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FEEDBACK STABILIZATION OF A FLUID-STRUCTURE MODEL*

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Abstract. We study a system coupling the incompressible Navier–Stokes equations in a 2D rectangular-type domain with a damped Euler–Bernoulli beam equation, where the beam is a part of the upper boundary of the domain occupied by the fluid. Due to the deformation of the beam, the fluid domain depends on time. We prove that this system is exponentially stabilizable, locally about the null solution, with any prescribed decay rate, by a feedback control corresponding to a force term in the beam equation. The feedback is applied on the whole structure, and it is determined, via a Riccati equation, by solving an infinite time horizon control problem for the linearized model. A crucial step in this analysis consists of showing that this linearized system can be rewritten thanks to an analytic semigroup of which the infinitesimal generator has a compact resolvent.

Key words. fluid-structure interaction, feedback control, stabilization, Navier–Stokes equations, beam equation

AMS subject classifications. 93B52, 93C20, 93D15, 35Q30, 76D55, 76D05, 74F10

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1. Setting of the problem.

Let Ω be the rectangular domain $(0,L)\times(0,1)\subset\mathbb{R}^2$, with boundary Γ. Let us set $\Gamma_s=(0,L)\times\{1\}$, the upper part of the boundary of Ω, and $\Gamma_0=\Gamma\setminus\Gamma_s$. For a given function η from $\Gamma_s\times(0,\infty)$ into $(-1,\infty)$ we denote by $\Omega_{\eta(t)}$ and $\Gamma_{s,\eta(t)}$ the sets

$$\Omega_{\eta(t)}=\Big\{(x,y)\mid x\in(0,L),\ 0<y<1+\eta(x,t)\Big\},$$
$$\Gamma_{s,\eta(t)}=\Big\{(x,y)\mid x\in(0,L),\ y=1+\eta(x,t)\Big\}.$$

For $0<T<\infty$ or $T=\infty$ we also use the notation

$$\Sigma_T^0=\Gamma_0\times(0,T),\quad \Sigma_T=\Gamma\times(0,T),$$
$$Q_T=\Omega\times(0,T),\quad \widetilde{Q}_T=\textstyle\bigcup_{t\in(0,T)}\Omega_{\eta(t)}\times\{t\},$$
$$\Sigma_T^s=\Gamma_s\times(0,T),\quad \widetilde{\Sigma}_T^s=\textstyle\bigcup_{t\in(0,T)}\Gamma_{s,\eta(t)}\times\{t\}.$$

We consider the following fluid-structure model coupling the Navier–Stokes equations with a damped Euler–Bernoulli beam equation:

$$\begin{aligned}
&\mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}-\operatorname{div}\sigma(\mathbf{u},p)=0,\quad \operatorname{div}\mathbf{u}=0\quad \text{in } \widetilde{Q}_\infty,\\
&\mathbf{u}(x,1+\eta(x,t),t)=\eta_t(x,t)\vec{e}_2\quad \text{for } (x,t)\in(0,L)\times(0,\infty),\\
&\mathbf{u}=0\ \text{ on } \Sigma_\infty^0,\quad \mathbf{u}(0)=\mathbf{u}^0 \text{ in } \Omega_{\eta(0)}=\Omega_{\eta_1^0},\\
&\eta_{tt}-\beta\eta_{xx}-\delta\eta_{txx}+\alpha\eta_{xxxx}=\rho_1 p+H(\mathbf{u},\eta)+f\ \text{ on } \Sigma_\infty^s,\\
&\eta=0\ \text{ and }\ \eta_x=0\ \text{ on } \big\{0,L\big\}\times(0,\infty),\\
&\eta(0)=\eta_1^0\ \text{ and }\ \eta_t(0)=\eta_2^0\ \text{ in } \Gamma_s,
\end{aligned}\tag{1.1}$$

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with

$$H(\mathbf{u},\eta) = -\rho_2\nu(\nabla\mathbf{u}+\nabla\mathbf{u}^T)\big|_{\Gamma_{s,\eta(t)}}\,(-\eta_x\vec{e}_1+\vec{e}_2)\cdot\vec{e}_2,$$
$$\sigma(\mathbf{u},p) = \nu(\nabla\mathbf{u}+\nabla\mathbf{u}^T) - p\,I, \quad \vec{e}_1=(1,0), \quad \vec{e}_2=(0,1).$$

In this setting $\nu > 0$ is the fluid viscosity; $\alpha > 0$, $\beta \geq 0$, and $\delta > 0$ are the adimensional rigidity, stretching, and friction coefficients of the beam; ρ_1 and ρ_2 are positive constants related to the density of the fluid and the density of the structure (see [3]); and f is a control function. The vertical force F exerted by the fluid on the beam can be defined by the variational formulation

$$\int_{\Gamma_s} F(x,t)\,\phi(x,1+\eta(x,t)) = \int_{\Gamma_{s,\eta(t)}} \big((\rho_1 p\,\mathbf{n} - \rho_2\nu(\nabla\mathbf{u}+\nabla\mathbf{u}^T)\,\mathbf{n}\big)\cdot\phi(x,y)\vec{e}_2,$$

for all regular functions ϕ, where $\mathbf{n} = \big(-\frac{\eta_x}{\sqrt{1+\eta_x^2}}\vec{e}_1 + \frac{1}{\sqrt{1+\eta_x^2}}\vec{e}_2\big)$ is the unit normal to $\Gamma_{s,\eta(t)}$ exterior to $\Omega_{\eta(t)}$. This leads to setting $F = \rho_1 p + H(\mathbf{u},\eta)$ in the right-hand side of the equation satisfied by η (see (1.1)).

Our objective is to determine f in feedback form, able to stabilize the system (1.1) (in an appropriate space) with a prescribed exponential decay rate $-\omega < 0$, locally about $(\mathbf{u},p,\eta,\eta_t) = (\mathbf{0},0,0,0)$. For that we look for strong solutions to the closed loop system associated with (1.1), that is, when f is replaced by a feedback law. Despite its apparent simplicity (2D (two-dimensional) model, rectangular-type domain) very few results are known about the existence of strong solutions to this type of system. Existence of a local strong solution for system (1.1) with $f = 0$ has been proved in [3] (with periodic boundary conditions on the lateral boundary of Ω), under smallness conditions on the data, while existence of Hopf solutions for a slightly different model is proved in [13] (see also [7] and [12] for other models for the beam equation and for existence results in the 3D case). To the author's knowledge, nothing is known about control and stabilization of such a system.

As already mentioned, system (1.1) suffers from several limitations. The model is stated in two dimensions, the domain occupied by the fluid is of rectangular type, the boundary condition

$$\mathbf{u}(x,1+\eta(x,t),t) = \eta_t(x,t)\vec{e}_2 \quad \text{for } (x,t)\in(0,L)\times(0,\infty)$$

is not necessarily the most relevant one for describing the interaction between the fluid and the structure (see, e.g., [13] and section A.5), and the damping term $-\delta\eta_{txx}$ simplifies the analysis of the coupled system. In order to clarify why these limitations are essential or not, we have indicated in the appendix what results can be extended to other models. But we can state right now that the 2D setting plays a crucial role at several stages of the paper. It is used in section 12 to prove the existence of a unique strong solution to the closed loop system. Indeed the proof is based on a fixed point argument, and there we use Sobolev imbeddings in one dimension for the structure. The 2D setting is also used to prove the stabilizability result of the linearized model in section 5 (a similar result is not known in three dimensions).

The technique used here to study the control system (1.1) is not new; it consists first of stabilizing the linearized system with a feedback law obtained by solving a linear quadratic optimal control problem with an infinite time horizon and next of proving that the linear feedback law, applied in the nonlinear system, is able to stabilize the nonlinear system, provided that the initial condition is small enough in

an appropriate norm. However, the analysis that we do for the linearized system is completely new for this type of fluid-structure system.

The paper is structured into three parts. In part 1 (sections 2–5) we analyze the linearized model and its stabilizability. In part 2 (sections 6–9) we study two feedback laws, their properties, and their relationships. These results are next used in an essential way in part 3 (sections 10–12) to study the closed loop nonlinear system. We end the paper with an appendix in which we analyze how assumptions made for system (1.1) are essential.

Let us first describe how we obtain the linearized model. As in [3], we make a change of variables in order to rewrite system (1.1) in the cylindrical domain $\Omega \times (0,\infty)$, and we denote by $(\hat{\mathbf{u}},\hat{p})$ the image of $(\mathbf{u},p)$ achieved by this transformation. Since we are looking for solutions satisfying a prescribed exponential decay rate $-\omega$, we rewrite the system as a first order system by setting $\eta=\eta_1$ and $\eta_t=\eta_2$, and we study the control system satisfied by $(\tilde{\mathbf{u}},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2)=e^{\omega t}(\hat{\mathbf{u}},\hat{p},\eta_1,\eta_2)$. We linearize the system satisfied by $(\tilde{\mathbf{u}},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2)$ about $(\mathbf{0},0,0,0)$. The linearized model reads as follows:

$$
\begin{aligned}
&\mathbf{v}_t-\operatorname{div}\sigma(\mathbf{v},p)-\omega\mathbf{v}=0,\\
&\operatorname{div}\mathbf{v}=0 \quad \text{in } Q_\infty,\\
&\mathbf{v}=\eta_2\vec{e}_2 \ \text{ on } \Sigma^s_\infty, \quad \mathbf{v}=0 \ \text{ on } \Sigma^0_\infty, \quad \mathbf{v}(0)=\mathbf{v}^0 \text{ in } \Omega,\\
&\eta_{1,t}=\eta_2+\omega\eta_1 \quad \text{on } \Sigma^s_\infty,\\
&\eta_{2,t}-\omega\eta_2-\beta\eta_{1,xx}-\delta\eta_{2,xx}+\alpha M_s\eta_{1,xxxx}=M_s(\rho_1 p+f) \ \text{ on } \Sigma^s_\infty,\\
&\eta_1=0 \quad \text{and} \quad \eta_{1,x}=0 \ \text{ on } \{0,L\}\times(0,\infty),\\
&\eta_1(0)=\eta_1^0 \quad \text{and} \quad \eta_2(0)=\eta_2^0 \ \text{ in } \Gamma_s,
\end{aligned}
\tag{1.2}
$$

where M_s is the orthogonal projection in $L^2(\Gamma_s)$ onto $L^2_0(\Gamma_s)=\{\eta\in L^2(\Gamma_s)\mid \int_{\Gamma_s}\eta=0\}$ (see section 2 and system (2.7)). In order to use the control theory, we need to rewrite system (1.2) in the form of an evolution equation. For that, we have to eliminate the pressure in the fluid and structure equations. The classical way to eliminate the pressure in the fluid equation consists of using the so-called Leray or Helmholtz projector $P \ : \ L^2(\Omega;\mathbb{R}^2)\longmapsto \mathbf{V}^0_n(\Omega)=\{\mathbf{y}\in L^2(\Omega;\mathbb{R}^2)\mid \operatorname{div}\mathbf{y}=0,\ \mathbf{y}\cdot\mathbf{n}=\mathbf{0} \text{ on } \Gamma\}$. It can be shown (see Lemma 3.1) that the equation satisfied by $\mathbf{v}$ in system (1.2) is equivalent to the system

$$
\begin{aligned}
&P\mathbf{v}'=A_0P\mathbf{v}+(-A_0)PD(\eta_2\vec{e}_2\chi_{\Gamma_s}) \quad \text{in } (0,T), \qquad \mathbf{v}(0)=\mathbf{v}^0 \quad \text{in } \Omega,\\
&(I-P)\mathbf{v}(t)=(I-P)D(\eta_2(t)\vec{e}_2\chi_{\Gamma_s}) \quad \text{in } (0,T),
\end{aligned}
$$

where A_0 is the extension of the Stokes operator $\nu P\Delta$ and D is the Dirichlet operator associated with the stationary Stokes equation (A_0 and its extensions to $(D(A_0^*))'$ and D are precisely defined in section 3). This type of decomposition of velocity fields, into $P\mathbf{v}$ and $(I-P)\mathbf{v}$, has already been introduced in [24] for the Navier–Stokes equations with nonhomogeneous boundary conditions. Finding again this decomposition for system (1.2) is not totally obvious because the pressure, which is eliminated in the Navier–Stokes equations thanks to the projector P, also appears in the beam equation. We also state in Lemma 3.1 that the pressure p in system (1.2) is equal to $\pi-q_t$, where q and π are the solutions of two Neumann problems (see Lemma 3.1). The splitting of the pressure p into the two terms π and $-q_t$ is crucial in our analysis. It is used to eliminate the pressure in the structure equation. The term $-q_t$ can be

expressed in terms of $\eta_{2,t}$, while the term π can be expressed in terms of $P\mathbf{v}$. This allows us to write system (1.2) in the form

(1.3)
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}_\omega \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} + \mathcal{B}\begin{pmatrix} \underline{0} \\ 0 \\ f \end{pmatrix}, \qquad \begin{pmatrix} P\mathbf{v}(0) \\ \eta_1(0) \\ \eta_2(0) \end{pmatrix} = \begin{pmatrix} P\mathbf{v}^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix},$$
$$(I-P)\mathbf{v}(t) = (I-P)D(\eta_2(t)\,\vec{e}_2\,\chi_{\Gamma_s}),$$

where $\mathcal{A}_\omega$ and $\mathcal{B}$ are defined in section 4. This rewriting of the system satisfied by $(\mathbf{v}, p, \eta_1, \eta_2)$ in the form (1.3) is crucial to proving the stabilizability of this system. Indeed, we show that the operator $(\mathcal{A}_\omega, D(\mathcal{A}_\omega))$ is the infinitesimal generator of an analytic semigroup on the space $\mathbf{H} = \mathbf{V}_n^0(\Omega) \times (H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)) \times L_0^2(\Gamma_s)$ and has a compact resolvent in this space. We show that the stabilizability of system (1.3) reduces to proving an approximate controllability result for a projected system. Such an approximate controllability result can be deduced from [21] in the case of a rectangular domain (see also [22, 23] for supplementary approximate controllability results). Let us emphasize the fact that, in order to use the approximate controllability results from [21] or [22, 23], it is essential that the control f be applied on the whole structure.

The plan of the paper is as follows. Section 2 is devoted to rewriting system (1.1) in a fixed domain and to obtaining a linearized system. We study the semigroup of the linearized system and properties of its infinitesimal generator in section 3. Existence and regularity results for the linearized system are stated in section 4. We study the stabilizability of the linearized system in section 5. Two feedback control laws for the linearized system (1.3) are introduced in section 6: The first one is a feedback law for system (1.2), and the second one is a feedback law obtained by solving a Riccati equation of the form

$$\widehat{\Pi} \in \mathcal{L}(\widehat{\mathbf{H}}), \quad \widehat{\Pi} = \widehat{\Pi}^* \geq 0, \quad \widehat{\Pi}\mathcal{A}_\omega + \mathcal{A}_\omega^\sharp \widehat{\Pi} - \widehat{\Pi}\mathcal{B}\mathcal{B}^\sharp\widehat{\Pi} + I = 0,$$

where $\widehat{\mathbf{H}}$ is the space $\mathbf{H}$ equipped with another inner product (see section 3.5), $\mathcal{A}_\omega^\sharp \in \mathcal{L}(\widehat{\mathbf{H}})$ is the adjoint of $\mathcal{A}_\omega \in \mathcal{L}(\widehat{\mathbf{H}})$, and $\mathcal{B}^\sharp \in \mathcal{L}(\widehat{\mathbf{H}}, L_0^2(\Gamma_s))$ is the adjoint of $\mathcal{B} \in \mathcal{L}(L_0^2(\Gamma_s), \widehat{\mathbf{H}})$. The main interest of this approach is that $\mathcal{A}_\omega^\sharp$ can be interpreted in terms of partial differential operators (which can be helpful for numerical calculations). Moreover, we are able to establish the precise relationship between the feedback operators obtained by the two approaches.

The optimal control problems corresponding to the first approach are studied in detail in sections 7 and 8.1. However, the feedback law corresponding to the first approach is expressed in terms of an operator Π, which is not, at that stage, characterized by a Riccati equation. This is why the second approach is helpful even if in that case the representation of the state and adjoint systems via $\mathcal{A}_\omega$ and $\mathcal{A}_\omega^\sharp$ cannot be avoided.

To deal with the nonlinear closed loop system, we first study the nonhomogeneous linearized closed loop system in section 9. The main results of the paper are stated in section 10 (Theorems 10.2 and 10.3). Some Lipschitz properties of the nonlinear terms in the nonlinear system are established in section 11. These properties are next used in section 12 in the proofs of the main results.

Finally let us give some references which are connected to the present work. The control of a channel flow with periodic boundary conditions has been studied

in [4, 32, 33, 34]. We think that the results in those papers may be very useful in studying the control of a channel flow coupled with a beam equation, with periodic boundary conditions at the lateral boundary $\{0\} \times [0, L] \cup \{L\} \times [0, L]$ (see section A.4). Let us also mention some controllability results obtained for systems coupling the Navier–Stokes equations with finite dimensional solid-structure models [5, 14, 28] (see also [27] for a simplified model). These controllability results are mainly based on results first obtained for the Navier–Stokes equations in [10]. In those models the controls act in the fluid equation and not in the structure equation as in (1.1). Thus the problems are quite different. The feedback stabilization of the Navier–Stokes equations in the 3D case is studied in [26]; it can be a starting point for studying the stabilization of systems similar to (1.1) in the 3D case (see also section A.3).

2. The linearized system. The solutions to system (1.1) obey

$$0 = \int_{\Omega_{\eta(t)}} \operatorname{div} \mathbf{u}(t) = \int_{\Gamma_{s,\eta(t)}} \mathbf{u}(t) \cdot \mathbf{n}(t) = \int_{\Gamma_s} \eta_t(t) = \int_0^L \eta_t(x,t)\, dx,$$

since the unit normal to $\Gamma_{s,\eta(t)}$ exterior to $\Omega_{\eta(t)}$ is

$$\mathbf{n}(t) = \left(\frac{-\eta_x(t)}{\sqrt{1+\eta_x^2(t)}}, \frac{1}{\sqrt{1+\eta_x^2(t)}} \right)^T.$$

Thus we must choose η_2^0 in the space

$$L_0^2(\Gamma_s) = \left\{ \eta \in L^2(\Gamma_s) \mid \int_{\Gamma_s} \eta = 0 \right\}.$$

Let us consider the system

$$\begin{aligned} &\eta_{tt} - \beta\eta_{xx} - \delta\eta_{txx} + \alpha\eta_{xxxx} = 0 \quad \text{on } \Sigma_\infty^s, \\ &\eta = 0 \quad \text{and} \quad \eta_x = 0 \quad \text{on } \big\{0, L\big\} \times (0, \infty), \\ &\eta(0) = \eta_1^0 \quad \text{and} \quad \eta_t(0) = \eta_2^0 \quad \text{in } \Gamma_s, \end{aligned} \tag{2.1}$$

and, for $t \geq 0$, let $\mathcal{S}(t)$ be the mapping $(\eta_1^0, \eta_2^0) \longmapsto (\eta(t), \eta_t(t))$, where (η, η_t) is the solution to (2.1). The family $(\mathcal{S}(t))_{t\geq 0}$ is a strongly continuous semigroup on $H_0^2(\Gamma_s) \times L^2(\Gamma_s)$. Since we look for solutions to system (1.1) such that $\eta_t(t)$ belongs to $L_0^2(\Gamma_s)$, we have to consider the restriction of $(\mathcal{S}(t))_{t\geq 0}$ to $(H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)) \times L_0^2(\Gamma_s)$, which is actually a strongly continuous semigroup on $(H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)) \times L_0^2(\Gamma_s)$.

Thus everywhere throughout the paper we shall choose η_1^0 in $H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$. If we denote by M_s the orthogonal projection in $L^2(\Gamma_s)$ onto $L_0^2(\Gamma_s)$, the equation satisfied by η in system (1.1) must be written in the form

$$\eta_{tt} - \beta\eta_{xx} - \delta\eta_{txx} + \alpha M_s(\eta_{xxxx}) = M_s(\rho_1 p + H(\mathbf{u}, \eta) + f) \quad \text{on } \Sigma_\infty^s.$$

Observe that due to the boundary conditions

$$\eta = 0 \quad \text{and} \quad \eta_x = 0 \quad \text{on } \big\{0, L\big\} \times (0, \infty),$$

we have (for solutions regular enough and when η_1^0 and η_2^0 belong to $L_0^2(\Gamma_s)$)

$$\int_{\Gamma_s} \eta_{tt} = 0, \quad \int_{\Gamma_s} \eta_{xx} = 0, \quad \text{and} \quad \int_{\Gamma_s} \eta_{txx} = 0,$$

but in general $\int_{\Gamma_s} \eta_{xxxx}$ is different from zero. This is why, in the equation satisfied by η, we have to write $M_s(\eta_{xxxx})$ in place of η_{xxxx}. But for simplicity we shall skip writing M_s in the different equations, except if we want to stress the role of the operator M_s (which is, for example, the case when we shall define the operator $(\mathcal{A}_\omega, D(\mathcal{A}_\omega))$).

For $\mathbf{u} \in L^2(\Omega; \mathbf{R}^2)$, we shall denote by $\mathbf{u}_1$ and $\mathbf{u}_2$ the components of $\mathbf{u}$. We consider system (1.1) for initial conditions $\mathbf{u}^0$ such that $\operatorname{div} \mathbf{u}^0 = \mathbf{u}^0_{1,x} + \mathbf{u}^0_{2,y} = 0$ and obeying the compatibility condition

$$\mathbf{u}^0 = 0 \ \text{ on } \Gamma_0, \quad \mathbf{u}^0(x, 1+\eta(x,0)) = \mathbf{u}^0(x, 1+\eta^0_1(x)) = \eta^0_2(x)\vec{e}_2 \ \text{ for } x \in (0, L). \tag{2.2}$$

As in [3], for a given function $\eta : (0, L) \times (0, T) \mapsto \mathbb{R}$ satisfying $\eta > -1$, we consider the changes of variables

$$\begin{aligned} \mathcal{T}_\eta : (x, y, t) \longmapsto (x, z, t) &= \left(x, \frac{y}{1+\eta(x,t)}, t\right) \quad \text{and} \\ \mathcal{T}_{\eta(t)} : (x, y) \longmapsto (x, z) &= \left(x, \frac{y}{1+\eta(x,t)}\right). \end{aligned} \tag{2.3}$$

The mapping $\mathcal{T}_{\eta^0_1}$ is defined in a similar way. The mapping $\mathcal{T}_{\eta(t)}$ transforms $\Omega_{\eta(t)}$ into $\Omega = (0, L) \times (0, 1)$. Setting

$$\hat{\mathbf{u}}(x, z, t) = \mathbf{u}(x, y, t), \quad \hat{p}(x, z, t) = p(x, y, t),$$

the nonlinear system (1.1) is rewritten in the form

$$\begin{aligned} &\hat{\mathbf{u}}_t + (\hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{u}} - \nu\Delta\hat{\mathbf{u}} - \nabla\hat{p} = \hat{F}(\hat{\mathbf{u}}, \hat{p}, \eta), \quad \operatorname{div} \hat{\mathbf{u}} = \hat{G}(\hat{\mathbf{u}}, \eta) \quad \text{in } Q_\infty, \\ &\hat{\mathbf{u}} = \eta_t \vec{e}_2 \ \text{ on } \Sigma^s_\infty, \quad \hat{\mathbf{u}} = 0 \ \text{ on } \Sigma^0_\infty, \quad \hat{\mathbf{u}}(0) = \hat{\mathbf{u}}^0 \text{ in } \Omega, \\ &\eta_{tt} - \beta\eta_{xx} - \delta\eta_{txx} + \alpha\eta_{xxxx} = \rho_1\hat{p} + \hat{H}(\hat{\mathbf{u}}, \eta) + f \ \text{ on } \Sigma^s_\infty, \\ &\eta = 0 \quad \text{and} \quad \eta_x = 0 \ \text{ on } \{0, L\} \times (0, \infty), \\ &\eta(0) = \eta^0_1 \quad \text{and} \quad \eta_t(0) = \eta^0_2 \ \text{ in } \Gamma_s, \end{aligned} \tag{2.4}$$

where $\hat{\mathbf{u}}^0(x, z) = \mathbf{u}^0(x, y) = \mathbf{u}^0(x, z(1+\eta(x,0))) = \mathbf{u}^0(x, z(1+\eta^0_1(x))) = \mathbf{u}^0 \circ \mathcal{T}^{-1}_{\eta^0_1}(x, z)$,

$$\begin{aligned} \hat{F}(\hat{\mathbf{u}}, \hat{p}, \eta) = &- \eta\hat{\mathbf{u}}_t + \left(z\eta_t + \nu z\left(\frac{\eta_x^2}{1+\eta} - \eta_{xx}\right)\right)\hat{\mathbf{u}}_z \\ &+ \nu\left(-2z\eta_x\hat{\mathbf{u}}_{xz} + \eta\hat{\mathbf{u}}_{xx} + \left(\frac{z^2\eta_x^2 - \eta}{1+\eta}\right)\hat{\mathbf{u}}_{zz}\right) \\ &+ z(\eta_x\hat{p}_z - \eta\hat{p}_x)\vec{e}_1 - (1+\eta)\hat{\mathbf{u}}_1\hat{\mathbf{u}}_x + (z\eta_x\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2)\hat{\mathbf{u}}_z, \end{aligned}$$

$$\hat{G}(\hat{\mathbf{u}}, \eta) = -\eta\hat{\mathbf{u}}_{1,x} + z\eta_x\hat{\mathbf{u}}_{1,z} = \operatorname{div}(\hat{\mathbf{w}}) \quad \text{with } \hat{\mathbf{w}} = -\eta\hat{\mathbf{u}}_1\vec{e}_1 + z\eta_x\hat{\mathbf{u}}_1\vec{e}_2,$$

and

$$\begin{aligned} \hat{H}(\hat{\mathbf{u}}, \eta) &= \rho_2\nu\left(\frac{\eta_x}{1+\eta}\hat{\mathbf{u}}_{1,z} + \eta_x\hat{\mathbf{u}}_{2,x} - \frac{2+\eta_x^2}{1+\eta}\hat{\mathbf{u}}_{2,z}\right) \\ &= -2\rho_2\nu\hat{\mathbf{u}}_{2,z} + \rho_2\nu\left(\frac{\eta_x}{1+\eta}\hat{\mathbf{u}}_{1,z} + \eta_x\hat{\mathbf{u}}_{2,x} - \frac{\eta_x^2 - 2\eta}{1+\eta}\hat{\mathbf{u}}_{2,z}\right). \end{aligned}$$

Due to (2.2), we can see that

(2.5)
$$\operatorname{div}(\hat{\mathbf{u}}^0 - \hat{\mathbf{w}}(0)) = 0 \ \text{ in } \Omega, \quad \hat{\mathbf{u}}^0 - \hat{\mathbf{w}}(0) = 0 \ \text{ on } \Gamma_0, \quad \hat{\mathbf{u}}^0 - \hat{\mathbf{w}}(0) = \eta_2^0 \vec{e}_2 \ \text{ on } \Gamma_s.$$

For $-\omega < 0$, we make the following change of variables:

$$\tilde{\mathbf{u}} = e^{\omega t}\hat{\mathbf{u}}, \quad \tilde{p} = e^{\omega t}\hat{p}, \quad \tilde{\eta}_1 = e^{\omega t}\eta, \quad \tilde{\eta}_2 = e^{\omega t}\eta_t.$$

