of $[X_{(n/2)}, X_{(n/2+1)}]$ and then any point of this interval is an MLE. If n is odd a unique MLE is the mediane but there is no point where the differential vanishes.

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MLE

- Cauchy law $f(x) = \frac{1}{\pi(1+(x-\theta)^2)}$.
  The critical point study is not explicit, in general there exists many critical point and then many MLE.
- Consider a model of the form

  $$f(x, \theta) = f_0(x - \theta)$$

  with

  $$f_0(x) = \frac{e^{-|x|/2}}{2\sqrt{2\pi|x|}}.$$

  then the likelihood converges towards $+\infty$ when $\theta \to X_i$ for all i then tehre is no MLE.

MLE

- Normal case $\mathcal{N}(\mu, \sigma^2)$
- Bernoulli case: $\mathcal{B}(\theta)$
- Uniform law case: $\mathcal{U}([0, \theta])$

MLE

- What can we say about the asymptotic behaviour of the MLE
- First we shall address the consistency
- To this end we introduce an assumption

$$\int |\ln f_\theta(x)| f_{\theta^*}(x) d\mu(x) < \infty, \ \forall \theta \in \Theta. \tag{7}$$

- This means that the r.v

$$-\ln(f_\theta(X_1)) \in L^1$$

and then we can applied the LLN to get that

$$l_n(\theta) \overset{\mathbb{P}_{\theta^*}\, a.s}{\longrightarrow} J(\theta) := -\int f(x, \theta^*) \ln f(x, \theta) d\mu$$

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1. We have $J(\theta) \geqslant J(\theta^*)$.
2. If moreover the model is identifiable the inequality is strict as soon as $\theta \neq \theta^*$.
3. Now we know that $l_n(\theta)$ converges towards $J(\theta)$ we can hope that the argmin of $l_n(\theta)$ converges towards the argmin of $J(\theta)$ which appears to be θ^* under the hypotheses of identifiability.

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Theorem

Suppose that Θ *is an open set of* $\mathbf{R}$ *and*

1. *that for all* x *the density* $f(x, \theta)$ *is continuous in* θ*,*
2. *that the model is identifiable*
3. *that the Hypothesis (7) is satisfied*
4. *that for all* n $\hat{\theta}_n^{MV}$ *exists and that the set of local minima of* $l_n(\theta)$ *is a bounded closed interval include in* θ*.*

then $\hat{\theta}_n^{MV}$ *is a consistant estimator (which converges in probability with respect to* $\mathbb{P}_{\theta^*}$*).*

Weibull Model of density $f(x, \theta) = \theta x^{\theta-1} \exp(-x^{\theta}) \mathbf{1}_{x>0}$. We then obtain

$$
\begin{aligned}
l_n(\theta) &= -\ln\theta - (\theta - 1)\frac{1}{n}\sum_{i=1}^{n} \ln X_i + \frac{1}{n}\sum_{i=1}^{n} X_i^{\theta} \\
l_n'(\theta) &= -\frac{1}{\theta} - \frac{1}{n}\sum_{i=1}^{n} \ln X_i + \frac{1}{n}\sum X_i^{\theta} \ln X_i \\
l_n''(\theta) &= \frac{1}{\theta^2} + \frac{1}{n}\sum X_i^{\theta} (\ln X_i)^2 > 0.
\end{aligned}
$$

a study of the function shows that there exists only one critival point which is then a global minimum, we have then existence and uniqueness $\hat{\theta}_n^{MV}$. It remains just to verify that

$$\mathbf{E}_{\theta^*}\left(\left|\ln f_{\theta}(X)\right|\right) < +\infty.$$

and then we conclude that $\hat{\theta}_n^{MV}$ is consistent.

MLE

We shall say that a model is ML regular if

1. The model is dominated
2. Θ is an open set of $\mathbf{R}$ and $f(x,\theta) > 0 \iff f(x,\theta') > 0$
3. The functions f and $l = \ln f$ are C^2 in θ.
4. $\forall \theta^*$ there exists a neighborhood of θ^* denoted by U and a function $\Lambda(x)$ such that $|l''(x,\theta)| \leqslant \Lambda(x),\ |l'(x,\theta)| \leqslant \Lambda(x),\ |l'(x,\theta)|^2 \leqslant \Lambda(x)$ for all $\theta \in U$ and μ almost surely in x and
$$\int \Lambda(x) \sup_{\theta \in U} f(x,\theta) d\mu < \infty.$$
5. $I(\theta) := \mathbf{E}_{\theta^*}\left[l'(X,\theta^*)l'(X,\theta^*)^t\right] = -\mathbf{E}_{\theta^*}\left[l''(X,\theta^*)\right] > 0,\ \forall \theta \in \Theta.$

MLE

Theorem (T.C.L pour $\hat{\theta}_n^{MV}$)

