UNIVERSITÉ PAUL SABATIER Année 2022/2023

M1-ESR. Stochastic Processes. .

TD - MARTINGALES

Exercice 1.

Consider the process (Z_n) defined by

$$Z_0 = 1,\ Z_{n+1} = \sum_{i=1}^{Z_n} Y_{n,i},$$

where $(Y_{n,i})$ are i.i.d with law μ. We suppose that $\mu(\{0\}) \neq 0$ and we note $m = \mathbb{E}[Y_n, i]$.

1. Show that (Z_n) is a Markov chain.
2. Identify the large time behavior of (Z_n) ?
3. Show that the process
$$\frac{Z_n}{m^n}$$
is a martingale.
4. Let $g(s) = \mathbb{E}[s^{Y_{n,i}}]$. For $m < 1$, suppose there exists a unique $s \in]0,1[$ such that $g(s) = s$. Show that s^{Z_n} is a martingale that converges almost surely and in L^p for all p.
5. Deduce the asymptotic behavior of Z_n in the cases $m > 1$ and $m < 1$.
6. Calculate the extinction probability in these two cases.
7. Case $m = 1$. Show that Z_∞ is in L^1 and deduce its value. Do we have L^1 convergence of (Z_n) ? Do we have uniform integrability ?

Exercice 1.

Let $(X_i)_{i\geq 1}$ be a sequence of independent random variables such that $\mathbb{E}(X_i) = 0$ and $\mathbb{E}(X_i^2) = \sigma_i^2$. We define $S_n = \sum_{i=1}^n X_i$, $M_n = S_n^2 - \sum_{i=1}^n \sigma_i^2$ and $\mathcal{F}_n$ as the natural filtration of X.

1. Show that S_n and M_n are $\mathcal{F}_n$ martingales.
2. Show that if $\sum_{i=1}^{+\infty} \sigma_i^2 < +\infty$, then S_n converges almost surely and in $\mathbb{L}^2$.

Exercice 2.

Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with $\mathcal{N}(0,1)$ distribution, and define $S_n = \sum_{i=1}^n X_i$. For $t \in \mathbb{R}$, we define

$$M_n(t) = e^{t\,S_n - \frac{nt^2}{2}}\ .$$

1. Show that $M_n(t)$ is a positive martingale that converges almost surely to $M_\infty(t)$.
2. Show that if $t \neq 0$, $tS_n - \frac{nt^2}{2}$ converges almost surely to $-\infty$. Deduce the value of $M_\infty(t)$, and then show that for $t \neq 0$, $M_n(t)$ does not converge in $\mathbb{L}^1$.

Exercice 3.

Let $(U_n)_n$ be a sequence of i.i.d. random variables uniformly distributed on $[0,1]$, and let $a \in [0,1]$. We define $X_0 = a$ and recursively, $X_{n+1} = U_{n+1} + (1 - U_{n+1})\,X_n^2$.

1. Show that X_n is a submartingale with respect to the natural filtration of U.

2. Show that almost surely $0 \leq X_n \leq 1$. Deduce that X_n converges almost surely and in $\mathbb{L}^p$ $(1 \leq p < +\infty)$ to a variable X_∞.
3. Show that $X_\infty = 1$ almost surely.

Exercise 4.
Let $(X_i)_{i\geq 1}$ be a sequence of independent random variables such that $\mathbb{E}(X_i) = 0$ and $\mathbb{E}(X_i^2) = 1$. We define $Y_n = \frac{1}{n}\prod_{i=1}^n X_i$ and $Z_n = \sum_{k=1}^n Y_k$. Show that Z_n converges almost surely and in $\mathbb{L}^2$. Replace $1/n$ by a_n. Under what condition can we assert that the result remains true ?

Exercise 5.
Let $x \in \mathbb{R}_+^*$ and let $\alpha \in \mathbb{R}$ such that $0 < |\alpha| < 1$. We define a sequence of random variables $(S_n)_{n\geq 0}$ by

$$S_0 = x, \quad S_{n+1} = S_n + \alpha\epsilon_{n+1}S_n,$$

for all $n \geq 0$ where $(\epsilon_n)_{n\geq 1}$ is a sequence of i.i.d. random variables such that

$$\mathbb{P}[\epsilon_1 = 1] = P[\epsilon_1 = -1] = \frac{1}{2}.$$

We denote by $(\mathcal{F}_n)$ the natural filtration of (S_n), that is,

$$\mathcal{F}_n = \sigma(S_0, \ldots, S_n), \quad n \in \mathbb{N}.$$

1. Show that (S_n) is a $(\mathcal{F}_n)$ martingale.
2. Show by induction that $S_n > 0$ for all $n \geq 0$.
3. Deduce that (S_n) converges almost surely as n tends to $+\infty$.
4. Define for all $n \geq 0$
$$Z_n = \log(S_n).$$
Show that $Z_n = Z_{n-1} + \log(1 + \alpha\epsilon_n)$ for all $n \geq 1$.
5. Deduce by induction that for all $n \geq 1$
$$Z_n = \log x + \sum_{k=1}^n \log(1 + \alpha\epsilon_k).$$
6. Calculate $\mathbb{E}(\log(1 + \alpha\epsilon_1))$ and deduce the limit of $\left(\frac{Z_n}{n}\right)$.
7. Deduce that S_n converges almost surely to zero. Does the convergence occur in L^1 norm ?