The system (2.4) is transformed into

(2.6)
$$\begin{aligned}
&\tilde{\mathbf{u}}_t + e^{-\omega t}(\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}} - \nu\Delta\tilde{\mathbf{u}} - \nabla\tilde{p} - \omega\tilde{\mathbf{u}} = e^{-\omega t}\tilde{F}(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2),\\
&\operatorname{div}\tilde{\mathbf{u}} = e^{-\omega t}\tilde{G}(\tilde{\eta}_1, \tilde{\mathbf{u}}) \quad \text{in } Q_\infty,\\
&\tilde{\mathbf{u}} = \tilde{\eta}_2\vec{e}_2 \ \text{ on } \Sigma_\infty^s, \quad \tilde{\mathbf{u}} = 0 \ \text{ on } \Sigma_\infty^0, \quad \tilde{\mathbf{u}}(0) = \hat{\mathbf{u}}^0 \text{ in } \Omega,\\
&\tilde{\eta}_{1,t} = \tilde{\eta}_2 + \omega\tilde{\eta}_1 \quad \text{on } \Sigma_\infty^s,\\
&\tilde{\eta}_{2,t} - \omega\tilde{\eta}_2 - \beta\tilde{\eta}_{1,xx} - \delta\tilde{\eta}_{2,xx} + \alpha\tilde{\eta}_{1,xxxx} = \rho_1\tilde{p} - 2\nu\rho_2\tilde{\mathbf{u}}_{2,z} + e^{-\omega t}\tilde{H}(\tilde{\mathbf{u}}, \tilde{\eta}_1) + \tilde{f} \ \text{ on } \Sigma_\infty^s,\\
&\tilde{\eta}_1 = 0 \quad \text{and} \quad \tilde{\eta}_{1,x} = 0 \ \text{ on } \{0, L\} \times (0, \infty),\\
&\tilde{\eta}_1(0) = \eta_1^0 \quad \text{and} \quad \tilde{\eta}_2(0) = \eta_2^0 \ \text{ in } \Gamma_s,
\end{aligned}$$

with

$$\begin{aligned}
&\tilde{f} = e^{\omega t} f,\\
&\tilde{F}(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2) = -\tilde{\eta}_1(\tilde{\mathbf{u}}_t - \omega\tilde{\mathbf{u}}) + \left(z\tilde{\eta}_2 + \nu z\left(\frac{\tilde{\eta}_{1,x}^2}{e^{\omega t} + \tilde{\eta}_1} - \tilde{\eta}_{1,xx}\right)\right)\tilde{\mathbf{u}}_z\\
&\quad + \nu\left(-2z\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_{xz} + \tilde{\eta}_1\tilde{\mathbf{u}}_{xx} + \left(\frac{z^2\tilde{\eta}_{1,x}^2 - e^{-\omega t}\tilde{\eta}_1}{e^{\omega t} + \tilde{\eta}_1}\right)\tilde{\mathbf{u}}_{zz}\right)\\
&\quad + z(\tilde{\eta}_{1,x}\tilde{p}_z - \tilde{\eta}_1\tilde{p}_x)\vec{e}_1 - (1 + e^{-\omega t}\tilde{\eta}_1)\tilde{\mathbf{u}}_1\tilde{\mathbf{u}}_x + (ze^{-\omega t}\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2)\tilde{\mathbf{u}}_z,\\
&\tilde{G}(\tilde{\eta}_1, \tilde{\mathbf{u}}) = -\tilde{\eta}_1\tilde{\mathbf{u}}_{1,x} + z\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_{1,z} = \operatorname{div}\left(-\tilde{\eta}_1\tilde{\mathbf{u}}_1\vec{e}_1 + z\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_1\vec{e}_2\right),\\
&\tilde{H}(\tilde{\mathbf{u}}, \tilde{\eta}_1) = \nu\left(\frac{e^{-\omega t}\tilde{\eta}_{1,x}}{e^{\omega t} + \tilde{\eta}_1}\tilde{\mathbf{u}}_{1,z} + \tilde{\eta}_{1,x}\tilde{\mathbf{u}}_{2,x} - \frac{\tilde{\eta}_{1,x}^2}{e^{\omega t} + \tilde{\eta}_1}\tilde{\mathbf{u}}_{2,z} + \frac{2e^{-\omega t}\tilde{\eta}_1}{e^{\omega t} + \tilde{\eta}_1}\tilde{\mathbf{u}}_{2,z}\right).
\end{aligned}$$

If we linearize (2.6) about $(\mathbf{0}, 0, 0, 0)$, we obtain the system

(2.7)
$$\begin{aligned}
&\mathbf{v}_t - \operatorname{div}\sigma(\mathbf{v}, p) - \omega\mathbf{v} = 0,\\
&\operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_\infty,\\
&\mathbf{v} = \eta_2\vec{e}_2 \ \text{ on } \Sigma_\infty^s, \quad \mathbf{v} = 0 \ \text{ on } \Sigma_\infty^0, \quad \mathbf{v}(0) = \mathbf{v}^0 \text{ in } \Omega,\\
&\eta_{1,t} = \eta_2 + \omega\eta_1 \quad \text{on } \Sigma_\infty^s,\\
&\eta_{2,t} - \omega\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha M_s\eta_{1,xxxx} = M_s(\rho_1 p - 2\nu\mathbf{v}_{2,z} + f) \ \text{ on } \Sigma_\infty^s,\\
&\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \{0, L\} \times (0, \infty),\\
&\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \ \text{ in } \Gamma_s.
\end{aligned}$$

Since

$$\operatorname{div}\mathbf{v} = \mathbf{v}_{1,x} + \mathbf{v}_{2,z} = 0 \quad \text{in } Q_\infty,$$

if $\mathbf{v}$ belongs to $L^2(0,\infty;\mathbf{H}^2(\Omega))$, this identity holds to be true on $\Gamma_s \times (0,\infty)$. Thus we have $(\mathbf{v}_{1,x}+\mathbf{v}_{2,z})|_{\Gamma_s}=0$. Due to the boundary condition $\mathbf{v}_1=0$ on $\Gamma_s\times(0,\infty)$, it follows that $\mathbf{v}_{1,x}|_{\Gamma_s}=0$, which implies $\mathbf{v}_{2,z}|_{\Gamma_s}=0$. This is why the term $-2\nu\mathbf{v}_{2,z}$ will be dropped out of the equation satisfied by η_2. Let us notice that $\tilde{\mathbf{u}}_{2,z}$ cannot be dropped out in system (2.6).

Remark 2.1. Let us notice that the regular solutions to system (2.7), with $\omega=0$, obey the following energy identity:

$$\begin{aligned}
&\frac{\rho_1}{2}\int_\Omega |\mathbf{v}(t)|^2+\nu\,\rho_1\int_{Q_t}|\nabla\mathbf{v}|^2+\frac{1}{2}\int_{\Gamma_s}|\eta_2(t)|^2+\frac{\beta}{2}\int_{\Gamma_s}|\eta_{1,x}(t)|^2\\
&\qquad+\delta\int_0^t\int_{\Gamma_s}|\eta_{1,tx}|^2+\frac{\alpha}{2}\int_{\Gamma_s}|\eta_{1,xx}(t)|^2\\
&=\frac{\rho_1}{2}\int_\Omega|\mathbf{v}^0|^2+\frac{1}{2}\int_{\Gamma_s}|\eta_2^0|^2+\frac{\beta}{2}\int_{\Gamma_s}|\eta_{1,x}^0|^2+\frac{\alpha}{2}\int_{\Gamma_s}|\eta_{1,xx}^0|^2+\int_0^t\int_{\Gamma_s}f\,\eta_{1,t}.
\end{aligned}$$

A similar result is established in the proof of Theorem 4.1 in the more general setting where $\omega>0$. This energy identity could be the starting point proving the existence of weak solutions to system (2.7). But it cannot be used, at least in an easy way, to establish regularity results for systems (2.7) and (9.1). (System (9.1) is a nonhomogeneous linear closed loop system in which the nonhomogeneous terms take the place of the nonlinear terms of the closed loop nonlinear system.) For that we need to rewrite system (2.7) as an evolution equation, and we have to show that the underlying semigroup is analytic. Next it will be very easy to apply [1, Chapter 1, Theorem 3.1] in order to prove Theorem 9.1.

3. Definition of an analytic semigroup.

3.1. Transformation of system (2.7). Let us recall that $\mathbf{L}^2(\Omega)=L^2(\Omega;\mathbb{R}^2)$ admits the orthogonal decomposition

$$\mathbf{L}^2(\Omega)=\mathbf{V}_n^0(\Omega)\oplus\operatorname{grad}H^1(\Omega),$$

with

$$\mathbf{V}_n^0(\Omega)=\Big\{\mathbf{y}\in\mathbf{L}^2(\Omega)\mid\ \operatorname{div}\mathbf{y}=0,\ \mathbf{y}\cdot\mathbf{n}=\mathbf{0}\text{ on }\Gamma\Big\},$$

and let us recall that $P\ :\ \mathbf{L}^2(\Omega)\longmapsto\mathbf{V}_n^0(\Omega)$ is the so-called Leray or Helmholtz projector. We also introduce the notation

$$\mathbf{V}^0(\Omega)=\Big\{\mathbf{y}\in\mathbf{L}^2(\Omega)\mid\ \operatorname{div}\mathbf{y}=0\Big\},\quad \mathbf{H}_0^1(\Omega)=H_0^1(\Omega;\mathbb{R}^2),\quad \mathbf{H}^2(\Omega)=H^2(\Omega;\mathbb{R}^2),$$

$$\mathbf{V}^2(\Omega)=\mathbf{H}^2(\Omega)\cap\mathbf{V}^0(\Omega),\quad \mathbf{V}_0^1(\Omega)=\mathbf{H}_0^1(\Omega)\cap\mathbf{V}_n^0(\Omega),\quad \mathbf{V}^{-1}(\Omega)=(\mathbf{V}_0^1(\Omega))',$$

$$\mathbf{V}^0(\Gamma)=\left\{\mathbf{y}\in L^2(\Gamma;\mathbb{R}^2)\mid\int_\Gamma\mathbf{y}\cdot\mathbf{n}=0\right\},$$

$$L_0^2(\Omega)=\left\{p\in L^2(\Omega)\mid\int_\Omega p=0\right\},\quad \mathcal{H}^\sigma(\Omega)=H^\sigma(\Omega)\cap L_0^2(\Omega),$$

$$\mathbf{V}_n^\sigma(\Omega)=\mathbf{H}^\sigma(\Omega)\cap\mathbf{V}_n^0(\Omega)\quad\text{for }\sigma\geq 0,$$

for $\sigma<0$, $\quad\mathcal{H}^\sigma(\Omega)=(\mathcal{H}^{-\sigma}(\Omega))'$,

$(\mathcal{H}^{-\sigma}(\Omega))'$ is the dual of $\mathcal{H}^{-\sigma}(\Omega)$ with $L_0^2(\Omega)$ as pivot space,

$$L_0^2(\Gamma_s)=\left\{\eta\in L^2(\Gamma_s)\mid\int_{\Gamma_s}\eta=0\right\},\quad L_0^2(\Gamma)=\left\{\pi\in L^2(\Gamma)\mid\int_\Gamma\pi=0\right\},$$

$$\mathcal{H}^\sigma(\Gamma_s) = H^\sigma(\Gamma_s) \cap L^2_0(\Gamma_s) \quad \text{and} \qquad \mathcal{H}^\sigma(\Gamma) = H^\sigma(\Gamma) \cap L^2_0(\Gamma) \quad \text{for } \sigma \geq 0,$$

for $\sigma < 0$:

$\mathcal{H}^\sigma(\Gamma) = (\mathcal{H}^{-\sigma}(\Gamma))'$, where $(\mathcal{H}^{-\sigma}(\Gamma))'$ is the dual of $\mathcal{H}^{-\sigma}(\Gamma)$ with $L^2_0(\Gamma)$ as pivot space,

$\mathcal{H}^\sigma(\Gamma_s) = (\mathcal{H}^{-\sigma}(\Gamma_s))'$, $(\mathcal{H}^{-\sigma}(\Gamma_s))'$ is the dual of $\mathcal{H}^{-\sigma}(\Gamma_s)$ with $L^2_0(\Gamma_s)$ as pivot space.

We also use notation similar to that introduced in [18]:

$$\mathbf{H}^{\sigma,\tilde\sigma}(Q_T) = L^2(0,T;H^\sigma(\Omega;\mathbb{R}^2)) \cap H^{\tilde\sigma}(0,T;L^2(\Omega;\mathbb{R}^2)) \quad \text{for } s,\, \sigma \geq 0,$$
$$\mathbf{L}^2(Q_T) = \mathbf{H}^{0,0}(Q_T),$$
$$\mathbf{H}^{\sigma,\tilde\sigma}(\Sigma^s_T) = L^2(0,T;H^\sigma(\Gamma_s;\mathbb{R}^2)) \cap H^{\tilde\sigma}(0,T;L^2(\Gamma_s;\mathbb{R}^2)) \quad \text{for } s,\, \sigma \geq 0.$$

We denote by $A_0 = \nu P\Delta$ the Stokes operator in $\mathbf{V}^0_n(\Omega)$ with domain

$$D(A_0) = \mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega).$$

It is well known that, by the extrapolation method, the Stokes operator can be extended as an unbounded operator in $(\mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega))'$ with domain $\mathbf{V}^0_n(\Omega)$ (see, e.g., [17]). This extension will be still denoted by A_0, and we shall see that it does not lead to confusion. The operator P may also be extended to a bounded operator from $\mathbf{H}^{-1}(\Omega)$ (the dual of $\mathbf{H}^1_0(\Omega)$ with $\mathbf{L}^2(\Omega)$ as pivot space) to $\mathbf{V}^{-1}(\Omega)$ (the dual of $\mathbf{V}^1_0(\Omega)$ with $\mathbf{V}^0_n(\Omega)$ as pivot space) by the formula

$$\langle P\mathbf{u}, \mathbf{\Phi}\rangle_{\mathbf{V}^{-1}(\Omega),\mathbf{V}^1_0(\Omega)} = \langle \mathbf{u}, \mathbf{\Phi}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)} \quad \text{for all} \quad \mathbf{\Phi} \in \mathbf{V}^1_0(\Omega).$$

In that case P is a projector in $\mathbf{H}^{-1}(\Omega)$ but no longer an orthogonal projector. Let us introduce the operator $D \in \mathcal{L}(\mathbf{V}^0(\Gamma), \mathbf{V}^0(\Omega))$ defined by $D\mathbf{g} = \mathbf{w}$, where $(\mathbf{w}, q)$ is the solution to the Dirichlet problem

$$-\nu\Delta\mathbf{w} + \nabla q = 0 \quad \text{and} \quad \operatorname{div}\mathbf{w} = 0 \quad \text{in } \Omega, \quad \mathbf{w} = \mathbf{g} \quad \text{on } \Gamma.$$

We shall also set

$$D_s\eta_2 = D(\eta_2\, \vec{e}_2\, \chi_{\Gamma_s}),$$

where χ_{Γ_s} denotes the characteristic function of Γ_s.

To define the semigroup corresponding to system (2.7), we need only consider this system in the case when $\omega = 0$. Following [24], it is convenient to rewrite the equation satisfied by $\mathbf{v}$ in system (2.7) (for $\omega = 0$) as two equations, one satisfied by $P\mathbf{v}$ and the other by $(I-P)\mathbf{v}$. More precisely we have the following result.

LEMMA 3.1. *A pair* $(\mathbf{v}, p) \in \mathbf{H}^{2,1}(Q_T) \times L^2(0,T;\mathcal{H}^1(\Omega))$ *obeys the first two equations of* (2.7) *with* $\omega = 0$ *if and only if*

$$P\mathbf{v}' = A_0P\mathbf{v} + (-A_0)PD(\eta_2\vec{e}_2\chi_{\Gamma_s}) \quad \textit{in } (0,T), \quad \mathbf{v}(0) = \mathbf{v}^0 \quad \textit{in } \Omega,$$
$$(I-P)\mathbf{v}(t) = (I-P)D(\eta_2(t)\vec{e}_2\chi_{\Gamma_s}) \quad \textit{in } (0,T),$$
$$p = \pi - q_t,$$

where $q \in H^1(0,T;\mathcal{H}^1(\Omega))$ *is the solution to the Neumann problem*

(3.1)
$$\Delta q(t) = 0 \quad \textit{in } \Omega, \quad \frac{\partial q(t)}{\partial \mathbf{n}} = \eta_2(t) \quad \textit{on } \Gamma_s, \quad \frac{\partial q(t)}{\partial \mathbf{n}} = 0 \quad \textit{on } \Gamma_0, \quad \textit{for all } t \in (0,T),$$

and $\pi \in L^2(0,T;\mathcal{H}^1(\Omega))$ *is the solution of the other Neumann problem*

$$\Delta\pi(t) = 0 \quad \text{in } \Omega, \quad \frac{\partial \pi(t)}{\partial \mathbf{n}} = \nu \Delta P\mathbf{v}(t)\cdot \mathbf{n} \quad \text{on } \Gamma, \quad \text{for all } t \in (0,T). \tag{3.2}$$

For the proof we refer to [24, Proof of Proposition 2.2]. Let us explain why we have $p = \pi - q_t$. In Lemma 3.1, A_0 is the Stokes operator in $(\mathbf{V}^2(\Omega)\cap \mathbf{V}_0^1(\Omega))'$ with domain $\mathbf{V}_n^0(\Omega)$. By using the Leray projector, we have eliminated the pressure in the equation satisfied by $\mathbf{v}$. However, since the pressure p also appears in the equation satisfied by η_2, we need to express p in terms of $P\mathbf{v}$ and $(I-P)\mathbf{v}$. Observing that $(I-P)\mathbf{v} = \nabla q$, we are able to characterize q by (3.1). The equation satisfied by $\pi = p - \nu\Delta q + q_t = p + q_t$ follows from the equations $(I-P)\mathbf{v} = \nabla q$ and

$$\frac{\partial P\mathbf{v}}{\partial t} - \nu\Delta P\mathbf{v} + \nabla\pi = 0 \quad \text{and} \quad \operatorname{div} P\mathbf{v} = 0 \quad \text{in } Q_T.$$

Since $\Delta P\mathbf{v}$ belongs to $L^2(0,T;\mathbf{L}^2(\Omega))$ and $\operatorname{div}\Delta P\mathbf{v} = 0 \in L^2(0,T;\mathbf{L}^2(\Omega))$, it follows that $\Delta P\mathbf{v}(\cdot)\cdot\mathbf{n}$ is well defined in $L^2(0,T;H^{-1/2}(\Gamma))$. Moreover, $\langle \Delta P\mathbf{v}(t)\cdot \mathbf{n}, 1\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0$ (see [29, Chapter 1, Theorem 1.2]). Therefore if the solution to system (2.7) is such that $P\mathbf{v} \in L^2(0,\infty;\mathbf{V}^2(\Omega))$, then the solution π to (3.2) belongs to $L^2(0,\infty;\mathcal{H}^1(\Omega))$.

In the following, we denote by $N_s \in \mathcal{L}(L_0^2(\Gamma_s),\mathcal{H}^{3/2}(\Omega))$ the operator defined by $N_s\eta_2(t) = q(t)$, and by $N_0 \in \mathcal{L}(\mathcal{H}^{-1/2}(\Gamma),\mathcal{H}^1(\Omega))$ the operator defined by $N_0(\nu\Delta P\mathbf{v}(t)\cdot \mathbf{n}) = \pi(t)$, when $\Delta P\mathbf{v}(t)\cdot\mathbf{n} \in \mathcal{H}^{-1/2}(\Gamma)$.

We denote by γ_s the modified trace operator on Γ_s defined by

$$\gamma_s p = M_s(p|_{\Gamma_s}) = p|_{\Gamma_s} - \frac{1}{|\Gamma_s|}\int_{\Gamma_s} p \quad \text{for all } p \in H^\sigma(\Omega) \quad \text{with } \sigma > \frac{1}{2}.$$

Thus we have

$$M_s(p(t)|_{\Gamma_s}) = M_s\big((\pi(t) - q_t(t))|_{\Gamma_s}\big) = \nu\,\gamma_s N_0 \Delta P\mathbf{v}(t)\cdot\mathbf{n} - \gamma_s N_s \eta_{2,t}(t).$$

We can now rewrite the equation satisfied by η_2 in (2.7) in the form

$$(I+\rho_1\gamma_s N_s)\eta_{2,t} - \omega\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha M_s\eta_{1,xxxx} = \rho_1\nu\gamma_s N_0\Delta P\mathbf{v}(t)\cdot\mathbf{n} + M_s f \ \text{ on } \Sigma_\infty^s.$$

LEMMA 3.2. *The operator* $I + \rho_1\gamma_s N_s$ *is an automorphism in* $L_0^2(\Gamma_s)$.

Proof. The operator $\gamma_s N_s$, considered as an operator belonging to $\mathcal{L}(L_0^2(\Gamma_s))$, is symmetric, positive, and compact. Indeed if $q = N_s\eta$ and $\tilde q = N_s\tilde\eta$, we have

$$0 = \int_\Omega \Delta q\,\tilde q = \int_{\Gamma_s} \eta\,\gamma_s N_s\tilde\eta - \int_{\Gamma_s}\gamma_s N_s\eta\,\tilde\eta$$

for all η, $\tilde\eta \in L_0^2(\Gamma_s)$. Thus $\gamma_s N_s$ is symmetric. Moreover,

$$0 = \int_\Omega \Delta q\, q = -\int_\Omega |\nabla q|^2 + \int_{\Gamma_s}\eta\,\gamma_s N_s\eta,$$

from which we deduce that $\gamma_s N_s$ is nonnegative. If

$$0 = \int_{\Gamma_s}\eta\,\gamma_s N_s\eta = \int_\Omega |\nabla q|^2,$$

we have $q = C = 0$ and $\frac{\partial q}{\partial \mathbf{n}} = \eta = 0$, which proves that $\gamma_s N_s$ is positive. Since $\gamma_s N_s \in \mathcal{L}(L^2_0(\Gamma_s), \mathcal{H}^1(\Gamma))$, it is clear that $\gamma_s N_s$ is a compact operator in $L^2_0(\Gamma_s)$. Thus $I + \rho_1 \gamma_s N_s$ is symmetric and positive and is an automorphism in $L^2_0(\Gamma_s)$. ∎

In order to write the system satisfied by $(P\mathbf{v}, \eta_1, \eta_2)$ as an evolution equation, we introduce the unbounded operator $(A_{\alpha,\beta}, D(A_{\alpha,\beta}))$ in $L^2_0(\Gamma_s)$ defined by

$$D(A_{\alpha,\beta}) = H^4(\Gamma_s) \cap H^2_0(\Gamma_s) \cap L^2_0(\Gamma_s), \quad A_{\alpha,\beta}\eta = \beta\eta_{xx} - \alpha M_s \eta_{xxxx}.$$

Let us notice that $(A_{\alpha,\beta}, D(A_{\alpha,\beta}))$ is a self-adjoint operator in $L^2_0(\Gamma_s)$. Since $A_{\alpha,\beta}$ is an isomorphism from $D(A_{\alpha,\beta})$ to $L^2_0(\Gamma_s)$, it can be extended as an isomorphism from $L^2_0(\Gamma_s)$ to $(D(A_{\alpha,\beta}))'$ (the dual of $D(A_{\alpha,\beta})$ with $L^2_0(\Gamma_s)$ as pivot space), and from $H^2_0(\Gamma_s) \cap L^2_0(\Gamma_s)$ into $(H^2_0(\Gamma_s) \cap L^2_0(\Gamma_s))'$. The space

$$\mathbf{H} = \mathbf{V}^0_n(\Omega) \times (H^2_0(\Gamma_s) \cap L^2_0(\Gamma_s)) \times L^2_0(\Gamma_s)$$

will be equipped with the inner product

$$\big((\mathbf{v}, \eta_1, \eta_2), (\mathbf{w}, \zeta_1, \zeta_2)\big)_{\mathbf{H}} = \rho_1 (\mathbf{v}, \mathbf{w})_{\mathbf{V}^0_n(\Omega)} + (\eta_1, \zeta_1)_{H^2_0(\Gamma_s)} + (\eta_2, \zeta_2)_{L^2_0(\Gamma_s)}$$

(where the inner product in $\mathbf{V}^0_n(\Omega)$ is inherited from $\mathbf{L}^2(\Omega)$) and

$$(\eta_1, \zeta_1)_{H^2_0(\Gamma_s)} = \int_{\Gamma_s} (-A_{\alpha,\beta})^{1/2} \eta_1 \, (-A_{\alpha,\beta})^{1/2} \zeta_1 = \int_{\Gamma_s} (\beta \eta_{1,x} \zeta_{1,x} + \alpha \eta_{1,xx} \zeta_{1,xx}) \, dx.$$

We define the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$ in $\mathbf{H}$ by

$$D(\mathcal{A}) = \Big\{ (P\mathbf{v}, \eta_1, \eta_2) \in \mathbf{V}^2_n(\Omega) \times (H^4 \cap H^2_0 \cap L^2_0)(\Gamma_s) \times (H^2_0 \cap L^2_0)(\Gamma_s) \mid \\ P\mathbf{v} - PD_s\eta_2 \in \mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega) \Big\}$$

and

$$\mathcal{A} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (I + \rho_1 \gamma_s N_s)^{-1} \end{pmatrix} \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & I \\ \rho_1 \, \nu \, \gamma_s N_0(\Delta(\cdot) \cdot \mathbf{n}) & A_{\alpha,\beta} & \delta \Delta_s \end{pmatrix},$$

where $\Delta_s = \frac{\partial^2}{\partial x_s^2}$. We define the unbounded operator $(A_s, D(A_s))$ in $H_s = (H^2_0(\Gamma_s) \cap L^2_0(\Gamma_s)) \times L^2_0(\Gamma_s)$ by

$$A_s = \begin{pmatrix} 0 & I \\ A_{\alpha,\beta} & \delta \Delta_s \end{pmatrix}, \quad D(A_s) = (H^4(\Gamma_s) \cap H^2_0(\Gamma_s) \cap L^2_0(\Gamma_s)) \times (H^2_0(\Gamma_s) \cap L^2_0(\Gamma_s)).$$

It can be easily shown that A_s is an isomorphism from $D(A_s)$ into H_s.

Now, it is clear that, for $\omega = 0$ and $f = 0$, we can rewrite system (2.7) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A} \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, \qquad \begin{pmatrix} P\mathbf{v}(0) \\ \eta_1(0) \\ \eta_2(0) \end{pmatrix} = \begin{pmatrix} P\mathbf{v}^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix},$$
$$(I - P)\mathbf{v}(t) = (I - P) D(\eta_2(t) \, \vec{e}_2 \, \chi_{\Gamma_s}).$$

The rewriting of system (2.7) when $\omega \neq 0$ and $f \neq 0$ is done in (4.1).

PROPOSITION 3.3. *The norm*

$$(P\mathbf{v}, \eta_1, \eta_2) \longmapsto \|(P\mathbf{v}, \eta_1, \eta_2)\|_{\mathbf{H}} + \|A_0 P\mathbf{v} + (-A_0)PD_s\eta_2\|_{\mathbf{V}_n^0(\Omega)} + \|A_s(\eta_1, \eta_2)\|_{H_s}$$

is a norm on $D(\mathcal{A})$ equivalent to the norm

$$(P\mathbf{v}, \eta_1, \eta_2) \longmapsto \|P\mathbf{v}\|_{\mathbf{V}_n^2(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H_0^2(\Gamma_s)}.$$

Proof. For $\lambda > 0$, $\lambda I - A_s$ is an isomorphism from $D(A_s)$ to H_s (see, e.g., section 3.4). Thus $(\eta_1, \eta_2) \mapsto \|(\eta_1, \eta_2)\|_{H_s} + \|A_s(\eta_1, \eta_2)\|_{H_s}$ is a norm equivalent to $(\eta_1, \eta_2) \mapsto \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H_0^2(\Gamma_s)}$. Since $(-A_0)$ is an isomorphism from $\mathbf{V}^2(\Omega) \cap \mathbf{V}_0^1(\Omega)$ to $\mathbf{V}_n^0(\Omega)$, there exist positive constants C_1 and C_2 such that

$$C_1\|P\mathbf{v} - PD_s\eta_2\|_{\mathbf{V}_n^2(\Omega)} \leq \|A_0 P\mathbf{v} + (-A_0)PD_s\eta_2\|_{\mathbf{V}_n^0(\Omega)} \leq C_2\|P\mathbf{v} - PD_s\eta_2\|_{\mathbf{V}_n^2(\Omega)}.$$

Moreover, $D_s \in \mathcal{L}(H_0^{3/2}(\Gamma_s), \mathbf{V}^2(\Omega))$ (see Lemma 3.11) and $A_s \in \mathcal{L}(D(A_s), H_s)$; therefore we have

$$\begin{aligned}
&\|(P\mathbf{v}, \eta_1, \eta_2)\|_{\mathbf{H}} + \|A_0 P\mathbf{v} + (-A_0)PD_s\eta_2\|_{\mathbf{V}_n^0(\Omega)} + \|A_s(\eta_1, \eta_2)\|_{H_s} \\
&\leq C(\|P\mathbf{v}\|_{\mathbf{V}_n^2(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H_0^2(\Gamma_s)}).
\end{aligned}$$

To prove the reverse inequality we write

$$\begin{aligned}
&\|P\mathbf{v}\|_{\mathbf{V}_n^2(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H_0^2(\Gamma_s)} \\
&\leq \frac{1}{C_1}\|A_0(P\mathbf{v} - PD_s\eta_2)\|_{\mathbf{V}_n^0(\Omega)} + \|PD_s\eta_2\|_{\mathbf{V}_n^2(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H_0^2(\Gamma_s)} \\
&\leq \frac{1}{C_1}\|A_0(P\mathbf{v} - PD_s\eta_2)\|_{\mathbf{V}_n^0(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + C\|\eta_2\|_{H_0^2(\Gamma_s)}.
\end{aligned}$$

The proof is complete. ∎

THEOREM 3.4. *The operator $(\mathcal{A}, D(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{H}$, and the resolvent of $\mathcal{A}$ is compact.*

To prove this theorem, we rewrite $\mathcal{A}$ in the form $\mathcal{A} = \mathcal{A}_1 + B_0$, with

$$\mathcal{A}_1 = \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & I \\ 0 & A_{\alpha,\beta} & \delta\Delta_s \end{pmatrix}$$

and

$$B_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \rho_1\nu(I + \rho_1\gamma_s N_s)^{-1}\gamma_s N_0(\Delta(\cdot)\cdot\mathbf{n}) & K_s\, A_{\alpha,\beta} & \delta K_s\, \Delta_s \end{pmatrix},$$

with $K_s = (I + \rho_1\gamma_s N_s)^{-1} - I$.

