Estimation of excursion sets under functional uncertainties

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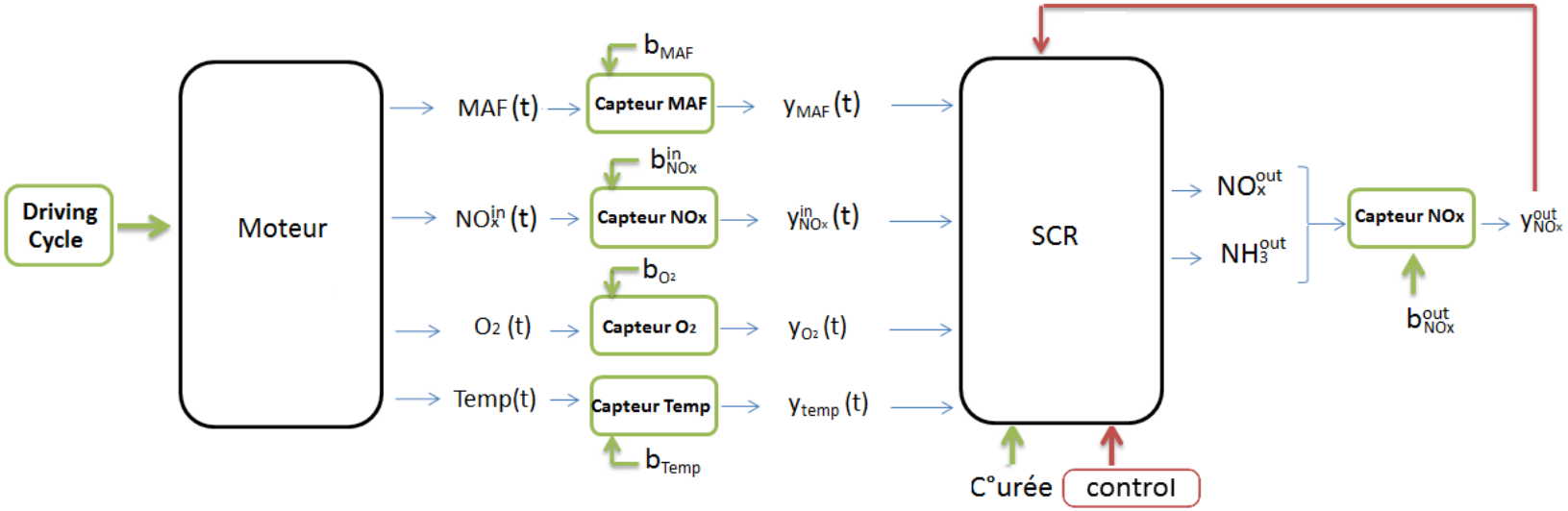
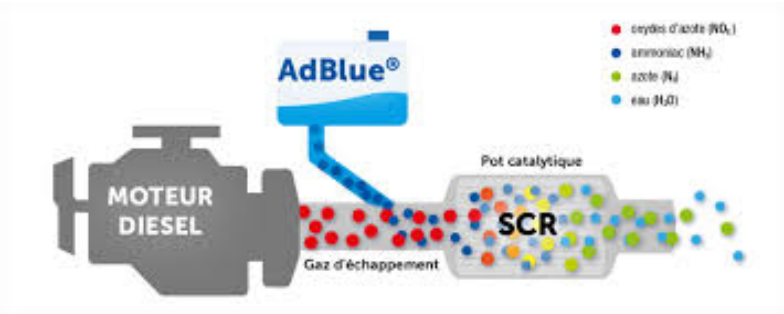
Joint work with Mohamed Reda El Amri (IRT Saint-Exupéry), Clémentine Prieur (UGA/LJK), Delphine Sinoquet (IFPEN) Miguel Munoz Zuniga (IFPEN) et Olivier Lepreux (IFPEN).

IFPEN test case: an automotive $\mathrm{NO_x}$ depollution system





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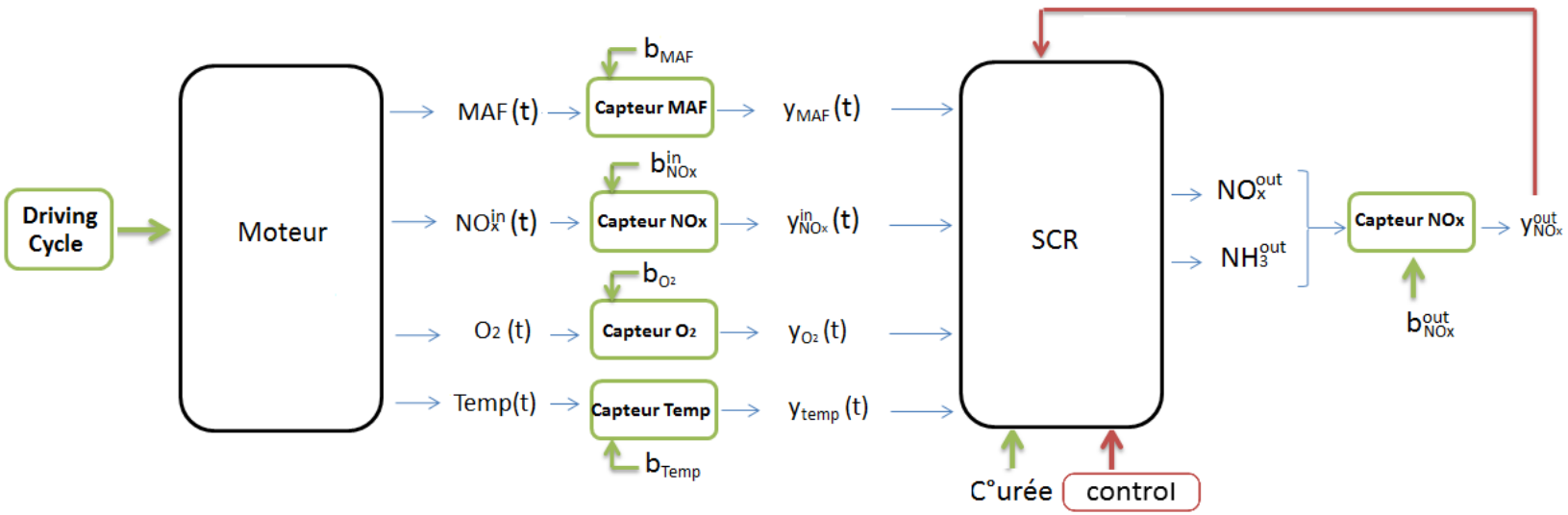
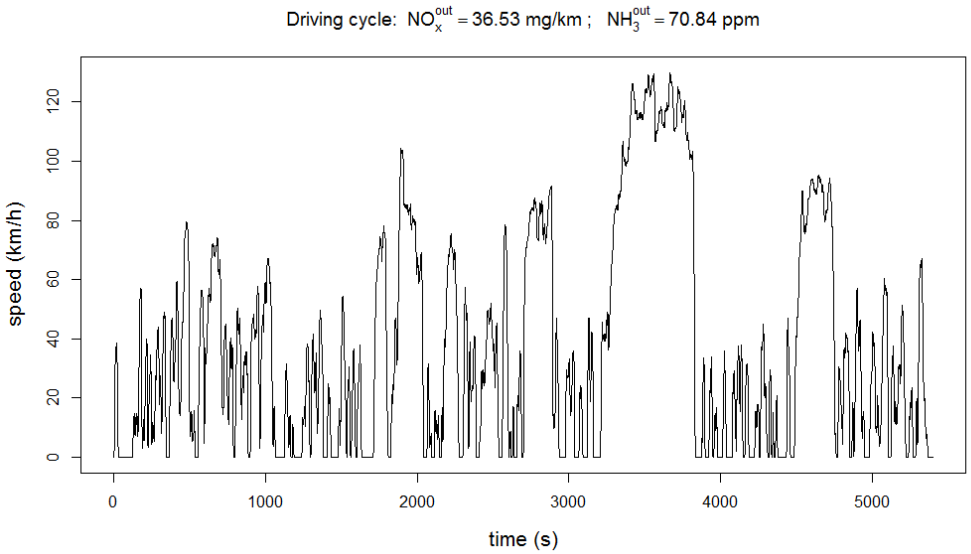
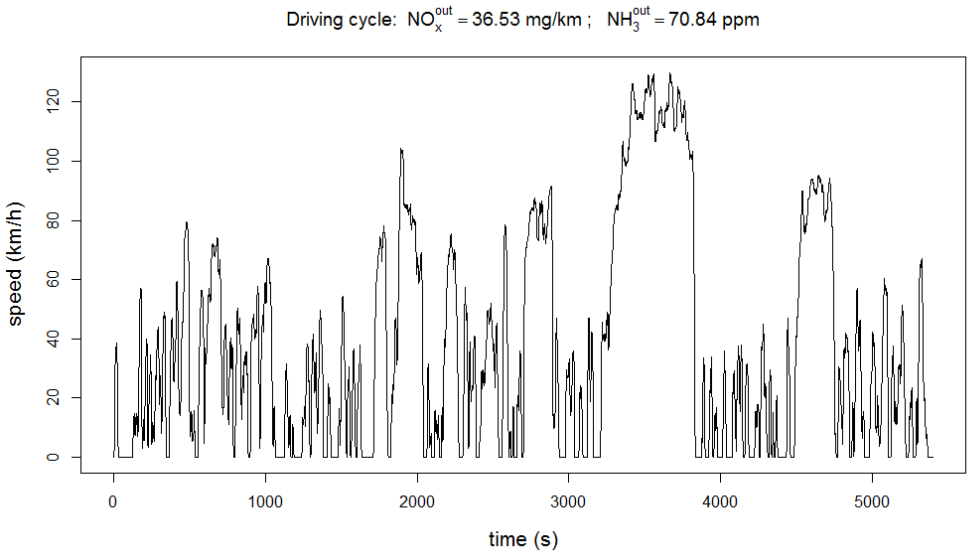


European emission standards

Scalar outputs and thresholds $\begin{cases} \mathrm{NO_x^{out}} & \leq & 80\ mg.km^{-1} \\ \mathrm{NH_3^{out}} & \leq & 30\ ppm \end{cases}$


