UNIVERSITÉ PAUL SABATIER Année 2022/2023

M1-ESR. Stochastic Processes. .

TD - 5 - MARTINGALES

Exercice 1.
Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be continuous and bounded. We suppose that there exist θ^* such that $f(\theta^*) = 0$ and we suppose that for all $\theta \neq \theta^*$

$$\langle (\theta - \theta^*), f(\theta) \rangle > 0$$

We define

$$\theta_n = \theta_{n-1} - \gamma_n f(\theta_{n-1})$$

We suppose

$$\gamma_n > 0, \quad \sum_n \gamma_n = \infty, \quad \sum_n \gamma_n^2 < \infty$$

1. Show that
$$\|\theta_n - \theta^*\|^2 \leq \|\theta_{n-1} - \theta^*\|^2 + \gamma_n^2 \|f(\theta_{n-1})\|^2$$
2. Deduce that
$$x_n = \|\theta_n - \theta^*\|^2 - \sum_{k=1}^{n} \gamma_k^2 \|f(\theta_{k-1})\|^2$$
is non increasing
3. Show that
$$x_n \geq -\|f\|_\infty^2 \sum_k \gamma_k^2$$
4. Show that (x_n) is convergent towards x_∞ and that
$$l = \lim_n \|\theta_n - \theta^*\|^2 = x_\infty + \sum_k \gamma_k^2 \|f(\theta_{k-1})\|^2$$
5. Show that
$$\sum_k \langle \gamma_k f(\theta_{k-1}), \theta_{k-1} - \theta^* \rangle = \frac{1}{2} \sum_k \gamma_k^2 \|f(\theta_{k-1})\|^2 - l + \|\theta_0 - \theta^*\|^2 < \infty$$
6. Suppose that $l \neq 0$ and put
$$\eta = \inf_{\delta < \|\theta - \theta^*\| < 2l} \langle \theta - \theta^*, f(\theta) \rangle > 0$$
Show that there exists N
$$\sum_k \langle \gamma_k f(\theta_{k-1}), \theta_{k-1} - \theta^* \rangle \geq \eta \sum_{k \leq N} \gamma_k$$
and conclude

7. Suppose that

$$f(\theta) = \mathbb{E}[F(\theta, X)]$$

where X is a r.v and F is bounded. Suppose that $(X_n)_n$ is a sequence of i.i.d r.v of law X. consider the natural filtration of (X_n). We put

$$\theta_{n+1} = \theta_n - \gamma_{n+1} Y_{n+1}, \quad Y_{n+1} = f(\theta_n) + \zeta_{n+1}$$

where

$$\zeta_{n+1} = F(\theta_n, X_{n+1}) - f(\theta_n).$$

Show that

$$\sup \|\zeta_n\|_\infty < \infty, \quad \mathbb{E}[\zeta_{n+1} \| \mathcal{F}_n] = 0$$

8. Introduce

$$S_n = \sum_{k=1}^{n-1} \gamma_{k+1}^2 \mathbb{E}[\|Y_{k+1}\|^2 \| \mathcal{F}_k]$$

and show that S_n is almost surely convergent.

9. Introduce

$$X_n = \|\theta_n - \theta^*\|^2 - S_n$$

Show that X_n is a surmartingale

10. Show that (X_n) is bounded by below by a non positive constant.
11. Show that (X_n) is almost surely convergent.
12. Assume $L = X_\infty + S_\infty \neq 0$ and note that

$$A_\delta = \{L > \delta\}$$

Show that

$$\sum_k \langle \gamma_k f(\theta_{k-1}), \theta_{k-1} - \theta^* \rangle = \infty$$

on A_δ

13. Show that

$$\sum_k \mathbb{E}[\langle \gamma_k f(\theta_{k-1}), \theta_{k-1} - \theta^* \rangle] < \infty$$

and conclude that $\mathbb{P}[A_\delta] = 0$.

14. Conclude that $L = 0$ almost surely.
15. Application : describe an algorithme which can estimate q such that (converge towards)

$$\int_{-\infty}^{q} f(t) = \alpha$$

where f is a probability density.