Suppose that the model M.V. is regular and Let $\hat{\theta}_n^{MV}$ be a sequence of consistent de square root of $l'_n(\theta) = 0$. Then $\forall \theta^* \in \theta$

$$\sqrt{n}(\hat{\theta}_n^{MV} - \theta^*) \to \mathcal{N}(0, 1/I(\theta^*)).$$

The quantity

$$I(\theta) := \mathbf{E}_{\theta^*}\left[l'(X, \theta^*) l'(X, \theta^*)^t\right] = -\mathbf{E}_{\theta^*}\left[l''(X, \theta^*)\right]$$

is usually called the Fisher information

- Why are we interested by unbiased estimator?
- Let (T_n) an estimator of θ, we have the quadratic risk defined by

$$\mathbb{E}((T_n-\theta)^2)$$

which corresponds to the L^2 distance between our estimator T_n and the target θ

- One can write

$$\begin{aligned}
& \mathbb{E}((T_n-\theta)^2) \\
= \; & \mathbb{E}((T_n-\mathbb{E}(T_n)+\mathbb{E}(T_n)-\theta)^2) \\
= \; & \mathbb{E}((T_n-\mathbb{E}(T_n))^2+2\mathbb{E}((T_n-\mathbb{T}_n)(\mathbb{E}(T_n)-\theta))+(\mathbb{E}(T_n)-\theta)^2 \\
= \; & {\color{red}\mathbb{E}((T_n-\mathbb{E}(T_n))^2}+{\color{blue}(\mathbb{E}(T_n)-\theta)^2}
\end{aligned}$$

which is called the variance-bias decomposition. The bias makes the distance larger.

Confidence set

Confidence set

- In this section we shall follow an example to make clear the idea behind the confidence set
- Essentially when we make an estimation we are forced to make an error. Confidence set are here to control this error.
- The idea is to construct a random interval (or set in higher dimension) who contains the true parameter with high probability.
- For example if $\bar{\mu}$ is an estimation we want to determine ϵ such that a true parameter satisfies

$$\mathbb{P}[\mu \in [-\epsilon + \bar{\mu}, \epsilon + \bar{\mu}]] = 1 - \alpha$$

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- Let consider the guiding example of $(X_1, \dots, X_n)$ a n-sample of Bernoulli law of parameter θ^*: $\mathcal{B}(\theta^*)$
- As we have seen a good estimator is

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

- We know that

$$\bar{X}_n \xrightarrow{\mathbb{P}_{\theta^*}} \theta^*$$

- Let us try to estimate

$$\mathbb{P}[\theta^* \in [\bar{X}_n - \epsilon, \bar{X}_n + \epsilon]] = \mathbb{P}_{\theta^*}[|\bar{X}_n - \theta^*| \leqslant \epsilon]$$

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- First let us check that

$$\mathbb{E}[\bar{X}_n] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \mathbb{E}[X_1] = \theta^*$$

and

$$Var_\theta\left(\bar{X}_n\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var[X_i] = \frac{\theta^*(1-\theta^*)}{n}.$$

- Then we can apply Bienaymé Chebyschev

$$\begin{aligned}
\mathbb{P}[|\bar{X}_n - \theta^*| > \epsilon] &\leqslant \frac{Var(\bar{X}_n)}{\epsilon^2} && (8)\\
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- Now one can see that for all $x \in [0,1]$

$$x(1-x) \leqslant \frac{1}{4}$$

then

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$$\alpha = \frac{1}{4n\epsilon^2}$$

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$$\mathbb{P}[|\bar{X}_n - \theta^*| > \frac{1}{\sqrt{4n\alpha}}] \leqslant \alpha$$

which finally yields

$$\mathbb{P}[\theta^* \in [\bar{X}_n - \frac{1}{\sqrt{4n\alpha}}, \bar{X}_n + \frac{1}{\sqrt{4n\alpha}}]] \geqslant 1 - \alpha$$

- As we can see through this approach we can adjust the parameter α to make the above probability close to 1. This parameter represents a risk.
- Often we choose $\alpha = 0,05 = 5.10^{-2}$

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Confidence set

- The confidence interval is then
$$[\bar{X}_n - \frac{1}{\sqrt{4n\alpha}}, \bar{X}_n + \frac{1}{\sqrt{4n\alpha}}]$$
- Assume you want a small interval this imposes
$$\frac{1}{\sqrt{4n\alpha}}$$
to be small
- For example for $\alpha = 0,05$ if you want $\frac{1}{\sqrt{4n\alpha}} = 0,1$ you need $n =$
- For example for $\alpha = 0,05$ if you want $\frac{1}{\sqrt{4n\alpha}} = 0,01$ you need $n =$
- Note that since this is $\sqrt{n}$ which is involved, when you want to obtain a smallr interval (gaining a significative number you need a sample 100 times bigger).