Driving cycle: $NO_x^{out} = 36.53$ mg/km ; $NH_3^{out} = 70.84$ ppm
speed (km/h)
0
20
40
60
80
100
120
0
1000
2000
3000
4000
5000
time (s)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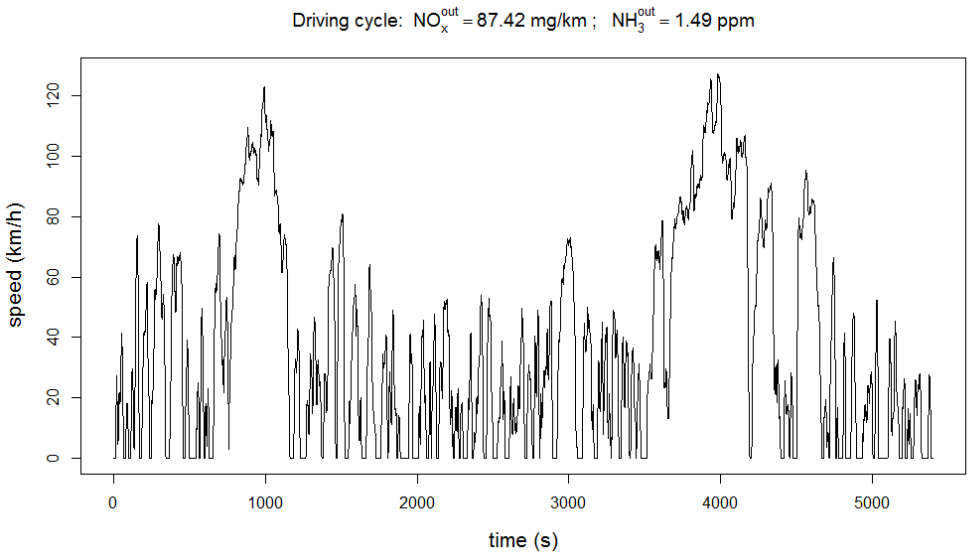
Driving cycle: NO_x^{out} = 87.42 mg/km ; NH_3^{out} = 1.49 ppm
speed (km/h)
0
20
40
60
80
100
120
0
1000
2000
3000
4000
5000
time (s)

The framework

$$\begin{array}{rcccc} f & : & \mathbb{X} \times \mathcal{V} & \rightarrow & \mathbb{R} \\ & & (\mathbf{x}, v) & \mapsto & f(\mathbf{x}, v) \end{array}$$

where $\mathbb{X} \subset \mathbb{R}^p$ is a compact and $\mathcal{V}$ functional space. For a fixed $c \in \mathbb{R}$,

$$\Gamma^* := \{\mathbf{x} \in \mathbb{X} \;\; s.t. \;\; g(\mathbf{x}) = \mathbb{E}[f(\mathbf{x}, \mathbf{V})] \leq c\} \quad := g^{-1}(C) \text{ , where } C = (-\infty, c].$$

Objective

- Estimate Γ^*, the excursion set of g below c.

The framework

- Each evaluation $f(\mathbf{x}, v)$ is expensive-to-compute,

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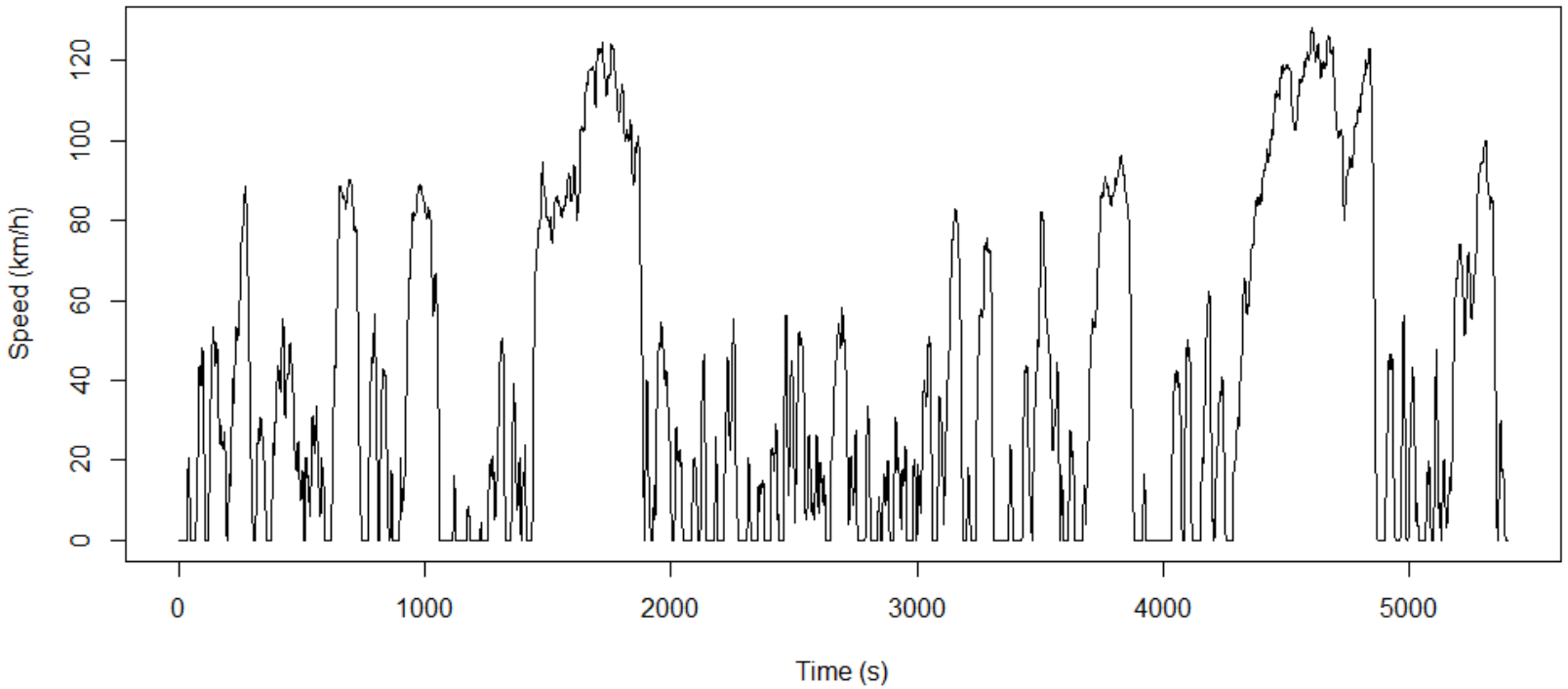
- $\mathbf{V}$ is only known through N realizations $\Xi = \{v_1, v_2, ..., v_N\}$
  - redundant information,
  - rough curves.



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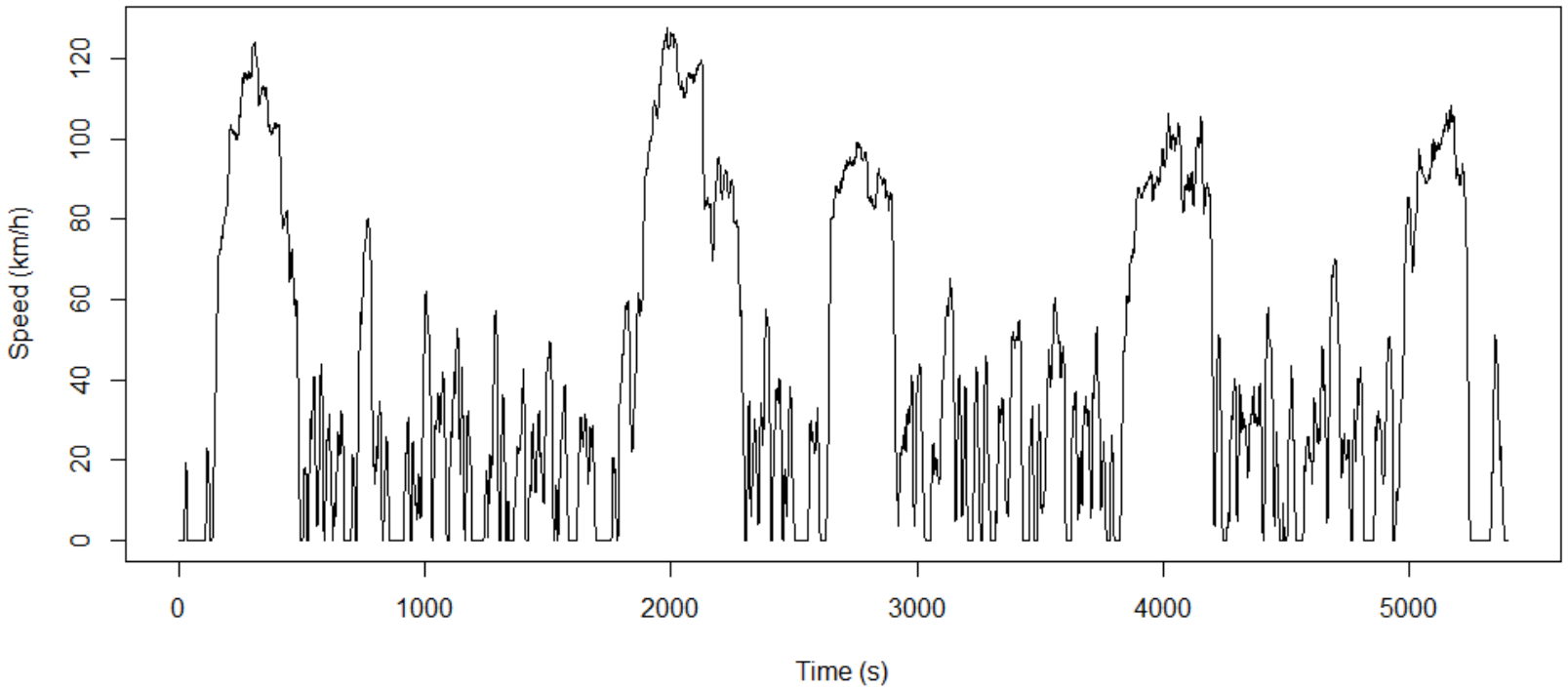

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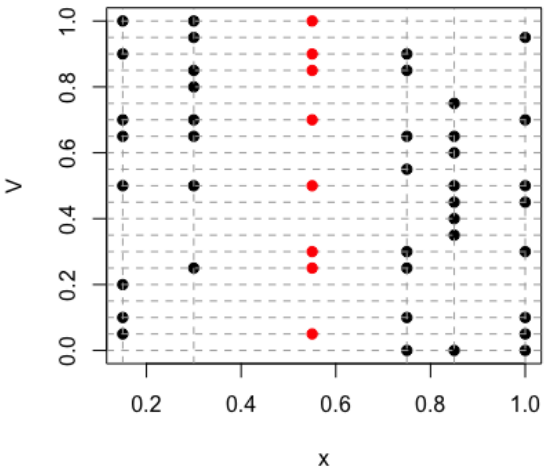
Objective

- Estimate $\Gamma^* := \{\mathbf{x} \in \mathbb{X} \;\; s.t. \;\; g(\mathbf{x}) = \mathbb{E}[f(\mathbf{x}, \mathbf{V})] \leq c\}$, with a limited budget of f simulations.