THEOREM 3.5. *The operator $(\mathcal{A}_1, D(\mathcal{A}_1))$ is the infinitesimal generator of a strongly continuous semigroup on $\mathbf{H}$.*

Proof. Step 1. We first show that the unbounded operator $(\widetilde{\mathcal{A}}_1, D(\widetilde{\mathcal{A}}_1))$ in $\mathbf{V}^{-1}(\Omega) \times H_s$, defined by

$$D(\widetilde{\mathcal{A}}_1) = \Big\{(P\mathbf{v}, \eta_1, \eta_2) \in \mathbf{V}^1_n(\Omega) \times (H^4 \cap H^2_0 \cap L^2_0)(\Gamma_s) \times (H^2_0 \cap L^2_0)(\Gamma_s) \\ \mid P\mathbf{v} - PD_s\eta_2 \in \mathbf{V}^1_0(\Omega)\Big\}$$

and

$$\widetilde{\mathcal{A}}_1 = \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & I \\ 0 & A_{\alpha,\beta} & \delta\Delta_s \end{pmatrix},$$

is the infinitesimal generator of a strongly continuous semigroup on $\mathbf{V}^{-1}(\Omega) \times H_s$. We endow $\mathbf{V}^{-1}(\Omega)$ with the norm

$$\mathbf{v} \longmapsto \Big(\langle (-A_0)^{-1}\mathbf{v}, \mathbf{v}\rangle_{\mathbf{V}^1_0(\Omega), \mathbf{V}^{-1}(\Omega)}\Big)^{1/2},$$

and H_s with the norm $\|\cdot\|_{H^2_0(\Gamma_s)\times L^2_0(\Gamma_s)}$. For $\lambda > 0$ we have

$$\Big((\widetilde{\mathcal{A}}_1 - \lambda I)(P\mathbf{v}, \eta_1, \eta_2), (P\mathbf{v}, \eta_1, \eta_2)\Big)_{\mathbf{V}^{-1}(\Omega)\times H_s} \\ = -\|P\mathbf{v}\|^2_{\mathbf{V}^0_n(\Omega)} + (PD_s\eta_2, P\mathbf{v})_{\mathbf{V}^0_n(\Omega)} - \lambda\|P\mathbf{v}\|^2_{\mathbf{V}^{-1}(\Omega)} - \lambda\|(\eta_1, \eta_2)\|^2_{H_s} - \delta\|\eta_2\|_{L^2_0(\Gamma_s)}.$$

Thus, for $\lambda > 0$ large enough, $(\widetilde{\mathcal{A}}_1 - \lambda I, D(\widetilde{\mathcal{A}}_1))$ is dissipative in $\mathbf{V}^{-1}(\Omega) \times H_s$. It can also be shown that it is maximal. Thus, for $\lambda > 0$ large enough, $(\widetilde{\mathcal{A}}_1 - \lambda I, D(\widetilde{\mathcal{A}}_1))$ is the infinitesimal generator of a semigroup of contractions on $\mathbf{V}^{-1}(\Omega) \times H_s$, and $(\widetilde{\mathcal{A}}_1, D(\widetilde{\mathcal{A}}_1))$ is the infinitesimal generator of a strongly continuous semigroup on $\mathbf{V}^{-1}(\Omega) \times H_s$.

Step 2. Let us consider the evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \widetilde{\mathcal{A}}_1 \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, \qquad \begin{pmatrix} P\mathbf{v}(0) \\ \eta_1(0) \\ \eta_2(0) \end{pmatrix} = \begin{pmatrix} P\mathbf{v}^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix}. \tag{3.3}$$

Let us recall that $(A_s, D(A_s))$ is the infinitesimal generator of an analytic semigroup on H_s (see, e.g., [8, 30]). Let us notice that the solution $(P\mathbf{v}, \eta_1, \eta_2)$ to (3.3) can be solved by determining first (η_1, η_2) and next $P\mathbf{v}$. Thus, if $(P\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{V}^{-1}(\Omega)\times H_s$, the solution $(P\mathbf{v}, \eta_1, \eta_2)$ to (3.3) is such that $\eta_1 \in H^{3,3/2}(\Sigma^s_T)$ and $\eta_2 \in H^{1,1/2}(\Sigma^s_T)$ for all $T > 0$ (see, e.g., [1, Chapter 3, Corollary 2.1]). From [24, Theorem 2.7] it follows that if $(P\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}$, then $P\mathbf{v} \in \mathbf{H}^{1,1/2}(Q_T) \cap C([0,T]; \mathbf{V}^0_n(\Omega))$ and $(P\mathbf{v}, \eta_1, \eta_2) \in C([0,T]; \mathbf{H})$. Therefore the restriction of the semigroup $(e^{t\widetilde{\mathcal{A}}_1})_{t\in\mathbb{R}^+}$ to $\mathbf{H}$ is a strongly continuous semigroup on $\mathbf{H}$. It is easy to verify that its domain is $D(\mathcal{A}_1) = D(\mathcal{A})$. ∎

We are going to prove the following two theorems.

THEOREM 3.6. *The operator $(\mathcal{A}_1, D(\mathcal{A}_1))$, with $D(\mathcal{A}_1) = D(\mathcal{A})$, is the infinitesimal generator of an analytic semigroup on $\mathbf{H} = \mathbf{V}^0_n(\Omega)\times(H^2_0(\Gamma_s)\cap L^2_0(\Gamma_s))\times L^2_0(\Gamma_s)$.*

THEOREM 3.7. *The operator $(B_0, D(\mathcal{A}_1))$ is $\mathcal{A}_1$-bounded with relative bound zero.*

The first claim in Theorem 3.4 clearly follows from Theorems 3.6 and 3.7 (see [15, Chapter 9, Corollary 2.5]). The second claim is proved in section 3.4.

3.2. Proof of Theorem 3.6. Now we are going to estimate the resolvent of $\mathcal{A}_1$. We have

$$(\lambda I - \mathcal{A}_1)^{-1} = \begin{pmatrix} (\lambda I - A_0)^{-1} & 0 & \left((\lambda I - A_0)^{-1}(-A_0)PD_s\right)(\lambda I - A_s)^{-1} \\ 0 & & (\lambda I - A_s)^{-1} \end{pmatrix}.$$

Since $(\lambda I - A_0)^{-1}(-A_0)PD_s = -\lambda(\lambda I - A_0)^{-1}PD_s + PD_s$, we obtain

$$(\lambda I - \mathcal{A}_1)^{-1} = \begin{pmatrix} (\lambda I - A_0)^{-1} & 0 & \left(-\lambda(\lambda I - A_0)^{-1}PD_s + PD_s\right)(\lambda I - A_s)^{-1} \\ 0 & & (\lambda I - A_s)^{-1} \end{pmatrix}.$$

From [8] (see also [30, section 2.2] and [9]), we know that there exist $a \in \mathbb{R}$ and $\pi/2 < \theta_0 < \pi$ such that

$$\|(\lambda I - A_s)^{-1}\|_{\mathcal{L}(H_s)} \leq \frac{C_s}{|\lambda - a|} \quad \text{for all } \lambda \in S_{a,\theta_0}, \tag{3.4}$$

where

$$S_{a,\theta_0} = \Big\{\lambda \in \mathbb{C} \mid \lambda \neq a, \quad |\arg(\lambda - a)| < \theta_0\Big\}.$$

For the Stokes resolvent we have

$$\|(\lambda I - A_0)^{-1}\mathbf{f}\|_{\mathbf{V}_n^0(\Omega)} \leq \frac{C_0}{|\lambda|}\|\mathbf{\Theta}\|_{\mathbf{V}_n^0(\Omega)} \quad \text{for all } \lambda \in S_{0,\theta_1}, \tag{3.5}$$

with $\pi/2 < \theta_1 < \pi$. We can choose $\theta_0 = \theta_1$ and $a > 0$. Thus if $(\mathbf{f}, \mathbf{\Theta}) \in \mathbf{V}_n^0(\Omega) \times H_s$, we have

$$\begin{aligned} &(\lambda I - \mathcal{A}_1)^{-1}\begin{pmatrix} \mathbf{f} \\ \mathbf{\Theta} \end{pmatrix} \\ &= \begin{pmatrix} (\lambda I - A_0)^{-1}\mathbf{f} - \lambda(\lambda I - A_0)^{-1}PD_s\left((\lambda I - A_s)^{-1}\mathbf{\Theta}\right)_2 + PD\left((\lambda I - A_s)^{-1}\mathbf{\Theta}\right)_2 \\ (\lambda I - A_s)^{-1}\mathbf{\Theta} \end{pmatrix}. \end{aligned}$$

From (3.4) and (3.5), it follows that

$$\|(\lambda I - A_s)^{-1}\mathbf{\Theta}\|_{H_s} \leq \frac{C_s}{|\lambda - a|}\|\mathbf{\Theta}\|_{H_s}, \quad \|PD\left((\lambda I - A_s)^{-1}\mathbf{\Theta}\right)_2\|_{\mathbf{V}_n^0(\Omega)} \leq \frac{C_{PD}\,C_s}{|\lambda - a|}\|\mathbf{\Theta}\|_{H_s},$$

$$\|\lambda(\lambda I - A_0)^{-1}PD\left((\lambda I - A_s)^{-1}\mathbf{\Theta}\right)_2\|_{\mathbf{V}_n^0(\Omega)} \leq \frac{C_0\,C_{PD}\,C_s}{|\lambda - a|}\|\mathbf{\Theta}\|_{H_s} \quad \text{for all } \lambda \in S_{a,\theta_0}.$$

By combining the previous estimates, we obtain

$$\begin{aligned} &\left\|(\lambda I - \mathcal{A}_1)^{-1}\begin{pmatrix} \mathbf{f} \\ \mathbf{\Theta} \end{pmatrix}\right\|_{\mathbf{V}_n^0(\Omega)\times H_s} \\ &\leq \frac{C_0}{|\lambda|}\|\mathbf{f}\|_{\mathbf{V}_n^0(\Omega)} + \frac{C_0\,C_{PD}\,C_s}{|\lambda - a|}\|\mathbf{\Theta}\|_{H_s} + \frac{C_{PD}\,C_s}{|\lambda - a|}\|\mathbf{\Theta}\|_{H_s} + \frac{C_s}{|\lambda - a|}\|\mathbf{\Theta}\|_{H_s} \end{aligned}$$

for all $\lambda \in S_{a,\theta_0}$, which proves the analyticity of the semigroup generated by $\mathcal{A}_1$.

3.3. Proof of Theorem 3.7. We set

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \rho_1 \nu\, (I + \rho_1 \gamma_s N)^{-1} \gamma_s N_0(\Delta(\cdot)\cdot \mathbf{n}) & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & K_s\, A_{\alpha,\beta} & 0 \end{pmatrix},$$

and

$$B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta K_s\, \Delta_s \end{pmatrix}.$$

LEMMA 3.8. *The operator* $(B_1, D(\mathcal{A}_1))$ *is* $\mathcal{A}_1$*-bounded with relative bound zero.*

Proof. Let us prove that, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\|\gamma_s N_0(\Delta \mathbf{v}\cdot \mathbf{n})\|_{L^2_0(\Gamma_s)} \le \varepsilon \|\mathbf{v}\|_{\mathbf{V}^2_n(\Omega)} + C_\varepsilon \|\mathbf{v}\|_{\mathbf{V}^0_n(\Omega)} \tag{3.6}$$

for all $\mathbf{v} \in \mathbf{V}^2_n(\Omega)$. To prove (3.6), we argue by contradiction. We assume that there exists a sequence $(\mathbf{v}_k)_k \subset \mathbf{V}^2_n(\Omega)$ such that

$$\|\gamma_s N_0(\Delta \mathbf{v}_k\cdot \mathbf{n})\|_{L^2_0(\Gamma_s)} = 1, \quad \|\mathbf{v}_k\|_{\mathbf{V}^0_n(\Omega)} \longrightarrow 0, \quad \text{and} \quad \|\mathbf{v}_k\|_{\mathbf{V}^2_n(\Omega)} \le M$$

for some $M > 0$. Therefore, without loss of generality, we can assume that there exists $\mathbf{v} \in \mathbf{V}^2_n(\Omega)$ such that

$$\mathbf{v}_k \rightharpoonup \mathbf{0} \text{ in } \mathbf{V}^2_n(\Omega), \quad \Delta \mathbf{v}_k \cdot \mathbf{n} \rightharpoonup \mathbf{0} \text{ in } H^{-1/2}(\Gamma), \quad \text{and } \Delta \mathbf{v}_k \cdot \mathbf{n} \longrightarrow 0 \text{ in } H^{-1/2-\varepsilon}(\Gamma)$$

for all $0 < \varepsilon \le 1/2$. From [6, Lemma A.5], we know that $\gamma_s N_0$ is bounded from $H^{-1}(\Gamma_s)$ to $L^2_0(\Gamma_s)$. Thus

$$\gamma_s N_0(\Delta \mathbf{v}_k \cdot \mathbf{n}) \longrightarrow 0 \quad \text{in } L^2_0(\Gamma_s),$$

which is in contradiction with

$$\|\gamma_s N_0(\Delta \mathbf{v}_k \cdot \mathbf{n})\|_{L^2_0(\Gamma_s)} = 1.$$

Thus (3.6) is proved. The lemma is a direct consequence of (3.6), Lemma 3.2, and Proposition 3.3. ∎

LEMMA 3.9. *There exists* $0 < \theta_2 < 1$ *such that* B_2 *is bounded from* $D((-\mathcal{A}_1)^{\theta_2})$ *into* $\mathbf{H}$.

Proof. Let $(\phi_k)_{k\ge 1}$ be an orthonormal basis in $L^2_0(\Gamma_s)$ constituted of eigenvectors of the operator $\rho_1 \gamma_s N$, and let $\lambda_k > 0$ be the eigenvalue associated with ϕ_k. For all $f = \sum_{k=1}^\infty \lambda_k f_k\, \phi_k \in L^2_0(\Gamma_s)$ we have

$$(I + \rho_1 \gamma_s N_s) f = \sum_{k=1}^\infty (1 + \lambda_k) f_k\, \phi_k.$$

Thus

$$(I + \rho_1 \gamma_s N_s)^{-1} f = \sum_{k=1}^\infty \frac{f_k}{1 + \lambda_k}\, \phi_k$$

and

$$K_s f = (I - (I + \rho_1 \gamma_s N_s)^{-1}) f = \sum_{k=1}^{\infty} \frac{\lambda_k}{1+\lambda_k} f_k \, \phi_k.$$

Since the operator $A_{\alpha,\beta}$ is an isomorphism from $H^4(\Gamma_s) \cap H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$ into $L_0^2(\Gamma_s)$ and from $L_0^2(\Gamma_s)$ into $(H^4(\Gamma_s) \cap H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s))'$, by interpolation it is also continuous from $H^{4-\varepsilon}(\Gamma_s) \cap H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$ into $\mathcal{H}^{-\varepsilon}(\Gamma_s)$ for all $0 \le \varepsilon \le 1$.

Denoting by $(A_{\alpha,\beta} f)_k$ the coefficient of $A_{\alpha,\beta} f$ in the basis $(\phi_k)_{k \ge 1}$, we have

$$\begin{aligned} \|K_s A_{\alpha,\beta} f\|^2_{L_0^2(\Gamma_s)} &= \sum_{k=1}^{\infty} \frac{\lambda_k^2}{(1+\lambda_k)^2} (A_{\alpha,\beta} f)_k^2 \le \sum_{k=1}^{\infty} \lambda_k^2 (A_{\alpha,\beta} f)_k^2 \\ &= \|\rho_1 \gamma_s N_s A_{\alpha,\beta} f\|^2_{L_0^2(\Gamma_s)} \le C_\varepsilon \|A_{\alpha,\beta} f\|^2_{\mathcal{H}^{-\varepsilon}(\Gamma_s)} \le C_\varepsilon \|f\|^2_{H^{4-\varepsilon}(\Gamma_s)} \end{aligned}$$

for all $f \in H^{4-\varepsilon}(\Gamma_s) \cap H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$ and all $0 \le \varepsilon < 1/2$. Indeed, $\gamma_s N_s$ is continuous from $\mathcal{H}^{-\varepsilon}(\Gamma_s)$ into $L_0^2(\Gamma_s)$ if $0 \le \varepsilon < 1/2$ (see, e.g., [6, Lemma A.5]). Since

$$H^{4-\varepsilon}(\Gamma_s) \cap H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s) \supset D((-\mathcal{A}_1)^{(4-\varepsilon')/4})$$

for all $0 \le \varepsilon' < \varepsilon < 1/2$, the proof is complete. ∎

LEMMA 3.10. *There exists* $0 < \theta_3 < 1$ *such that* B_3 *is bounded from* $D((-\mathcal{A}_1)^{\theta_3})$ *into* $\mathbf{H}$.

Proof. The proof is very similar to that of the previous lemma and is left to the reader. ∎

Theorem 3.7 is a direct consequence of Lemmas 3.8, 3.9, and 3.10.

3.4. Resolvent of $\mathcal{A}$. In this section we want to show that the resolvent of $\mathcal{A}$ is compact. For that we study the stationary problem

$$\begin{aligned} &\lambda \mathbf{v} - \operatorname{div} \sigma(\mathbf{v}, p) = \mathbf{f} \quad \text{and} \quad \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega, \\ &\mathbf{v} = \eta_2 \vec{e}_2 \ \text{ on } \Gamma_s, \quad \mathbf{v} = 0 \ \text{ on } \Gamma_0, \\ &\lambda \eta_1 - \eta_2 = g \quad \text{in } \Gamma_s, \\ &\lambda \eta_2 - \beta \eta_{1,xx} - \delta \eta_{2,xx} + \alpha M_s \eta_{1,xxxx} = M_s(\rho_1 p + h) \ \text{ in } \Gamma_s, \\ &\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \{0, L\}, \end{aligned} \tag{3.7}$$

where $\mathbf{f} \in \mathbf{V}_n^0(\Omega)$, $g \in H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$, $h \in L_0^2(\Gamma_s)$, $\lambda \in \mathbb{R}$, and $\lambda > 0$. This system is equivalent to

$$\begin{aligned} &\lambda \mathbf{v} - \operatorname{div} \sigma(\mathbf{v}, p) = \mathbf{f} \quad \text{and} \quad \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega, \\ &\mathbf{v} = (\lambda \eta_1 - g) \vec{e}_2 \ \text{ on } \Gamma_s, \quad \mathbf{v} = 0 \ \text{ on } \Gamma_0, \\ &\lambda \eta_1 - \eta_2 = g \quad \text{in } \Gamma_s, \\ &\lambda^2 \eta_1 - \beta \eta_{1,xx} - \delta \lambda \eta_{1,xx} + \alpha M_s \eta_{1,xxxx} = M_s(\rho_1 p + h + \lambda g - \delta \lambda g_{xx}) \ \text{ in } \Gamma_s, \\ &\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \{0, L\}. \end{aligned} \tag{3.8}$$

We denote by L the unbounded operator in $L_0^2(\Gamma_s)$ with domain $H^4(\Gamma_s) \cap H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$ defined by

$$L\eta = \lambda^2 \eta - \beta \eta_{xx} - \delta \lambda \eta_{xx} + \alpha M_s \eta_{xxxx}.$$

The operator L is also an isomorphism from $H^4(\Gamma_s) \cap H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$ into $L_0^2(\Gamma_s)$ and from $H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$ into $(H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s))'$. Thus, we can rewrite the system (3.8) in the form

$$\begin{aligned}
&\lambda \mathbf{v} - \operatorname{div} \sigma(\mathbf{v}, p) = \mathbf{f} \quad \text{and} \quad \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega,\\
&\mathbf{v} = (\lambda L^{-1} M_s(\rho_1 \gamma_s p + h + \lambda g - \delta \lambda g_{xx}) - g)\vec{e}_2 \ \text{ on } \Gamma_s, \quad \mathbf{v} = 0 \ \text{ on } \Gamma_0,\\
&\lambda \eta_1 - \eta_2 = g \quad \text{in } \Gamma_s,\\
&\lambda^2 \eta_1 - \beta \eta_{1,xx} - \delta \lambda \eta_{1,xx} + \alpha M_s \eta_{1,xxxx} = \rho_1 \gamma_s p + h + \lambda g - \delta \lambda g_{xx} \ \text{ in } \Gamma_s,\\
&\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \{0, L\}.
\end{aligned} \tag{3.9}$$

We consider the system

$$\begin{aligned}
&\lambda \mathbf{v} - \operatorname{div} \sigma(\mathbf{v}, p) = \mathbf{f} \quad \text{and} \quad \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega,\\
&\mathbf{v} = \lambda \rho_1 L^{-1}(\gamma_s p)\vec{e}_2 + f\vec{e}_2 \ \text{ on } \Gamma_s, \quad \mathbf{v} = 0 \ \text{ on } \Gamma_0,
\end{aligned} \tag{3.10}$$

where $f \in H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$ stands for $\lambda L^{-1} M_s(h + \lambda g - \delta \lambda g_{xx}) - g$. We set

$$\mathbf{E} = \Big\{ \mathbf{w} \in \mathbf{V}^1(\Omega) \mid \mathbf{v} = 0 \text{ on } \Gamma_0,\ \mathbf{v}_1 = 0 \text{ on } \Gamma_s,\ \mathbf{v}_2|_{\Gamma_s} \in H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s) \Big\}.$$

The space $\mathbf{E}$, equipped with the norm

$$\|\mathbf{v}\|_{\mathbf{E}} = \Big(\|\mathbf{v}\|^2_{\mathbf{V}^1(\Omega)} + \|L^{1/2}\mathbf{v}_2|_{\Gamma_s}\|^2_{L_0^2(\Gamma_s)} \Big)^{1/2},$$

is a Hilbert space because $L^{1/2}$ is an isomorphism from $H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$ onto $L_0^2(\Gamma_s)$.

Multiplying the first equation in (3.10) by $\mathbf{w} \in \mathbf{E}$, after integration we obtain

$$\int_\Omega (\lambda \mathbf{v} \cdot \mathbf{w} + \nu \nabla \mathbf{v} : \nabla \mathbf{w}) + \int_{\Gamma_s} p\, \mathbf{w}_2 = \int_\Omega \mathbf{f}\, \mathbf{w}.$$

Using

$$\lambda \rho_1 \gamma_s p = L\mathbf{v}_2 - Lf \quad \text{in } (H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s))',$$

we obtain

$$\int_\Omega (\lambda \mathbf{v} \cdot \mathbf{w} + \nu \nabla \mathbf{v} : \nabla \mathbf{w}) + \frac{1}{\lambda \rho_1} \int_{\Gamma_s} L^{1/2}\mathbf{v}_2\, L^{1/2}\mathbf{w}_2 = \int_\Omega \mathbf{f}\, \mathbf{w} + \frac{1}{\lambda \rho_1} \int_{\Gamma_s} L^{1/2} f\, L^{1/2}\mathbf{w}_2.$$

Next, we set

$$a(\mathbf{v}, \mathbf{w}) = \int_\Omega (\lambda \mathbf{v} \cdot \mathbf{w} + \nu \nabla \mathbf{v} : \nabla \mathbf{w}) + \frac{1}{\lambda \rho_1} \int_{\Gamma_s} L^{1/2}\mathbf{v}_2\, L^{1/2}\mathbf{w}_2$$

and

$$\ell(\mathbf{w}) = \int_\Omega \mathbf{f}\, \mathbf{w} + \frac{1}{\lambda \rho_1} \int_{\Gamma_s} L^{1/2} f\, L^{1/2}\mathbf{w}_2.$$

Thus system (3.10) is equivalent to

$$\begin{aligned}
&a(\mathbf{v}, \mathbf{w}) = \ell(\mathbf{w}) \quad \text{for all } \mathbf{w} \in \mathbf{E},\\
&\lambda \rho_1 \gamma_s p = L\mathbf{v}_2 - Lf \quad \text{in } (H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s))'.
\end{aligned} \tag{3.11}$$

With the Lax–Milgram theorem, we can prove that the variational problem

$$\text{Find } \mathbf{v} \in \mathbf{E} \text{ such that} \quad a(\mathbf{v}, \mathbf{w}) = \ell(\mathbf{w}) \quad \text{for all } \mathbf{w} \in \mathbf{E} \tag{3.12}$$

has a unique solution. Indeed, for all $H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$, we have

$$\int_{\Gamma_s} L^{1/2}\eta\, L^{1/2}\eta = \int_{\Gamma_s} \left(\lambda^2|\eta|^2 + \beta|\eta_x|^2 + \alpha|\eta_{xx}|^2\right) \geq \rho\|\eta\|^2_{H_0^2(\Gamma_s)}$$

for some $\rho > 0$.

The solution $\mathbf{v} \in \mathbf{E}$ to the above variational problem obeys

$$\|\mathbf{v}\|_{\mathbf{E}} \leq C(\|\mathbf{f}\|_{\mathbf{V}_0^n(\Omega)} + \|L^{1/2}f\|_{L^2(\Gamma_s)}).$$

Since $f = \lambda L^{-1}M_s(h + \lambda g - \delta\lambda g_{xx}) - g$, we have

$$\begin{aligned}\|\mathbf{v}\|_{\mathbf{E}} &\leq C(\|\mathbf{f}\|_{\mathbf{V}_0^n(\Omega)} + \|L^{-1/2}h\|_{L_0^2(\Gamma_s)} + \|L^{1/2}g\|_{L_0^2(\Gamma_s)})\\ &\leq C(\|\mathbf{f}\|_{\mathbf{V}_0^n(\Omega)} + \|h\|_{L_0^2(\Gamma_s)} + \|g\|_{H_0^2(\Gamma_s)}).\end{aligned}$$

Therefore

$$\|\mathbf{v}_2|_{\Gamma_s}\|_{H_0^2(\Gamma_s)} \leq C(\|\mathbf{f}\|_{\mathbf{V}_0^n(\Omega)} + \|h\|_{L_0^2(\Gamma_s)} + \|g\|_{H_0^2(\Gamma_s)}).$$

By taking $\mathbf{w} \in \mathbf{V}_0^1(\Omega)$ in the variational problem, we prove that $\mathbf{v} \in \mathbf{E}$ is the unique solution to the problem

$$\begin{aligned}&\text{Find } \mathbf{v} \in \mathbf{E} \text{ such that} \quad \int_\Omega (\lambda\mathbf{v}\cdot\mathbf{w} + \nabla\mathbf{v} : \nabla\mathbf{w}) = \int_\Omega \mathbf{f}\,\mathbf{w} \quad \text{for all } \mathbf{w} \in \mathbf{V}_0^1(\Omega),\\ &\mathbf{v} = 0 \quad \text{on } \Gamma_0, \qquad \mathbf{v} = \mathbf{v}_2|_{\Gamma_s}\vec{e}_2 \quad \text{on } \Gamma_s.\end{aligned}$$

Since $\mathbf{v}|_{\Gamma_0} = \mathbf{0}$, $\mathbf{v}_1|_{\Gamma_s} = 0$, and $\mathbf{v}_2|_{\Gamma_s} \in H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$, due to Lemma 3.11 below, it follows that $\mathbf{v} \in \mathbf{V}^2(\Omega) \cap \mathbf{E}$ and

$$\|\mathbf{v}\|_{\mathbf{V}^2(\Omega)} \leq C(\|\mathbf{f}\|_{\mathbf{V}_n^0(\Omega)} + \|g\|_{H_0^2(\Gamma_s)} + \|h\|_{L_0^2(\Gamma_s)}).$$

From the equation satisfied by $\mathbf{v}$ we also deduce that $p \in \mathcal{H}^1(\Omega)$ and

$$\|p\|_{\mathcal{H}^1(\Omega)} \leq C(\|\mathbf{f}\|_{\mathbf{V}_n^0(\Omega)} + \|g\|_{H_0^2(\Gamma_s)} + \|h\|_{L_0^2(\Gamma_s)}).$$

Finally, with the equation satisfied by η_1 and η_2 in (3.9), we have shown that system (3.7) admits a unique solution $(\mathbf{v}, p, \eta_1, \eta_2) \in \mathbf{V}^2(\Omega) \times \mathcal{H}^1(\Omega) \times (H^4(\Gamma_s) \cap H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)) \times (H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s))$ and

$$\|\mathbf{v}\|_{\mathbf{V}^2(\Omega)} + \|p\|_{\mathcal{H}^1(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H_0^2(\Gamma_s)} \leq C(\|\mathbf{f}\|_{\mathbf{V}_n^0(\Omega)} + \|g\|_{H_0^2(\Gamma_s)} + \|h\|_{L_0^2(\Gamma_s)}).$$

Thus the resolvent of $\mathcal{A}$ is compact in $\mathbf{H}$.