Confidence set

- Using this approach you can see that you can need a large number n. But when n is large enough you can use the Central Limit Theorem.
- Recall that

$$\sqrt{n}\left(\frac{\bar{X}_n - \theta}{\sqrt{\theta(1-\theta^*)}}\right) \xrightarrow{\mathcal{L}_{\theta^*}} \mathcal{N}(0,1)$$

- Since

$$\bar{X}_n \xrightarrow{\mathbb{P}_{\theta^*}} \theta^*,$$

then by Slutsky we have

$$\sqrt{n}\left(\frac{\bar{X}_n - \theta^*}{\sqrt{\bar{X}_n^*(1-\bar{X}_n^*)}}\right) = \frac{\sqrt{\theta(1-\theta^*)}}{\sqrt{\bar{X}_n^*(1-\bar{X}_n^*)}}\sqrt{n}\left(\frac{\bar{X}_n - \theta}{\sqrt{\theta(1-\theta^*)}}\right) \xrightarrow{\mathcal{L}_{\theta^*}} \mathcal{N}(0,1)$$

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Confidence set

- Keep in mind that for n large enough we have

$$\sqrt{n}\left(\frac{\bar{X}_n - \theta}{\sqrt{\bar{X}_n^*(1-\bar{X}_n^*)}}\right) \overset{\mathcal{L}_{\theta^*}}{\simeq} \mathcal{N}(0,1)$$

- We can say that

$$\mathbb{P}_{\theta^*}\left(\left[\bar{X}_n - q_{1-\alpha/2}\sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n}}\ ;\ \bar{X}_n + q_{1-\alpha/2}\sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n}}\ \right] \ni \theta^*\right)$$

$$= \mathbb{P}_{\theta^*}\left(\left|\bar{X}_n - \theta^*\right| \leqslant q_{1-\alpha/2}\sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n}}\right)$$

$$= \mathbb{P}_{\theta^*}\left(\left|\sqrt{n}\,\frac{\hat{\theta}^*{}_n - \theta^*}{\sqrt{\bar{X}_n^*(1-\bar{X}_n^*)}}\right| \leqslant q_{1-\alpha/2}\right) \simeq \mathbb{P}[|X| \leqslant q_{1-\alpha/2}], \qquad (10)$$

where $X \sim \mathcal{N}(0,1)$.

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$$\begin{aligned}
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&= \mathbb{P}_{\theta^*}\left(\left|\bar{X}_n - \theta^*\right| \leqslant q_{1-\alpha/2}\sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n}}\right) \\
&= \mathbb{P}_{\theta^*}\left(\left|\sqrt{n}\,\frac{\hat{\theta}^*{}_n - \theta^*}{\sqrt{\bar{X}_n^*(1-\bar{X}_n^*)}}\right| \leqslant q_{1-\alpha/2}\right) \simeq \mathbb{P}[|X| \leqslant q_{1-\alpha/2}], \qquad (10)
\end{aligned}$$

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Confidence set

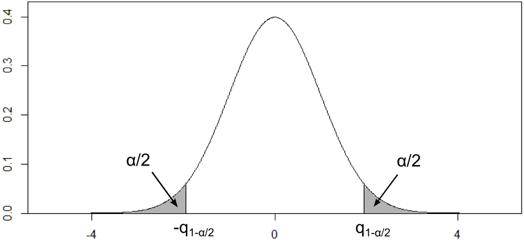
- So far we have

$$\mathbb{P}_{\theta^*}\left(\left[\bar{X}_n - q_{1-\alpha/2}\sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n}}\ ;\ \bar{X}_n + q_{1-\alpha/2}\sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n}}\ \right] \ni \theta^*\right) \tag{11}$$

$$\simeq \mathbb{P}[|X| \leqslant q_{1-\alpha/2}], \tag{12}$$

- Now we can say what is $q_{1-\alpha/2}$,

$$\mathbb{P}(|X| \leqslant q_{1-\alpha/2}) = 1 - (\alpha/2 + \alpha/2) = 1 - \alpha$$



Confidence set

- This way we have construct a confidence interval

$$\left[\bar{X}_n - q_{1-\alpha/2}\sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n}}\ ;\ \bar{X}_n + q_{1-\alpha/2}\sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n}}\ \right]$$

- For example for $\alpha = 0,05$, we get $q_{1-\alpha/2} = 1,96$. This can be read on table of the $\mathcal{N}(0,1)$ law.

Confidence set

- Can we compare the two interval that we have constructed. In fact we can show that

$$\lim_{n\to\infty} \mathbb{P}_{\theta^*}\left(\left[\bar{X}_n - \frac{1}{\sqrt{4n\alpha}}\ ;\ \bar{X}_n + \frac{1}{\sqrt{4n\alpha}}\right] \ni \theta\right) \geqslant 1 - \exp\left(-\frac{1}{2\alpha}\right) = 1 - \circ(\alpha)$$

- Essentially this means that for large n, we have

$$\left[\bar{X}_n - q_{1-\alpha/2}\sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n}}\ ;\ \bar{X}_n + q_{1-\alpha/2}\sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n}}\right] \tag{13}$$

$$\subset \left[\bar{X}_n - \frac{1}{\sqrt{4n\alpha}}\ ;\ \bar{X}_n + \frac{1}{\sqrt{4n\alpha}}\right] \tag{14}$$

then for large n the confidence interval obtained via the CLT is better than the one obtained by Bienaymé Chebychev

- The interest of Bienaymé Tchebychev is that it is true for all n. This can give information for small sample.

