UNIVERSITÉ PAUL SABATIER Année 2022/2023
M1-ESR. Stochastic Processes. .

TD - 6 - MARTINGALES

1 Exercice

Let $X_1, \ldots, X_n$ n r.v of exponential law $\mathcal{E}(\lambda)$ independant. Compute the law of

$$S = X_1 + \ldots + X_n$$

Let $Z \sim \Gamma(m, \lambda)$ and $W \sim \Gamma(p, \lambda)$ two independent random variables. What is the law of $Z + W$?

2 Exercice

Let (N_t) be a Poisson process of intensity 2. Determine the conditionnal distribution of τ_1 the first jumping time, knowing that at time 1 $N_1 = 1$, that is

3 Exercice

Let (N_t) and (M_t) two independent Poisson processes of parameter λ and μ. Show that $(N_t + M_t)$ is a Poisson process.

Let (X_n) be a sequence of i.i.d r.v of Bernoulli law with parameter p. Let N_t be a Poisson process of parameter λ. Define

$$P_t = \sum_{i=1}^{N_t} X_i \; and \; F_t = \sum_{i=1}^{N_t} (1 - X_i)$$

Show that (P_t) and (N_t) are independent Poisson processes.

4 Exercice

Suppose that the number of customers visiting a fast food restaurant in a given time interval I is $N \sim \mathcal{P}(\mu)$. Assume that each customer purchases a drink with probability p, independently from other customers, and independently from the value of N. Let X be the number of customers who purchase drinks in that time interval. Also, let Y be the number of customers that do not purchase drinks ; so $X + Y = N$.

1. Find the marginal of X and Y.
2. Find the joint law of X and Y.
3. Are X and Y independent ?

5 Exercice

Let (X_n) be a sequence of independent random variables, all with the same exponential distribution of parameter $\lambda > 0$. We set $S_0 = 0$ and define, for $n \in \mathbb{N}^*$,

$$S_n = \sum_{i=1}^{n} X_i, \quad n \in \mathbb{N}^*.$$

We define the process (N_t) by

$$N_t = \sum_{i=1}^{\infty} \mathbf{1}_{S_n \leq t}$$

where $\mathbf{1}_A$ is the indicator function of the event A.
The task consists of several parts :

1. For $n \in \mathbb{N}^*$, show that the random vector $(S_1, \ldots, S_n)$ has density f_n given by
$$f_n(t_1, \ldots, t_n) = \begin{cases} \lambda^n e^{-\lambda t_n} & \text{si} \quad 0 < t_1 < t_2 < \ldots < t_n \\ 0 & \text{sinon} \end{cases}$$
2. Deduce the distribution of S_n for $n \in \mathbb{N}$.
3. Show that N_t follows a Poisson distribution and express its parameter in terms of λ and t.
4. Show that N_t is a finite almost surely random variable.
5. For $0 \leq s \leq t$ and $k, l \in \mathbb{N}$, express the event $\{N_s = k, N_t - N_s = l\}$ in terms of the random variables (S_n), and calculate its probability.
6. Deduce that (N_t) is a Poisson process.

6 Exercice

A point process (T_n) is an increasing sequence of random variables (T_n) taking values in $\mathbb{R}^+$, $0 = T_0 < T_1 < T_2 < \ldots < T_n < \ldots$ such that

$$\lim_{n \to \infty} T_n = +\infty.$$

A counting process (M_t) is a process associated with a punctual process (T_n) defined by

$$M_t = \sup n ; T_n \leq t.$$

1. Show that $t \to M_t$ defines a function that is right-continuous at every point and has a left limit at every point. We now consider a counting process M_t such that :
   — a) $\forall k \geq 2, \forall 0 \leq t_1 < \ldots < t_k$, the random variables $M_{t_1}, M_{t_2} - M_{t_1}, \ldots, M_{t_k} - M_{t_{k-1}}$ are independent.
   — b) If $0 \leq s < t$, the distribution of $M_t - M_s$ depends only on $t - s$.
   The goal is to show that such a process is necessarily a Poisson process.
2. Let $t > 0$. Let h a bounded continuous function. Show that $\lim_{s \to 0, s > 0} \mathbb{E}[h(M_{t+s} - M_s)] = \mathbb{E}(h(M_t))$.

3. Deduce that for $t, s \in \mathbb{R}^+$, the random variables $M_{t+s} - M_s$ and M_t have the same distribution.
4. For $u \in]0,1]$ and $t \geq 0$, let $f_u(t) = \mathbb{E}[u^{M_t}]$. For $t, s \geq 0$, compute $f_u(t+s)$ and find a functional equation for the function f_u.
5. Let $u \in]0,1]$. Justify that f_u is decreasing and does not vanish.
6. Let $u \in]0,1]$. Show that f_u is right-continuous. Deduce that there exists a real number $\lambda(u) > 0$ such that
$$f_u(t) = e^{-\lambda(u)t}, \quad t \in \mathbb{R}^+.$$
7. We put (T_n) the point process whose (M_t) is the counting process, justify $\bigcup_{n\in\mathbb{N}}\{M_{nt} = 0, M_{(n+1)t} \geq 2\} \subset \{T_2 < T_1 + t\}$
8. Deduce that $\frac{1}{1-\mathbb{P}(M_t=0)}\mathbb{P}(M_t \geq 2) \leq \mathbb{P}(T_2 < T_1 + t)$ and then $\frac{\mathbb{P}(M_t \geq 2)}{t}$ converges to 0 when t goes to 0.
9. Deduce that there exists $c > 0$ such that for all $u \in]0,1]$, $\lambda(u) = \lim_{t\to 0} \frac{1-f_t(u)}{t} = c(1-u)$
10. Conclude