LEMMA 3.11. *If* $\mathbf{f} \in \mathbf{V}_n^0(\Omega)$ *and* $g \in H_0^{3/2}(\Gamma_s) \cap L_0^2(\Gamma_s)$ *(with* $H_0^{3/2}(\Gamma_s) = [H_0^1(\Gamma_s), H_0^2(\Gamma_s)]_{1/2}$*), then the solution* $\mathbf{v}$ *to*

$$\lambda\mathbf{v} - \operatorname{div}\sigma(\mathbf{v}, p) = \mathbf{f} \quad \textit{and} \quad \operatorname{div}\mathbf{v} = 0 \quad \textit{in } \Omega, \quad \mathbf{v} = 0 \quad \textit{on } \Gamma_0, \quad \mathbf{v} = g\,\vec{e}_2 \quad \textit{on } \Gamma_s$$

belongs to $\mathbf{V}^2(\Omega)$ *and*

$$\|\mathbf{v}\|_{\mathbf{V}^2(\Omega)} \leq C(\|\mathbf{f}\|_{\mathbf{V}_n^0(\Omega)} + \|g\|_{H_0^{3/2}(\Gamma_s)}).$$

Proof. With a localization argument and the regularity results in [11], we can show that $\mathbf{v}|_{(0,L)\times(0,1-\epsilon)}$ belongs to $\mathbf{V}^2((0,L)\times(0,1-\epsilon))$ and that $\mathbf{v}|_{(\epsilon,L-\epsilon)\times(0,1)}$ belongs to $\mathbf{V}^2((\epsilon,L-\epsilon)\times(0,1))$ for all $0<\epsilon<\min(1,L)$. Thus the only difficulty is at the corners $(0,1)$ and $(L,1)$. To prove the lemma, we look for $\mathbf{v}=(\mathbf{v}_1,\mathbf{v}_2)^T$ in the form $\mathbf{v}=\widetilde{\mathbf{v}}+\widehat{\mathbf{v}}$, with $\widetilde{\mathbf{v}}=(0,\widetilde{\mathbf{v}}_2)^T$, where $\widetilde{\mathbf{v}}_2$ is the solution to the Laplace equation

$$\lambda\widetilde{\mathbf{v}}_2-\nu\Delta\widetilde{\mathbf{v}}_2=0 \quad \text{in } \Omega, \quad \widetilde{\mathbf{v}}_2=0 \quad \text{on } \Gamma_0, \quad \widetilde{\mathbf{v}}_2=g \quad \text{on } \Gamma_s,$$

and $\widehat{\mathbf{v}}$ is the solution to

$$\lambda\widehat{\mathbf{v}}-\operatorname{div}\sigma(\widehat{\mathbf{v}},p)=\mathbf{f} \quad \text{and} \quad \operatorname{div}\widehat{\mathbf{v}}=-\widetilde{\mathbf{v}}_{2,y} \quad \text{in } \Omega, \qquad \widehat{\mathbf{v}}=0 \quad \text{on } \Gamma.$$

We set $h=-\widetilde{\mathbf{v}}_{2,y}$. We notice that $\widetilde{\mathbf{v}}_2\in H^2(\Omega)$, $h\in H^1(\Omega)$, $h|_{\{0\}\times(0,1)}=0$, and $h|_{\{L\}\times(0,1)}=0$. We look for $\widehat{\mathbf{v}}$ in the form $\widehat{\mathbf{v}}=\zeta+\mathbf{w}$, where ζ is the solution to

$$\operatorname{div}\zeta=h \quad \text{in } \Omega, \quad \zeta=0 \quad \text{on } \Gamma,$$

and $\mathbf{w}$ is the solution to

$$\lambda\mathbf{w}-\operatorname{div}\sigma(\mathbf{w},p)=\mathbf{f}-\lambda\zeta+\nu\Delta\zeta \quad \text{and} \quad \operatorname{div}\mathbf{w}=0 \quad \text{in } \Omega, \quad \mathbf{w}=0 \quad \text{on } \Gamma.$$

It is clear that if $\zeta\in\mathbf{H}^2(\Omega)$, then the lemma is proved (we can use a regularity result from [16]). We have to study the regularity of ζ at the corners $(0,0)$, $(0,1)$, $(L,1)$ and $(L,0)$. We study the regularity at the corners $(0,0)$ and $(0,1)$; the others can be studied in the same way. To study the regularity of $\zeta=(\zeta_1,\zeta_2)^T$, we extend the equation satisfied by ζ by using a symmetry argument. Let us set

$$\tilde{\Omega}=(-L,L)\times(0,1), \quad \tilde{\Gamma}=\partial(\tilde{\Omega}), \quad \tilde{\zeta}_1(x,z)=\begin{cases}\zeta_1(x,z) & \text{if } x\in(0,L),\\ -\zeta_1(-x,z) & \text{if } x\in(-L,0),\end{cases}$$

$$\tilde{\zeta}_2(x,z)=\begin{cases}\zeta_2(x,z) & \text{if } x\in(0,L),\\ \zeta_2(-x,z) & \text{if } x\in(-L,0),\end{cases} \qquad \tilde{h}(x,z)=\begin{cases}h(x,z) & \text{if } x\in(0,L),\\ h(-x,z) & \text{if } x\in(-L,0).\end{cases}$$

Since $h|_{\{0\}\times(0,1)}=0$, then $\tilde{h}\in H^1(\tilde{\Omega})$. Thus $\tilde{\zeta}=(\tilde{\zeta}_1,\tilde{\zeta}_2)$ is solution to the equation

$$\operatorname{div}\tilde{\zeta}=\tilde{h} \quad \text{in } \tilde{\Omega}, \quad \tilde{\zeta}=0 \quad \text{on } \tilde{\Gamma},$$

and $\tilde{\zeta}|_{(-L+\epsilon,L-\epsilon)\times(0,1)}$ belongs to $\mathbf{H}^2((-L+\epsilon,L-\epsilon)\times(0,1))$ for all $0<\epsilon<L$. This completes the proof. ∎

3.5. Adjoint of $(\mathcal{A},D(\mathcal{A}))$.

THEOREM 3.12. *The adjoint of* $(\mathcal{A},D(\mathcal{A}))$ *in* $\mathbf{H}$ *is defined by* $D(\mathcal{A}^*)=D(\mathcal{A})$ *and*

$$\mathcal{A}^*=\begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & -I \\ \rho_1\nu\gamma_s N_0(\Delta(\cdot)\cdot\mathbf{n}) & -A_{\alpha,\beta} & \delta\Delta_s \end{pmatrix}\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (I+\rho_1\gamma_s N_s)^{-1}\end{pmatrix}.$$

Remark 3.13. Let us notice that

$$\begin{pmatrix} 0 & -I \\ -A_{\alpha,\beta} & \delta\Delta_s \end{pmatrix} \quad \text{is the adjoint of} \quad \begin{pmatrix} 0 & I \\ A_{\alpha,\beta} & \delta\Delta_s \end{pmatrix}$$

in $(H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)) \times L_0^2(\Gamma_s)$. Next, we have to consider the unbounded operator $(\mathcal{A}_f, D(\mathcal{A}_f))$ in $\mathbf{V}_n^0(\Omega) \times L_0^2(\Gamma_s)$ defined by

$$D(\mathcal{A}_f) = \Big\{ (P\mathbf{v}, \eta_2) \in \mathbf{V}_n^2(\Omega) \times (H_0^2 \cap L_0^2)(\Gamma_s) \mid P\mathbf{v} - PD_s\eta_2 \in \mathbf{V}^2(\Omega) \cap \mathbf{V}_0^1(\Omega) \Big\}$$

and

$$\mathcal{A}_f = \begin{pmatrix} A_0 & (-A_0)PD_s \\ \rho_1\, \nu\, \gamma_s N_0(\Delta(\cdot)\cdot \mathbf{n}) & 0 \end{pmatrix}.$$

We expect that

$$\begin{aligned} &\rho_1\, (A_0(P\mathbf{v} - PD_s\eta_2), P\mathbf{\Phi})_{\mathbf{V}_n^0(\Omega)} + \rho_1\, (\nu\, \gamma_s N_0(\Delta(P\mathbf{v})\cdot \mathbf{n}), k_2)_{L^2(\Gamma_s)} \\ &= \rho_1\, (P\mathbf{v}, A_0(P\mathbf{\Phi} - PD_s k_2))_{\mathbf{V}_n^0(\Omega)} + \rho_1\, (\nu\, \gamma_s N_0(\Delta(P\mathbf{\Phi})\cdot \mathbf{n}), \eta_2)_{L^2(\Gamma_s)} \end{aligned} \tag{3.13}$$

for all $(P\mathbf{v}, \eta_2) \in D(\mathcal{A}_f)$ and all $(P\mathbf{\Phi}, k_2) \in D(\mathcal{A}_f)$, which means that $(\mathcal{A}_f, D(\mathcal{A}_f))$ is self-adjoint in $\mathbf{V}_n^0(\Omega) \times L_0^2(\Gamma_s)$.

The reader must be careful. We may write that

$$A_0(P\mathbf{v} - PD_s\eta_2) = \nu\, \Delta(P\mathbf{v} - PD_s\eta_2) = \nu\, \Delta\, P\mathbf{v} - \nu\, \Delta\, PD_s\eta_2,$$

because $P\mathbf{v} - PD_s\eta_2$ belongs to $\mathbf{V}^2(\Omega) \cap \mathbf{V}_0^1(\Omega)$. But we cannot write that $A_0\, P\mathbf{v} = \nu\, \Delta\, P\mathbf{v}$ and $A_0\, PD_s\eta_2 = \nu\, \Delta\, PD_s\eta_2$ separately, because $P\mathbf{v}$ and $PD_s\eta_2$ do not belong to $\mathbf{V}^2(\Omega) \cap \mathbf{V}_0^1(\Omega)$. The two terms $A_0\, P\mathbf{v}$ and $A_0\, PD_s\eta_2$ have a meaning only in $(\mathbf{V}^2(\Omega) \cap \mathbf{V}_0^1(\Omega))'$.

To verify identity (3.13), we set $\mathbf{f} = -\nu P\Delta P\mathbf{v} \in \mathbf{V}_n^0(\Omega)$, $\mathbf{\Theta} = -\nu P\Delta P\mathbf{\Phi} \in \mathbf{V}_n^0(\Omega)$, $q = N_s\, \eta_2 \in L_0^2(\Omega)$, and $\rho = N_s\, k_2 \in L_0^2(\Omega)$. We can verify that $\nabla q = (I-P)D_s\eta_2$ and $\nabla\rho = (I-P)D_s k_2$. We define $(I-P)\mathbf{v}$ and $(I-P)\mathbf{\Phi}$ by $(I-P)\mathbf{v} = \nabla q$ and $(I-P)\mathbf{\Phi} = \nabla\rho$. Thus $\mathbf{v}|_\Gamma = P\mathbf{v}|_\Gamma + (I-P)\mathbf{v}|_\Gamma = (PD_s\eta_2 + (I-P)D_s\eta_2)|_\Gamma = \eta_2\, \vec{e_2}\, \chi_{\Gamma_s}$. Similarly $\mathbf{\Phi}|_\Gamma = k_2\, \vec{e_2}\, \chi_{\Gamma_s}$. Finally, we set $p = N_0(\nu\Delta(P\mathbf{v})\cdot \mathbf{n})$ and $\psi = N_0(\nu\Delta(P\mathbf{\Phi})\cdot \mathbf{n})$. We notice that $\nabla p = \nu(I-P)\Delta P\mathbf{v}$ and $\nabla\psi = \nu(I-P)\Delta P\mathbf{\Phi}$. We have

$$-\nu\Delta\mathbf{v} = -\nu P\Delta P\mathbf{v} - \nu(I-P)\Delta P\mathbf{v} - \nu\Delta(I-P)\mathbf{v} = \mathbf{f} - \nabla p$$

because $\Delta(I-P)\mathbf{v} = \Delta\nabla q = 0$ and $\nu(I-P)\Delta P\mathbf{v} = \nabla p$. Thus $\mathbf{v}$ is the solution to the boundary value problem

$$-\nu\Delta\mathbf{v} + \nabla p = \mathbf{f} \quad \text{and} \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega, \qquad \mathbf{v} = \eta_2\vec{e}_2 \ \text{ on } \Gamma_s, \quad \mathbf{v} = 0 \ \text{ on } \Gamma_0. \tag{3.14}$$

Similarly, $\mathbf{\Phi}$ is the solution to the boundary value problem

$$-\nu\Delta\mathbf{\Phi} + \nabla\psi = \mathbf{\Theta} \quad \text{and} \quad \operatorname{div}\mathbf{\Phi} = 0 \quad \text{in } \Omega, \qquad \mathbf{\Phi} = k_2\vec{e}_2 \ \text{ on } \Gamma_s, \quad \mathbf{\Phi} = 0 \ \text{ on } \Gamma_0. \tag{3.15}$$

Since $\mathbf{v} = D_s\eta_2 + (-A_0)^{-1}\mathbf{f}$ and $\mathbf{\Phi} = D_s k_2 + (-A_0)^{-1}\mathbf{\Theta}$, we have $P\mathbf{v} - PD_s\eta_2 = (-A_0)^{-1}\mathbf{f}$ and $A_0(P\mathbf{v} - PD_s\eta_2) = -\mathbf{f}$. In a similar way $A_0(P\mathbf{\Phi} - PD_s k_2) = -\mathbf{\Theta}$. Thus (3.13) is nothing but

$$-\,(\mathbf{f}, \mathbf{\Phi})_{\mathbf{L}^2(\Omega)} + (p, k_2)_{L^2(\Gamma_s)} = -\,(\mathbf{v}, \mathbf{\Theta})_{\mathbf{L}^2(\Omega)} + (\psi, \eta_2)_{L^2(\Gamma_s)}\,.$$

This identity can be deduced by integrations by parts from (3.14) and (3.15). Thus we have drawn a short proof of Theorem 3.12. We give below another proof, which is based only on Green's formula and in which the calculations are easier to follow.

Proof. Let $\mathbf{f}$ belong to $\mathbf{V}_n^0(\Omega)$, g belong to $H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$, and h belong to $L_0^2(\Gamma_s)$. Let $(\mathbf{v}, p, \eta_1, \eta_2)$ be the solution to (3.7). Let $\mathbf{\Theta}$ belong to $\mathbf{V}_n^0(\Omega)$, ζ belong to $H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$, and ξ belong to $L_0^2(\Gamma_s)$, and let $(\mathbf{\Phi}, \psi, k_1, k_2)$ be the solution to

$$
\begin{aligned}
&\lambda \mathbf{\Phi} - \operatorname{div} \sigma(\mathbf{\Phi}, \psi) = \mathbf{\Theta} \quad \text{and} \quad \operatorname{div} \mathbf{\Phi} = 0 \quad \text{in } \Omega,\\
&\mathbf{\Phi} = k_2 \vec{e}_2 \ \text{ on } \Gamma_s, \quad \mathbf{\Phi} = 0 \ \text{ on } \Gamma_0,\\
&\lambda k_1 + k_2 = \zeta \quad \text{in } \Gamma_s,\\
&\lambda k_2 + \beta k_{1,xx} - \delta k_{2,xx} - \alpha M_s k_{1,xxxx} = M_s(\rho_1 \psi) + \xi \ \text{ in } \Gamma_s,\\
&k_1 = 0 \quad \text{and} \quad k_{1,x} = 0 \ \text{ on } \{0, L\}.
\end{aligned}
\tag{3.16}
$$

With integration by parts we have

$$
\begin{aligned}
\int_\Omega \mathbf{f} \cdot \mathbf{\Phi} &= \int_\Omega (\lambda \mathbf{v} - \operatorname{div} \sigma(\mathbf{v}, p))\, \mathbf{\Phi}\\
&= \int_\Omega \mathbf{v}\, (\lambda \mathbf{\Phi} - \operatorname{div} \sigma(\mathbf{\Phi}, \psi)) - \int_{\Gamma_s} \sigma(\mathbf{v}, p) \mathbf{n} \cdot \mathbf{\Phi} + \int_{\Gamma_s} \sigma(\mathbf{\Phi}, \psi) \mathbf{n} \cdot \mathbf{v}\\
&= \int_\Omega \mathbf{v} \cdot \mathbf{\Theta} + \int_{\Gamma_s} p\, \mathbf{\Phi}_2 - \int_{\Gamma_s} \psi\, \mathbf{v}_2\\
&= \int_\Omega \mathbf{v} \cdot \mathbf{\Theta} + \int_{\Gamma_s} p\, k_2 - \int_{\Gamma_s} \psi\, \eta_2,
\end{aligned}
$$

$$
\begin{aligned}
\int_{\Gamma_s} \zeta\, (-A_{\alpha,\beta}) \eta_1 &= \int_{\Gamma_s} (\lambda k_1 + k_2)\, (-A_{\alpha,\beta}) \eta_1\\
&= \int_{\Gamma_s} (\lambda (-A_{\alpha,\beta}) k_1\, \eta_1 + k_2\, (-A_{\alpha,\beta}) \eta_1)\\
&= \int_{\Gamma_s} \big((-\beta k_{1,xx} + \alpha k_{1,xxxx})(\eta_2 + g) + k_2 (-\beta \eta_{1,xx} + \alpha \eta_{1,xxxx}) \big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\Gamma_s} (\xi + \rho_1 \psi)\, \eta_2 = \int_{\Gamma_s} (\lambda k_2 + \beta k_{1,xx} - \delta k_{2,xx} - \alpha k_{1,xxxx})\, \eta_2\\
&= \int_{\Gamma_s} (\lambda k_2\, \eta_2 + (\beta k_{1,xx} - \delta k_{2,xx} - \alpha k_{1,xxxx})\, \eta_2)\\
&= \int_{\Gamma_s} \big(k_2 (\beta \eta_{1,xx} + \delta \eta_{2,xx} - \alpha \eta_{1,xxxx} + \rho_1 p + h) + (\beta k_{1,xx} - \delta k_{2,xx} - \alpha k_{1,xxxx})\, \eta_2 \big)\\
&= \int_{\Gamma_s} \big(k_2 (\beta \eta_{1,xx} - \alpha \eta_{1,xxxx} + \rho_1 p + h) + (\beta k_{1,xx} - \alpha k_{1,xxxx})\, \eta_2 \big).
\end{aligned}
$$

By combining the three identities, we obtain

$$\begin{aligned}
&\rho_1 \int_\Omega \mathbf{f}\cdot\mathbf{\Phi} + \int_{\Gamma_s} g\,(-A_{\alpha,\beta})k_1 + \int_{\Gamma_s} k_2\, h \\
&= \rho_1 \int_\Omega \mathbf{v}\cdot\mathbf{\Theta} + \rho_1 \int_{\Gamma_s} p\, k_2 - \rho_1 \int_{\Gamma_s} \psi\, \eta_2 \\
&\quad + \int_{\Gamma_s} \zeta\,(-A_{\alpha,\beta})\eta_1 + \int_{\Gamma_s} \big((\beta k_{1,xx} - \alpha k_{1,xxxx})\eta_2 - k_2(-\beta\eta_{1,xx} + \alpha\eta_{1,xxxx})\big) \\
&\quad + \int_{\Gamma_s} (\xi + \rho_1\psi)\,\eta_2 + \int_{\Gamma_s} \big(k_2(-\beta\eta_{1,xx} + \alpha\eta_{1,xxxx} - \rho_1 p) - (\beta k_{1,xx} - \alpha k_{1,xxxx})\,\eta_2\big) \\
&= \rho_1 \int_\Omega \mathbf{v}\cdot\mathbf{\Theta} + \int_{\Gamma_s} \zeta\,(-A_{\alpha,\beta})\eta_1 + \int_{\Gamma_s} \xi\,\eta_2.
\end{aligned}$$

To prove the theorem, we have to interpret the identity

$$\rho_1 \int_\Omega \mathbf{f}\cdot\mathbf{\Phi} + \int_{\Gamma_s} g\,(-A_{\alpha,\beta})k_1 + \int_{\Gamma_s} k_2\, h = \rho_1 \int_\Omega \mathbf{v}\cdot\mathbf{\Theta} + \int_{\Gamma_s} \zeta\,(-A_{\alpha,\beta})\eta_1 + \int_{\Gamma_s} \xi\,\eta_2. \tag{3.17}$$

For that we introduce the unbounded operator $(\mathcal{A}^\sharp, D(\mathcal{A}^\sharp))$ in $\mathbf{H}$ defined by $D(\mathcal{A}^\sharp) = D(\mathcal{A})$ and

$$\mathcal{A}^\sharp = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (I + \rho_1\gamma_s N_s)^{-1} \end{pmatrix} \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & -I \\ \gamma_s N_0(\rho_1\nu\Delta(\cdot)\cdot\mathbf{n}) & -A_{\alpha,\beta} & \delta\Delta_s \end{pmatrix}.$$

We first notice that $(\mathbf{v}, p, \eta_1, \eta_2)$ is the solution to (3.7) if and only if it satisfies

$$(\lambda I_s - \mathcal{A}) \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ g \\ (I + \rho_1\gamma_s N_s)^{-1} h \end{pmatrix}, \quad I_s = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (I + \rho_1\gamma_s N_s)^{-1} \end{pmatrix},$$

$$(I - P)\mathbf{v} = (I - P)D_s(\eta_2).$$

Similarly, we can show that $(\mathbf{\Phi}, \psi, k_1, k_2)$ is the solution to system (3.16) if and only if

$$(\lambda I_s - \mathcal{A}^\sharp) \begin{pmatrix} P\mathbf{\Phi} \\ k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \mathbf{\Theta} \\ \zeta \\ (I + \rho_1\gamma_s N_s)^{-1}\xi \end{pmatrix}, \qquad (I - P)\mathbf{\Phi} = (I - P)D_s(k_2).$$

Thus, identity (3.17) is equivalent to

$$\begin{aligned}
&\big((\lambda I - \mathcal{A})(P\mathbf{v}, \eta_1, \eta_2), (\mathbf{\Phi}, k_1, (I + \rho_1\gamma_s N_s)k_2)\big)_{\mathbf{H}} \\
&= \big((\lambda I - \mathcal{A}^\sharp)(P\mathbf{\Phi}, k_1, k_2), (\mathbf{v}, \eta_1, (I + \rho_1\gamma_s N_s)\eta_2)\big)_{\mathbf{H}}
\end{aligned} \tag{3.18}$$

for all $(P\mathbf{v}, \eta_1, \eta_2) \in D(\mathcal{A})$ and all $(P\mathbf{\Phi}, k_1, k_2) \in D(\mathcal{A})$. Let us denote by $\widehat{\mathbf{H}}$ the space $\mathbf{H}$ equipped with the inner product

$$\begin{aligned}
&\big((\mathbf{v}^0, \eta_1^0, \eta_2^0), (\mathbf{w}^0, \zeta_1^0, \zeta_2^0)\big)_{\widehat{\mathbf{H}}} \\
&\quad = \rho_1 \big(\mathbf{v}^0, \mathbf{w}^0\big)_{\mathbf{V}_n^0(\Omega)} + \big(\eta_1^0, \zeta_1^0\big)_{H_0^2(\Gamma_s)} + \big(\eta_2^0, (I + \rho_1\gamma_s N_s)\zeta_2^0\big)_{L_0^2(\Gamma_s)}.
\end{aligned}$$

Thus identity (3.18) means that $(\mathcal{A}^\sharp, D(\mathcal{A}^\sharp))$ is the adjoint of $(\mathcal{A}, D(\mathcal{A}))$ in $\widehat{\mathbf{H}}$. We can easily deduce the theorem from this result. $\square$

4. Regularity of solutions to the linearized system.

4.1. Studying system (2.7). We introduce the operator $(\mathcal{A}_\omega, D(\mathcal{A}_\omega))$ defined by $D(\mathcal{A}_\omega) = D(\mathcal{A})$ and

$$\mathcal{A}_\omega = \mathcal{A} + \begin{pmatrix} \omega I & 0 & 0 \\ 0 & \omega I & 0 \\ 0 & 0 & \omega(I + \rho_1\gamma_s N_s)^{-1} \end{pmatrix}.$$

From calculations in section 3.1, it follows that, if $f \in L^2(0,\infty; L^2_0(\Gamma_s))$, system (2.7) can be rewritten in the following equivalent form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}_\omega \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} + \mathcal{B}f, \qquad \begin{pmatrix} P\mathbf{v}(0) \\ \eta_1(0) \\ \eta_2(0) \end{pmatrix} = \begin{pmatrix} P\mathbf{v}^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix}, \tag{4.1}$$
$$(I-P)\mathbf{v}(t) = (I-P)D(\eta_2(t)\vec{e}_2\,\chi_{\Gamma_s}),$$

where $\mathcal{B} \in \mathcal{L}(L^2_0(\Gamma_s), \mathbf{H})$ is defined by

$$\mathcal{B}f = \begin{pmatrix} \mathbf{0} \\ 0 \\ (I + \rho_1\gamma_s N_s)^{-1} f \end{pmatrix}.$$

We have to study solutions to system (4.1) when $(P\mathbf{v}^0, \eta_1^0, \eta_2^0) \in [D(\mathcal{A}), \mathbf{H}]_{1/2}$. From the definition of $D(\mathcal{A})$ and $\mathbf{H}$, we can deduce that

$$[D(\mathcal{A}), \mathbf{H}]_{1/2} = \Big\{(P\mathbf{v}, \eta_1, \eta_2) \in \mathbf{V}^1_n(\Omega) \times (H^3 \cap H^2_0 \cap L^2_0)(\Gamma_s) \times (H^1_0 \cap L^2_0)(\Gamma_s) \mid \\ P\mathbf{v} - PD_s\eta_2 \in \mathbf{V}^1_0(\Omega)\Big\}.$$

Equipped with the norm

$$(P\mathbf{v}, \eta_1, \eta_2) \longrightarrow \Big(\|\mathbf{v}\|^2_{\mathbf{V}^1_n(\Omega)} + \|\eta_1\|^2_{H^3(\Gamma_s)} + \|\eta_2\|^2_{H^1(\Gamma_s)}\Big)^{1/2},$$

$[D(\mathcal{A}), \mathbf{H}]_{1/2}$ is a Hilbert space.

If $(P\mathbf{v}^0, \eta_1^0, \eta_2^0)$ belongs to $\mathbf{H}$, no compatibility condition between $P\mathbf{v}^0$ and η_2^0 is needed to define weak solutions of the evolution equation (4.1). However, the mapping $t \mapsto (I-P)\mathbf{v}(t)$, which satisfies the second equation in (4.1), will be continuous at time $t = 0$ only if $(I-P)\mathbf{v}^0$ and η_2^0 satisfy $(I-P)\mathbf{v}^0 \cdot \mathbf{n} = \eta_2^0\,\chi_{\Gamma_s}$. Notice that if $\mathbf{v}^0 \in \mathbf{V}^0(\Omega)$, then $\operatorname{div}(I-P)\mathbf{v}^0 = 0$ and $(I-P)\mathbf{v}^0 \cdot \mathbf{n}$ is well defined in $\mathcal{H}^{-1/2}(\Gamma)$. We define a space of initial conditions, satisfying the compatibility condition needed for the continuity of the mapping $t \mapsto (I-P)\mathbf{v}(t)$, as follows:

$$\mathbf{H}_{cc} = \Big\{(\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{V}^0(\Omega) \times H_s \mid \mathbf{v}^0 \cdot \mathbf{n} = \eta_2^0\,\chi_{\Gamma_s}\Big\}.$$

Recall that $H_s = (H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)) \times L_0^2(\Gamma_s)$ (see section 3.2). We equip $\mathbf{H}_{cc}$ with the inner product

$$\big((\mathbf{v}^0, \eta_1^0, \eta_2^0), (\mathbf{w}_0, \zeta_1^0, \zeta_2^0)\big)_{\mathbf{H}_{cc}} = \rho_1(\mathbf{v}^0, \mathbf{w}_0)_{\mathbf{L}^2(\Omega)} + (\eta_1^0, \zeta_1^0)_{H_0^2(\Gamma_s)} + (\eta_2^0, \zeta_2^0)_{L_0^2(\Gamma_s)}.$$

THEOREM 4.1. (i) *If* $(P\mathbf{v}^0, \eta_1^0, \eta_2^0) \in [D(\mathcal{A}), \mathbf{H}]_{1/2}$, $(\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}_{cc}$, *and* $f \in L^2(0,T; L_0^2(\Gamma_s))$, *then system* (4.1) *admits a unique strict solution satisfying*

$$\begin{aligned}&\|P\mathbf{v}\|_{\mathbf{H}^{2,1}(Q_T)} + \|\eta_1\|_{H^{4,2}(\Sigma_T^s)} + \|\eta_2\|_{H^{2,1}(\Sigma_T^s)}\\ &\qquad \le C(\|(P\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{[D(\mathcal{A}),\mathbf{H}]_{1/2}} + \|f\|_{L^2(0,T;L_0^2(\Gamma_s))}),\end{aligned}$$

$$\begin{aligned}&\|(I-P)\mathbf{v}\|_{L^2(0,T;\mathbf{H}^2(\Omega))} + \|(I-P)\mathbf{v}\|_{H^1(0,T;\mathbf{H}^{1/2}(\Omega))}\\ &\qquad \le C(\|(P\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{[D(\mathcal{A}),\mathbf{H}]_{1/2}} + \|f\|_{L^2(0,T;L_0^2(\Gamma_s))}).\end{aligned}$$

(ii) *If* $(\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}_{cc}$ *and* $f \in L^2(0,T; L_0^2(\Gamma_s))$, *then system* (4.1) *admits a unique weak solution* (*in the sense of semigroup theory*) *satisfying*

$$\begin{aligned}&\|P\mathbf{v}\|_{W(0,T;\mathbf{V}^1(\Omega),\mathbf{V}^{-1}(\Omega))} + \|\eta_1\|_{H^{2,1}(\Sigma_T^s)} + \|\eta_2\|_{L^2(0,T;H^1(\Gamma_s))}\\ &\qquad \le C(\|(P\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{\mathbf{H}} + \|f\|_{L^2(0,T;L_0^2(\Gamma_s))}),\end{aligned}$$

$$\|(I-P)\mathbf{v}\|_{L^2(0,T;\mathbf{H}^{3/2}(\Omega))} \le C(\|(P\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{\mathbf{H}} + \|f\|_{L^2(0,T;L_0^2(\Gamma_s))}).$$

(*Here we use the terminology* strict solution *and* weak solution *in the sense of semigroup theory for the evolution equation satisfied by* $(P\mathbf{v}, \eta_1, \eta_2)$ *and not for the equation satisfied by* $(I-P)\mathbf{v}$.)