Confidence set

- Can we compare the two interval that we have constructed. In fact we can show that

$$\lim_{n\to\infty} \mathbb{P}_{\theta^*}\left(\left[\bar{X}_n - \frac{1}{\sqrt{4n\alpha}}\ ;\ \bar{X}_n + \frac{1}{\sqrt{4n\alpha}}\right] \ni \theta\right) \geqslant 1 - \exp\left(-\frac{1}{2\alpha}\right) = 1 - \circ(\alpha)$$

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- The interest of Bienaymé Tchebychev is that it is true for all n. This can give information for small sample.

Confidence set

- In general for a n-sample $(X_1, \ldots, X_n)$ of a law $\mathbb{P}_{\theta^*}$ for using Bienaymé Tchebychev we need to control the variance independently of θ^*. Here for $\mathcal{B}(\theta^*)$ we have used

$$Var(\bar{X}_n) = \frac{\theta^*(1-\theta^*)}{n} \leqslant \frac{1}{4n}$$

- For Poisson random variable $\mathcal{P}(\theta^*)$ we have

$$Var(\bar{X}_n) = \frac{\theta^*}{n}$$

  and conditions on θ^* have to be known to construct a confidence interval with B-T (example you know that $\theta^* \leqslant M$ for a known value M.
- For using CLT one can use the same trick by replacing the variance in terms of $\bar{X}_n$ and justify it via Slustsky theorem.

Confidence set

- In general if we are not in such a situation, in order to use the CLT, we have to estimate the variance. To this end we have the following estimator
$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$$
- and the corresponding confidence interval is
$$\left[\bar{X}_n - q_{1-\alpha/2}\sqrt{\frac{\hat{\sigma}_n^2}{n}}\ ;\ \bar{X}_n + q_{1-\alpha/2}\sqrt{\frac{\hat{\sigma}_n^2}{n}}\right]$$

- Let us concentrate on this estimator

$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i)^2 - (\bar{X}_n)^2$$

- As we said it is an estimator of the variance. If you come back to the previous chapter, let us adress the usual question, bias, consistency....

Confidence set

- Let us start with the bias

$$
\begin{aligned}
\mathbb{E}[\sigma_n^2] &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i^2] - \mathbb{E}[(\bar{X}_n)^2] \\
&= \mathbb{E}(X_1^2) - \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)^2\right] \\
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Confidence set

- Let us start with the bias

$$\mathbb{E}[\sigma_n^2] \quad = \quad \frac{n-1}{n} Var(X_1)$$

- Then considering

$$S_n^2 = \frac{n}{n-1}\sigma_n^2 = \frac{1}{n-1}\sum_i^n (X_i - \bar{X}_n)^2,$$

we have an unbiased estimator.

Confidence set

- Let assume that $(X_1, \ldots, X_n)^t$ be a Gaussian vector of law $\mathcal{N}(m, \sigma^2)$. We have
$$\frac{n-1}{\sigma^2} S_n^2 \sim \chi^2(n-1)$$
- Indeed note that $Y = \frac{1}{\sigma}(X_1 - m, \ldots, X_n - m)^t \sim \mathcal{N}_n(0, I_n)$
- Define $F = Vect(1_n)$ where $1_n = (1, \ldots, 1)^t$. We easily have $dim(F) = 1$ and $dim(F^\perp) = n - 1$.
- Now note that $P_F(X) = \left\langle \frac{1_n}{\sqrt{n}}, X \right\rangle \frac{1_n}{\sqrt{n}} = \frac{1}{\sigma}(\bar{X}_n - m, \ldots \bar{X}_n - m)^t$ and then
$$P_{F^\perp}(X) = X - P_F(X) = \frac{1}{\sigma}(X_1 - \bar{X}_n, \ldots, X_n - \bar{X}_n)^t$$
- The Cochran Theorem then says that $||P_{F^\perp}(X)||^2 \sim \chi^2(n-1)$. Now it is easy to see that
$$||P_{F^\perp}(X)||^2 = \frac{n-1}{\sigma^2} S_n^2$$

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Confidence set

- This allows to construct confidence interval for the variance of a Gaussian law. Let denote $\chi^k_{1-\alpha}$ the quantile of the $\chi^2(k)$ law that is if $T \sim \chi^2(k)$ then

$$\mathbb{P}[\chi^k_{\alpha/2} \leqslant T \leqslant \chi^k_{1-\alpha/2}] = 1 - \alpha$$

- Then we have

$$\mathbb{P}\left[\chi^k_{\alpha/2} \leqslant \frac{n-1}{\sigma^2} S_n^2 \leqslant \chi^{n-1}_{1-\alpha/2}\right] = 1 - \alpha$$

- This implies

$$\mathbb{P}\left[\frac{n-1}{\chi^{n-1}_{1-\alpha/2}} S_n^2 \leqslant \sigma^2 \leqslant \frac{n-1}{\chi^{n-1}_{\alpha/2}} S_n^2\right] = 1 - \alpha$$

  and then the interval

$$\left[\frac{n-1}{\chi^{n-1}_{1-\alpha/2}} S_n^2, \frac{n-1}{\chi^{n-1}_{\alpha/2}} S_n^2\right]$$

  is a confidence interval of level α for the variance σ^2 of X_1.