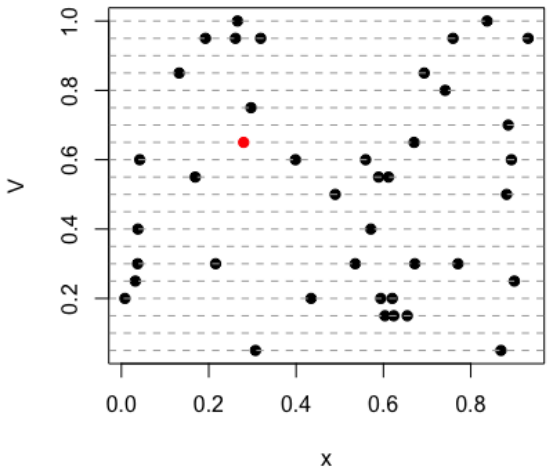
Approach I

- Build a meta-model for g and choose $\mathbf{x}_{n+1} \in \mathbb{X}$,
- estimate $\mathbb{E}[f(\mathbf{x}_{n+1}, \mathbf{V})]$ (requires l evaluations from Ξ).



Approach II

- Build a meta-model for f,
- choose $(\mathbf{x}_{n+1}, \mathbf{v}_{n+1}) \in \mathbb{X} \times \Xi$ (only one evaluation of f is required).



Outline

1. Data-driven stochastic inversion via functional quantization
2. Stochastic inversion via meta-modelling in the joint space
3. IFPEN test case
4. Conclusion and outlook

Algorithm Data-driven stochastic inversion via functional quantization

Create an initial DoE $\mathcal{X}_n$ of n points in the control space $\mathbb{X}$
Evaluate $\mathbf{g}_{\mathcal{X}_n} = \mathbf{g}(\mathcal{X}_n)$
Gaussian process regression from $\mathcal{X}_n$ and $\mathbf{g}_{\mathcal{X}_n}$
Vorob'ev estimation of $\Gamma^* : Q_n$
while Stopping criterion not met **do**
    Choose $\mathbf{x}_{n+1}$ through a Stepwise Uncertainty Reduction (SUR) strategy
    Evaluate $\mathbf{g}(\mathbf{x}^{n+1}) = \mathbb{E}\left(f(\mathbf{x}^{n+1}, \mathbf{V})\right)$
    Update DoE, Meta-model and estimation of Γ^*
    Set $n = n + 1$
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Bayesian framework

Assume: g is seen as one realization of a Gaussian Process $(Z_{\mathbf{x}})_{\mathbf{x}\in\mathbb{X}}$ with prior mean m and covariance kernel k.

Given the function evaluations $\mathbf{g}_{\mathcal{X}_n}$ the posterior has a Gaussian distribution

$$Z_{\mathbf{x}}|Z_{\mathcal{X}_n} = \mathbf{g}_{\mathcal{X}_n}$$

with mean and covariance kernel

$$m_n(\mathbf{x}) = m(\mathbf{x}) - k(\mathbf{x},\mathcal{X}_n)k(\mathcal{X}_n,\mathcal{X}_n)^{-1}(\mathbf{g}_{\mathcal{X}_n} - m(\mathcal{X}_n)),$$
$$k_n(\mathbf{x},\mathbf{x}') = k(\mathbf{x},\mathbf{x}') - k(x,\mathcal{X}_n)k(\mathcal{X}_n,\mathcal{X}_n)^{-1}k(\mathcal{X}_n,\mathbf{x}').$$

Γ^* is a realization of $\Gamma := \{\mathbf{x}\in\mathbb{X} \ : \ Z_{\mathbf{x}}|Z_{\mathcal{X}_n} \leq c\}$.

Bayesian framework

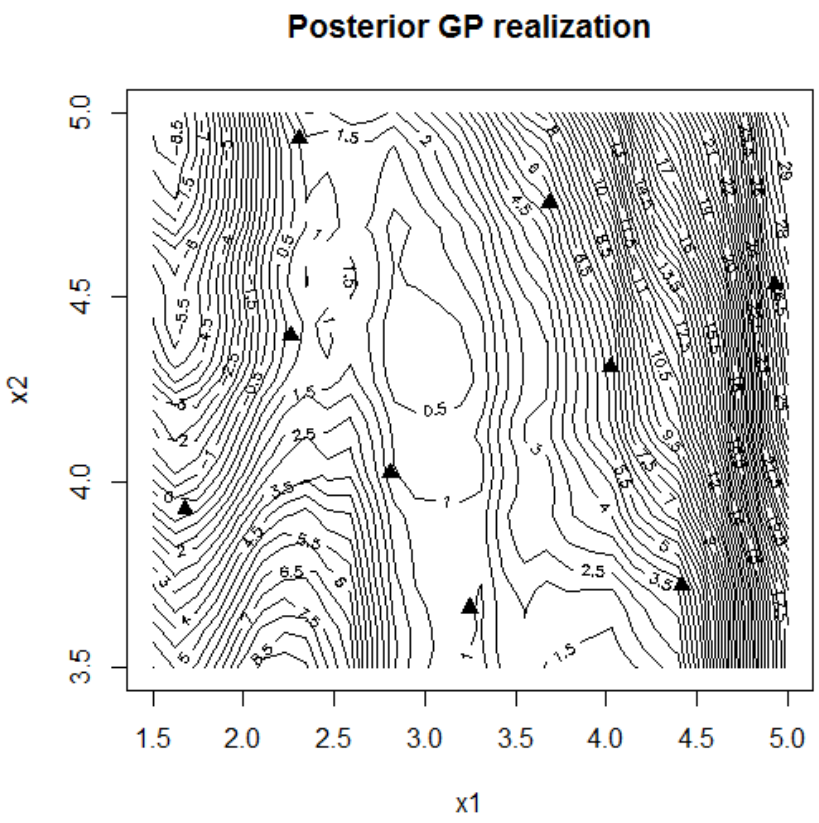
Assume: g is realization of a Gaussian Process $(Z_{\mathbf{x}})_{\mathbf{x}\in\mathbb{X}}$

Prior: $(Z_{\mathbf{x}})_{\mathbf{x}\in\mathbb{X}}$ with

- a.s. continuous paths,
- covariance kernel k,
- constant mean function m.

Given n evaluations $\mathbf{g}_{\mathcal{X}_n}$ at $\mathcal{X}_n$

The closed form formulas for m_n and k_n allow us to generate realizations of the **posterior GP**.



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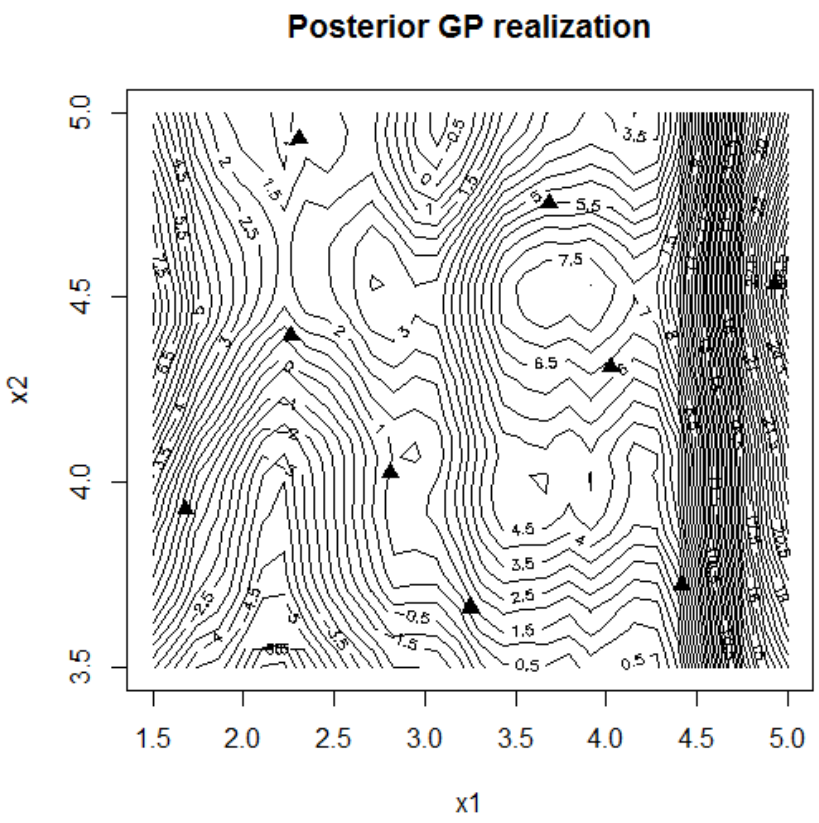
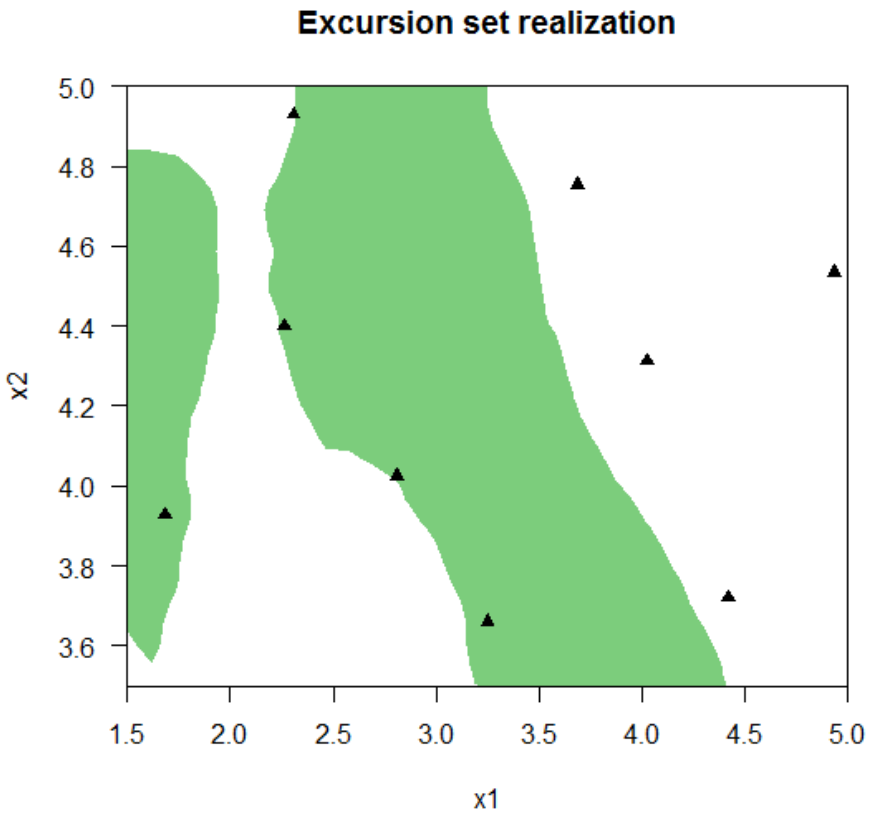


Distribution of excursion sets

The **posterior GP** induces a **posterior distribution on excursion sets**.