Proof. (i) If $(P\mathbf{v}^0, \eta_1^0, \eta_2^0) \in [D(\mathcal{A}), \mathbf{H}]_{1/2}$ and $f \in L^2(0,T; L_0^2(\Gamma_s))$, the estimate of $(P\mathbf{v}, \eta_1, \eta_2)$ follows from [1, Chapter 1, Theorem 3.1]. The estimate of $(I-P)\mathbf{v}$ in $L^2(0,T;\mathbf{H}^2(\Omega))$ follows from Lemma 3.11 and from the estimate of η_2 in $H^{2,1}(\Sigma_T^s)$. The estimate of $(I-P)\mathbf{v}$ in $H^1(0,T;\mathbf{H}^{1/2}(\Omega))$ follows from the property of the operator D.

(ii) If $(P\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}$ and $f \in L^2(0,T; L_0^2(\Gamma_s))$, we know that system (4.1) admits a unique weak solution in $L^2(0,T;\mathbf{H})$ satisfying

$$\|(P\mathbf{v}, \eta_1, \eta_2)\|_{C([0,T];\mathbf{H})} \le C(\|(P\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{\mathbf{H}} + \|f\|_{L^2(0,T;L_0^2(\Gamma_s))}).$$

With this estimate and the equation $\eta_{1,t} = \eta_2 + \omega\eta_1$, we obtain

$$\|\eta_1\|_{H^1(0,T;L_0^2(\Gamma_s))} \le C(\|(P\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{\mathbf{H}} + \|f\|_{L^2(0,T;L_0^2(\Gamma_s))}).$$

To prove the other estimates, we have to write an energy estimate for strict solutions to system (2.7). We substitute η_2 by $\eta_{1,t} - \omega\eta_1$ in the equation of η_2:

$$\eta_{1,tt} - 2\omega\eta_{1,t} + \omega^2\eta_1 - \beta\eta_{1,xx} - \delta\eta_{1,txx} + \delta\omega\eta_{1,xx} + \alpha\eta_{1,xxxx} = \rho_1 p + f.$$

We multiply this equation by $\eta_{1,t} - \omega\eta_1$, and by $\rho_1\mathbf{v}$ the equation satisfied by $\mathbf{v}$. After integration and by adding the two identities, we obtain

$$\begin{aligned}&\frac{\rho_1}{2}\int_\Omega |\mathbf{v}(t)|^2 + \nu\,\rho_1\int_{Q_t}|\nabla\mathbf{v}|^2 + \frac{1}{2}\int_{\Gamma_s}|(\eta_{1,t}-\omega\eta_1)(t)|^2 - \omega\int_0^t\int_{\Gamma_s}|\eta_{1,t}-\omega\eta_1|^2\\ &+\tfrac{\beta}{2}\textstyle\int_{\Gamma_s}|\eta_{1,x}(t)|^2 - \beta\omega\displaystyle\int_0^t\int_{\Gamma_s}|\eta_{1,x}|^2 + \delta\int_0^t\int_{\Gamma_s}|\eta_{1,tx}-\omega\eta_{1,x}|^2\\ &+\tfrac{\alpha}{2}\textstyle\int_{\Gamma_s}|\eta_{1,xx}(t)|^2 - \alpha\omega\int_0^t\int_{\Gamma_s}|\eta_{1,xx}|^2 + \omega\int_0^t\int_{\Gamma_s}\eta_1\,p\\ &= \frac{\rho_1}{2}\int_\Omega|\mathbf{v}^0|^2 + \frac{\beta}{2}\int_{\Gamma_s}|\eta_{1,x}^0|^2 + \frac{\alpha}{2}\int_{\Gamma_s}|\eta_{1,xx}^0|^2 + \frac{1}{2}\int_{\Gamma_s}|\eta_2^0-\omega\eta_1^0|^2 + \int_0^t\int_{\Gamma_s}f(\eta_{1,t}-\omega\eta_1).\end{aligned}$$

We also have

$$\omega \int_0^t \int_{\Gamma_s} \eta_1\, p = \omega \int_{\Gamma_s} \eta_1(t)\, \eta_{1,t}(t) - \omega \int_{\Gamma_s} \eta_1^0\, \eta_2^0 - \omega \int_0^t \int_{\Gamma_s} |\eta_{1,t}|^2 - \omega \int_{\Gamma_s} |\eta_1(t)|^2$$
$$+ \omega \textstyle\int_{\Gamma_s} |\eta_1^0|^2 + \omega^3 \int_0^t \int_{\Gamma_s} |\eta_1|^2 + \beta\omega \displaystyle\int_0^t \int_{\Gamma_s} |\eta_{1,x}|^2 + \frac{\omega\delta}{2} \int_{\Gamma_s} |\eta_{1,x}(t)|^2$$
$$- \tfrac{\omega\delta}{2} \textstyle\int_{\Gamma_s} |\eta_{1,x}^0|^2 - \delta\omega^2 \int_0^t \int_{\Gamma_s} |\eta_{1,x}|^2 + \alpha\omega \displaystyle\int_0^t \int_{\Gamma_s} |\eta_{1,xx}|^2 - \omega \int_0^t \int_{\Gamma_s} f\, \eta_1.$$

From these identities and the previous estimates we deduce that

$$\|\mathbf{v}\|_{L^2(0,T;\mathbf{H}^1(\Omega))} + \|\eta_2\|_{L^2(0,T;H^1(\Gamma_s))} \le C(\|(\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{\mathbf{H}_{cc}} + \|f\|_{L^2(0,T;L_0^2(\Gamma_s))}),$$

not only for strict solutions but also for weak solutions. Next we obtain

$$\|(I-P)\mathbf{v}\|_{L^2(0,T;\mathbf{H}^{3/2}(\Omega))} \le C_\varepsilon \|\eta_2\|_{L^2(0,T;H^1(\Gamma_s))}$$

from the properties of the operator D (see, e.g., [24]; we can also adapt the proof of Lemma 3.11). Thus we have

$$\|P\mathbf{v}\|_{L^2(0,T;\mathbf{V}^1(\Omega))} + \|(I-P)\mathbf{v}\|_{L^2(0,T;\mathbf{H}^{3/2}(\Omega))} \le C(\|(\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{\mathbf{H}_{cc}} + \|f\|_{L^2(0,T;L_0^2(\Gamma_s))}).$$

Finally, using that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mathbf{v} \cdot \mathbf{\Phi} = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega P\mathbf{v} \cdot \mathbf{\Phi} = -\nu \int_\Omega \nabla \mathbf{v} : \nabla \mathbf{\Phi} + \omega \int_\Omega \mathbf{v} \cdot \mathbf{\Phi}$$

for all $\mathbf{\Phi} \in \mathbf{V}_0^1(\Omega)$, we deduce that

$$\|P\mathbf{v}\|_{H^1(0,T;\mathbf{V}^{-1}(\Omega))} \le C\|\mathbf{v}\|_{L^2(0,T;\mathbf{V}^1(\Omega))} \le C(\|(\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{\mathbf{H}_{cc}} + \|f\|_{L^2(0,T;L_0^2(\Gamma_s))}),$$

and the proof is complete. ∎

4.2. Another nonhomogeneous linear system. We now consider the system

$$\begin{aligned}
&\mathbf{v}_t - \operatorname{div} \sigma(\mathbf{v}, p) - \omega \mathbf{v} = F \quad \text{and} \quad \operatorname{div} \mathbf{v} = 0 \quad \text{in } Q_\infty,\\
&\mathbf{v} = \eta_2 \vec{e}_2 \ \text{ on } \Sigma_\infty^s, \quad \mathbf{v} = 0 \ \text{ on } \Sigma_\infty^0, \quad \mathbf{v}(0) = \mathbf{v}^0 \text{ in } \Omega,\\
&\eta_{1,t} = \eta_2 + \omega\eta_1 \quad \text{on } \Sigma_\infty^s,\\
&\eta_{2,t} - \omega\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha M_s \eta_{1,xxxx} = M_s(\rho_1 p + f) \ \text{ on } \Sigma_\infty^s,\\
&\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \{0, L\} \times (0, \infty),\\
&\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \ \text{ in } \Gamma_s,
\end{aligned} \tag{4.2}$$

where F belongs to $L^2(0,\infty;\mathbf{L}^2(\Omega))$. We shall need to write this system in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}_\omega \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} + \mathcal{B}f + \mathcal{C}F, \qquad \begin{pmatrix} P\mathbf{v}(0) \\ \eta_1(0) \\ \eta_2(0) \end{pmatrix} = \begin{pmatrix} P\mathbf{v}^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix}, \tag{4.3}$$
$$(I-P)\mathbf{v}(t) = (I-P)D(\eta_2(t)\vec{e}_2\, \chi_{\Gamma_s}),$$

where $\mathcal{C} \in \mathcal{L}(\mathbf{L}^2(\Omega), \mathbf{H})$ is to be determined. For that we decompose $F = PF + (I-P)F$, and we denote by $\pi_F \in L^2(0,\infty;\mathcal{H}^1(\Omega))$ the function defined by $\nabla\pi_F = (I-P)F$. We have

$$p = \pi - q_t + \pi_F,$$

where q is the solution to (3.1), π is the solution to (3.2), and $\pi_F = \pi_1 + \pi_2$ with

$$\pi_1 \in H_0^1(\Omega), \quad \Delta\pi_1 = \mathrm{div} F \text{ in } \Omega \quad \text{and} \quad \Delta\pi_2 = 0 \text{ in } \Omega, \quad \frac{\partial \pi_2}{\partial \mathbf{n}} = (F - \nabla\pi_1)\cdot\mathbf{n} \text{ on } \Gamma.$$

If we set $\pi_1 = -(-\Delta_D)^{-1}(\mathrm{div}F)$, we have $\pi_2 = N((F + \nabla(-\Delta_D)^{-1}(\mathrm{div}F))\cdot\mathbf{n})$. Thus the term $M_s p$ in the equation satisfied by η_2 in system (4.2) is

$$M_s p = \nu\,\gamma_s N_0 \Delta P\mathbf{v}(t)\cdot\mathbf{n} - \gamma_s N_s \eta_{2,t}(t) + \gamma_s N\left(F + \nabla\left((-\Delta_D)^{-1}(\mathrm{div}F)\right)\cdot\mathbf{n}\right).$$

Therefore

$$\mathcal{C}F = \begin{pmatrix} PF \\ 0 \\ \rho_1\left(I + \rho_1\gamma_s N_s\right)^{-1}\left(\gamma_s N\left(F + \nabla\left((-\Delta_D)^{-1}(\mathrm{div}F)\right)\cdot\mathbf{n}\right)\right) \end{pmatrix}.$$

The rewriting of system (4.2) in the form (4.3) is needed in section 9 to prove Theorem 9.1.

5. Approximate controllability and stabilizability. In this section, we study the approximate controllability of the system coupling the Stokes equation with the beam equation. Next we prove that system (2.7) is exponentially stabilizable.

Recall that the linearized system is

$$\begin{aligned}
&\mathbf{v}_t - \mathrm{div}\,\sigma(\mathbf{v},p) = 0 \quad \text{and} \quad \mathrm{div}\,\mathbf{v} = 0 \quad \text{in } Q_T,\\
&\mathbf{v} = \eta_2\vec{e}_2 \ \text{ on } \Sigma_T^s, \quad \mathbf{v} = 0 \ \text{ on } \Sigma_T^0, \quad \mathbf{v}(0) = \mathbf{v}^0 \text{ in } \Omega,\\
&\eta_{1,t} = \eta_2,\\
&\eta_{2,t} - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx} = \rho_1 p + f \ \text{ on } \Sigma_T^s,\\
&\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \{0, L\}\times(0,T),\\
&\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \ \text{ in } \Gamma_s.
\end{aligned} \tag{5.1}$$

THEOREM 5.1. *System* (5.1) *is approximately controllable, in time $T > 0$, in the space $\mathbf{H}_{cc}$ by controls f belonging to $L^2(0,T;L_0^2(\Gamma_s))$.*

Proof. To prove the above approximate controllability result in $\mathbf{H}_{cc}$ we have to show that if $(\mathbf{v}^0, \eta_0, \eta_1) = (\mathbf{0}, 0, 0)$, then the reachable set $R(T)$ at time T, when the control f describes $L^2(0,T;L_0^2(\Gamma_s))$, is dense in $\mathbf{H}_{cc}$. To prove that result we assume that $(\mathbf{\Phi}^T, k_1^T, k_2^T) \in R(T)^\perp$. We want to show that $(\mathbf{\Phi}^T, k_1^T, k_2^T) = 0$.

We introduce the adjoint system

$$
\begin{aligned}
&-\mathbf{\Phi}_t - \operatorname{div}\sigma(\mathbf{\Phi},\psi) = \mathbf{0} \quad \text{and} \quad \operatorname{div}\mathbf{\Phi} = 0 \quad \text{in } Q_T,\\
&\mathbf{\Phi} = k_2\vec{e}_2 \ \text{ on } \Sigma_T^s, \quad \mathbf{\Phi} = 0 \ \text{ on } \Sigma_T^0, \quad \mathbf{\Phi}(T) = \mathbf{\Phi}^T \text{ in } \Omega,\\
&-k_{1,t} = -k_2,\\
&-k_{2,t} + \beta k_{1,xx} - \delta k_{2,xx} - \alpha k_{1,xxxx} = \rho_1\psi \ \text{ on } \Sigma_T^s,\\
&k_1 = 0 \quad \text{and} \quad k_{1,x} = 0 \ \text{ on } \{0, L\} \times (0,\infty),\\
&k_1(T) = k_1^T \quad \text{and} \quad k_2(T) = k_2^T \ \text{ in } \Gamma_s.
\end{aligned}
\tag{5.2}
$$

With an integration by parts we obtain

$$
\rho_1 \int_\Omega \mathbf{v}(T)\cdot\mathbf{\Phi}^T + \int_{\Gamma_s} (-A_{\alpha,\beta})^{1/2}\eta_1(T)\,(-A_{\alpha,\beta})^{1/2}k_1^T + \int_{\Gamma_s}\eta_2(T)\,k_2^T = \int_0^T\int_{\Gamma_s} f\,k_2. \tag{5.3}
$$

If $(\mathbf{\Phi}^T, k_1^T, k_2^T) \in R(T)^\perp$, we deduce that

$$
\int_0^T\int_{\Gamma_s} f\,k_2 = 0
$$

for all $f \in L^2(0,T;L_0^2(\Gamma_s))$; that is, $k_2 = 0$. Thus we must show that if $k_2 = 0$ and if $(\mathbf{\Phi}, k_1, k_2)$ is solution to (5.2), then $(\mathbf{\Phi}^T, k_1^T, k_2^T) = 0$.

By taking the time derivative in the equation

$$
k_{2,t} - \beta k_{1,xx} + \delta k_{2,xx} - \alpha M_s k_{1,xxxx} = -\rho_1 M_s\psi,
$$

we deduce that $\psi_t|_{\Sigma_s} = C(t)$. Thus, using an expansion of the solution $\mathbf{\Phi}$ to

$$
\begin{aligned}
&-\mathbf{\Phi}_t - \operatorname{div}\sigma(\mathbf{\Phi},\psi) = 0, \quad \operatorname{div}\mathbf{\Phi} = 0 \quad \text{in } Q_T,\\
&\mathbf{\Phi} = 0 \ \text{ on } \Sigma_T, \quad \mathbf{\Phi}(T) = \mathbf{\Phi}^T \text{ in } \Omega,
\end{aligned}
\tag{5.4}
$$

in terms of the eigenfunctions of the Stokes operator, as in Osses and Puel [21], the approximate controllability problem reduces to showing that if

$$
\begin{aligned}
&-\nu\Delta\mathbf{v} + \nabla p = \mu\mathbf{v}, \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega,\\
&\mathbf{v} = 0 \ \text{ on } \Gamma, \quad \text{and} \quad p = C \text{ on } \Gamma_s,
\end{aligned}
$$

with $\mu \in \mathbb{R}$, then $\mathbf{v} = 0$. Therefore we can use results from [21, 22] to complete the proof. (See also [18].) $\square$

THEOREM 5.2. *For all $\omega > 0$ and all $(\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}_{cc}$, there exists $f \in L^2(0,\infty;L_0^2(\Gamma_s))$ such that the solution to system* (2.7) *obeys*

$$
\|(\mathbf{v},\eta_1,\eta_2)\|_{L^2(0,\infty;\mathbf{H}_{cc})} < \infty.
$$

Proof. Without loss of generality, we can choose ω in the resolvent set of $\mathcal{A}$. Due to Theorem 3.4, we know that the spectrum of $-\mathcal{A}$ is only a pointwise spectrum constituted of a countable number of distinct eigenvalues, that we can order as follows:

$$
\Re\lambda_1 \geq \Re\lambda_2 \geq \cdots \geq \Re\lambda_N > -\omega > \Re\lambda_{N+1} \geq \cdots.
$$

Moreover, the generalized eigenspace of each eigenvalue is of finite dimension (see [15]). Let us denote by $G(\lambda_i)$ the real generalized eigenspace associated with λ_i if $\lambda_i \in \mathbb{R}$ and with the pair $(\lambda_i, \bar{\lambda}_i)$ if $\Im\lambda_i \neq 0$, and let us set $\mathbf{H}_u = \bigoplus_{i=1}^{N} G(\lambda_i)$ and $\mathbf{H}_s = \bigoplus_{i=N+1}^{\infty} G(\lambda_i)$. If $E(\lambda_i)$ denotes the complex generalized eigenspace associated with λ_i and if $(e_j(\lambda_i))_{1\leq j\leq m(\lambda_i)}$ is a basis of $E(\lambda_i)$, then $G(\lambda_i)$ is nothing else than the space generated by the family $\{\Re e_j(\lambda_i), \Im e_j(\lambda_i) \mid 1 \leq j \leq m(\lambda_i)\}$. Let us observe that $\mathbf{H}_u$ is the unstable subspace of system (2.7), while $\mathbf{H}_s$ is the stable space. Let us denote by P_ω the projection onto the finite dimensional unstable subspace $\mathbf{H}_u$ (parallel to the stable subspace $\mathbf{H}_s$). If we project system (5.1) onto $\mathbf{H}_u$, we obtain

(5.5)
$$\frac{\mathrm{d}}{\mathrm{d}t} P_\omega \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}\, P_\omega \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} + P_\omega \mathcal{B} f, \qquad P_\omega \begin{pmatrix} P\mathbf{v}(0) \\ \eta_1(0) \\ \eta_2(0) \end{pmatrix} = P_\omega \begin{pmatrix} P\mathbf{v}^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix}.$$

Due to Theorem 5.1, system (5.1) is approximately controllable in time $T > 0$. Thus the projected system (5.5) is also approximately controllable. Since it is of finite dimension, it is also controllable. Let $f_0 \in L^2(0, T; L_0^2(\Gamma_s))$ be a control such that $P_\omega(P\mathbf{v}, \eta_1, \eta_2)(T) = (\mathbf{0}, 0, 0)$, and still denote by f_0 its extension by zero to (T, ∞). Now, we notice that $P_\omega(P\mathbf{v}, \eta_1, \eta_2)$ is the solution of system (5.5) corresponding to f if and only if $P_\omega(P\hat{\mathbf{v}}, \hat{\eta}_1, \hat{\eta}_2) = e^{\omega t} P_\omega(P\mathbf{v}, \eta_1, \eta_2)$ is the solution of the system

(5.6)
$$\frac{\mathrm{d}}{\mathrm{d}t} P_\omega \begin{pmatrix} P\hat{\mathbf{v}} \\ \hat{\eta}_1 \\ \hat{\eta}_2 \end{pmatrix} = \mathcal{A}_\omega\, P_\omega \begin{pmatrix} P\hat{\mathbf{v}} \\ \hat{\eta}_1 \\ \hat{\eta}_2 \end{pmatrix} + P_\omega \mathcal{B} \hat{f}, \qquad P_\omega \begin{pmatrix} P\hat{\mathbf{v}}(0) \\ \hat{\eta}_1(0) \\ \hat{\eta}_2(0) \end{pmatrix} = P_\omega \begin{pmatrix} P\mathbf{v}^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix},$$

corresponding to the control $\hat{f} = e^{\omega t f}$. Thus system (5.6) is stabilizable. System (5.6) is the projection of system (2.7) onto its unstable subspace. Due to [31, 20], system (2.7) is stabilizable by a control f belonging in $L^2(0, \infty; L_0^2(\Gamma_s))$ if and only if its projection onto its finite dimensional unstable subspace is stabilizable. The proof is complete. ∎

6. Feedback stabilization of system (2.7). In this section, we study the feedback stabilization of system (2.7). There are several ways to do that. One way consists of studying the infinite time horizon control problem

$(\mathcal{P}^{\infty}_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$
$$\inf\Big\{ I(\mathbf{v}, \eta_1, \eta_2, f) \ \mid\ (\mathbf{v}, \eta_1, \eta_2, f) \text{ satisfies (2.7)},\ f \in L^2(0, \infty; L_0^2(\Gamma_s)) \Big\},$$

where

$$I(\mathbf{v}, \eta_1, \eta_2, f) = \frac{\rho_1}{2} \int_0^\infty \int_\Omega |\mathbf{v}|^2 \,\mathrm{d}x\mathrm{d}z\mathrm{d}t + \frac{1}{2} \int_0^\infty \|\eta_1(t)\|^2_{H_0^2(\Gamma_s)} \,\mathrm{d}t + \frac{1}{2} \int_0^\infty \int_{\Gamma_s} |\eta_2|^2 \,\mathrm{d}x\mathrm{d}t + \frac{1}{2} \int_0^\infty |f(t)|^2_{L^2(\Gamma_s)} \,\mathrm{d}t$$

and (see section 3)

$$\|\eta_1\|^2_{H_0^2(\Gamma_s)} = \int_{\Gamma_s} |(-A_{\alpha,\beta})^{1/2} \eta_1|^2.$$

From Theorem 5.2 we know that system (2.7) is stabilizable in $\mathbf{H}_{cc}$. Thanks to this stabilizability result, and following the approach in [25], the next theorem can be proved.

THEOREM 6.1. *For all* $(\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}_{cc}$, *problem* $(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ *admits a unique solution* $(\mathbf{v}_{\mathbf{v}^0,\eta_1^0,\eta_2^0}, \eta_{1,\mathbf{v}^0,\eta_1^0,\eta_2^0}, \eta_{2,\mathbf{v}^0,\eta_1^0,\eta_2^0}, f_{\mathbf{v}^0,\eta_1^0,\eta_2^0})$. *There exists* $\Pi \in \mathcal{L}(\mathbf{H}_{cc})$, *obeying* $\Pi = \Pi^* \geq 0$, *such that the optimal cost is given by*

$$\inf(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0}) = \frac{1}{2}\Big(\Pi(\mathbf{v}^0, \eta_1^0, \eta_2^0), (\mathbf{v}^0, \eta_1^0, \eta_2^0)\Big)_{\mathbf{H}_{cc}}. \tag{6.1}$$

Theorem 6.1 will be proved in section 8.1.

The operator $\Pi \in \mathcal{L}(\mathbf{H}_{cc})$, which defines the value function of $(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ through formula (6.1), is obtained as the limit of the operator $\Pi(T) \in \mathcal{L}(\mathbf{H}_{cc})$ when T tends to infinity, where $\Pi(T) \in \mathcal{L}(\mathbf{H}_{cc})$ is the operator defining the value function of the corresponding finite time horizon control problem

$(\mathcal{P}^T_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$

$$\inf\Big\{I_0^T(\mathbf{v}, \eta_1, \eta_2, f) \ \mid \ (\mathbf{v}, \eta_1, \eta_2, f) \text{ satisfies (2.7)}, \ f \in L^2(0,T; L_0^2(\Gamma_s))\Big\},$$

where

$$\begin{aligned} I_0^T(\mathbf{v}, \eta_1, \eta_2, f) = \frac{\rho_1}{2}\int_0^T\int_\Omega |\mathbf{v}|^2 \,\mathrm{d}x\mathrm{d}z\mathrm{d}t + \frac{1}{2}\int_0^T \|\eta_1(t)\|^2_{H_0^2(\Gamma_s)}\,\mathrm{d}t \\ + \frac{1}{2}\int_0^T\int_{\Gamma_s} |\eta_2|^2\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^T |f(t)|^2_{L_0^2(\Gamma_s)}\,\mathrm{d}t. \end{aligned}$$

We are going to see in section 8.1 that the solution $(\mathbf{v}_{\mathbf{v}^0,\eta_1^0,\eta_2^0}, \eta_{1,\mathbf{v}^0,\eta_1^0,\eta_2^0}, \eta_{2,\mathbf{v}^0,\eta_1^0,\eta_2^0}, f_{\mathbf{v}^0,\eta_1^0,\eta_2^0})$ of problem $(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ obeys the feedback law

$$f_{\mathbf{v}^0,\eta_1^0,\eta_2^0}(t) = -\Pi_3\left(\mathbf{v}_{\mathbf{v}^0,\eta_1^0,\eta_2^0}(t), \eta_{1,\mathbf{v}^0,\eta_1^0,\eta_2^0}(t), \eta_{2,\mathbf{v}^0,\eta_1^0,\eta_2^0}(t)\right),$$

where $\Pi_3 \in \mathcal{L}(\mathbf{H}_{cc}, L_0^2(\Gamma_s))$ is the third component of the mapping Π:

$$\Pi = \begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \end{pmatrix} = \begin{pmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} \\ \Pi_{21} & \Pi_{22} & \Pi_{23} \\ \Pi_{31} & \Pi_{32} & \Pi_{33} \end{pmatrix} \in \mathcal{L}(\mathbf{H}_{cc}). \tag{6.2}$$

We would like to find an equation characterizing the operator Π. Because system (2.7) is not an evolution equation (indeed, $(I-P)\mathbf{v}$ does not obey an evolution equation), the operator Π is not characterized by a classical algebraic Riccati equation. To address this issue we introduce a second problem leading to another feedback law that we can link with the one expressed with Π. We consider the problem

$(\mathcal{R}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$

$$\inf\Big\{\widehat{I}(P\mathbf{v}, \eta_1, \eta_2, f) \ \mid \ (P\mathbf{v}, \eta_1, \eta_2, f) \text{ satisfies (4.1)}, \ f \in L^2(0,\infty; L_0^2(\Gamma_s))\Big\},$$

where

$$\begin{aligned} \widehat{I}(P\mathbf{v}, \eta_1, \eta_2, f) = \frac{\rho_1}{2}\int_{Q_\infty} |P\mathbf{v}|^2 + \frac{1}{2}\int_0^\infty \|\eta_1\|^2_{H_0^2(\Gamma_s)} \\ + \frac{1}{2}\int_{\Sigma^s_\infty} |(I + \rho_1\gamma_s N_s)\eta_2|^2 + \frac{1}{2}\int_{\Sigma^s_\infty} |f|^2. \end{aligned}$$

Observe that

$$\widehat{I}(P\mathbf{v},\eta_1,\eta_2,f)=\frac{1}{2}\int_0^\infty \|(P\mathbf{v}(t),\eta_1(t),\eta_2(t))\|^2_{\widehat{\mathbf{H}}}\,dt+\frac{1}{2}\int_{\Sigma^s_\infty}|f|^2.$$

(See end of section 3.5 for the definition of $\widehat{\mathbf{H}}$.)

THEOREM 6.2. *For all* $(P\mathbf{v}^0,\eta_1^0,\eta_2^0)\in\widehat{\mathbf{H}}$, *problem* $(\mathcal{R}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$ *admits a unique solution* $(P\mathbf{v}_{P\mathbf{v}^0,\eta_1^0,\eta_2^0},\eta_{1,P\mathbf{v}^0,\eta_1^0,\eta_2^0},\eta_{2,P\mathbf{v}^0,\eta_1^0,\eta_2^0},f_{P\mathbf{v}^0,\eta_1^0,\eta_2^0})$. *There exists* $\widehat{\Pi}\in\mathcal{L}(\widehat{\mathbf{H}})$, *obeying* $\widehat{\Pi}=\widehat{\Pi}^*\geq 0$, *such that the optimal cost is given by*

$$\inf(\mathcal{R}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})=\frac{1}{2}\Big(\widehat{\Pi}(P\mathbf{v}^0,\eta_1^0,\eta_2^0),(P\mathbf{v}^0,\eta_1^0,\eta_2^0)\Big)_{\widehat{\mathbf{H}}}.$$

Moreover, $\widehat{\Pi}$ *is the solution to the algebraic Riccati equation*

$$\widehat{\Pi}\in\mathcal{L}(\widehat{\mathbf{H}}),\quad \widehat{\Pi}=\widehat{\Pi}^*\geq 0,\qquad \widehat{\Pi}\mathcal{A}_\omega+\mathcal{A}^\sharp_\omega\widehat{\Pi}-\widehat{\Pi}\mathcal{B}\mathcal{B}^\sharp\widehat{\Pi}+I=0,$$

where $(\mathcal{A}^\sharp_\omega,D(\mathcal{A}^\sharp_\omega))$ *is the adjoint of* $(\mathcal{A}_\omega,D(\mathcal{A}_\omega))$ *in* $\widehat{\mathbf{H}}$ *and* $\mathcal{B}^\sharp\in\mathcal{L}(\widehat{\mathbf{H}},L^2_0(\Gamma_s))$ *is the adjoint of* $\mathcal{B}\in\mathcal{L}(L^2_0(\Gamma_s),\widehat{\mathbf{H}})$.