Confidence set

- Other possible interesting result when $X_1, \ldots, X_n$ are Gaussian $\mathcal{N}(m, \sigma^2)$

$$\sqrt{n}\left(\frac{\bar{X}_n - m}{\sigma}\right) \sim \mathcal{N}(0, 1)$$

then if σ^2 is known this allows to construct a confidence interval for μ

- If σ^2 is not known replace σ by S_n and we have

$$\sqrt{n}\left(\frac{\bar{X}_n - m}{S_n}\right) \sim \mathcal{T}_{n-1}$$

where $\mathcal{T}_{n-1}$ is a r.v distributed along a Student law of $n-1$ degree of freedom.

Confidence set

- In the above example the confidence interval are bounded but we can also consider bounds which are infinite (only one of course)

Definition

Let $\alpha \in [0, 1]$ fixé and let $\theta^* \in \mathbb{R}^k$

1. When $k = 1$, we call confidence interval of level $1 - \alpha$ for θ^* all random interval I of the form $[a(X_1, \ldots, X_n), b(X_1, \ldots, X_n)]$ xhere $a(X_1, \ldots, X_n)$ and $b(X_1, \ldots, X_n)$ are statistics (independent of θ^*) satisfying

$$\mathbf{P}_\theta(\theta \in [a(X_1, \ldots, X_n), b(X_1, \ldots, X_n)]) = 1 - \alpha.$$

   1. if $a(X_1, \ldots, X_n) > -\infty$ and $b(X_1, \ldots, X_n) < \infty$ we speak about bilateral interval
   2. if $a(X_1, \ldots, X_n) = -\infty$ we speak about left unilateral interval
   3. if $b(X_1, \ldots, X_n) = \infty$ we speak about right unilateral interval

2. When $k > 1$ we speak about confidence set of level $1 - \alpha$ for θ all random subset $R(X_1, \ldots, X_n)$ of $\mathbf{R}^k$ which depends on $(X_1, \ldots, X_n)$ in a measurable way and is independent of θ satisfying

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Confidence set

- We can relax the previous definition by allowing $\geqslant$ instead of $=$

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$$\mathbf{P}_\theta\left(\theta \in R(X_1,\dots,X_n)\right) \geqslant 1-\alpha.$$

Confidence set

- We can also have asymptotic confidence set

Definition

Let $\alpha \in [0,1]$ fixé and let $\theta^* \in \mathbb{R}^k$

1. When $k = 1$, we call confidence interval of level $1-\alpha$ for θ^* all random interval I of the form $[a(X_1,\dots,X_n), b(X_1,\dots,X_n)]$ xhere $a(X_1,\dots,X_n)$ and $b(X_1,\dots,X_n)$ are statistics (independent of θ^*) satisfying

$$\lim_n \mathbf{P}_\theta\left(\theta \in [a(X_1,\dots,X_n), b(X_1,\dots,X_n)]\right) = 1-\alpha.$$

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   3. if $b(X_1,\dots,X_n) = \infty$ we speak about right unilateral interval

2. When $k > 1$ we speak about confidence set of level $1-\alpha$ for θ all random subset $R(X_1,\dots,X_n)$ of $\mathbf{R}^k$ which depends on $(X_1,\dots,X_n)$ in a measurable way and is independent of θ satisfying

$$\lim_n \mathbf{P}_\theta\left(\theta \in R(X_1,\dots,X_n)\right) = 1-\alpha.$$

Confidence set

- One can also use open set for confidence set
- In general there is an infinity of confidence interval. For example with the CLT we can choose

$$\left]-\infty, \bar{X}_n - q_{1-\alpha}\sqrt{\frac{\sigma_n^2}{n}}\right]$$

- Can it be interested to have a interval bound which is infinite? It looks like not sharp!
- Imagine that you known that the unknow quantity is non negative (decibel of a night club, number of student attending the summer school in France); then the part $]-\infty, 0]$ is useless and the interval

$$\left]0, \bar{X}_n - q_{1-\alpha}\sqrt{\frac{\sigma_n^2}{n}}\right] \subset \left]0, \bar{X}_n - q_{1-\alpha/2}\sqrt{\frac{\sigma_n^2}{n}}\right]$$

which makes the interval $\left]0, \bar{X}_n - q_{1-\alpha}\sqrt{\frac{\sigma_n^2}{n}}\right]$ more relevant.