Exercise 6.
Let $\theta \in]0, 1[$ and $x \in]0, 1[$. We consider the sequence (X_n) defined by

$$X_0 = x, \quad X_{n+1} = \theta X_n + (1 - \theta)\varepsilon_{n+1}$$

where (ε_n) is a sequence of random variables taking values in $\{0, 1\}$ satisfying

$$\mathbb{E}[f(\varepsilon_{n+1})|\mathcal{F}_n] = X_n f(1) + (1 - X_n)f(0)$$

for all bounded measurable functions f. Here $(\mathcal{F}_n)$ denotes the natural filtration of (X_n), that is, for all $n \in \mathbb{N}$

$$\mathcal{F}_n = \sigma(X_0, \ldots, X_n).$$

1. Show that $0 < X_n < 1$ for all $n \in \mathbb{N}$.
2. Show that (X_n) is a $(\mathcal{F}_n)$ martingale.
3. Show that (X_n) converges in L^2 to a random variable X_∞.
4. Does the previous convergence occur almost surely ? Does it occur in L^1 ?
5. Show that for all $n \in \mathbb{N}$
$$\mathbb{E}\left[(X_{n+1} - X_n)^2\right] = (1-\theta)^2 \mathbb{E}[X_n(1-X_n)].$$
6. Deduce the value of $\mathbb{E}[X_\infty(1 - X_\infty)]$.
7. Determine the distribution of X_∞.

Exercise 7.
(A proof of Kolmogorov's 0-1 Law using martingale theory). Let $(X_n)_{n\geq 0}$ be a sequence of independent random variables. We define :

$$\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n), \quad \mathcal{F}_\infty = \sigma\left(\bigcup_{n\geq 1} \mathcal{F}_n\right)$$
$$\mathcal{F}^n = \sigma(X_n, \ldots, X_{n+1}, \ldots), \quad \mathcal{F}^\infty = \bigcap_{n\geq 1} \mathcal{F}^n$$

Let $A \in \mathcal{F}^\infty$, show using $M_n = \mathbb{E}[\mathbf{1}_A | \mathcal{F}_n]$ that $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

Exercice 8.
The aim of this exercise is to show by a probabilistic approach that any lipschitzian function is a primitive of a bounded measurable function. Let X a random variable with a uniform distribution on $[0, 1]$ and $f : [0, 1] \to \mathbb{R}$ a Lipschitz function with constant $L > 0$. For all $n \geq 0$, we put

$$X_n = \lfloor 2^n X \rfloor 2^{-n}$$

and

$$Z_n = 2^n(f(X_n + 2^{-n}) - f(X_n)).$$

1. Show that for all n $n \geq 0$
$$\mathcal{F}_n := \sigma(X_0, X_1, ..., X_n) = \sigma(X_n) \text{ et } \bigcap_{n\geq 0} \sigma(X_n, X_{n+1}, ...) = \sigma(X)$$
(We could prove that for $0 \leq k \leq n$: $X_k = 2^{-k}\lfloor 2^k X_n \rfloor$)
2. Let $x \in]0, 1[$ and let $n \geq 0$, we put
$$x_n = \lfloor 2^n x \rfloor 2^{-n}.$$
Lett $n \geq 0$, what are the values taken by x_n ? Suppose that the value of x_n is fixed, what are the possible values of x_{n+1} ?
3. Let $h : [0, 1] \to \mathbb{R}$ a bounded measurable function. Show that for all n $n \geq 0$ and all $0 \leq k \leq 2^{n-1}$
$$\mathbb{E}[h(X)\mathbf{1}_{X_n = \frac{k}{2^n}}] = \int_{k2^{-n}}^{(k+1)2^{-n}} h(x)dx$$

4. Deduce that for all $n \geq 0$

$$\mathbb{E}[h(X)|X_n] = 2^n \int_{X_n}^{X_n+2^{-n}} h(x)dx$$

5. Show that for all n $n \geq 0$ and all $0 \leq k \leq 2^{n-1}$

$$\mathbb{E}[h(X_{n+1})\mathbf{1}_{X_n=\frac{k}{2^n}}] = 2^{-(n+1)} \left(h\left(\frac{k}{2^n}\right) + h\left(\frac{2k+1}{2^{n+1}}\right)\right)$$

6. Deduce that for all $n \geq 0$

$$\mathbb{E}[h(X_{n+1})|X_n] = \frac{1}{2}\left(h\left(X_n\right) + h\left(X_n + 2^{-(n+1)}\right)\right)$$

7. Show that (Z_n) is a $(\mathcal{F}_n)$ bounded martingale
8. Show that (Z_n) converges almost surely and in L^1 towards a random variable Z.
9. Show that Z is $\sigma(X)$ mesurable and then that there exists g mesurable such that $Z = g(X)$. Show that g can be chosen bounded.
10. We recall that $Z_n = \mathbb{E}[Z|\mathcal{F}_n]$, show that

$$Z_n = 2^n \int_{X_n}^{X_n+2^{-n}} g(x)dx$$

11. Show that for all $x \in [0,1]$ we have

$$f(x) = f(0) + \int_0^x g(u)du$$