- Z continuous paths,
- $(-\infty, c]$ is a closed set.

The set $\Gamma := \{\mathbf{x} \in \mathbb{X} \ : \ Z_{\mathbf{x}} | Z_{\mathcal{X}_n} \leq c\}$ is a random closed set.



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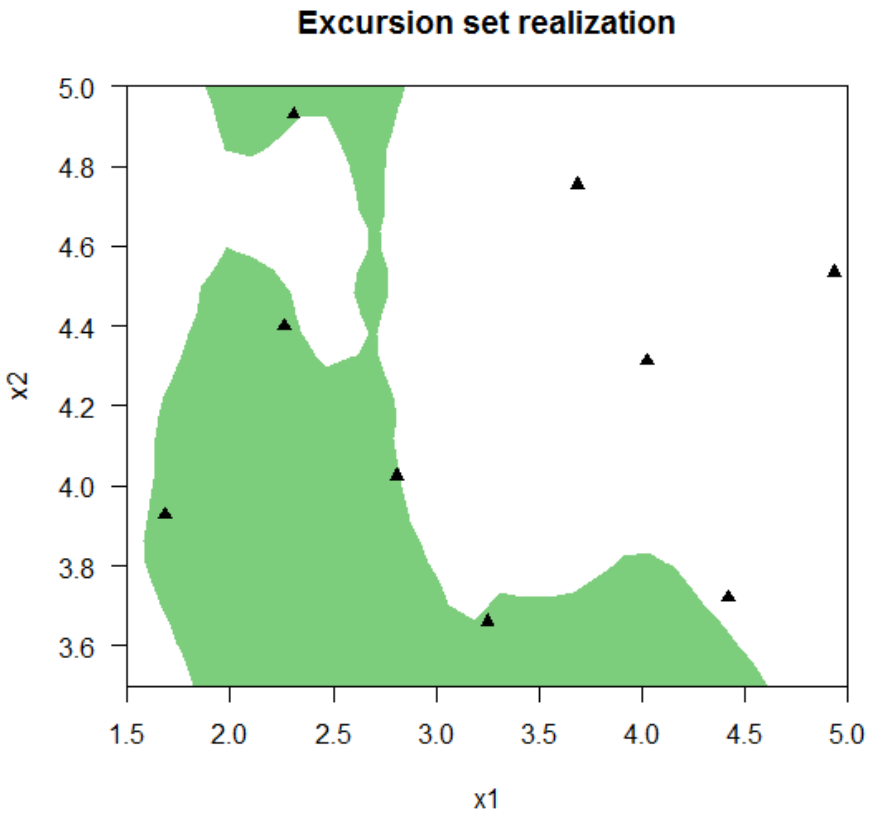

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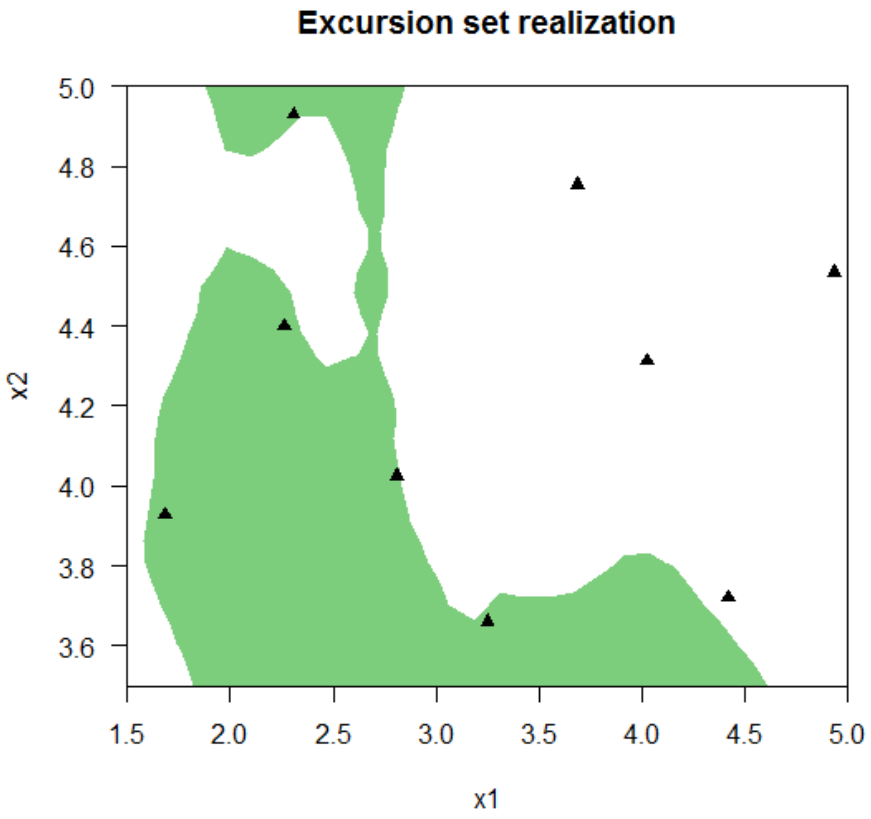
How to summarize the distribution on sets ?

- Estimate for Γ^* with **Expectation** of random closed sets.

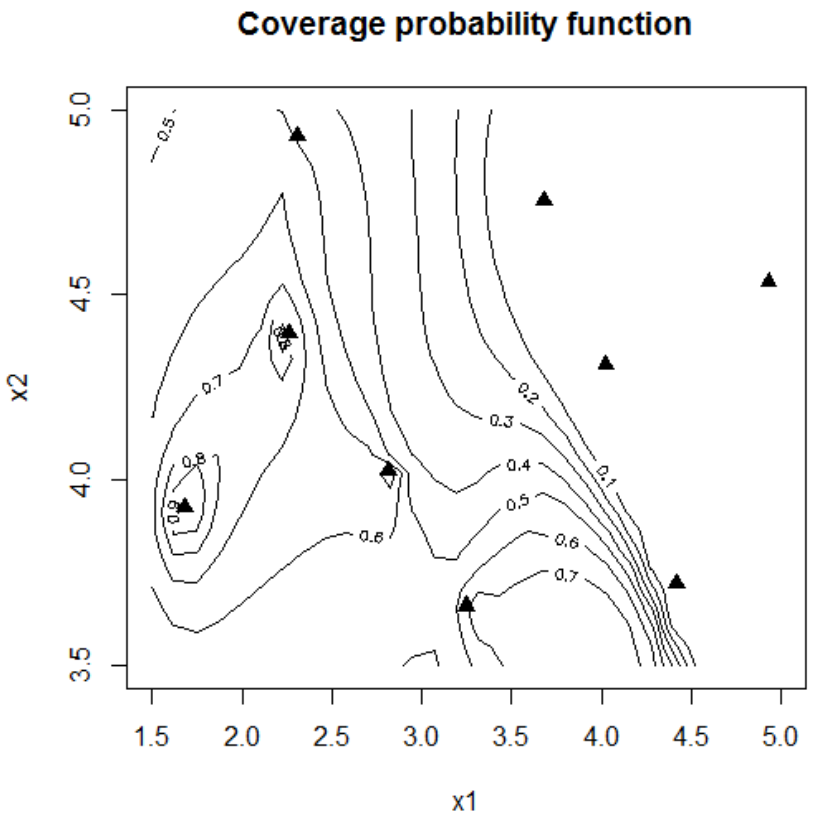


Vorob'ev quantiles

The coverage function of Γ is given by

$$p_n : \begin{cases} \mathbb{X} & \to [0,1] \\ \mathbf{x} & \mapsto \mathbb{P}(\mathbf{x} \in \Gamma \mid Z_{\mathcal{X}_n} = \mathbf{g}_{\mathcal{X}_n}) = \phi\left(\frac{c - m_n(\mathbf{x})}{\sqrt{k_n(\mathbf{x},\mathbf{x})}}\right) \end{cases}$$

- Creates a family of set $Q_{n,\alpha} = \{\mathbf{x} \in \mathbb{X} \ : \ p_n(\mathbf{x}) \geq \alpha\}$.