Basic of Regression

Basic of Regression

- First let us start with a simple situation. Let Y be a L^2 r.v. You want to approximate Y by a constant a by minimizing the quadratic error that is you want to find
$$argmin_{a\in\mathbb{R}}\mathbb{E}[(Y-a)^2]$$
- In fact it is easy to check that
$$\min_{a\in\mathbb{R}}\mathbb{E}[(Y-a)^2]$$
is reached for $a=\mathbb{E}[Y]$.
- Indeed one can think in terms of projection of Y onto the subspace of constant function.
- If you do not have the possibility to consider the L^2 norma, one could have thought
$$argmin_{a\in\mathbb{R}}\mathbb{E}[|Y-a|]$$
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- Now imagine you have a couple (X, Y) whose you know the joint distribution. Suppose that X and Y are L^2.
- Consider the situation where you only observe a realization of X let say x. You want to estimate Y knowing this realization. Without further information it is not possible since Y knowing x is random.
- An idea is to approximate Y as an affine function of X, i.e $Y = aX + b$ and you to minimise

$$\min_{a,b} \mathbb{E}[(Y - aX + b)^2]$$

- Here, you see that, you need to find the orthogonal projection onto the subspace of affine function of X. Computations give

$$a = \frac{Cov(X, Y)}{\sigma^2(X)}, \quad b = \mathbb{E}[Y] - \frac{Cov(X, Y)}{\sigma^2(X)}\mathbb{E}[X]$$

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- At this stage let us introduce the so called correlation coefficient

$$\rho = \frac{Cov(X,Y)}{\sigma(X)\sigma(Y)}, \quad |\rho| \leqslant 1$$

- Note that X and Y independent implies $\rho = 0$
- In terms of ρ one can check

$$\min_{a,b} \mathbb{E}[(Y - aX + b)^2] = \sigma^2(Y)(1 - \rho^2)$$

- The error is small when $|\rho|$ is close to 1
- When $\rho = 0$ the error is maximum. In this case the best approximation is $\mathbb{E}[Y]$

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Basic of Regression

- In statistics, i.e in the true life we do not know the law of the couple (X, Y). We have n realizations $((X_1, Y_1), \ldots, (X_n, Y_n))$ and you want to minimize
$$\min_{a,b} \sum_{i=1}^{n} (Y_i - (aX_i + b))^2$$
- In terms of realizations, in concrete terms you want to minimize
$$\min_{a,b} \sum_{i=1}^{n} (y_i - (ax_i + b))^2$$
- Concretely, you replace
$$a = \frac{Cov(X, Y)}{\sigma^2(X)}, \quad b = \mathbb{E}[Y] - \frac{Cov(X, Y)}{\sigma^2(X)}\mathbb{E}[X]$$

  by their empirical versions (variance, covariance, expectation...)

- More generally you can ask to approximate Y as a function $u(X)$ and then minimize

$$\min_u \mathbb{E}[(Y - u(X))^2]$$

- As we already seen this quantity is obtained by using the conditional expectation that is

$$\mathbb{E}[Y|X]$$

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Basic of Regression

- Example of a couple (X, Y) with density

$$f(x,y) = 2e^{-(x+y)}\mathbf{1}_{0\leqslant x\leqslant y}$$

- The conditional expectation is then $f_{Y|X=x} = f_{x,y}(x,y)/f_X(x)$ where

$$f_X(x) = 2e^{-2x}\mathbf{1}_{0\leqslant x}, \quad (exponential \quad law)$$

- We then have

$$f_{Y|X=x}(y) = e^{x-y}\mathbf{1}_{0\leqslant x\leqslant y}$$

- We can then compute

$$\mathbb{E}[Y|X=x] = \int y f_{Y|X=x}(y)dy = \int_x^{+\infty} e^x y e^{-y} dy = x+1$$

- Come back to the Gaussian case
- Let (X, Y) be a Gaussian vector, one can check

$$\mathbb{E}[Y|X] = \mathbb{E}[Y] + \frac{Cov(X,Y)}{Var(X)}(X - \mathbb{E}[X])$$

Theorem

In the Gaussian world the regression curve and the regression line are the same!

- $\mathbb{E}[Y|X]$ is supposed to be the orthogonal projection of Y onto

$$L^2(X) = \{f(X), \mathbb{E}[f(X)^2] < \infty\}$$

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$$Vect\{1, X\}$$

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$$1, \frac{X - \mathbb{E}[X]}{\sqrt{Var(X)}}$$

is an orthonormal basis.

- One can then check

$$\mathbb{E}[Y|X] = \langle 1, Y \rangle 1 + \left\langle \frac{X - \mathbb{E}[X]}{\sqrt{Var(X)}}, Y \right\rangle \frac{X - \mathbb{E}[X]}{\sqrt{Var(X)}}$$

which is exactly another way of writting

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Regression Hyperplan

- Let $X = (X_1, \ldots, X_n)$ be a random vector, we aim to approximate Y by a hyperlan which minimizes

$$\min_{a_1,\ldots,a_n,b} \mathbb{E}\left[\left(Y - \left(b + \sum_{i=1}^{n} a_i X_i\right)\right)^2\right]$$

- We suppose that the dispersion matrix

$$\Gamma_X = \mathbb{E}[(X - \mathbb{E}[X])(X - \mathbb{E}[X])^t]$$

- The regression hyperplan is given by

$$\pi_H(Y) = \mathbb{E}[Y] + \Gamma_{Y,X}\Gamma_X^{-1}(X - \mathbb{E}[X]),$$

where $\Gamma_{Y,X} = \mathbb{E}[(Y - \mathbb{E}(Y))(X - \mathbb{E}(X))]$ is the covariance line matrix $(Cov(Y, X_1), \ldots, Cov(Y, X_n))$

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Regression Hyperplan

- We can also compute the quadratic error

$$\mathbb{E}[(Y - \pi_H(Y))^2] = \Gamma_Y - \Gamma_{Y,X}\Gamma_X^{-1}\Gamma_{X,Y}$$

- Gaussian situation

Theorem

In the Gaussian world if $(X_1, \ldots, X_n, Y)$ is a Gaussian vector, we have

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then the Hyperplan of regression is equal to the conditional expectation.