Proof. The theorem follows from [2, Part III, Chapter 1, Theorem 3.1] (see also [17, Chapter 2]). Indeed, for the control system (4.1), the operator $\mathcal{B}$ is bounded from the control space $L^2_0(\Gamma_s)$ into the state space $\widehat{\mathbf{H}}$, and the observation operator in the cost functional $\widehat{I}$ is the identity in $\widehat{\mathbf{H}}$. ∎

One can verify that $D(\mathcal{A}^\sharp_\omega)=D(\mathcal{A}_\omega)=D(\mathcal{A})$ and

$$\mathcal{A}^\sharp_\omega=\mathcal{A}^\sharp+\begin{pmatrix}\omega I & 0 & 0\\ 0 & \omega I & 0\\ 0 & 0 & \omega(I+\rho_1\gamma_sN_s)^{-1}\end{pmatrix}.$$

Moreover,

$$\mathcal{B}^\sharp(\mathbf{f},g,h)^T=h.$$

We are able to prove the following relationship between Π and $\widehat{\Pi}$.

THEOREM 6.3. *The operator* $\Pi\in\mathcal{L}(\mathbf{H}_{cc})$ *can be expressed in terms of*

$$\widehat{\Pi}=\begin{pmatrix}\widehat{\Pi}_1\\ \widehat{\Pi}_2\\ \widehat{\Pi}_3\end{pmatrix}=\begin{pmatrix}\widehat{\Pi}_{11} & \widehat{\Pi}_{12} & \widehat{\Pi}_{13}\\ \widehat{\Pi}_{21} & \widehat{\Pi}_{22} & \widehat{\Pi}_{23}\\ \widehat{\Pi}_{31} & \widehat{\Pi}_{32} & \widehat{\Pi}_{33}\end{pmatrix}\in\mathcal{L}(\widehat{\mathbf{H}})$$

as follows:

$$P\Pi_1(\mathbf{v}^0,\eta_1^0,\eta_2^0)=\widehat{\Pi}_1(P\mathbf{v}^0,\eta_1^0,\eta_2^0),\quad \Pi_2(\mathbf{v}^0,\eta_1^0,\eta_2^0)=\widehat{\Pi}_2(P\mathbf{v}^0,\eta_1^0,\eta_2^0),$$
$$\Pi_3(\mathbf{v}^0,\eta_1^0,\eta_2^0)=\widehat{\Pi}_3(P\mathbf{v}^0,\eta_1^0,\eta_2^0),\quad (I-P)\Pi_1(\mathbf{v}^0,\eta_1^0,\eta_2^0)=(I-P)D_s\widehat{\Pi}_3(P\mathbf{v}^0,\eta_1^0,\eta_2^0)$$

for all $(\mathbf{v}^0,\eta_1^0,\eta_2^0)\in\mathbf{H}_{cc}$.

The main interest of problem $(\mathcal{R}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$ is that its optimality system is the same as for problem $(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ (see section 8.2).

7. Studying problem $(\mathcal{P}^T_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$.

THEOREM 7.1. *For all* $(\mathbf{v}^0,\eta_1^0,\eta_2^0)\in\mathbf{H}_{cc}$, *Problem* $(\mathcal{P}^T_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ *admits a unique solution* $(\bar{\mathbf{v}},\bar\eta_1,\bar\eta_2,\bar f)$, *and the optimal control is*

$$\bar f=-k_2,$$

where $(\mathbf{\Phi},k_1,k_2)$ *is the solution of the following adjoint system:*

$$\begin{aligned}
&-\mathbf{\Phi}_t-\operatorname{div}\sigma(\mathbf{\Phi},\psi)-\omega\mathbf{\Phi}=\bar{\mathbf{v}}\quad \textit{and}\quad \operatorname{div}\mathbf{\Phi}=0\quad \textit{in}\ \ Q_T,\\
&\mathbf{\Phi}=k_2\vec{e}_2\ \ \textit{on}\ \ \Sigma_T^s,\quad \mathbf{\Phi}=0\ \ \textit{on}\ \ \Sigma_T^0,\quad \mathbf{\Phi}(T)=0\ \ \textit{in}\ \ \Omega,\\
&-k_{1,t}=-k_2+\omega k_1+\bar\eta_1,\\
&-k_{2,t}-\omega k_2+\beta k_{1,xx}-\delta k_{2,xx}-\alpha k_{1,xxxx}=\rho_1\psi+\bar\eta_2\quad \textit{on}\ \ \Sigma_T^s,\\
&k_1=0\quad \textit{and}\quad k_{1,x}=0\ \ \textit{on}\ \ \big\{0,L\big\}\times(0,\infty),\\
&k_1(T)=0\quad \textit{and}\quad k_2(T)=0\ \ \textit{in}\ \ \Gamma_s.
\end{aligned}\tag{7.1}$$

Conversely, the system

$$\begin{aligned}
&\mathbf{v}_t-\operatorname{div}\sigma(\mathbf{v},p)-\omega\mathbf{v}=0\quad \textit{and}\quad \operatorname{div}\mathbf{v}=0\quad \textit{in}\ \ Q_T,\\
&\mathbf{v}=\eta_2\vec{e}_2\ \ \textit{on}\ \ \Sigma_T^s,\quad \mathbf{v}=0\ \ \textit{on}\ \ \Sigma_T^0,\quad \mathbf{v}(0)=\mathbf{v}^0\ \ \textit{in}\ \ \Omega,\\
&\eta_{1,t}=\eta_2+\omega\eta_1,\\
&\eta_{2,t}-\omega\eta_2-\beta\eta_{1,xx}-\delta\eta_{2,xx}+\alpha\eta_{1,xxxx}=\rho_1 p-k_2\quad \textit{on}\ \ \Sigma_T^s,\\
&\eta_1=0\quad \textit{and}\quad \eta_{1,x}=0\ \ \textit{on}\ \ \big\{0,L\big\}\times(0,\infty),\\
&\eta_1(0)=\eta_1^0\quad \textit{and}\quad \eta_2(0)=\eta_2^0\ \ \textit{in}\ \Gamma_s,\\
&-\mathbf{\Phi}_t-\operatorname{div}\sigma(\mathbf{\Phi},\psi)-\omega\mathbf{\Phi}=\mathbf{v}\quad \textit{and}\quad \operatorname{div}\mathbf{\Phi}=0\quad \textit{in}\ \ Q_T,\\
&\mathbf{\Phi}=k_2\vec{e}_2\ \ \textit{on}\ \ \Sigma_T^s,\quad \mathbf{\Phi}=0\ \ \textit{on}\ \ \Sigma_T^0,\quad \mathbf{\Phi}(T)=0\ \ \textit{in}\ \ \Omega,\\
&-k_{1,t}=-k_2+\omega k_1+\eta_1,\\
&-k_{2,t}-\omega k_2+\beta k_{1,xx}-\delta k_{2,xx}-\alpha k_{1,xxxx}=\rho_1\psi+\eta_2\quad \textit{on}\ \ \Sigma_T^s,\\
&k_1=0\quad \textit{and}\quad k_{1,x}=0\ \ \textit{on}\ \ \big\{0,L\big\}\times(0,\infty),\\
&k_1(T)=0\quad \textit{and}\quad k_2(T)=0\ \ \textit{in}\ \ \Gamma_s
\end{aligned}\tag{7.2}$$

admits a unique solution $(\mathbf{v},p,\eta_1,\eta_2,\mathbf{\Phi},\psi,k_1,k_2)$, *and now the optimal solution to* $(\mathcal{P}^T_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ *is*

$$f=-k_2.$$

The operator $\Pi(T)\in\mathcal{L}(\mathbf{H}_{cc})$ *defined by*

$$\Pi(T)(\mathbf{v}^0,\eta_1^0,\eta_2^0)=(\mathbf{\Phi}(0),k_1(0),k_2(0))$$

is linear and continuous in $\mathbf{H}_{cc}$, *it is symmetric and semidefinite positive, and the optimal cost is given by*

$$\inf(\mathcal{P}^T_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})=\frac{1}{2}\big(\Pi(T)(\mathbf{v}^0,\eta_1^0,\eta_2^0),(\mathbf{v}^0,\eta_1^0,\eta_2^0)\big)_{\mathbf{H}_{cc}}.$$

Proof. The proof is classical, and thus omitted for brevity. ∎

8. Studying problems $(\mathcal{P}^{\infty}_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ and $(\mathcal{R}^{\infty}_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$.

8.1. Problem $(\mathcal{P}^{\infty}_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$.

Proof of Theorem 6.1. The existence of admissible controls follows from Theorem 5.2. Next the existence of an optimal control can be proved in a classical way. The operator Π is obtained as the limit of $\Pi(T)$ when T tends to infinity (see, e.g., [25, Theorem 4.1]). ∎

Following the approach of [25, Lemma 4.2], we can obtain an optimality system for problem $(\mathcal{P}^{\infty}_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ in the form

$$
\begin{aligned}
&\mathbf{v}_t - \operatorname{div}\sigma(\mathbf{v},p) - \omega\mathbf{v} = 0 \quad \text{and} \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_\infty,\\
&\mathbf{v} = \eta_2\vec{e}_2 \ \text{ on } \Sigma^s_\infty, \quad \mathbf{v} = 0 \ \text{ on } \Sigma^0_\infty, \quad \mathbf{v}(0) = \mathbf{v}^0 \text{ in } \Omega,\\
&\eta_{1,t} = \eta_2 + \omega\eta_1,\\
&\eta_{2,t} - \omega\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx} = \rho_1 p - k_2 \ \text{ on } \Sigma^s_\infty,\\
&\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \{0,L\}\times(0,\infty),\\
&\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \ \text{ in } \Gamma_s,\\
&-\mathbf{\Phi}_t - \operatorname{div}\sigma(\mathbf{\Phi},\psi) - \omega\mathbf{\Phi} = \mathbf{v} \quad \text{and} \quad \operatorname{div}\mathbf{\Phi} = 0 \quad \text{in } Q_\infty,\\
&\mathbf{\Phi} = k_2\vec{e}_2 \ \text{ on } \Sigma^s_\infty, \quad \mathbf{\Phi} = 0 \ \text{ on } \Sigma^0_\infty, \quad \mathbf{\Phi}(\infty) = 0 \text{ in } \Omega,\\
&-k_{1,t} = -k_2 + \omega k_1 - \eta_1,\\
&-k_{2,t} - \omega k_2 + \beta k_{1,xx} - \delta k_{2,xx} - \alpha k_{1,xxxx} = \rho_1\psi + \eta_2 \ \text{ on } \Sigma^s_\infty,\\
&k_1(\infty) = 0 \quad \text{and} \quad k_2(\infty) = 0 \ \text{ in } \Gamma_s,\\
&(\mathbf{\Phi}(t), k_1(t), k_2(t)) = \Pi(\mathbf{v}(t), \eta_1(t), \eta_2(t)).
\end{aligned}
\tag{8.1}
$$

More precisely, the following theorem can be proved by adapting the proof of [25, Lemma 4.2] to problem $(\mathcal{P}^{\infty}_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$.

THEOREM 8.1. *For all* $(\mathbf{v}^0,\eta_1^0,\eta_2^0) \in \mathbf{H}_{cc}$, *system* (8.1) *admits a unique solution* $(\mathbf{v},p,\eta_1,\eta_2,\mathbf{\Phi},\psi,k_1,k_2)$ *in* $W(0,\infty;\mathbf{V}^1(\Omega),\mathbf{V}^{-1}(\Omega)) \times L^2(0,\infty;L^2_0(\Omega)) \times H^{2,1}(\Sigma^s_\infty) \times L^2(0,\infty;\mathcal{H}^1(\Gamma_s)) \times \mathbf{V}^{2,1}(Q_\infty) \times L^2(0,\infty;\mathcal{H}^1(\Omega)) \times H^{4,2}(\Sigma^s_\infty) \times H^{2,1}(\Sigma^s_\infty)$, *and the optimal control to* $(\mathcal{P}^{\infty}_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ *is*

$$f = -k_2.$$

Therefore the solution $(\mathbf{v}_{\mathbf{v}^0,\eta_1^0,\eta_2^0}, \eta_{1,\mathbf{v}^0,\eta_1^0,\eta_2^0}, \eta_{2,\mathbf{v}^0,\eta_1^0,\eta_2^0}, f_{\mathbf{v}^0,\eta_1^0,\eta_2^0})$ *to problem* $(\mathcal{P}^{\infty}_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ *obeys the feedback law*

$$f_{\mathbf{v}^0,\eta_1^0,\eta_2^0}(t) = -\Pi_3\left(\mathbf{v}_{\mathbf{v}^0,\eta_1^0,\eta_2^0}(t), \eta_{1,\mathbf{v}^0,\eta_1^0,\eta_2^0}(t), \eta_{2,\mathbf{v}^0,\eta_1^0,\eta_2^0}(t)\right),$$

where $\Pi_3 \in \mathcal{L}(\mathbf{H}_{cc}, L^2_0(\Gamma_s))$ *is the third component of the operator* Π (*see* (6.2)) *and* Π *is the operator defined in Theorem* 6.1.

THEOREM 8.2. *If* $(P\mathbf{v}^0,\eta_1^0,\eta_2^0) \in [D(\mathcal{A}),\mathbf{H}]_{1/2}$, $(\mathbf{v}^0,\eta_1^0,\eta_2^0) \in \mathbf{H}_{cc}$, *then the optimal solution to problem* $(\mathcal{P}^{\infty}_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ *belongs to* $\mathbf{H}^{2,1}(Q_\infty) \times H^{4,2}(\Sigma^s_\infty) \times H^{2,1}(\Sigma^s_\infty)$ *and*

$$
\begin{aligned}
&\|\mathbf{v}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|\eta_1\|_{H^{4,2}(\Sigma^s_\infty)} + \|\eta_2\|_{H^{2,1}(\Sigma^s_\infty)}\\
&\le C(\|P\mathbf{v}^0\|_{\mathbf{V}^1_n(\Omega)} + \|\eta_1^0\|_{H^3(\Gamma_s)\cap H^2_0(\Gamma_s)} + \|\eta_2^0\|_{H^1_0(\Gamma_s)}).
\end{aligned}
$$

The proof of Theorem 8.2 is postponed to subsection 8.3.

8.2. Problem ($\mathcal{R}^{\infty}_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0}$). In order to prove Theorem 6.3, we first need to compare the solutions to ($\mathcal{P}^T_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0}$) and ($\mathcal{R}^T_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0}$), where

($\mathcal{R}^T_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0}$)

$$\inf\left\{\widehat{I}_0^T(P\mathbf{v},\eta_1,\eta_2,f)\ \mid\ (P\mathbf{v},\eta_1,\eta_2,f)\text{ satisfies (4.1)},\ f\in L^2(0,T;L^2_0(\Gamma_s))\right\}$$

and

$$\begin{aligned}&\widehat{I}_0^T(P\mathbf{v},\eta_1,\eta_2,f)\\&=\frac{\rho_1}{2}\int_{Q_T}|P\mathbf{v}|^2+\frac{1}{2}\int_0^T\|\eta_1(t)\|^2_{H^2_0(\Gamma_s)}+\frac{1}{2}\int_{\Sigma_T^s}|(I+\rho_1\gamma_sN_s)\eta_2|^2+\frac{1}{2}\int_{\Sigma_T^s}|f|^2.\end{aligned}$$

The following theorem is a classical result in control theory.

THEOREM 8.3. *For all* $(P\mathbf{v}^0,\eta_1^0,\eta_2^0)\in\widehat{\mathbf{H}}$, *problem* ($\mathcal{R}^T_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0}$) *admits a unique solution.*

The system

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}P\mathbf{v}\\ \eta_1\\ \eta_2\end{pmatrix}=\mathcal{A}_\omega\begin{pmatrix}P\mathbf{v}\\ \eta_1\\ \eta_2\end{pmatrix}-\mathcal{B}\mathcal{B}^\sharp\begin{pmatrix}P\mathbf{\Phi}\\ k_1\\ k_2\end{pmatrix},\qquad\begin{pmatrix}P\mathbf{v}(0)\\ \eta_1(0)\\ \eta_2(0)\end{pmatrix}=\begin{pmatrix}P\mathbf{v}^0\\ \eta_1^0\\ \eta_2^0\end{pmatrix},\\&-\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}P\mathbf{\Phi}\\ k_1\\ k_2\end{pmatrix}=\mathcal{A}^\sharp_\omega\begin{pmatrix}P\mathbf{\Phi}\\ k_1\\ k_2\end{pmatrix}+\begin{pmatrix}P\mathbf{v}\\ \eta_1\\ \eta_2\end{pmatrix},\qquad\begin{pmatrix}P\mathbf{\Phi}(T)\\ k_1(T)\\ k_2(T)\end{pmatrix}=\begin{pmatrix}\mathbf{0}\\ 0\\ 0\end{pmatrix}\end{aligned}\tag{8.2}$$

admits a unique solution $(P\mathbf{v},\eta_1,\eta_2,P\mathbf{\Phi},k_1,k_2)$, *and now the optimal control to* ($\mathcal{R}^T_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0}$) *is*

$$f(t)=-\mathcal{B}^\sharp(P\mathbf{\Phi}(t),k_1(t),k_2(t))=-k_2(t).$$

The operator $\widehat{\Pi}(T)\in\mathcal{L}(\widehat{\mathbf{H}})$, *defined by*

$$\widehat{\Pi}(T)(P\mathbf{v}^0,\eta_1^0,\eta_2^0)=(P\mathbf{\Phi}(0),k_1(0),k_2(0))\,,$$

is linear and continuous in $\widehat{\mathbf{H}}$, *it is symmetric and semidefinite positive, and the optimal cost is given by*

$$\inf(\mathcal{R}^T_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})=\frac{1}{2}\big(\widehat{\Pi}(T)(P\mathbf{v}^0,\eta_1^0,\eta_2^0),(P\mathbf{v}^0,\eta_1^0,\eta_2^0)\big)_{\widehat{\mathbf{H}}}.$$

Using the expression of $\mathcal{A}^\sharp_\omega$ determined in section 3.5, it can be shown that the solution $(P\mathbf{v},\eta_1,\eta_2,P\mathbf{\Phi},k_1,k_2)$ to system (8.2) and the solution $(\bar{\mathbf{v}},\bar{p},\bar{\eta}_1,\bar{\eta}_2,\bar{\mathbf{\Phi}},\bar{\psi},\bar{k}_1,\bar{k}_2)$ to system (7.2) obey

$$(P\bar{\mathbf{v}},\bar{\eta}_1,\bar{\eta}_2,P\bar{\mathbf{\Phi}},\bar{k}_1,\bar{k}_2)=(P\mathbf{v},\eta_1,\eta_2,P\mathbf{\Phi},k_1,k_2).$$

Therefore we have

$$\begin{aligned}&\widehat{\Pi}(T)(P\mathbf{v}^0,\eta_1^0,\eta_2^0)=(P\mathbf{\Phi}(0),k_1(0),k_2(0))=(P\bar{\mathbf{\Phi}}(0),\bar{k}_1(0),\bar{k}_2(0))\\&=\begin{pmatrix}P&0&0\\0&I&0\\0&0&I\end{pmatrix}\Pi(T)(\mathbf{v}^0,\eta_1^0,\eta_2^0)\quad\text{for all }(\mathbf{v}^0,\eta_1^0,\eta_2^0)\in\mathbf{H}_{cc}.\end{aligned}\tag{8.3}$$

The following analogue of Theorem 8.1 can be proved for problem $(\mathcal{R}^{\infty}_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$.

THEOREM 8.4. *For all* $(P\mathbf{v}^0,\eta_1^0,\eta_2^0)\in\mathbf{H}$ *we consider the system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{v}\\ \eta_1\\ \eta_2\end{pmatrix}=\mathcal{A}_\omega\begin{pmatrix} P\mathbf{v}\\ \eta_1\\ \eta_2\end{pmatrix}-\mathcal{B}\mathcal{B}^\sharp\begin{pmatrix} P\mathbf{\Phi}\\ k_1\\ k_2\end{pmatrix},\qquad \begin{pmatrix} P\mathbf{v}(0)\\ \eta_1(0)\\ \eta_2(0)\end{pmatrix}=\begin{pmatrix} P\mathbf{v}^0\\ \eta_1^0\\ \eta_2^0\end{pmatrix},$$
$$-\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{\Phi}\\ k_1\\ k_2\end{pmatrix}=\mathcal{A}_\omega^\sharp\begin{pmatrix} P\mathbf{\Phi}\\ k_1\\ k_2\end{pmatrix}+\begin{pmatrix} P\mathbf{v}\\ \eta_1\\ \eta_2\end{pmatrix},\qquad \begin{pmatrix} P\mathbf{\Phi}(\infty)\\ k_1(\infty)\\ k_2(\infty)\end{pmatrix}=\begin{pmatrix} \mathbf{0}\\ 0\\ 0\end{pmatrix},\tag{8.4}$$
$$(P\mathbf{\Phi}(t),k_1(t),k_2(t))=\widehat{\Pi}(P\mathbf{v}(t),\eta_1(t),\eta_2(t)).$$

System (8.4) *admits a unique solution* $(P\mathbf{v},\eta_1,\eta_2,P\mathbf{\Phi},k_1,k_2)$ *in* $W(0,\infty;\mathbf{V}^1(\Omega),\mathbf{V}^{-1}(\Omega))\times H^{2,1}(\Sigma^s_\infty)\times L^2(0,\infty;\mathcal{H}^1(\Gamma_s))\times\mathbf{V}^{2,1}(Q_\infty)\times H^{4,2}(\Sigma^s_\infty)\times H^{2,1}(\Sigma^s_\infty)$, *and the optimal control to* $(\mathcal{R}^{\infty}_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$ *is*

$$f=-k_2.$$

This theorem may be proved, as in [25], by passing to the limit in the optimality system of the finite time horizon control problem $(\mathcal{R}^{T}_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$.

Proof of Theorem 6.3. Since Π and $\widehat{\Pi}$ are defined as the respective limits of $\Pi(T)$ and $\widehat{\Pi}(T)$ when T tends to infinity, with (8.3), we obtain

$$\begin{pmatrix} P & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{pmatrix}\Pi(\mathbf{v}^0,\eta_1^0,\eta_2^0)=\widehat{\Pi}(P\mathbf{v}^0,\eta_1^0,\eta_2^0)$$

for all $(\mathbf{v}^0,\eta_1^0,\eta_2^0)\in\mathbf{H}_{cc}$. This equality gives the expression for $P\Pi_1$, Π_2, and Π_3. The expression for $(I-P)\Pi_1$ follows from the equalities

$$\begin{aligned}(I-P)\Pi_1(\mathbf{v}^0,\eta_1^0,\eta_2^0)&=(I-P)\mathbf{\Phi}(0)=(I-P)D_sk_2(0)\\ &=(I-P)D_s\widehat{\Pi}_3(P\mathbf{v}^0,\eta_1^0,\eta_2^0).\qquad\square\end{aligned}$$

8.3. Proof of Theorem 8.2. The proof is based on the fact that system (8.1) is equivalent to system (8.4) with the additional equations $(I-P)\mathbf{v}=(I-P)D(\eta_2\vec{e}_2\chi_{\Gamma_s})$ and $(I-P)\mathbf{\Phi}=(I-P)D(k_2\vec{e}_2\chi_{\Gamma_s})$. Since we can use, for system (8.4), the maximal regularity result stated in [1, Chapter 1, Theorem 3.1], we can derive the same estimates for the solution to system (8.1).

We already know that

$$\begin{aligned}&\|P\mathbf{v}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))}+\|\eta_1\|_{L^2(0,\infty;H^2_0(\Gamma_s))}+\|\eta_2\|_{L^2(0,\infty;L^2_0(\Gamma_s))}\\ &\qquad+\|P\mathbf{\Phi}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))}+\|k_1\|_{L^2(0,\infty;H^2_0(\Gamma_s))}+\|k_2\|_{L^2(0,\infty;L^2_0(\Gamma_s))}\\ &\le C(\|P\mathbf{v}^0\|_{\mathbf{V}^0_n(\Omega)}+\|\eta_1^0\|_{H^2_0(\Gamma_s)}+\|\eta_2^0\|_{L^2_0(\Gamma_s)}).\end{aligned}\tag{8.5}$$

We can rewrite the adjoint equation of (8.4) in the form

$$
\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{\Phi} \\ k_1 \\ k_2 \end{pmatrix} = (\mathcal{A}_\omega^\sharp - \lambda I)\begin{pmatrix} P\mathbf{\Phi} \\ k_1 \\ k_2 \end{pmatrix} + \lambda \begin{pmatrix} P\mathbf{\Phi} \\ k_1 \\ k_2 \end{pmatrix} + \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, \\
&\begin{pmatrix} P\mathbf{\Phi}(\infty) \\ k_1(\infty) \\ k_2(\infty) \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \\ 0 \end{pmatrix}.
\end{aligned} \tag{8.6}
$$

We choose $\lambda > 0$ such that $(e^{t(\mathcal{A}_\omega^\sharp - \lambda I)})_{t\geq 0}$ is exponentially stable. From [1, Chapter 1, Theorem 3.1], with estimate (8.5), it can be shown that the solution $(P\mathbf{\Phi}, k_1, k_2)$ of system (8.6) obeys

$$
\begin{aligned}
&\|P\mathbf{\Phi}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|k_1\|_{H^{4,2}(\Sigma_\infty^s)} + \|k_2\|_{H^{2,1}(\Sigma_\infty^s)} \\
&\leq C(\|P\mathbf{\Phi}\|_{L^2(0,\infty;\mathbf{V}_n^0(\Omega))} + \|k_1\|_{L^2(0,\infty;H_0^2(\Gamma_s))} + \|k_2\|_{L^2(0,\infty;L_0^2(\Gamma_s))}) \\
&\leq C(\|P\mathbf{v}^0\|_{\mathbf{V}_n^0(\Omega)} + \|\eta_1^0\|_{H_0^2(\Gamma_s)} + \|\eta_2^0\|_{L_0^2(\Gamma_s)}).
\end{aligned} \tag{8.7}
$$

Next, with estimates (8.5) and (8.7), still with [1, Chapter 1, Theorem 3.1], and with [25], we can show that

$$
\begin{aligned}
&\|P\mathbf{v}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|\eta_1\|_{H^{4,2}(\Sigma_\infty^s)} + \|\eta_2\|_{H^{2,1}(\Sigma_\infty^s)} \\
&\qquad \leq C(\|P\mathbf{v}^0\|_{\mathbf{V}_n^1(\Omega)} + \|\eta_1^0\|_{(H^3\cap H_0^2)(\Gamma_s)} + \|\eta_2^0\|_{H_0^1(\Gamma_s)} + \|k_2\|_{L_0^2(\Sigma_\infty^s)}), \\
&\|(I-P)\mathbf{v}\|_{\mathbf{H}^{2,1}(Q_\infty)} = \|(I-P)D_s\eta_2\|_{\mathbf{H}^{2,1}(Q_\infty)} \leq C\|\eta_2\|_{H^{2,1}(\Sigma_\infty^s)}.
\end{aligned} \tag{8.8}
$$

This completes the proof. ∎

9. Nonhomogeneous system. We now consider the nonhomogeneous linear system

$$
\begin{aligned}
&\mathbf{v}_t - \operatorname{div}\sigma(\mathbf{v},p) - \omega\mathbf{v} = F \quad \text{and} \quad \operatorname{div}\mathbf{v} = G = \operatorname{div}\bar{\mathbf{w}} \quad \text{in } Q_\infty, \\
&\mathbf{v} = \eta_2\vec{e}_2 \ \text{ on } \Sigma_\infty^s, \quad \mathbf{v} = 0 \ \text{ on } \Sigma_\infty^0, \quad \mathbf{v}(0) = \mathbf{v}^0 \text{ in } \Omega, \\
&\eta_{1,t} = \eta_2 + \omega\eta_1 \quad \text{on } \Sigma_\infty^s, \\
&\eta_{2,t} - \omega\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx} = \rho_1 p - 2\nu\rho_2\mathbf{v}_{2,z} + H - \Pi_3(\mathbf{v},\eta_1,\eta_2) \ \text{ on } \Sigma_\infty^s, \\
&\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \{0,L\}\times(0,\infty), \\
&\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \ \text{ in } \Gamma_s,
\end{aligned} \tag{9.1}
$$

with $\bar{\mathbf{w}} \in \mathbf{H}^{2,1}(Q_\infty) \cap L^2(0,\infty;\mathbf{H}_0^1(\Omega))$. We can look for a solution to system (9.1) in the form $\mathbf{v} = \mathbf{w} + \bar{\mathbf{w}}$, where $(\mathbf{w}, p, \eta)$ is the solution to