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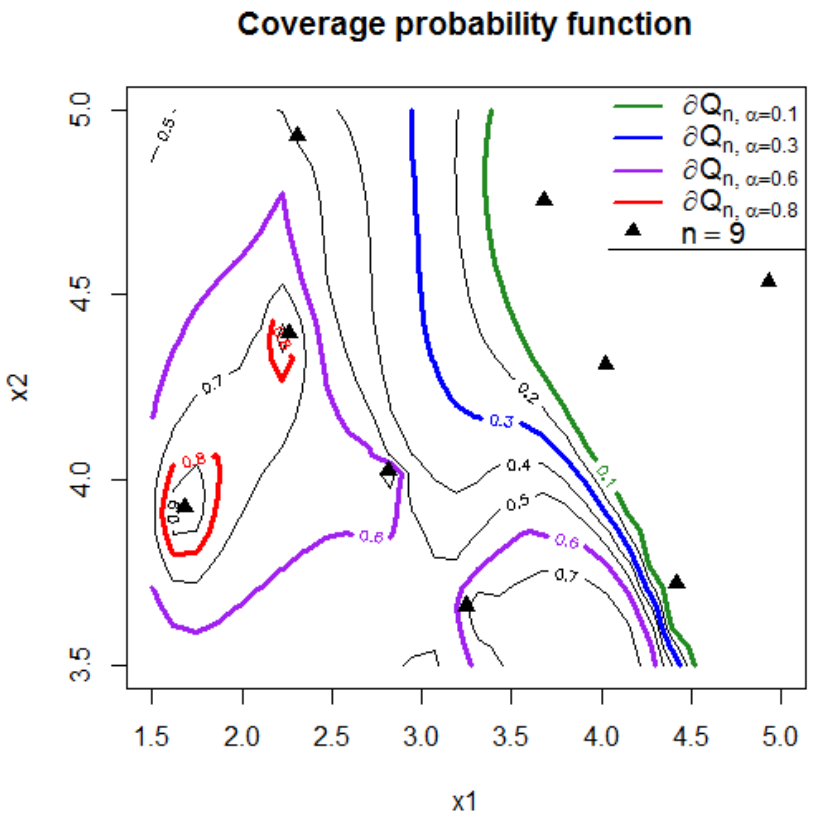

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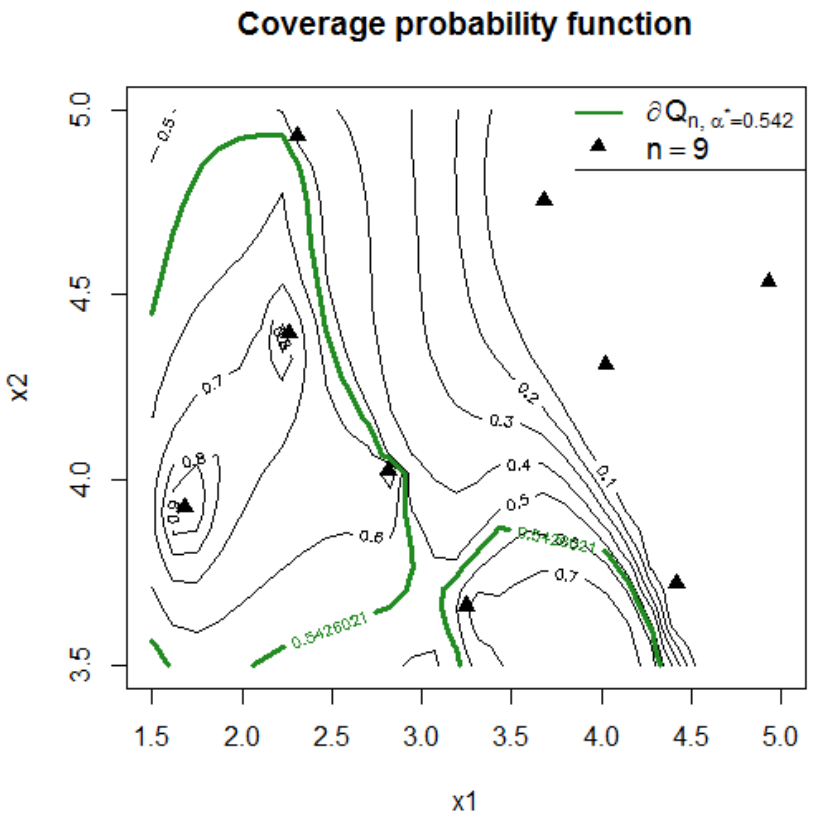

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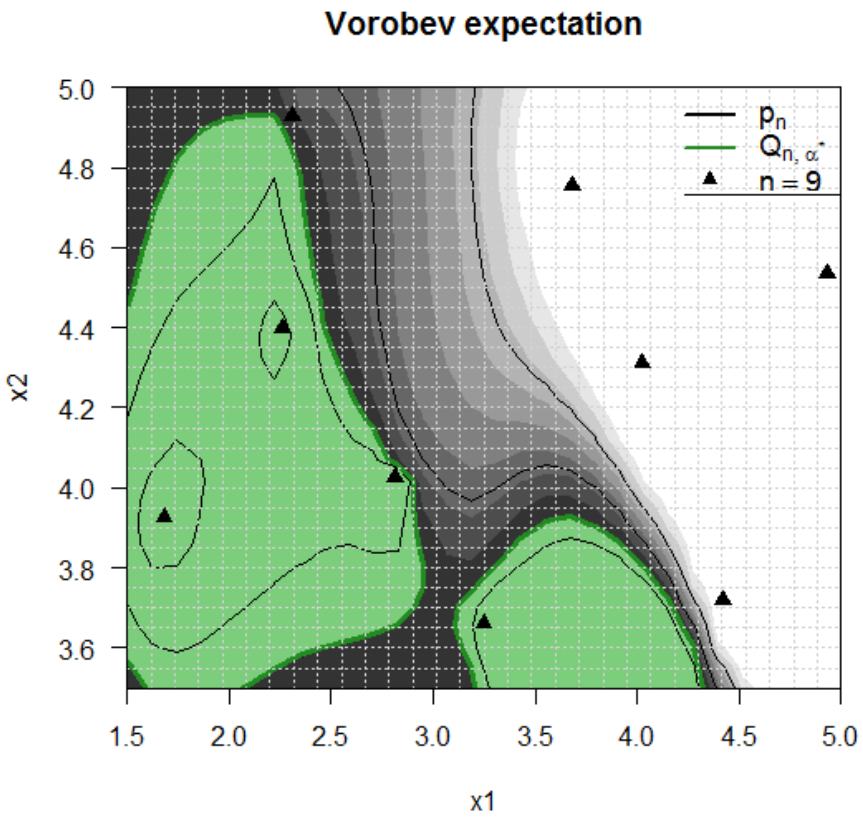

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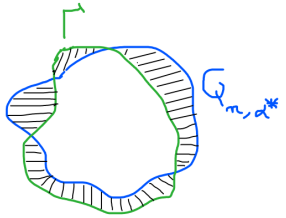
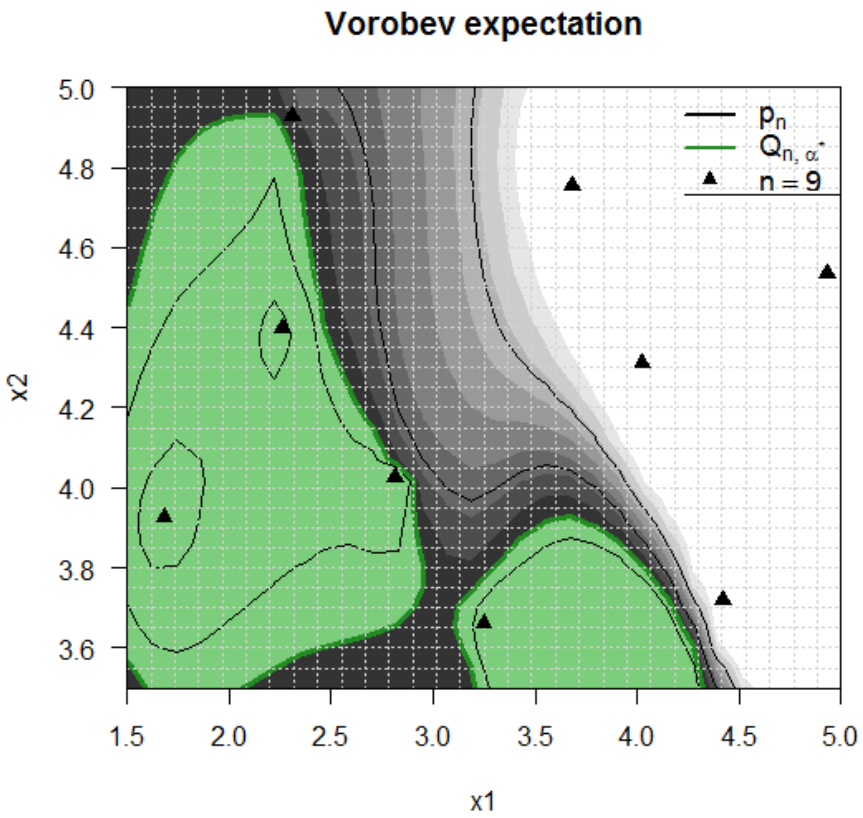
- **Deviation:** $\mathbb{E}[\mu(\Gamma \triangle Q_{n,\alpha^*})] = \mathcal{H}_n^{\mathtt{uncert}}$

  $\forall M \ s.t. \ \mu(M) = \mathbb{E}[\mu(\Gamma)]$,
  $\mathbb{E}[\mu(\Gamma \triangle Q_{n,\alpha^*})] \leq \mathbb{E}[\mu(\Gamma \triangle M)]$.





Sequential strategy for uncertainty reduction

Stepwise Uncertainty Reduction (SUR): find a sequence of evaluations points $\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{n+r}$ that optimally reduces the expected uncertainty on the future estimate, i.e. given an initial design $\mathcal{X}_n$, select

$$\mathbf{x}_{n+1} \in \arg\min_{\mathbf{x}\in\mathbb{X}} \mathbb{E}_{n,\mathbf{x}}[\mathcal{H}_{n+1}^{\mathtt{uncert}}(\mathbf{x})].$$

- $\mathcal{H}_{n+1}^{\mathtt{uncert}}$ can be computed quite easily,
- an efficient implementation of the optimal SUR criteria relies on closed-form formulas derived, again, from the kriging update formulas [1].