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Principal Component Analysis: Overview

PCA

- Will be developed in details in the 3rd week
- Assume you have access to p datas (age, sex, color of hair, rate of alcohol in the blood ...) of n people
- The parameter p can be huge and unless for $p \leqslant 3$ it is not possible to represent these datas on a graph
- We want to determine $q < p$ variables which explains the phenomena, we study, and which can be represented in a graph ($q = 2, 3$)

PCA

- The datas are grouped in a matrix X of size $n \times p$

$$X = \left(X^1, \ldots\ldots\ldots\ldots, X^p \right) \tag{15}$$

$$X = \begin{pmatrix} X_{1,1} & \ldots & \ldots & X_{1,p} \\ \vdots & \ldots & \ldots & \vdots \\ X_{i,1} & \ldots & \ldots & X_{i,p} \\ \vdots & \ldots & \ldots & \vdots \\ X_{n,1} & \ldots & \ldots & X_{n,p} \end{pmatrix} = \begin{pmatrix} X_1 \\ \vdots \\ X_i \\ \vdots \\ X_n \end{pmatrix} \tag{16}$$

- Introduce $\bar{X} = \begin{pmatrix} \bar{X}^1 & \ldots & \ldots & \bar{X}^p \end{pmatrix}$, where $\bar{X}^k$ is the mean of the variable X^k. Denote $s_k^2 = Var(X^k) = \frac{1}{n} \sum_{i=1}^{n} (X_{ik} - \bar{X}^k)^2$ the corresponding variance.
- The number of people belongs to $\mathbb{R}^n$ and the variables to $\mathbb{R}^p$ where the average is made by column

PCA

- The centered version

$$Y = \begin{pmatrix} X_{1,1} - \bar{X}^1 & \dots & \dots & X_{1,p} - \bar{X}^p \\ \vdots & \dots & \dots & \vdots \\ X_{j,1} - \bar{X}^1 & \dots & \dots & X_{j,p} - \bar{X}^p \\ \vdots & \dots & \dots & \vdots \\ X_{n,1} - \bar{X}^1 & \dots & \dots & X_{n,p} - \bar{X}^p \end{pmatrix} \tag{17}$$

- The centered and reduced version

$$Z = \begin{pmatrix} \frac{X_{1,1}-\bar{X}^1}{s_1} & \dots & \dots & \frac{X_{1,p}-\bar{X}^p}{s_p} \\ \vdots & \dots & \dots & \vdots \\ \frac{X_{j,1}-\bar{X}^1}{s_1} & \dots & \dots & \frac{X_{j,p}-\bar{X}^p}{s_p} \\ \vdots & \dots & \dots & \vdots \\ \frac{X_{n,1}-\bar{X}^p}{s_1} & \dots & \dots & \frac{X_{n,p}-\bar{X}^p}{s_p} \end{pmatrix}, \quad Var(Z^j) = 1, j = 1, \dots, p \tag{18}$$

PCA

- Let us speak about the distance between two people. To this end consider a symmetric definite positive matrix M of size $p \times p$ and denote
$$\langle x, y\rangle_M = \langle x, My\rangle = x^t My$$
and $||x||_M = \sqrt{\langle x, x\rangle_M}$ as well as
$$d_M(x, y) = ||x - y||_M$$

- Often we consider matrix M of diagonal form $M = diag(m_i)$ and in this case
$$\langle x, y\rangle_M = \sum_{i=1}^{p} m_i x_i y_i$$
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PCA

- Let us make the link between the matrix X, Y, Z and the above distance. Let us consider a diagonal matrix $M = diag(m_i)$

$$\|X_i\|_M^2 = \sum_{k=1}^{p} m_k X_{ik}^2, \quad d_M^2(X_i, X_j) = \sum_{k=1}^{p} m_k (X_{i,k} - X_{j,k})^2$$

- In the case where $M = I_p$ we have

$$d_{I_p}^2(X_i, X_j) = \sum_{k=1}^{p} (X_{i,k} - X_{j,k})^2 = d_{I_p}^2(Y_i, Y_j)$$

- In the case where $M = diag(1/s_1^2, \ldots, 1/s_p^2)$ we have

$$d_M^2(X_i, X_j) = d_{I_p}^2(Z_i, Z_j)$$

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- Now let us define the notion of inertia. Introducing the diagonal matrix $M = diag(m_i)$ allows to consider weight. We define the inertia as

$$I(X) = \sum_{k=1}^{p} m_i d^2(X_i, \bar{X}) = \sum_{k=1}^{p} m_i s_j^2$$

  It measures the dispersion of the data X_i with respect to the barycenter $\bar{X}$.
- In the case $M = diag(1/s_1^2, \ldots, 1/s_p^2)$ we have

$$I(Z) = p$$

PCA

- The p column of X represent a so-called scatter graph.
- Regarding the weight introduced before we shall concentrate on $m_j = 1$ in the context of PCA.
- If we analyze Y we shall say we do non-normalized PCA
- If we analyze Z we do normed PCA and we are going to focus on this case