(9.2)

$$\mathbf{w}_t - \operatorname{div}\sigma(\mathbf{w},p) - \omega\mathbf{w} = F - \bar{\mathbf{w}}_t + \nu\Delta\bar{\mathbf{w}} + \nu\nabla\operatorname{div}\bar{\mathbf{w}} + \omega\bar{\mathbf{w}} \quad \text{and} \quad \operatorname{div}\mathbf{w} = 0 \quad \text{in } Q_\infty,$$

$$\mathbf{w} = \eta_2\vec{e}_2 \ \text{ on } \Sigma^s_\infty, \quad \mathbf{w} = 0 \ \text{ on } \Sigma^0_\infty, \quad \mathbf{w}(0) = \mathbf{v}^0 - \bar{\mathbf{w}}(0) \text{ in } \Omega,$$

$$\eta_{1,t} = \eta_2 + \omega\eta_1 \quad \text{on } \Sigma^s_\infty,$$

$$\eta_{2,t} - \omega\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx}$$
$$= \rho_1 p - 2\nu\rho_2(\mathbf{w}_{2,z} + \bar{\mathbf{w}}_{2,z}) + H - \Pi_3(\bar{\mathbf{w}},0,0) - \Pi_3(\mathbf{w},\eta_1,\eta_2) \ \text{ on } \Sigma^s_\infty,$$

$$\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \{0,L\}\times(0,\infty),$$

$$\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \ \text{ in } \Gamma_s.$$

Since $\operatorname{div}\mathbf{w} = 0$, the term $2\nu\rho_2\mathbf{w}_{2,z}$ can be dropped out in the equation satisfied by η_2, but not the term $2\nu\rho_2\bar{\mathbf{w}}_{2,z}$. We introduce the operator unbounded operator $(\mathcal{A}_{\omega,\widehat{\Pi}}, D(\mathcal{A}_{\omega,\widehat{\Pi}}))$ in $\mathbf{H}$, defined by $D(\mathcal{A}_{\omega,\widehat{\Pi}}) = D(\mathcal{A})$ and

$$\mathcal{A}_{\omega,\widehat{\Pi}} = \mathcal{A}_\omega - \mathcal{B}\mathcal{B}^\sharp\widehat{\Pi}.$$

System (9.2) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{w} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}_{\omega,\widehat{\Pi}}\begin{pmatrix} P\mathbf{w} \\ \eta_1 \\ \eta_2 \end{pmatrix} + \mathcal{B}\bar{H} + \begin{pmatrix} P\bar{F} \\ 0 \\ 0 \end{pmatrix}, \tag{9.3}$$

$$\begin{pmatrix} P\mathbf{w}(0) \\ \eta_1(0) \\ \eta_2(0) \end{pmatrix} = \begin{pmatrix} P(\mathbf{v}^0 - \bar{\mathbf{w}}(0)) \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix},$$

$$(I-P)\mathbf{w} = (I-P)D(\eta_2\vec{e}_2\chi_{\Gamma_s}),$$

where

$$\bar{F} = F - \bar{\mathbf{w}}_t + \nu\Delta\bar{\mathbf{w}} + \nu\nabla\operatorname{div}\bar{\mathbf{w}} + \omega\bar{\mathbf{w}},$$

$$\bar{H} = -2\nu\rho_2\bar{\mathbf{w}}_{2,z} + H - \Pi_3(\bar{\mathbf{w}},0,0)$$
$$+\rho_1(I+\rho_1\gamma_s N_s)^{-1}\gamma_s N((\bar{F} + \nabla(-\Delta_D)^{-1}\operatorname{div}\bar{F})\cdot\mathbf{n}).$$

We assume that $\bar{\mathbf{w}}$ belongs to $\mathbf{H}^{2,1}(Q_\infty)$, $F \in L^2(0,\infty;\mathbf{L}^2(\Omega))$, and also $H \in L^2(0,\infty;L^2_0(\Gamma_s))$. Thus $P\bar{F}$ belongs to $L^2(0,\infty;\mathbf{V}^0_n(\Omega))$. Moreover, $(\bar{F} + \nabla(-\Delta_D)^{-1}\operatorname{div}\bar{F})\cdot\mathbf{n}$ belongs to $L^2(0,\infty;\mathcal{H}^{-1/2}(\Gamma))$, $\gamma_s N((\bar{F} + \nabla(-\Delta_D)^{-1}\operatorname{div}\bar{F})\cdot\mathbf{n})$ belongs to $L^2(0,\infty;\mathcal{H}^{1/2}(\Gamma_s))$, and $\bar{H}$ belongs to $L^2(0,\infty;L^2_0(\Gamma_s))$. Since the semigroup generated by $(\mathcal{A}_{\omega,\widehat{\Pi}}, D(\mathcal{A}_{\omega,\widehat{\Pi}}))$ is exponentially stable on $\mathbf{V}^0_n(\Omega)\times(H^2_0(\Gamma_s)\cap L^2_0(\Gamma_s))\times L^2_0(\Gamma_s)$, system (9.3) admits a unique solution $(P\mathbf{w},\eta_1,\eta_2)$ in $L^2(0,\infty;\mathbf{V}^0_n(\Omega)\times(H^2_0(\Gamma_s)\cap L^2_0(\Gamma_s))\times L^2_0(\Gamma_s))$.

THEOREM 9.1. *If* $(P\mathbf{v}^0 - P\bar{\mathbf{w}}(0),\eta_1^0,\eta_2^0) \in [D(\mathcal{A}),\mathbf{H}]_{1/2}$, $(\mathbf{v}^0 - \bar{\mathbf{w}}(0),\eta_1^0,\eta_2^0) \in \mathbf{H}_{cc}$, $F \in L^2(0,\infty;\mathbf{L}^2(\Omega))$, $\bar{\mathbf{w}} \in \mathbf{H}^{2,1}(Q_\infty)$, *and* $H \in L^2(0,\infty;L^2_0(\Gamma_s))$, *then system* (9.1) *admits a unique solution, which belongs to* $\mathbf{H}^{2,1}(Q_\infty)\times H^{4,2}(\Sigma^s_\infty)\times H^{2,1}(\Sigma^s_\infty)$ *and*

$$\|\mathbf{v}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|p\|_{L^2(0,1;H^1(\Omega))} + \|\eta_1\|_{H^{4,2}(\Sigma^s_\infty)} + \|\eta_1\|_{L^\infty(\Sigma^s_\infty)} + \|\eta_2\|_{H^{2,1}(\Sigma^s_\infty)}$$
$$\le C_1(\|P\mathbf{v}^0\|_{\mathbf{V}^1_n(\Omega)} + \|\eta_1^0\|_{H^3(\Gamma_s)} + \|\eta_2^0\|_{H^1(\Gamma_s)}$$
$$+\|F\|_{\mathbf{L}^2(Q_\infty)} + \|\bar{\mathbf{w}}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|H\|_{L^2(\Sigma^s_\infty)}).$$

Proof. We first consider system (9.3). We know that $(P\mathbf{v}^0 - P\bar{\mathbf{w}}(0), \eta_1^0, \eta_2^0) \in [D(\mathcal{A}_{\omega,\widehat{\Pi}}), \mathbf{H}]_{1/2}$, $(P\bar{F}, 0, \bar{H}) \in L^2(0,\infty;\mathbf{H})$, and that the semigroup generated by $(\mathcal{A}_{\omega,\widehat{\Pi}}, D(\mathcal{A}_{\omega,\widehat{\Pi}}))$ is exponentially stable on $\mathbf{H}$. Thus, arguing as for (8.6), with [1, Chapter 1, Theorem 3.1] and the continuous imbedding $H^{4,2}(\Sigma_\infty^s) \hookrightarrow L^\infty(\Sigma_\infty^s)$, we obtain

$$\begin{aligned}
&\|\mathbf{w}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|\eta_1\|_{H^{4,2}(\Sigma_\infty^s)} + \|\eta_1\|_{L^\infty(\Sigma_\infty^s)} + \|\eta_2\|_{H^{2,1}(\Sigma_\infty^s)} \\
&\le C_1(\|P\mathbf{v}^0\|_{\mathbf{V}_n^1(\Omega)} + \|\eta_1^0\|_{H^3(\Gamma_s)} + \|\eta_2^0\|_{H^1(\Gamma_s)} \\
&\qquad + \|F\|_{\mathbf{L}^2(Q_\infty)} + \|\bar{\mathbf{w}}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|H\|_{L^2(\Sigma_\infty^s)}).
\end{aligned}$$

Since $\mathbf{v} = \mathbf{w} + \bar{\mathbf{w}}$ and $\bar{\mathbf{w}} \in \mathbf{H}^{2,1}(Q_\infty)$, we recover the estimate for $\mathbf{v}$. The estimate for the pressure can be obtained from the estimate for $\mathbf{v}$ and from the first equation of system (9.1). ∎

10. Stabilization of the coupled system. In this section we study the nonlinear closed loop system

(10.1)
$$\begin{aligned}
&\tilde{\mathbf{u}}_t - \operatorname{div}\sigma(\tilde{\mathbf{u}}, \tilde{p}) - \omega\tilde{\mathbf{u}} = e^{-\omega t}\tilde{F}(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2), \quad \operatorname{div}\tilde{\mathbf{u}} = e^{-\omega t}\tilde{G}(\tilde{\eta}_1, \tilde{\mathbf{u}}) \quad \text{in } Q_\infty, \\
&\tilde{\mathbf{u}} = \tilde{\eta}_2\vec{e}_2 \ \text{ on } \Sigma_\infty^s, \quad \tilde{\mathbf{u}} = 0 \ \text{ on } \Sigma_\infty^0, \quad \tilde{\mathbf{u}}(0) = \hat{\mathbf{u}}^0 \text{ in } \Omega, \\
&\tilde{\eta}_{1,t} = \tilde{\eta}_2 + \omega\tilde{\eta}_1 \quad \text{on } \Sigma_\infty^s, \\
&\tilde{\eta}_{2,t} - \omega\tilde{\eta}_2 - \beta\tilde{\eta}_{1,xx} - \delta\tilde{\eta}_{2,xx} + \alpha\tilde{\eta}_{1,xxxx} \\
&\qquad\qquad = \rho_1\tilde{p} - 2\nu\rho_2\tilde{\mathbf{u}}_{2,z} + e^{-\omega t}\tilde{H}(\tilde{\mathbf{u}}, \tilde{\eta}_1) - \Pi_3(\tilde{\mathbf{u}}, \tilde{\eta}_1, \tilde{\eta}_2) \ \text{ on } \Sigma_\infty^s, \\
&\tilde{\eta}_1 = 0 \quad \text{and} \quad \tilde{\eta}_{1,x} = 0 \ \text{ on } \{0, L\} \times (0,\infty), \\
&\tilde{\eta}_1(0) = \eta_1^0 \quad \text{and} \quad \tilde{\eta}_2(0) = \eta_2^0 \ \text{ in } \Gamma_s,
\end{aligned}$$

with

$$\begin{aligned}
&\tilde{F}(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2) \\
&= -\tilde{\eta}_1(\tilde{\mathbf{u}}_t - \omega\tilde{\mathbf{u}}) + \left(z\tilde{\eta}_2 + \nu z\left(\frac{\tilde{\eta}_{1,x}^2}{e^{\omega t} + \tilde{\eta}_1} - \tilde{\eta}_{1,xx}\right)\right)\tilde{\mathbf{u}}_z - (\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}} \\
&\quad + \nu\left(-2z\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_{xz} + \tilde{\eta}_1\tilde{\mathbf{u}}_{xx} + \left(\frac{z^2\tilde{\eta}_{1,x}^2 - e^{-\omega t}\tilde{\eta}_1}{e^{\omega t} + \tilde{\eta}_1}\right)\tilde{\mathbf{u}}_{zz}\right) \\
&\quad + z(\tilde{\eta}_{1,x}\tilde{p}_z - \tilde{\eta}_1\tilde{p}_x)\vec{e}_1 - (1 + e^{-\omega t}\tilde{\eta}_1)\tilde{\mathbf{u}}_1\tilde{\mathbf{u}}_x + (ze^{-\omega t}\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2)\tilde{\mathbf{u}}_z,
\end{aligned} \tag{10.2}$$

$$\tilde{G}(\tilde{\mathbf{u}}, \tilde{\eta}_1) = -\tilde{\eta}_1\tilde{\mathbf{u}}_{1,x} + z\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_{1,z} = \operatorname{div}\tilde{\mathbf{w}}, \qquad \tilde{\mathbf{w}} = -\tilde{\eta}_1\tilde{\mathbf{u}}_1\vec{e}_1 + z\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_1\vec{e}_2, \tag{10.3}$$

and

(10.4)
$$\tilde{H}(\tilde{\mathbf{u}}, \tilde{\eta}_1) = \nu\rho_2\left(\frac{\tilde{\eta}_{1,x}}{e^{\omega t} + \tilde{\eta}_1}\tilde{\mathbf{u}}_{1,z} + e^{-\omega t}\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_{2,x} - \frac{\tilde{\eta}_{1,x}^2}{e^{\omega t} + \tilde{\eta}_1}\tilde{\mathbf{u}}_{2,z} + \frac{\tilde{\eta}_1}{e^{\omega t} + \tilde{\eta}_1}\tilde{\mathbf{u}}_{2,z}\right).$$

We want to show the following theorem.

THEOREM 10.1. *There exist $0 < \mu_0 < 1$ and an increasing function θ_0 from $\mathbb{R}^+$ into itself such that if $\mu \in (0, \mu_0)$, $(P(\hat{\mathbf{u}}^0 + \eta_1^0\hat{\mathbf{u}}_1^0\vec{e}_1 - z\eta_{1,x}^0\hat{\mathbf{u}}_1^0\vec{e}_2), \eta_1^0, \eta_2^0) \in [D(\mathcal{A}), \mathbf{H}]_{1/2}$,*

$(\hat{\mathbf{u}}^0 + \eta_1^0\hat{\mathbf{u}}_1^0\vec{e}_1 - z\eta_{1,x}^0\hat{\mathbf{u}}_1^0\vec{e}_2)|_\Gamma = \eta_2^0\,\vec{e}_2\,\chi_{\Gamma_s}$, *and* $\|P\hat{\mathbf{u}}^0\|_{\mathbf{V}_n^1(\Omega)} + \|\eta_1^0\|_{H^3(\Gamma_s)\cap H_0^2(\Gamma_s)} + \|\eta_2^0\|_{H_0^1(\Gamma_s)} \le \theta_0(\mu)$, *then system* (10.1) *admits a unique solution in the set*

$$\tilde{D}_\mu = \Big\{(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2) \mid \|\tilde{\mathbf{u}}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|\tilde{p}\|_{L^2(0,\infty;H^1(\Omega))} + \|\tilde{\eta}_1\|_{H^{4,2}(\Sigma_\infty^s)} + \|\tilde{\eta}_1\|_{L^\infty(\Sigma_\infty^s)} + \|\tilde{\eta}_2\|_{H^{2,1}(\Sigma_\infty^s)} \le \mu\Big\}.$$

Let us recall that the imbedding from $H^{4,2}(\Sigma_\infty^s)$ into $L^\infty(\Sigma_\infty^s)$ is continuous. Thus an estimate of $\tilde{\eta}_1$ in $H^{4,2}(\Sigma_\infty^s)$ also provides an estimate of $\tilde{\eta}_1$ in $L^\infty(\Sigma_\infty^s)$. But we look for solutions to system (10.1) such that $-1 < \tilde{\eta}_1$ in order that $\mathcal{T}_{\tilde{\eta}_1}$ be a diffeomorphism. This is why the condition $\|\tilde{\eta}_1\|_{L^\infty(\Sigma_\infty^s)} \le \mu \le \mu_0 < 1$ is added in the definition of $\tilde{D}_\mu$.

Next we consider the system

$$\begin{aligned}
&\hat{\mathbf{u}}_t - \operatorname{div}\sigma(\hat{\mathbf{u}}, \hat{p}) = \hat{F}(\hat{\mathbf{u}}, \hat{p}, \eta_1, \eta_2), \quad \operatorname{div}\hat{\mathbf{u}} = \hat{G}(\eta_1, \hat{\mathbf{u}}) \quad \text{in } Q_\infty,\\
&\hat{\mathbf{u}} = \eta_2\vec{e}_2 \ \text{ on } \Sigma_\infty^s, \quad \hat{\mathbf{u}} = 0 \ \text{ on } \Sigma_\infty^0, \quad \hat{\mathbf{u}}(0) = \hat{\mathbf{u}}^0 \text{ in } \Omega,\\
&\eta_{1,t} = \eta_2 \quad \text{on } \Sigma_\infty^s,\\
&\eta_{2,t} - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx} = \rho_1\hat{p} + \hat{H}(\hat{\mathbf{u}}, \eta_1) - \Pi_3(\hat{\mathbf{u}}, \eta_1, \eta_2) \ \text{ on } \Sigma_\infty^s,\\
&\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \big\{0, L\big\} \times (0, \infty),\\
&\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \ \text{ in } \Gamma_s,
\end{aligned} \tag{10.5}$$

where

$$\begin{aligned}
\hat{F}(\hat{\mathbf{u}}, \hat{p}, \eta_1, \eta_2) = {}& -\eta_1\hat{\mathbf{u}}_t + \left(z\eta_2 + \nu z\left(\frac{\eta_{1,x}^2}{1+\eta_1} - \eta_{1,xx}\right)\right)\hat{\mathbf{u}}_z - (\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{u}}\\
&+ \nu\left(-2z\eta_{1,x}\hat{\mathbf{u}}_{xz} + \eta_1\hat{\mathbf{u}}_{xx} + \left(\frac{z^2\eta_{1,x}^2 - \eta_1}{1+\eta_1}\right)\hat{\mathbf{u}}_{zz}\right)\\
&+ z(\eta_{1,x}\hat{p}_z - \eta_1\hat{p}_x)\vec{e}_1 - (1+\eta_1)\hat{\mathbf{u}}_1\hat{\mathbf{u}}_x + (z\eta_{1,x}\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2)\hat{\mathbf{u}}_z,
\end{aligned}$$

$$\hat{G}(\hat{\mathbf{u}}, \eta_1) = -\eta_1\hat{\mathbf{u}}_{1,x} + z\eta_{1,x}\hat{\mathbf{u}}_{1,z} = \operatorname{div}(\hat{\mathbf{w}}), \qquad \hat{\mathbf{w}} = (-\eta_1\hat{\mathbf{u}}_1\vec{e}_1 + z\eta_{1,x}\hat{\mathbf{u}}_1\vec{e}_2),$$

and

$$\hat{H}(\hat{\mathbf{u}}, \eta_1) = \nu\rho_2\left(\frac{\eta_{1,x}}{1+\eta_1}\hat{\mathbf{u}}_{1,z} + \eta_{1,x}\hat{\mathbf{u}}_{2,x} - \frac{2+\eta_{1,x}^2}{1+\eta_1}\hat{\mathbf{u}}_{2,z}\right).$$

From calculations in section 2 it follows that $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2)$ is a solution to system (10.1) if and only if

$$\hat{\mathbf{u}} = e^{-\omega t}\tilde{\mathbf{u}}, \quad \hat{p} = e^{-\omega t}\tilde{p}, \quad \eta_1 = e^{-\omega t}\tilde{\eta}_1, \quad \eta_2 = e^{-\omega t}\tilde{\eta}_2$$

is a solution to system (10.5). Therefore from Theorem 10.1, we deduce the following result.

THEOREM 10.2. *There exist* $0 < \mu_0 < 1$ *and an increasing function* θ_0 *from* $\mathbb{R}^+$ *into itself such that if* $\mu \in (0, \mu_0)$, $(P(\hat{\mathbf{u}}^0 + \eta_1^0\hat{\mathbf{u}}_1^0\vec{e}_1 - z\eta_{1,x}^0\hat{\mathbf{u}}_1^0\vec{e}_2), \eta_1^0, \eta_2^0) \in [D(\mathcal{A}), \mathbf{H}]_{1/2}$,

$(\hat{\mathbf{u}}^0 + \eta_1^0\hat{\mathbf{u}}_1^0\vec{e}_1 - z\eta_{1,x}^0\hat{\mathbf{u}}_1^0\vec{e}_2)|_{\Gamma_s} = \eta_2^0\,\vec{e}_2\,\chi_{\Gamma_s}$, *and* $\|P\hat{\mathbf{u}}^0\|_{\mathbf{V}_n^1(\Omega)} + \|\eta_1^0\|_{H^3(\Gamma_s)\cap H_0^2(\Gamma_s)} + \|\eta_2^0\|_{H_0^1(\Gamma_s)} \le \theta_0(\mu)$, *then system* (10.5) *admits a unique solution in the set*

$$D_\mu = \Big\{(\hat{\mathbf{u}}, \hat{p}, \eta_1, \eta_2) \mid \|e^{\omega\cdot}\hat{\mathbf{u}}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|e^{\omega\cdot}\hat{p}\|_{L^2(0,\infty;H^1(\Omega))} + \|e^{\omega\cdot}\eta_1\|_{H^{4,2}(\Sigma_\infty^s)} + \|e^{\omega\cdot}\eta_1\|_{L^\infty(\Sigma_\infty^s)} + \|e^{\omega\cdot}\eta_2\|_{H^{2,1}(\Sigma_\infty^s)} \le \mu\Big\}.$$

Still, from calculations in section 2 we know that $(\hat{\mathbf{u}}, \hat{p}, \eta_1, \eta_2)$ is a solution to system (10.5) if and only if $(\mathbf{u}, p, \eta, \eta_t) = (\hat{\mathbf{u}} \circ \mathcal{T}_{\eta_1}, \hat{p} \circ \mathcal{T}_{\eta_1}, \eta_1, \eta_2)$ is solution to system (1.1) with $\mathbf{u}^0 = \hat{\mathbf{u}}^0 \circ \mathcal{T}_{\eta_1^0}$. Thus from Theorem 10.2, we deduce the next claim.

THEOREM 10.3. *There exist* $0 < \mu_0 < 1$ *and an increasing function* θ_0 *from* $\mathbb{R}^+$ *into itself such that if* $\mu \in (0, \mu_0)$, $(P(\hat{\mathbf{u}}^0 + \eta_1^0\hat{\mathbf{u}}_1^0\vec{e}_1 - z\eta_{1,x}^0\hat{\mathbf{u}}_1^0\vec{e}_2), \eta_1^0, \eta_2^0) \in [D(\mathcal{A}), \mathbf{H}]_{1/2}$, $(\hat{\mathbf{u}}^0 + \eta_1^0\hat{\mathbf{u}}_1^0\vec{e}_1 - z\eta_{1,x}^0\hat{\mathbf{u}}_1^0\vec{e}_2)|_{\Gamma_s} = \eta_2^0\,\vec{e}_2\,\chi_{\Gamma_s}$, *and* $\|P\hat{\mathbf{u}}^0\|_{\mathbf{V}_n^1(\Omega)} + \|\eta_1^0\|_{H^3(\Gamma_s)\cap H_0^2(\Gamma_s)} + \|\eta_2^0\|_{H_0^1(\Gamma_s)} \le \theta_0(\mu)$, *where* $\hat{\mathbf{u}}^0 = (\hat{\mathbf{u}}_1^0, \hat{\mathbf{u}}_2^0) = \mathbf{u}^0 \circ \mathcal{T}_{\eta_1^0}^{-1}$, *then system* (1.1) *with the feedback law* $f = -\Pi_3(\mathbf{u} \circ \mathcal{T}_\eta^{-1}(x, z, t), \eta, \eta_t)$ *admits a unique solution in the set*

$$F_\mu = \Big\{(\mathbf{u}, p, \eta, \eta_t) \mid \|e^{\omega\cdot}\mathbf{u} \circ \mathcal{T}_\eta^{-1}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|e^{\omega\cdot}p \circ \mathcal{T}_\eta^{-1}\|_{L^2(0,\infty;H^1(\Omega))} + \|e^{\omega\cdot}\eta\|_{H^{4,2}(\Sigma_\infty^s)} + \|e^{\omega\cdot}\eta\|_{L^\infty(\Sigma_\infty^s)} + \|e^{\omega\cdot}\eta_t\|_{H^{2,1}(\Sigma_\infty^s)} \le \mu\Big\},$$

where $\mathcal{T}_\eta$ *is defined in* (2.3).

11. Some Lipschitz properties.

THEOREM 11.1. *The mapping*

$$(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2) \longmapsto (\tilde{F}(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2), \tilde{\mathbf{w}}(\tilde{\mathbf{u}}, \tilde{\eta}_1), \tilde{H}(\tilde{\mathbf{u}}, \tilde{\eta}_1)),$$

where $\tilde{F}$, $\tilde{\mathbf{w}}$, *and* $\tilde{H}$ *are respectively defined by* (10.2), (10.3), *and* (10.4), *is locally Lipschitz from* $\mathbf{H}^{2,1}(Q_\infty) \times L^2(0,\infty;\mathcal{H}^1(\Omega)) \times H^{4,2}(\Sigma_\infty^s) \times H^{2,1}(\Sigma_\infty^s)$ *into* $\mathbf{L}^2(Q_\infty) \times \mathbf{H}^{2,1}(Q_\infty) \times L^2(\Sigma_\infty^s)$. *More precisely, for all* $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2)$, $(\tilde{\mathbf{u}}^1, \tilde{p}^1, \tilde{\eta}_1^1, \tilde{\eta}_2^1)$, $(\tilde{\mathbf{u}}^2, \tilde{p}^2, \tilde{\eta}_1^2, \tilde{\eta}_2^2)$ *belonging to* $\mathbf{H}^{2,1}(Q_\infty) \times L^2(0,\infty;\mathcal{H}^1(\Omega)) \times H^{4,2}(\Sigma_\infty^s) \times H^{2,1}(\Sigma_\infty^s)$ *and such that* $\max(\|(1+\tilde{\eta}_1)^{-1}\|_{L^\infty(\Sigma_\infty^s)}, \|(1+\tilde{\eta}_1^1)^{-1}\|_{L^\infty(\Sigma_\infty^s)}, \|(1+\tilde{\eta}_1^2)^{-1}\|_{L^\infty(\Sigma_\infty^s)}) \le \mu_1$ *and* $\max(\|\tilde{\eta}_{1,x}\|_{L^\infty(\Sigma_\infty^s)}, \|\tilde{\eta}_{1,x}^1\|_{L^\infty(\Sigma_\infty^s)}, \|\tilde{\eta}_{1,x}^2\|_{L^\infty(\Sigma_\infty^s)}) \le 1$, *we have*

$$\begin{aligned}
&\|\tilde{F}(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2)\|_{L^2(0,\infty;\mathbf{L}^2(\Omega))} \\
&\le C_2(\mu_1)(\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma_\infty^s)}\|\tilde{\mathbf{u}}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|\tilde{\eta}_2\|_{H^{2,1}(\Sigma_\infty^s)}\|\tilde{\mathbf{u}}\|_{\mathbf{H}^{2,1}(Q_\infty)} \\
&\qquad + \|\tilde{\eta}_1\|_{H^{4,2}(\Sigma_\infty^s)}\|\tilde{p}\|_{L^2(0,\infty;\mathcal{H}^1(\Omega))} + \|\tilde{\mathbf{u}}\|_{\mathbf{H}^{2,1}(Q_\infty)}\|\tilde{\mathbf{u}}\|_{\mathbf{H}^{2,1}(Q_\infty)}),
\end{aligned} \tag{11.1}$$

$$\begin{aligned}
&\|\tilde{F}(\tilde{\mathbf{u}}^1, \tilde{p}^1, \tilde{\eta}_1^1, \tilde{\eta}_2^1) - \tilde{F}(\tilde{\mathbf{u}}^2, \tilde{p}^2, \tilde{\eta}_1^2, \tilde{\eta}_2^2)\|_{L^2(0,\infty;\mathbf{L}^2(\Omega))} \\
&\le C_2(\mu_1)\big(\|(\tilde{\mathbf{u}}^1, \tilde{p}^1, \tilde{\eta}_1^1, \tilde{\eta}_2^1)\|_{\mathbf{W}}\|(\tilde{\mathbf{u}}^1, \tilde{p}^1, \tilde{\eta}_1^1, \tilde{\eta}_2^1) - (\tilde{\mathbf{u}}^2, \tilde{p}^2, \tilde{\eta}_1^2, \tilde{\eta}_2^2)\|_{\mathbf{W}} \\
&\qquad + \|(\tilde{\mathbf{u}}^2, \tilde{p}^2, \tilde{\eta}_1^2, \tilde{\eta}_2^2)\|_{\mathbf{W}}\|(\tilde{\mathbf{u}}^1, \tilde{p}^1, \tilde{\eta}_1^1, \tilde{\eta}_2^1) - (\tilde{\mathbf{u}}^2, \tilde{p}^2, \tilde{\eta}_1^2, \tilde{\eta}_2^2)\|_{\mathbf{W}}\big),
\end{aligned} \tag{11.2}$$

with $\mathbf{W} = \mathbf{H}^{2,1}(Q_\infty) \times L^2(0,\infty;\mathcal{H}(\Omega)) \times H^{4,2}(\Sigma_\infty^s) \times H^{2,1}(\Sigma_\infty^s)$,

$$\|\tilde{\mathbf{w}}(\tilde{\mathbf{u}}, \tilde{\eta}_1)\|_{\mathbf{H}^{2,1}(Q_\infty)} \le C_2(\mu_1)\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma_\infty^s)}\|\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)}, \tag{11.3}$$

$$\begin{aligned}
&\|\tilde{\mathbf{w}}(\tilde{\mathbf{u}}^1, \tilde{\eta}_1^1) - \tilde{\mathbf{w}}(\tilde{\mathbf{u}}^2, \tilde{\eta}_1^2)\|_{\mathbf{H}^{2,1}(Q_\infty)} \\
&\le C_2(\mu_1)(\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma_\infty^s)}\|\tilde{\mathbf{u}}_1^1 - \tilde{\mathbf{u}}_1^2\|_{H^{2,1}(Q_\infty)} + \|\tilde{\eta}_1^1 - \tilde{\eta}_1^2\|_{H^{4,2}(\Sigma_\infty^s)}\|\tilde{\mathbf{u}}_1^2\|_{H^{2,1}(Q_\infty)}),
\end{aligned} \tag{11.4}$$

$$\|\tilde{H}(\tilde{\mathbf{u}},\tilde{\eta}_1)\|_{L^2(\Sigma^s_\infty)} \le C_2(\mu_1)\|\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)}\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma^s_\infty)}, \tag{11.5}$$

and

$$\begin{aligned}&\|\tilde{H}(\tilde{\mathbf{u}}^1,\tilde{\eta}^1_1) - \tilde{H}(\tilde{\mathbf{u}}^2,\tilde{\eta}^2_1)\|_{L^2(\Sigma^s_\infty)}\\ &\le C_2(\mu_1)(\|\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)}\|\tilde{\eta}^1_1-\tilde{\eta}^2_1\|_{H^{4,2}(\Sigma^s_\infty)} + \|\tilde{\mathbf{u}}^1_1-\tilde{\mathbf{u}}^2_1\|_{H^{2,1}(Q_\infty)}\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma^s_\infty)}).\end{aligned} \tag{11.6}$$