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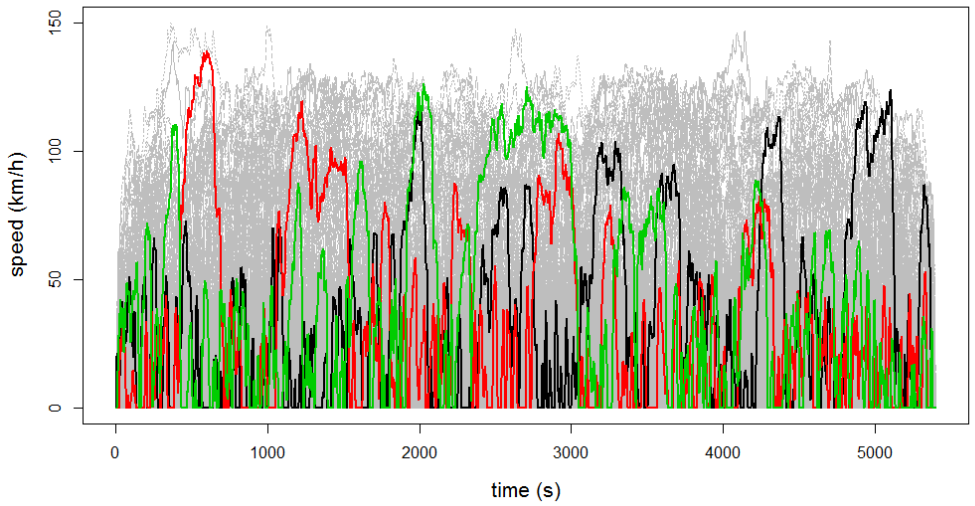
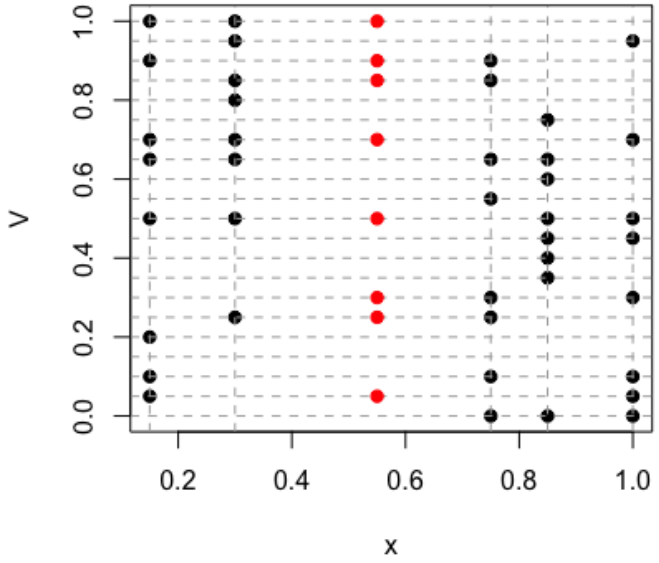
Sequential expectation estimation

Aim

$$\mathbb{E}[f(.,\mathbf{V})] \simeq \sum_{i=1}^{I} w_i f(.,\hat{v}_i), \quad \text{where} \quad \hat{v}_i \in \Xi \quad \text{and} \quad w_i \quad \text{the associated weight.}$$





Functional Quantization formulation

Let $\mathbf{V}$ be a stochastic process defined in the space $\mathcal{H} = \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{V})$ such that

$$||\mathbf{V}||_{\mathbb{L}^2} = \left(\mathbb{E}[||\mathbf{V}||^2]\right)^{1/2} = \left(\mathbb{E}\Big[\int_0^T \mathbf{V}^2 \mathrm{dt}\Big]\right)^{1/2},$$

- Objective: Replace $\mathbf{V}$ by a r.v. taking finite number of values, $\hat{\mathbf{V}}_l$, such that $\mathbb{E}||\mathbf{V} - \hat{\mathbf{V}}_l||^2$ is minimum.
- For a given 'grid' $\Theta_l = \{\hat{v}_1, \ldots, \hat{v}_l\} \subset \mathcal{V}$, the Voronoï quantizer associated to Θ_l is

$$\hat{\mathbf{V}}_l = \sum_{i=1}^{l} \hat{v}_i \mathbf{1}_{C_{\hat{v}_i}(\Theta_l)}(\mathbf{V})$$

  The weight associated to each $\hat{v}_i$ is

$$\mathbb{P}(\hat{\mathbf{V}}_l = \hat{v}_i) = \mathbb{P}(\mathbf{V} \in C_{\hat{v}_i}(\Theta_l)), \qquad i = 1 \ldots, l$$

- How to find the optimal quantizer, i.e. the best 'grid' ?

Numerical computation of quantizers

- **Vectorial case:**
  - Lloyd's algorithm, Competitive Learning Vector Quantization algorithm (Gradient based alg.),
  - Greedy vector quantization [2].
- **Functional case :**
  - Centroidal Voronoi Tessellation method (probabilistic approach of Lloyd's alg.)
  - Optimal quantization for Gaussian processes (e.g. Brownian motion) Available online at quantize.maths-fi.com

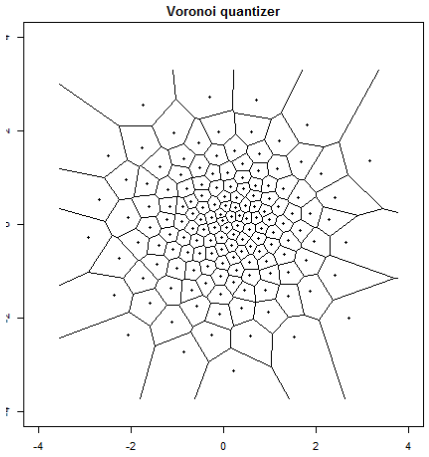


Figure: Quantizers for $\mathcal{N}(0, I_2)$ of size $l = 150$.

[2] Harald Luschgy and Gilles Pagès (2015). "Greedy vector quantization". In: *Journal of Approximation Theory* 198

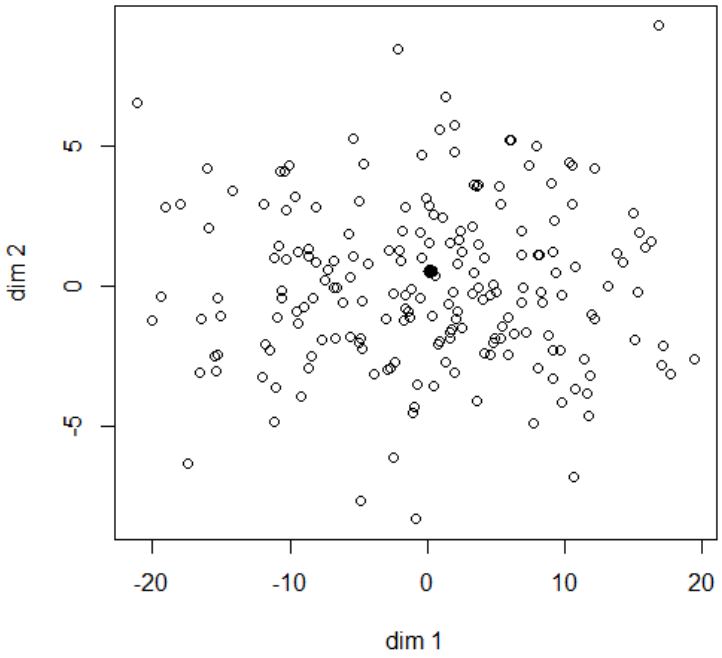
The Karhunen-Loève expansion

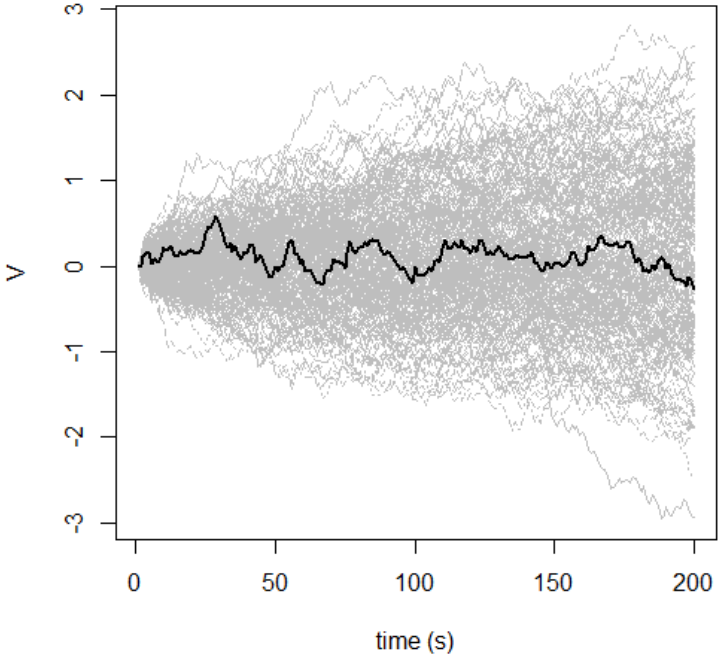
Let $\mathbf{V} \in \mathcal{H}$ be a random process with zero mean and continuous covariance function $C(t, s)$. Then

$$\mathbf{V}(t) \simeq \sum_{i=1}^{m} u_i \psi_i(t),$$

where $\{\psi_i\}_{i=1}^{m}$ are orthogonal and normalized eigenfunctions of the integral operator corresponding to C. The $\{u_i\}_{i=1}^{m}$ denotes a set of orthogonal random variables with zero mean and variance λ_i, where λ_i is the eigenvalue corresponding to the eigenfunction ψ_i.

Greedy Functional Quantization: $\mathbb{L}^2$-error





$\hat{D}_1 = \{\hat{u}_1\}$ where $\hat{u}_1$ is the optimal solution at step 1

$\forall l \geq 2\ ,\ \hat{D}_l = \hat{D}_{l-1} \cup \{\hat{u}_l\}$

where $\quad \hat{u}_l \in \arg\min_{u \in \mathcal{G}} \left(\mathbb{E}||\mathbf{U} - \mathbf{U}_l||^2\right)^{1/2},$

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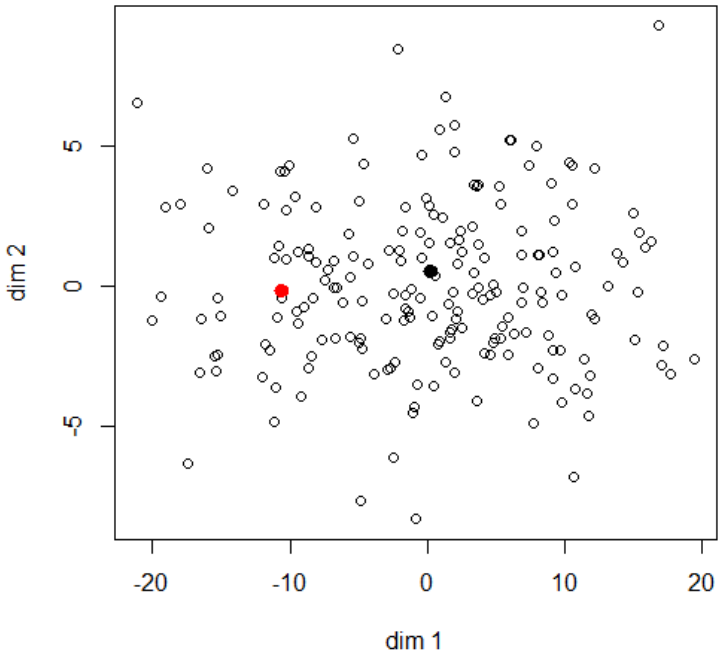