PCA

- In PCA you can have two points of view
  - Either you analyze the n point people and you will choose the metric with $M = I_p$
  - Or you analyze the p datas and you will choose the metric given by $N = \frac{1}{n} I_n$
- We already have seen the effect of $M = I_p$ on the line of the matrix
- The effect of the matrix N is on the column. Note that

$$Var(X^j) = \frac{1}{n}\sum_{i=1}^{n}(X_{i,j} - \bar{X}^i)^2 = \|Y^j\|_N^2$$

$$Var(Z^j) = \|Y^j\|_N^2 = 1$$

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PCA

- The covariance between X_j and $X_{j'}$ is given by

$$c_{jj'} = \frac{1}{n}\sum_{i=1}^{n}(X_{i,j} - \bar{X}^j)(X_{i,j'} - \bar{X}^{j'}) = \langle Y^i, Y^j \rangle_N$$

- In particular one can easily see that the covariance matrix

$$C = Y^t NY$$

- The correlation between X_j and $X_{j'}$ is given by

$$r_{jj'} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_{i,j} - \bar{X}^j}{s_j}\right)\left(\frac{X_{i,j'} - \bar{X}^{j'}}{s_{j'}}\right) = \langle Y^i, Y^j \rangle_N$$

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$$R = Z^t NZ$$

PCA

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- In particular one can easily see that the correlation matrix

$$R = Z^t N Z$$

PCA

- Let us start by concentrating on the people
- For example an reasonable objective is to find the projection plan such that the distance between the people are the better conserved.
- Let us speak about the projection of a guy. We are in the case $M = I_p$ and we want to project $Z_j \in \mathbb{R}^p$ for example on an axis defined by Δ_α which is directed by a vector v_α of norm 1. The coordinate will be given by

$$f_{j\alpha} = \langle Z_j, v_\alpha \rangle = Z_j^t v_\alpha$$

- Define now

$$f^\alpha := (f_{1\alpha}, \ldots, f_{n\alpha})^t = Zv_\alpha$$

  this the vector of each coordinate of each projection of the Z_j
- We can rewrite

$$f^\alpha = Zv_\alpha = \sum_{j=1}^{p} v_{j\alpha} Z^j$$

PCA

- Method: we are looking for an axis Δ_1 with generator v_1 such that

$$v_1 = argmax_{v_1/\|v_1\|=1} Var(Zv_1)$$

- We can show that this optimization problem can be written as

$$\max_{v/\|v\|=1} \|Rv\|^2$$

with $R = \frac{1}{n} Z^t Z$

- Then this maximum is reached for v_1 the eigenvector associated to the maximum eigenvalue of R

PCA

- Then $f_1 = Zv_1$ is the first principal coomponent
- If we want to find a plan we look for v_2 such that

$$v_2 = argmax_{v_2/v_2 \perp v_1 \|v_2\|=1} Var(Zv_2)$$

- v_2 appears as the second eigenvector corresponding to the second higher eigenvalue. The vector $f_2 = Zv_2$ is the second principal component
- and so on
- Note that f_1 and f_2 are orthogonal and then non correlated.
- Conclusion: to find the principal component we need to diagonalize R.

PCA

- If you denote $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_r$ the eigenvalues of R (here r corresponds to the rank of Z), we can show easily that

$$Var(f_i) = \lambda_i$$

- An important question is how many component shall we need. This can be quantified by looking at the quantity

$$\frac{\lambda_1 + \ldots + \lambda_q}{\lambda_1 + \ldots + \lambda_r} = \frac{\lambda_1 + \ldots + \lambda_q}{Tr(R)}$$

- You can fix a level $1 - \alpha$ and you stop to the first time (first q) where

$$\frac{\lambda_1 + \ldots + \lambda_q}{Tr(R)} \geqslant 1 - \alpha$$

- In practice to find the first eigenvector v_1 and the first eigenvalue λ_1 you can use the power method. Define

$$w_{n+1} = \frac{Rw_n}{||Rw_n||}$$

- We have

$$||Rw_n|| \to_n \lambda_1$$

and

$$w_n \to v_1$$

- In order to find the second eigenvector and the second eigenvalue you do the same job on the orthogonal $vect\, v_1{}^{\perp}$

PCA

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- You can also take the problem from the the p variable size by considering Z^t instead of Z and do the same job.

- Moment method for $\mathcal{N}(\mu, \sigma^2)$
- MLE for $\mathcal{U}([0, \theta])$. Consistency? Confidence set ?
- Consider the density

$$f_\theta(x) = \frac{|x-\theta|}{2} e^{-|x-\theta|} ,$$

Moment method ? Two type of confidence interval ?