UNIVERSITÉ PAUL SABATIER Année 2023/2024

M1-ESR. Stochastic Processes. .

TD - MARTINGALES

Exercice 1. Let (Z_n) a sequence of L^1 independent r.v. with mean $\mathbb{E}(Z_k) = m_k$. Show that

$$Y_n = \prod_{i=1}^{n} \frac{Z_i}{m_i}$$

is a martingale.

In the case where the r.v. are i.i.d, under which condition on $m = \mathbb{E}[Z_1]$ we have that

$$P_n = \prod_{i=1}^{n} Z_i$$

is a martingale (resp sub-martingale, resp super martingale) ?

Exercice 2. Let (M_n) be a L^2 martingale with respect to a filtration $(\mathcal{F}_n)$ and let (η_n) be a L^2 predictable process, that is η_n is $\mathcal{F}_{n-1}$ mesurable for all $n \geq 1$. Show that (N_n) defined by

$$N_n = \sum_{k=1}^{n} \eta_k (M_k - M_{k-1})$$

is a martingale.

Exercice 3.
Let $(X_n)_n$ a sequence of i.i.d. r.v. and let $u \in \mathbb{R}$ such that

$$g(u) = \mathbb{E}(e^{uX_1}) < \infty$$

Show that (Y_n) defined by

$$Y_n = g(u)^{-n} \exp\left(u \sum_{i=1}^{n} X_i\right)$$

is a martingale.

Exercice 4.
Let $(X_i)_{i\geq 1}$ a sequence of i.i.d. r.v. such that $\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = 1/2$. We put $S_n = \sum_{i=1}^{n} X_i$ and $\mathcal{F}_n$ the natural filtration of X.

1. Show that (S_n), $(S_n^2 - n)$, $(S_n^3 - 3n\, S_n)$ are martingales
2. Let $Q(x,t) = x^3 + axt + bx + ct + d$. For which values of (a,b,c,d) the sequence $Q(S_n, n)$ is a $\mathcal{F}_n$ martingale ?
3. Find α and β such that

$$(\exp(\alpha S_n + \beta n))$$

is a martingale

Exercice 5.
Let $(X_n)_{n\geq 2}$ a sequence of r.v. such that

$$\mathbb{P}[X_n = -n^2] = \frac{1}{n^2}, \quad \mathbb{P}[X_n = \frac{n^2}{n^2-1}] = 1 - \frac{1}{n^2}$$

1. Show that $S_n = X_2 + \ldots + X_n$ is a martingale
2. Show that S_n converge a.s towards infinity (use Borel Cantelli lemma).

Exercice 6.
Let $(X_i)_{i\geq 1}$ be a sequence of independent r.v such that $\mathbb{E}(X_i) = 0 =$ and $\mathbb{E}(X_i^2) = \sigma_i^2$. We put $S_n = \sum_{i=1}^n X_i$, $M_n = S_n^2 - \sum_{i=1}^n \sigma_i^2$ and $(\mathcal{F}_n)$ the natural filtration of X.

1. Show that S_n and M_n are $\mathcal{F}_n$ martingales.