TD1 : Conditional Calculus

Exercice 1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let X be a L^1 random variable. Let $\mathcal{B}$ be a sub sigma algebra of $\mathcal{A}$.

(1) Let $A \in \mathcal{A}$. Determine $\mathbb{E}[X|A] = \mathbb{E}[X|\sigma(A)]$
(2) Let $(A_i)_{i\in I}$ be a countable partition of Ω such that $A_i \in \mathcal{A}$ for all $i \in I$. Determine
$$\mathbb{E}[X|\sigma(A_i, i \in I)]$$
Deduce a formula for $\mathbb{E}[X|Z]$ where Z is a discrete random variable.
(3) Determine $\mathbb{E}[\mathbb{E}[X|\mathcal{B}]]$.
(4) Suppose that X is independent of $\mathcal{B}$. Determine $\mathbb{E}[X|\mathcal{B}]$.
(5) Let $\mathcal{F} \subset \mathcal{B}$ be a sub sigma algebra of $\mathcal{A}$. Show that
$$\mathbb{E}[\mathbb{E}[X|\mathcal{B}]|\mathcal{F}] = \mathbb{E}[X|\mathcal{F}]$$
(6) Suppose that $X > 0$ a.s. Show that $\mathbb{E}[X|\mathcal{B}] > 0$ a.s.

Exercice 2. Let X, Y be two independent Poisson random variables. Compute
$$\mathbb{E}[X|X+Y]$$

Exercice 3. Let N be a $\mathbb{N}$-valued r.v. and let $(X_i)_{i\in\mathbb{N}^*}$ be a sequence of i.i.d. r.v. in L^2. Let
$$S = \sum_{i=1}^{N} X_i$$

(1) Compute $\mathbb{E}[S|N]$ and then $\mathbb{E}[S]$
(2) Compute $\mathbb{E}[S^2|N]$ and then $\mathbb{E}[S^2]$. Show that
$$Var(S) = \mathbb{E}[N]Var(X_1) + Var(N)\mathbb{E}[X_1^2]$$

Exercice 4. Let U and V be two independent r.v. uniformly distributed on $\{1, 2, 3, 4, 5\}$. Let $X = \min(U, V)$ and $Y = \max(U, V)$.

(1) Determine the law of (U, Y)
(2) Determine $\mathbb{E}[U|Y = n]$
(3) Deduce $\mathbb{E}[U|Y]$
(4) Determine $\mathbb{E}[Y|U]$
(5) Same questions with X instead of Y.

Exercice 5. Let X uniformly distributed on $\{-1, 1, 2\}$. Compute
$$\mathbb{E}(X||X|)$$

Exercice 6. Let X, Y be two independent r.v. with law P_X and P_Y respectively. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a bounded Borel function. Show that $\mathbb{E}(\Phi(X, Y)|X) = h(X)$ where
$$h(x) = \int_{\mathbb{R}} \Phi(x, y) dP_Y(y).$$

Exercice 7.

(1) Let X be uniformly distributed on $\{-2,-1,1,2\}$, Determine $\mathbb{E}[X]$ and $\mathbb{E}[X|Y]$ where $Y=|X|$. Show that X and $|X|$ are not independent. Conclusion?
(2) Let X and Y be two Bernoulli r.v with parameter $1/2$. Determine $\mathbb{E}[X|\sigma(X+Y)]$. Conclusion?

Exercice 8. For $x\in\mathbb{R}$, we denote by $\lfloor x\rfloor$ the floor function and $F(x)=x-\lfloor x\rfloor$. Let X be a r.v. with exponential distribution with parameter 1 : $P_X(dt)=e^{-t}\mathbb{1}_{]0,+\infty[}(t)dt$.

(1) Determine the law of $\lfloor X\rfloor$.
(2) Compute the conditional expectations $\mathbb{E}(\lfloor X\rfloor|X)$ and $\mathbb{E}(X|\lfloor X\rfloor)$. Deduce $\mathbb{E}(F(X)|\lfloor X\rfloor)$.
(3) Determine the conditional distribution of X given $\lfloor X\rfloor$. Deduce that $F(X)$ and $\lfloor X\rfloor$ are independent.

Exercice 9. Let $X_1,...,X_n$ be independent Poisson random variables with parameters $\lambda_1,...,\lambda_n$.