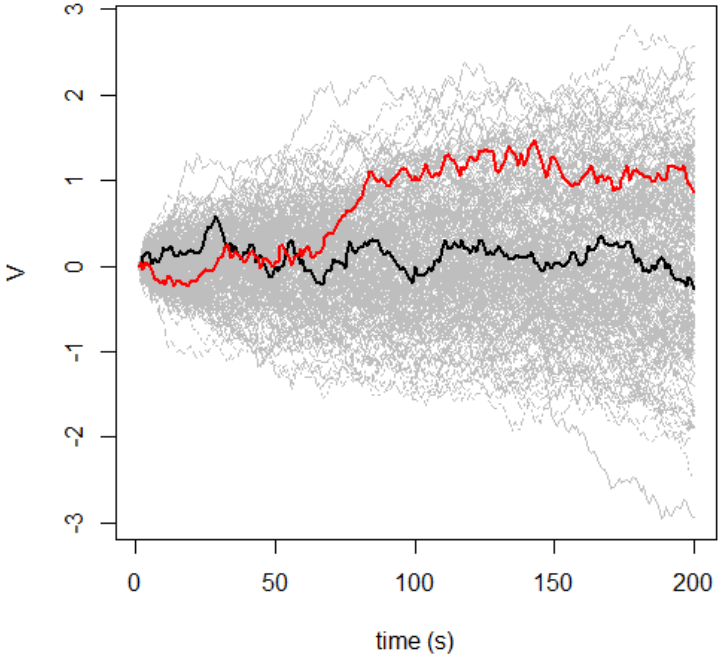

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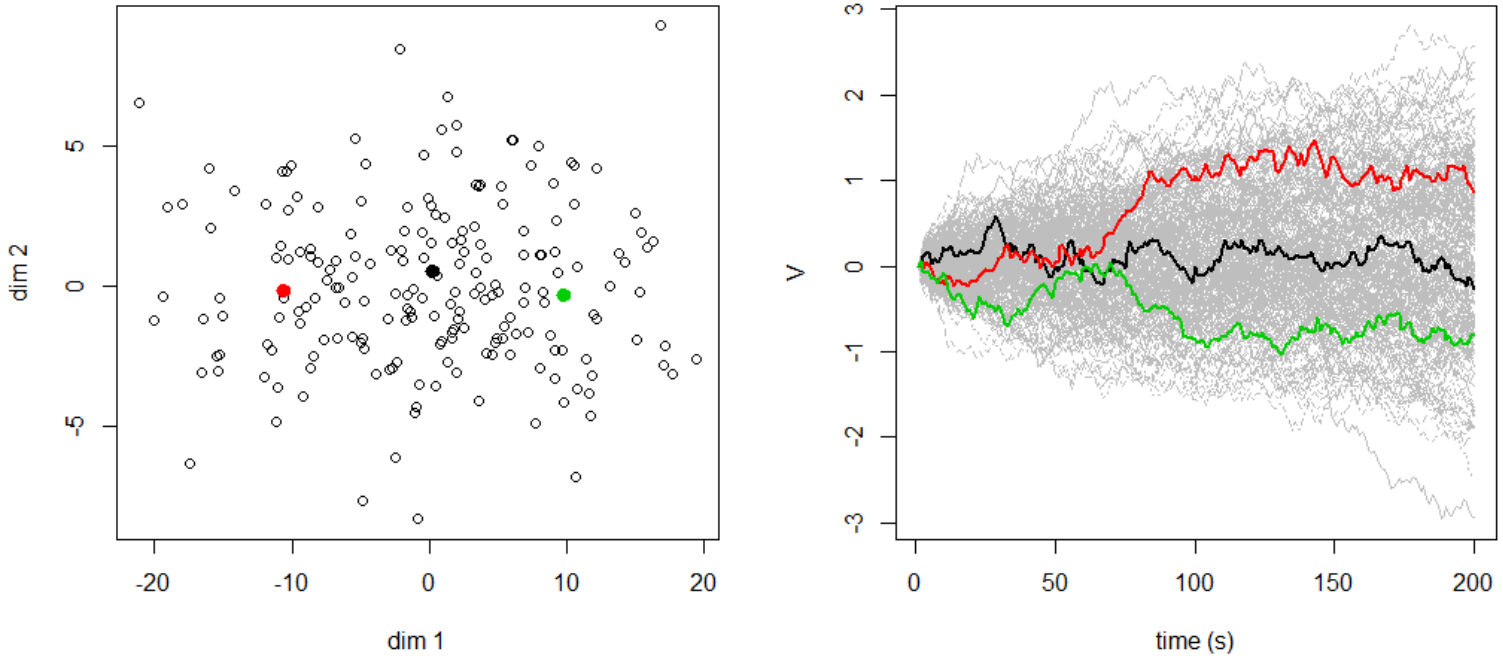

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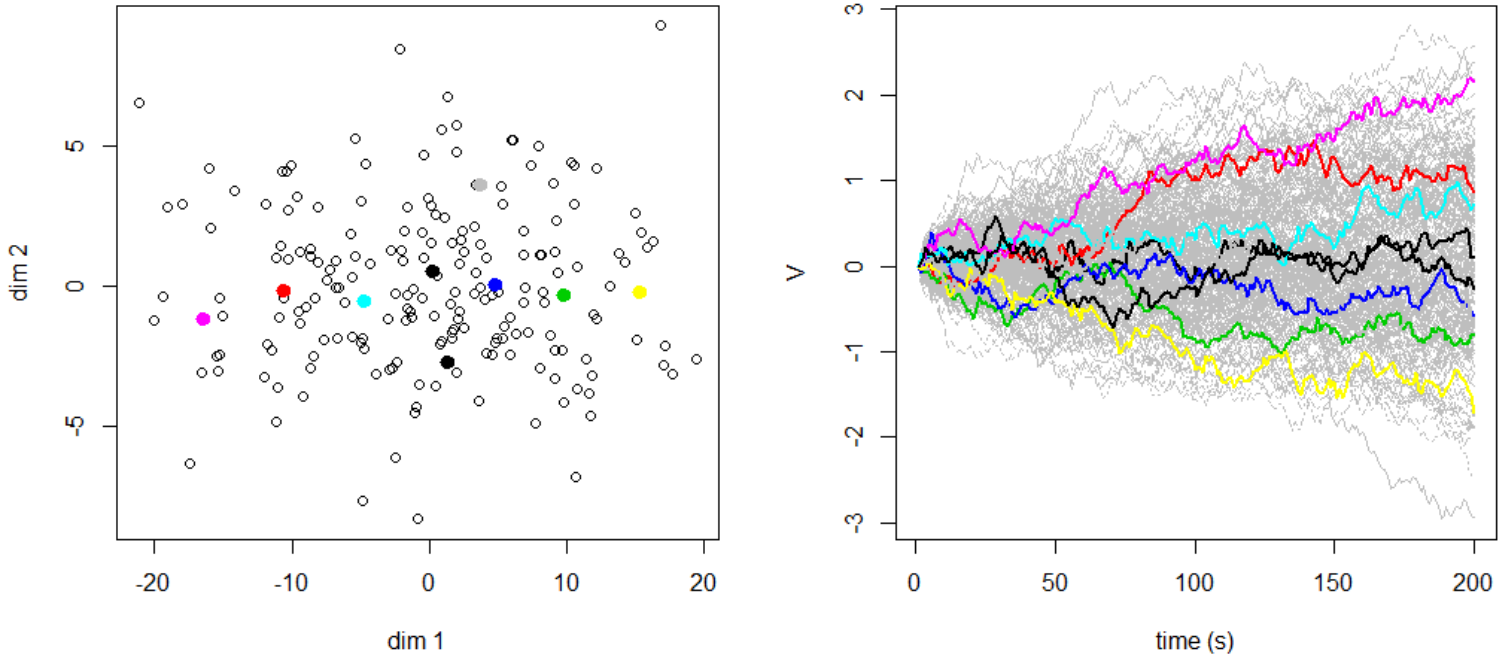

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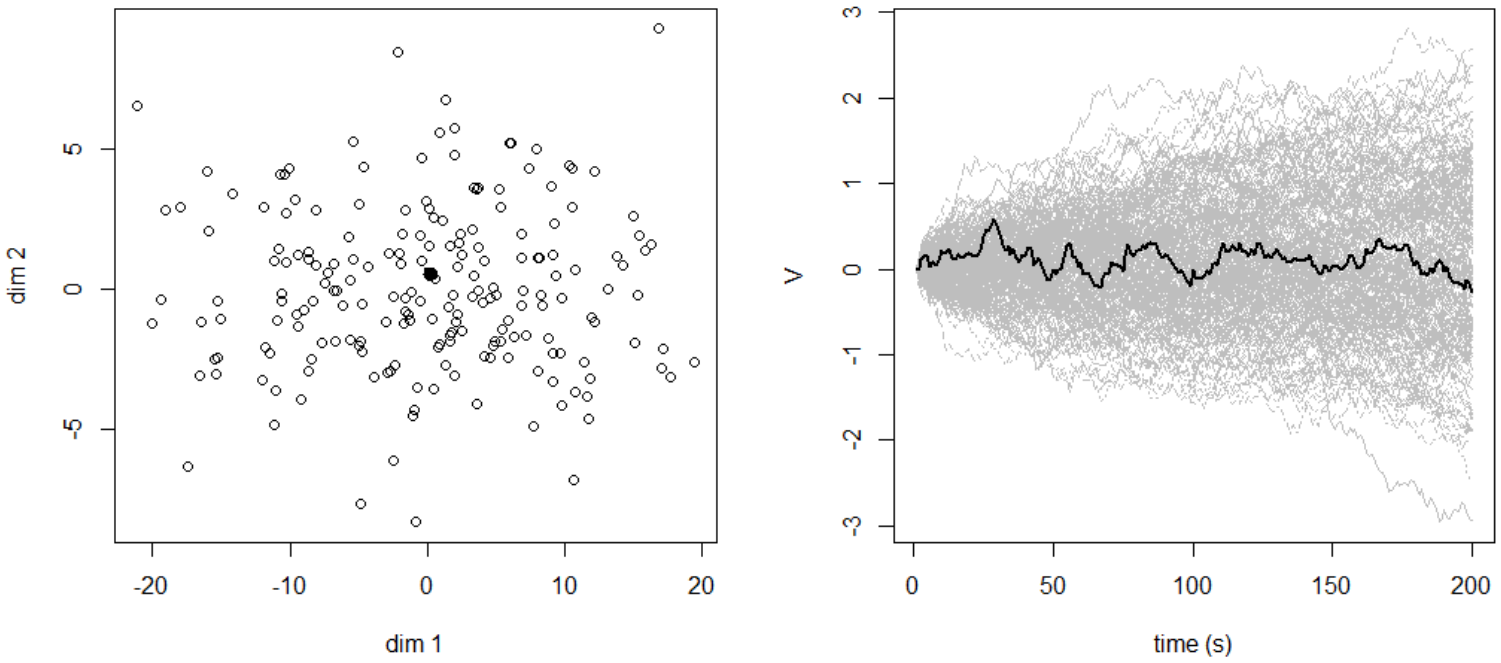
Greedy Functional Quantization: Maximin criterion