(*In these estimates the constant C_2 depends in an explicit manner on μ_1.*)

Proof. Step 1: *Proof of* (11.3) *and* (11.4). If $(\tilde{\mathbf{u}},\tilde{\eta}_1) \in \mathbf{H}^{2,1}(Q_\infty)\times H^{4,2}(\Sigma^s_\infty)$, then we have

$$\begin{aligned}&\|\tilde{\eta}_1\tilde{\mathbf{u}}_1\|_{L^2(0,\infty;H^2(\Omega))} + \|\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_1\|_{L^2(0,\infty;H^2(\Omega))}\\ &\le C(\|\tilde{\eta}_1\|_{L^\infty(0,\infty;H^2(\Gamma_s))}\|\tilde{\mathbf{u}}_1\|_{L^2(0,\infty;H^2(\Omega))} + \|\tilde{\eta}_1\|_{L^\infty(0,\infty;H^3(\Gamma_s))}\|\tilde{\mathbf{u}}_1\|_{L^2(0,\infty;H^2(\Omega))})\\ &\le C\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)}.\end{aligned}$$

We also have

$$\begin{aligned}&\|\tilde{\eta}_1\tilde{\mathbf{u}}_1\|_{H^1(0,\infty;L^2(\Omega))} + \|\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_1\|_{H^1(0,\infty;L^2(\Omega))}\\ &\le C(\|\tilde{\eta}_1\|_{H^1(0,\infty;L^\infty(\Gamma_s))}\|\tilde{\mathbf{u}}_1\|_{H^1(0,\infty;L^2(\Omega))} + \|\tilde{\eta}_1\|_{H^1(0,\infty;H^1(\Gamma_s))}\|\tilde{\mathbf{u}}_1\|_{H^1(0,\infty;L^2(\Omega))})\\ &\le C\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_1\|_{\mathbf{H}^{2,1}(Q_\infty)}.\end{aligned}$$

In these estimates we have used that

$$\|\tilde{\eta}_1\|_{H^{3/2}(0,\infty;H^1(\Gamma_s))} \le C\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma^s_\infty)}.$$

Thus we have

$$\|\tilde{\eta}_1\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)} + \|\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)} \le C_2\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)}.$$

Now, we assume that $(\tilde{\mathbf{u}}^1,\tilde{\eta}^1_1) \in \mathbf{H}^{2,1}(Q_\infty)\times H^{4,2}(\Sigma^s_\infty)$ and $(\tilde{\mathbf{u}}^2,\tilde{\eta}^2_1) \in \mathbf{H}^{2,1}(Q_\infty)\times H^{4,2}(\Sigma^s_\infty)$. Let us estimate

$$\tilde{\eta}^1_{1,x}\tilde{\mathbf{u}}^1_1 - \tilde{\eta}^2_{1,x}\tilde{\mathbf{u}}^2_1.$$

The other component, that is $\tilde{\eta}^1_1\tilde{\mathbf{u}}^1_1 - \tilde{\eta}^2_1\tilde{\mathbf{u}}^2_1$, can be estimated in the same way. We have

$$\tilde{\eta}^1_{1,x}\tilde{\mathbf{u}}^1_1 - \tilde{\eta}^2_{1,x}\tilde{\mathbf{u}}^2_1 = \tilde{\eta}^1_{1,x}(\tilde{\mathbf{u}}^1_1-\tilde{\mathbf{u}}^2_1) + (\tilde{\eta}^1_{1,x}-\tilde{\eta}^2_{1,x})\tilde{\mathbf{u}}^2_1.$$

As above, we estimate these terms as follows:

$$\begin{aligned}&\|\tilde{\eta}^1_{1,x}(\tilde{\mathbf{u}}^1_1-\tilde{\mathbf{u}}^2_1)\|_{H^{2,1}(Q_\infty)} + \|(\tilde{\eta}^1_{1,x}-\tilde{\eta}^2_{1,x})\tilde{\mathbf{u}}^2_1\|_{H^{2,1}(Q_\infty)}\\ &\le C_2(\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}^1_1-\tilde{\mathbf{u}}^2_1\|_{H^{2,1}(Q_\infty)} + \|\tilde{\eta}^1_1-\tilde{\eta}^2_1\|_{H^{4,2}(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}^2_1\|_{H^{2,1}(Q_\infty)}).\end{aligned}$$

Step 2: *Proof of* (11.1) *and* (11.2). To estimate the different terms in $\tilde{F}$, we first write

$$\begin{aligned}&\|\tilde{\eta}_1\tilde{\mathbf{u}}_t\|_{\mathbf{L}^2(Q_\infty)} \le \|\tilde{\eta}_1\|_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_t\|_{\mathbf{L}^2(Q_\infty)},\\ &\|\tilde{\eta}_1\omega\tilde{\mathbf{u}}\|_{\mathbf{L}^2(Q_\infty)} \le \|\tilde{\eta}_1\|_{L^\infty(\Sigma^s_\infty)}\|\omega\tilde{\mathbf{u}}\|_{\mathbf{L}^2(Q_\infty)},\\ &\|z\tilde{\eta}_2\tilde{\mathbf{u}}_z\|_{\mathbf{L}^2(Q_\infty)} \le \|\tilde{\eta}_2\|_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_z\|_{\mathbf{L}^2(Q_\infty)} \le C\|\tilde{\eta}_2\|_{L^\infty(0,\infty;H^1(\Gamma_s))}\|\tilde{\mathbf{u}}_z\|_{L^2(0,\infty;\mathbf{H}^1(\Omega))},\\ &\left\|\nu z\frac{\tilde{\eta}^2_{1,x}}{e^{\omega t}+\tilde{\eta}_1}\tilde{\mathbf{u}}_z\right\|_{\mathbf{L}^2(Q_\infty)} \le C\|(1+\tilde{\eta}_1)^{-1}\|_{L^\infty(\Sigma^s_\infty)}\|\tilde{\eta}_{1,x}\|^2_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_z\|_{\mathbf{L}^2(Q_\infty)}\\ &\qquad\qquad \le C\|\tilde{\eta}_{1,x}\|_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_z\|_{\mathbf{L}^2(Q_\infty)},\\ &\|\nu z\tilde{\eta}_{1,xx}\tilde{\mathbf{u}}_z\|_{\mathbf{L}^2(Q_\infty)} \le C\|\tilde{\eta}_{1,xx}\|_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_z\|_{\mathbf{L}^2(Q_\infty)}.\end{aligned}$$

In these estimates we have used that $\|(1+\tilde\eta_1)^{-1}\|_{L^\infty(\Sigma^s_\infty)} \le \mu_1$, $\|\tilde\eta_{1,x}\|_{L^\infty(\Sigma^s_\infty)} \le 1$ and that

$$\tilde\eta_{1,xx} \in H^{2,1}(\Sigma^s_\infty) \hookrightarrow L^\infty(0,\infty;H^1(\Gamma_s)) \hookrightarrow L^\infty(\Sigma^s_\infty)$$

because Γ_s is of dimension one.

We continue as follows:

$$\|\tilde\eta_{1,x}\tilde{\mathbf{u}}_{xz}\|_{\mathbf{L}^2(Q_\infty)} \le \|\tilde\eta_{1,x}\|_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_{xz}\|_{\mathbf{L}^2(Q_\infty)},$$

$$\|\tilde\eta_1\tilde{\mathbf{u}}_{xx}\|_{\mathbf{L}^2(Q_\infty)} \le \|\tilde\eta_1\|_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_{xx}\|_{\mathbf{L}^2(Q_\infty)},$$

$$\left\|\frac{z^2\tilde\eta_{1,x}^2}{e^{\omega t}+\tilde\eta_1}\tilde{\mathbf{u}}_{zz}\right\|_{\mathbf{L}^2(Q_\infty)} \le C\|(1+\tilde\eta_1)^{-1}\|_{L^\infty(\Sigma^s_\infty)}\|\tilde\eta_{1,x}\|^2_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_{zz}\|_{\mathbf{L}^2(Q_\infty)},$$

$$\left\|\frac{e^{-\omega t}\tilde\eta_1}{e^{\omega t}+\tilde\eta_1}\tilde{\mathbf{u}}_{zz}\right\|_{\mathbf{L}^2(Q_\infty)} \le C\|\tilde\eta_1\|_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_{zz}\|_{\mathbf{L}^2(Q)},$$

$$\|\tilde\eta_{1,x}\tilde p_z\|_{L^2(Q)} \le \|\tilde\eta_{1,x}\|_{L^\infty(\Sigma^s_\infty)}\|\tilde p_z\|_{L^2(Q)},$$

$$\|\tilde\eta_1\tilde p_x\|_{L^2(Q)} \le \|\tilde\eta_1\|_{L^\infty(\Sigma^s_\infty)}\|\tilde p_x\|_{L^2(Q)},$$

$$\|(1+e^{-\omega t}\tilde\eta_1)\tilde{\mathbf{u}}_1\tilde{\mathbf{u}}_x\|_{\mathbf{L}^2(Q_\infty)} \le C\|1+\tilde\eta_1\|_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_1\|_{L^\infty(0,\infty;H^1(\Omega))}\|\tilde{\mathbf{u}}_x\|_{L^2(0,\infty;\mathbf{H}^1(\Omega))},$$

$$\|e^{-\omega t}\tilde\eta_{1,x}\tilde{\mathbf{u}}_1\|_{\mathbf{L}^2(Q_\infty)} \le C\|\tilde\eta_{1,x}\|_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_1\|_{L^2(0,\infty;L^2(\Omega))},$$

$$\|\tilde{\mathbf{u}}_2\tilde{\mathbf{u}}_z\|_{\mathbf{L}^2(Q_\infty)} \le C\|\tilde{\mathbf{u}}_2\|_{L^\infty(0,\infty;H^1(\Omega))}\|\tilde{\mathbf{u}}_z\|_{L^2(0,\infty;\mathbf{H}^1(\Omega))},$$

$$\|(\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}}\|_{\mathbf{L}^2(Q_\infty)} \le C\|\tilde{\mathbf{u}}\|_{L^\infty(0,\infty;\mathbf{H}^1(\Omega))}\|\tilde{\mathbf{u}}\|_{L^2(0,\infty;\mathbf{H}^1(\Omega))}.$$

Thus

$$\begin{aligned}
&\|\tilde F(\tilde{\mathbf{u}},\tilde p,\tilde\eta_1,\tilde\eta_2)\|_{L^2(0,\infty;\mathbf{L}^2(\Omega))}\\
&\le C_2(\|\tilde\eta_1\|_{H^{4,2}(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)} + \|\tilde\eta_2\|_{H^{2,1}(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)}\\
&\qquad + \|\tilde\eta_1\|_{H^{4,2}(\Sigma^s_\infty)}\|\tilde p\|_{L^2(0,\infty;H^1(\Omega))} + \|\tilde{\mathbf{u}}\|_{\mathbf{H}^{2,1}(Q_\infty)}\|\tilde{\mathbf{u}}\|_{\mathbf{H}^{2,1}(Q_\infty)}).
\end{aligned}$$

Estimate (11.2) can be proved in the same way.

Step 3: *Proof of* (11.5) *and* (11.6). We have

$$\left\|\frac{\tilde\eta_{1,x}}{e^{\omega t}+\tilde\eta_1}\tilde{\mathbf{u}}_{1,z}\right\|_{L^2(\Sigma^s_\infty)} \le C\|(1+\tilde\eta_1)^{-1}\|_{L^\infty(\Sigma^s_\infty)}\|\tilde\eta_{1,x}\|_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_{1,z}\|_{L^2(\Sigma^s_\infty)},$$

$$\|\tilde\eta_{1,x}\tilde{\mathbf{u}}_{2,x}\|_{L^2(\Sigma^s_\infty)} \le C\|\tilde\eta_{1,x}\|_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_{2,x}\|_{L^2(\Sigma^s_\infty)},$$

$$\left\|\frac{\tilde\eta_{1,x}^2}{e^{\omega t}+\tilde\eta_1}\tilde{\mathbf{u}}_{2,z}\right\|_{L^2(\Sigma^s_\infty)} \le C\|(1+\tilde\eta_1)^{-1}\|_{L^\infty(\Sigma^s_\infty)}\|\tilde\eta_{1,x}\|^2_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_{2,z}\|_{L^2(\Sigma^s_\infty)},$$

$$\left\|\frac{\tilde\eta_1}{e^{\omega t}+\tilde\eta_1}\tilde{\mathbf{u}}_{2,z}\right\|_{L^2(\Sigma^s_\infty)} \le C\|\tilde\eta_1\|^2_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_{2,z}\|_{L^2(\Sigma^s_\infty)}.$$

(We have used that $\|(1+\tilde\eta_1)^{-1}\|_{L^\infty(\Sigma^s_\infty)} \le \mu_1$ and $\|\tilde\eta_{1,x}\|_{L^\infty(\Sigma^s_\infty)} \le 1$.) With these estimates we can show that

$$\|\tilde H(\tilde{\mathbf{u}},\tilde\eta_1)\|_{L^2(\Sigma^s_\infty)} \le C_2\|\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)}\|\tilde\eta_1\|_{H^{4,2}(\Sigma^s_\infty)}$$

and that

$$\begin{aligned}
&\|\tilde H(\tilde{\mathbf{u}}^1,\tilde\eta_1^1) - \tilde H(\tilde{\mathbf{u}}^2,\tilde\eta_1^2)\|_{L^2(\Sigma^s_\infty)}\\
&\le C_2(\|\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)}\|\tilde\eta_1^1-\tilde\eta_1^2\|_{H^{4,2}(\Sigma^s_\infty)} + \|\tilde{\mathbf{u}}_1^1-\tilde{\mathbf{u}}_1^2\|_{H^{2,1}(Q_\infty)}\|\tilde\eta_1\|_{H^{4,2}(\Sigma^s_\infty)}). \qquad \square
\end{aligned}$$

12. Proof of Theorem 10.1. To prove Theorem 10.1, we consider the nonhomogeneous closed loop linear system

(12.1)

$$\mathbf{v}_t - \operatorname{div}\sigma(\mathbf{v},p) - \omega\mathbf{v} = e^{-\omega t}\tilde{F} \quad \text{and} \quad \operatorname{div}\mathbf{v} = e^{-\omega t}\tilde{G} = e^{-\omega t}\operatorname{div}\tilde{\mathbf{w}} \quad \text{in } Q_\infty,$$

$$\mathbf{v} = \eta_2\vec{e}_2 \ \text{ on } \Sigma^s_\infty, \quad \mathbf{v} = 0 \ \text{ on } \Sigma^0_\infty, \quad \mathbf{v}(0) = \hat{\mathbf{u}}^0 \text{ in } \Omega,$$

$$\eta_{1,t} = \eta_2 + \omega\eta_1 \quad \text{on } \Sigma^s_\infty,$$

$$\begin{aligned}\eta_{2,t} - \omega\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx}\\ = \rho_1 p - 2\nu\rho_2\mathbf{v}_{2,z} + e^{-\omega t}\tilde{H} - \Pi_3(\mathbf{v},\eta_1,\eta_2) \ \text{ on } \Sigma^s_\infty,\end{aligned}$$

$$\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \{0,L\}\times(0,\infty),$$

$$\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \ \text{ in } \Gamma_s,$$

where $\tilde{F}$, $\tilde{G}$, and $\tilde{H}$ stand, respectively, for the mappings $\tilde{F}(\tilde{\eta}_1,\tilde{\eta}_2,\tilde{\mathbf{u}},\nabla\tilde{p})$, $\tilde{G}(\tilde{\eta}_1,\tilde{\mathbf{u}})$, and $\tilde{H}(\tilde{\mathbf{u}},\tilde{p},\tilde{\eta}_1)$ defined in (10.2), (10.3), and (10.4).

We first choose $1 < \mu_1$. Without loss of generality, we can assume that $C_1 \geq 1$ and $C_2(\mu_1) \geq 1$. We set

$$\mu_0 = \min\left(\frac{1}{6C_1\,C_2(\mu_1)}, 1 - \frac{1}{\mu_1}\right) \quad \text{and} \quad \theta_0(\mu) = \frac{\mu}{2C_1}.$$

Let us notice that if $(\tilde{\mathbf{u}},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2)$ belongs to $\tilde{D}_\mu$, then $\|(1+\tilde{\eta}_1)^{-1}\|_{L^\infty(\Sigma^s_\infty)} \leq \frac{1}{1-\mu} \leq \frac{1}{1-\mu_0} \leq \mu_1$ and $\|\tilde{\eta}_{1,x}\|_{L^\infty(\Sigma^s_\infty)} \leq \mu < 1$. Thus estimates of Theorem 11.1 may be used for elements in $\tilde{D}_\mu$.

We are going to prove that the mapping

$$\mathcal{F} \ : \ (\tilde{\mathbf{u}},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2) \longmapsto (\mathbf{v},p,\eta_1,\eta_2),$$

where $(\mathbf{v},p,\eta_1,\eta_2)$ is the solution to system (12.1), and in which $\tilde{F}$, $\tilde{G}$, and $\tilde{H}$ are the functions of $(\tilde{\mathbf{u}},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2)$ defined by (10.2), (10.3), (10.4), is a contraction in $\tilde{D}_\mu$.

If $(\mathbf{v},p,\eta_1,\eta_2) = \mathcal{F}(\tilde{\mathbf{u}},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2)$, due to Theorems 9.1 and 11.1, we have

$$\begin{aligned}&\|\mathbf{v}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|p\|_{L^2(0,\infty;H^1(\Omega))} + \|\eta_1\|_{H^{4,2}(\Sigma^s_\infty)} + \|\eta_1\|_{L^\infty(\Sigma^s_\infty)} + \|\eta_2\|_{H^{2,1}(\Sigma^s_\infty)}\\ &\leq C_1(\|P\mathbf{v}^0\|_{\mathbf{V}^1_0(\Omega)} + \|\eta_1^0\|_{H^3(\Gamma_s)\cap H^2_0(\Gamma_s)} + \|\eta_2^0\|_{H^1_0(\Gamma_s)}\\ &\qquad + \|e^{-\omega t}\tilde{F}\|_{L^2(Q_\infty)} + \|e^{-\omega t}\tilde{\mathbf{w}}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|e^{-\omega t}\tilde{H}\|_{L^2(\Sigma^s_\infty)})\\ &\leq C_1\left(\frac{1}{2C_1}\mu + 3C_2\mu^2\right) \leq \mu.\end{aligned}$$

Thus $\mathcal{F}$ is a mapping from $\tilde{D}_\mu$ into itself.

Let $(\tilde{\mathbf{u}}^1,\tilde{p}^1,\tilde{\eta}_1^1,\eta_2^1)$ and $(\tilde{\mathbf{u}}^2,\tilde{p}^2,\tilde{\eta}_1^2,\eta_2^2)$ belong to $\tilde{E}_\mu$. For $i = 1,2$, we set $(\mathbf{v}^i,p^i,\eta_1^i,\eta_2^i) = \mathcal{F}(\tilde{\mathbf{u}}^i,\tilde{p}^i,\tilde{\eta}_1^i,\eta_2^i)$. Due to Theorems 9.1 and 11.1, we also have

$$
\begin{aligned}
&\|\mathbf{v}^1-\mathbf{v}^2\|_{\mathbf{H}^{2,1}(Q_\infty)}+\|p^1-p^2\|_{L^2(0,\infty;H^1(\Omega))}+\|\eta_1^1-\eta_1^2\|_{H^{4,2}(\Sigma_\infty^s)}\\
&\qquad\qquad+\|\eta_1^1-\eta_1^2\|_{L^\infty(\Sigma_\infty^s)}+\|\eta_2^1-\eta_2^2\|_{H^{2,1}(\Sigma_\infty^s)}\\
&\le C_1(\|e^{-\omega t}(\tilde F^1-\tilde F^2)\|_{\mathbf{L}^2(Q_\infty)}+\|e^{-\omega t}(\tilde{\mathbf{w}}^1-\tilde{\mathbf{w}}^2)\|_{\mathbf{H}^{2,1}(Q_\infty)}+\|e^{-\omega t}(\tilde H^1-\tilde H^2)\|_{L^2(\Sigma_\infty^s)})\\
&\le 3C_1C_2\mu(\|\mathbf{v}^1-\mathbf{v}^2\|_{\mathbf{H}^{2,1}(Q_\infty)}+\|p^1-p^2\|_{L^2(0,\infty;H^1(\Omega))}+\|\eta_1^1-\eta_1^2\|_{H^{4,2}(\Sigma_\infty^s)}\\
&\qquad\qquad+\|\eta_2^1-\eta_2^2\|_{H^{2,1}(\Sigma_\infty^s)})\\
&\le \tfrac12(\|\mathbf{v}^1-\mathbf{v}^2\|_{\mathbf{H}^{2,1}(Q_\infty)}+\|p^1-p^2\|_{L^2(0,\infty;H^1(\Omega))}\\
&\qquad+\|\eta_1^1-\eta_1^2\|_{H^{4,2}(\Sigma_\infty^s)}+\|\eta_2^1-\eta_2^2\|_{H^{2,1}(\Sigma_\infty^s)}).
\end{aligned}
$$

Thus $\mathcal{F}$ is a contraction in $\tilde D_\mu$, and the proof is complete. ∎

Appendix. In this section we analyze what results can be extended to models slightly different from system (1.1).

A.1. The case when $\delta = 0$. In that case Theorem 3.4 is replaced by the following.

THEOREM A.1. *The operator $(\mathcal{A}, D(\mathcal{A}))$, with $\delta = 0$, is the infinitesimal generator of a strongly continuous semigroup on $\mathbf{H}$, and the resolvent of $\mathcal{A}$ is compact.*

Theorem 5.1 is still valid, but we cannot deduce the stabilizability of system (5.1) from the approximate controllability result stated in Theorem 5.1 because the semigroup generated by $(\mathcal{A}, D(\mathcal{A}))$ is no longer analytic. Therefore the assumption $\delta > 0$ is essential in sections 6–12.

Of course, since the control acts everywhere in the structure, if we want to control system (1.1) in the case when $\delta = 0$, we can artificially add a viscous term $-\delta\,\eta_{txx}$ in the model, determine the corresponding feedback, and take the sum of this feedback and of $\delta\,\eta_{txx}$ as feedback for the original system. In that case the total feedback law is no longer a bounded operator from $\widehat{\mathbf{H}}$ into H_s.

A.2. The case when Ω is not a rectangular domain. We can consider 2D domains Ω of class C^2 for which $\Gamma_s = (0, L) \times \{1\} \subset \Gamma$, where Γ is the boundary of Ω. We may assume in addition that $(0, L) \times (0, 1) \subset \Omega$. In that case Theorem 3.4 is still valid. But we do not know whether Theorem 5.1 is still valid. The unique continuation property, which is the main argument in the proof of Theorem 5.1, is in that case an open problem. (See, e.g., [22] to see for which domains the unique continuation property is established.)

Using results in [22], it is possible to extend results of the present paper to domains Ω which are not necessarily of rectangular type but which have a corner at the junction between the structure and the rigid part of the boundary Γ of Ω. In that case if $(P\mathbf{v}, \eta_1, \eta_2)$ belongs to $D(\mathcal{A})$, $P\mathbf{v}$ does not necessarily belong to $\mathbf{V}_n^2(\Omega)$. It is necessary to analyze the loss of regularity for $P\mathbf{v}$, due to the presence of the corner at the boundary of Ω, and the loss of regularity for the associated pressure, to see whether we still have $(\eta_1, \eta_2) \in (H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)) \times L_0^2(\Gamma_s)$ when $(P\mathbf{v}, \eta_1, \eta_2) \in D(\mathcal{A})$. Therefore this loss of regularity of elements belonging to $D(\mathcal{A})$ implies that the regularity results of Theorem 9.1 have to be weakened. Therefore, it may happen that the analysis of the nonlinear closed loop system made in section 10–12 fails.

A.3. 3D models. If we want to extend some results of the paper to 3D models, we have to replace the beam equation by a plate equation with a damping of the form $-\delta\Delta\eta_t$. Let us notice that this damping is different from that considered in [7]. We can consider either the case when Ω is the parallelepiped $(0, L_1) \times (0, L_2) \times (0, 1)$

and Γ_s, the reference configuration for the plate, is $(0, L_1) \times (0, L_2) \times \{1\}$ or the case when Ω is a domain of class C^2 and the reference configuration for the plate is a 2D domain with a boundary of class C^2. The analogue of Theorem 3.4 can be established in those cases, but it is out of the scope of the paper to give a precise definition of the corresponding operator $\mathcal{A}$ and its domain $D(\mathcal{A})$. As mentioned in section A.2, no approximate controllability result is known in those cases.

A.4. Periodic boundary conditions. As mentioned in the introduction, some results can be extended to systems of the form (1.1) in which the boundary conditions $\mathbf{u} = 0$ on $\{0\} \times (0, 1) \cup \{1\} \times (0, 1)$ are replaced by periodic boundary conditions as in [3]. Using the stabilizability results from [4] for a channel flow problem, it is possible to extend the results of our paper to these models. For that we have to consider a model with two beams, one beam occupying the upper part of the boundary of the rectangle $\Omega = (0, L) \times (0, 1)$ and the other occupying the lower part. The complete stabilizability results that we obtain in section 5 may be replaced, using [4], by stabilizability for some exponential decay rate. And therefore we have to adapt results that are stated for an arbitrary exponential decay rate $-\omega$ to the case where the decay rate is the one obtained in the paper by Barbu [4].

A.5. Other boundary conditions. It may be more relevant, from the physical viewpoint, to replace the boundary condition

$$\mathbf{u}(x, 1 + \eta(x, t), t) = \eta_t(x, t)\vec{e}_2 \quad \text{for } (x, t) \in (0, L) \times (0, \infty)$$

by the following (see [13, 1.1c]):

$$\mathbf{u}(x, 1 + \eta(x, t), t) \cdot \mathbf{n}(x, t)\,(1 + \eta_x^2)^{1/2} = \eta_t(x, t) \quad \text{for } (x, t) \in (0, L) \times (0, \infty). \tag{A.1}$$

In that case this boundary condition may be completed by a condition on the tangential component of the normal stress at the boundary,

$$(\sigma(\mathbf{u}(x, 1 + \eta(x, t), t), p(x, 1 + \eta(x, t), t))\mathbf{n}(x, t)) \cdot \tau(x, t) = 0 \quad \text{for } (x, t) \in (0, L) \times (0, \infty), \tag{A.2}$$

where

$$\tau(t) = \left(\frac{1}{\sqrt{1 + \eta_x^2(t)}}, \frac{\eta_x(t)}{\sqrt{1 + \eta_x^2(t)}} \right)^T.$$

(Such a model is considered in [13].) Due to (A.1) and (A.2), in the linearized model (1.2) the boundary condition

$$\mathbf{v} = \eta_2 \vec{e}_2 \quad \text{on } \Sigma_\infty^s$$

has to be replaced by

$$\mathbf{v} \cdot \vec{e}_2 = \eta_2 \quad \text{and} \quad (\sigma(\mathbf{v}, p)\vec{e}_2) \cdot \vec{e}_1 = 0 \quad \text{on } \Sigma_\infty^s. \tag{A.3}$$

We think that the results of sections 2–4 can be extended to that case. However, the regularity results that we use in section 3 for the Stokes equation with Dirichlet boundary conditions have to be recovered if the linearized model is written with the boundary conditions (A.3). This has to be done very carefully, and it is out of the scope of the present paper. The approximate controllability result of section 5 is an open problem in that case. In particular the unique continuation result for the

Stokes equation proved in [21] for Dirichlet boundary conditions is not known when the Dirichlet condition on Γ_s is replaced by

$$\mathbf{v} \cdot \vec{e}_2 = 0 \quad \text{and} \quad (\sigma(\mathbf{v}, p)\vec{e}_2) \cdot \vec{e}_1 = 0 \quad \text{on } \Gamma_s.$$

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