- Give the law of $S_n=X_1+....+X_n$.
- Compute $\mathbb{P}(X_1=k_1,....,X_n=k_n|S_n=k)$ for all integers $k_1,...,k_n,k$.
- Give the conditional distribution of $(X_1,...,X_n)$ given $S_n=k$.
- Give the conditional distribution of $(X_1,...,X_n)$ given S_n.

Exercice 10. Let (X,Y) be a random vector in $\mathbb{R}^2$ uniformly distributed on the domain T of $\mathbb{R}^2$ defined by $T=\{(x,y)\in[0,1]^2: y\leq x\}$.

(1) Determine the law of Y.
(2) Compute $\mathbb{E}(h(X)|Y)$ for all Borel positive function h.
(3) Give the conditional law of X given Y.
(4) Compute the law of $X+Y$. (*consider* $u=x$, $v=x+y$).
(5) Determine the conditional law of X given $X+Y$.

Exercice 11. Let (X,Y) whose density is given by $f(x,y)=\frac{1}{4}x^{-5/4}y^{-1/4}\mathbb{1}_D(x,y)$ where the domain D is given by $D=\{(x,y)\in\mathbb{R}^2: 0<y<x<1/y\}$. Compute the density $f_Y(y)$ of Y. Deduce the conditional density of X given $Y=y$, for $y\in]0,1[$.

Exercice 12. Let (X,Y) be a couple of random variables whose law is supported on $\mathbb{N}^*\times[0,1]$, and is given by

$$\forall k\in\mathbb{N}^*,\forall A\in\mathcal{B}([0,1]),\qquad \mathbb{P}(X=k,Y\in A)=\frac{k}{2^k}\int_0^1(1-y)^{k-1}\mathbf{1}_A(y)dy.$$

(1) Show that (X,Y) has a density with respect to $\mu\otimes\lambda$ where μ is the counting measure on $\mathbb{N}^*$ and where λ is the Lebesgue measure on $\mathbb{R}$. Give it ?
(2) What are the distributions of X and Y?
(3) Deduce the conditional distribution of X given $Y=y$ and that of Y given $X=k$.

Exercice 13. Let Y_1,Y_2 two i.i.d. r.v. with exponential distribution with parameter 1.

(1) For $\Psi:\mathbb{R}^2\to\mathbb{R}$ a Borel positive function, compute $\mathbb{E}\big(\Psi(Y_1,Y_2)\big|\min(Y_1,Y_2)\big)$.
(2) Deduce the conditional law of (Y_1,Y_2) given $\min(Y_1,Y_2)$. Interpret the result.

Exercice 14. Let (X,Y) be a couple of random variables such that X is uniformly distributed on $[0,1]$ and such that the conditional distribution of Y given X is given by

$$\mathcal{L}_{Y|X=x}=x\delta_1+(1-x)\delta_0.$$

(1) Determine $\mathbb{P}((X,Y) \in A)$, for $A \in \mathcal{B}(\mathbb{R}^2)$. Draw the support of the law $P_{(X,Y)}$.
(2) What is the law of Y?
(3) Compute the conditional law of X given Y.

Exercice 15. Let X, Y be two random variables on the same probability space. Suppose that X has a Poisson distribution $\mathcal{P}(\lambda) = \sum_{k\in\mathbb{N}} e^{-\lambda}\frac{\lambda^k}{k!}\delta_k$, and that the conditional law of Y given $X = x$ is given by the probability kernel

$$K(x, \cdot) = B(x, q) = \sum_{k=0}^{x} C_x^k q^k (1-q)^{x-k}\delta_k.$$

(1) Compute the law of the couple (X, Y).
(2) Show that Y has a classical law.

Exercice 16.

Let X and Y be two random variables on $\mathbb{N}$ and on $\mathbb{R}$ respectively such that, for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}_+$,

$$\mathbb{P}\left[X = n, Y \leq t\right] = 2\int_0^t \frac{y^n}{n!}e^{-3y}dy.$$

(1) Show that, for all $n \geq 0$, $\int_0^\infty \frac{y^n}{n!}e^{-3y}dy = \frac{1}{3^{n+1}}$.
(2) What is the law of X and what is the density of Y ?
(3) Compute $\mathbb{E}[Y|X]$.
(4) Compute $\mathbb{E}\left[\frac{Y}{(X+1)}\right]$.
(5) Compute $\mathbb{P}[X = n|Y] = \mathbb{E}[\mathbb{1}_{X=n}|Y]$ and give the conditional law of X given Y using a transition kernel.
(6) Compute $\mathbb{E}[X|Y]$.