Initialization: $\tilde{D}_1 = \{\tilde{\mathbf{u}}_1\}$ where $\tilde{\mathbf{u}}_1$ is randomly chosen

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$$\text{where} \qquad \tilde{\mathbf{u}}_l \in \arg\max_{\mathbf{u} \in \mathcal{G}} \phi_{\text{Maximin}}\left(\tilde{D}_{l-1} \cup \{\mathbf{u}\}\right).$$

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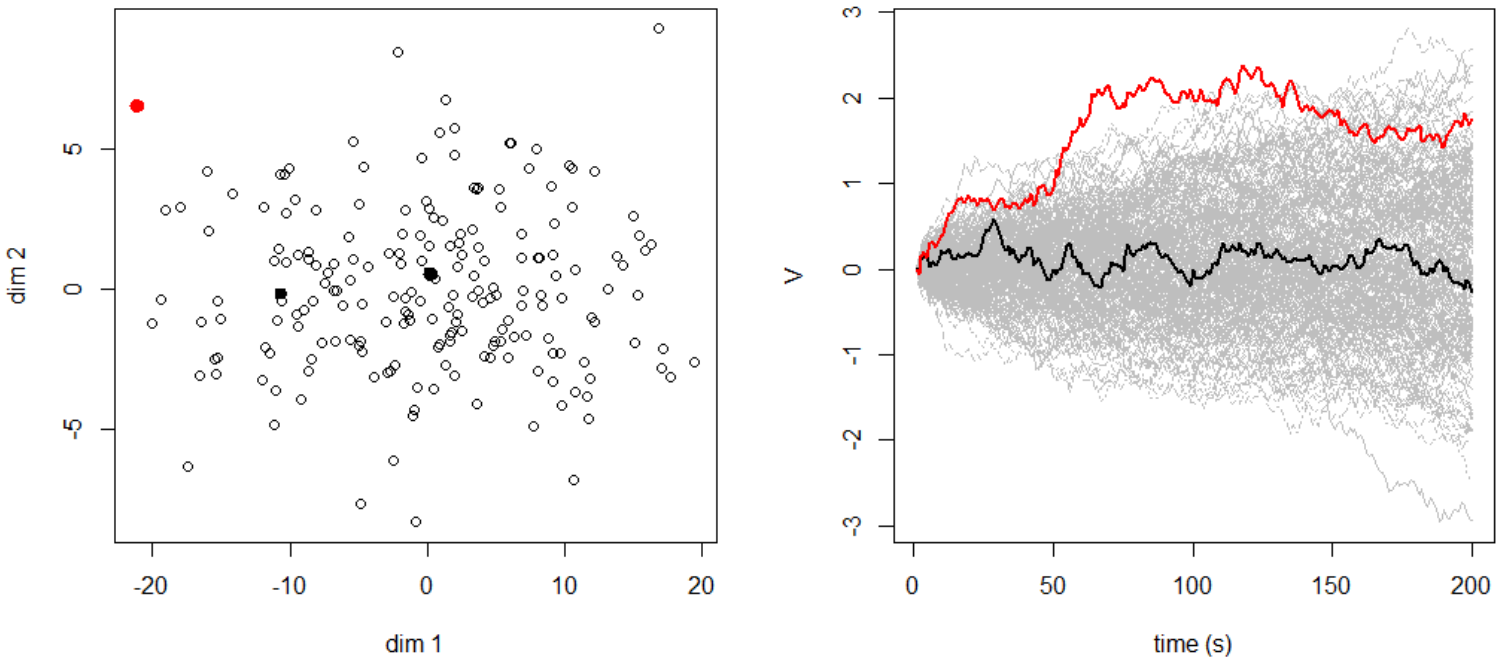

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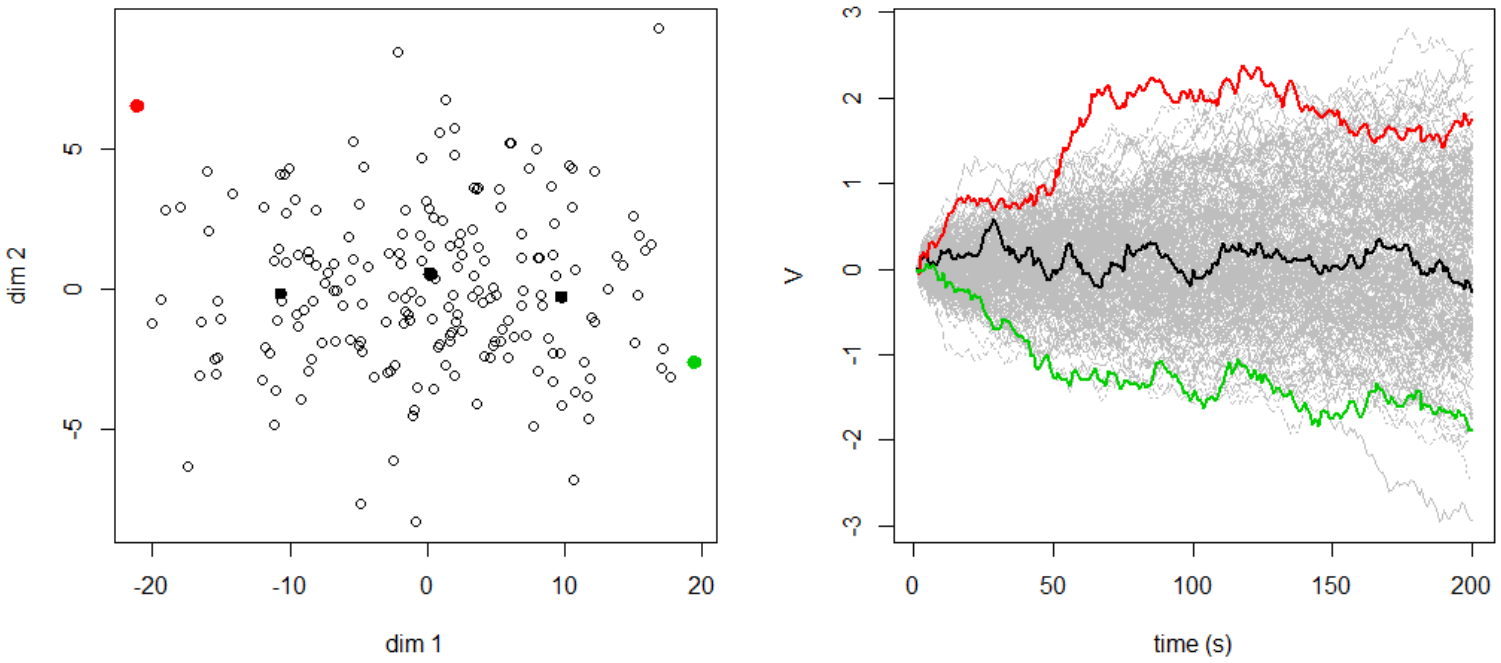

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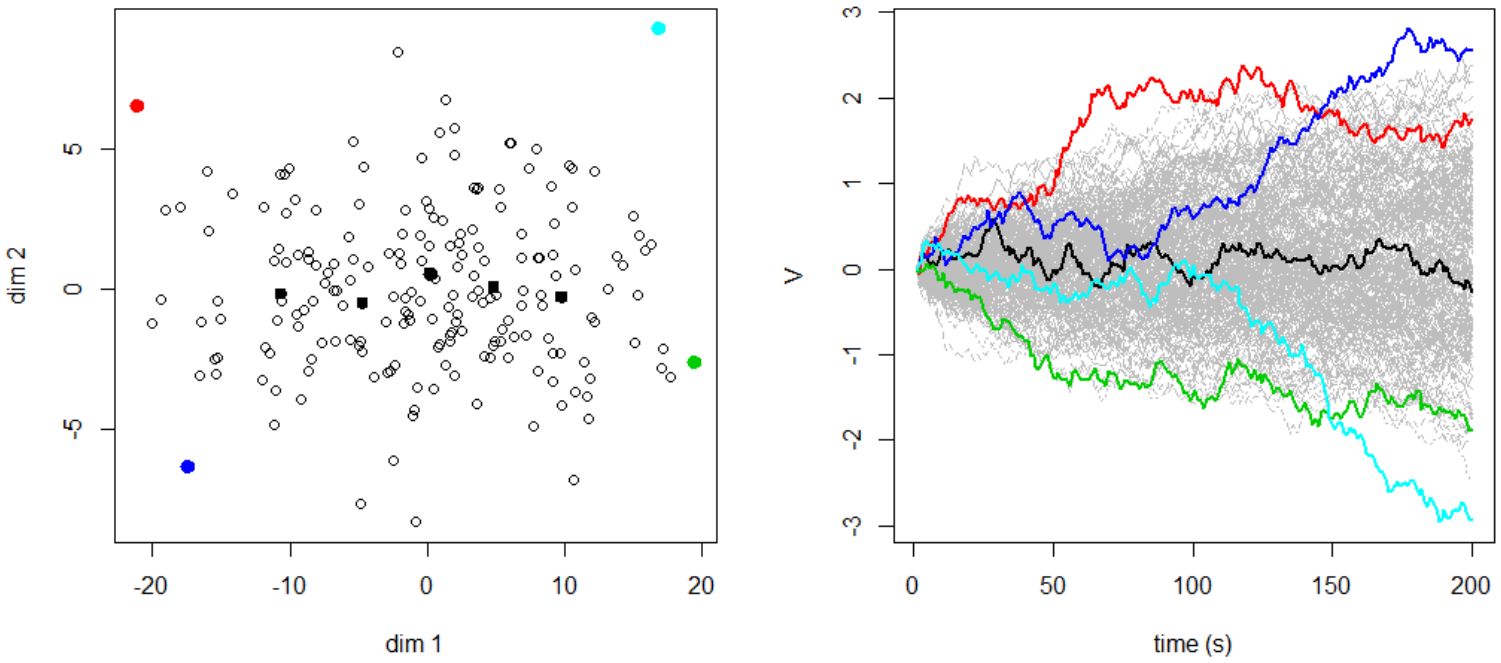

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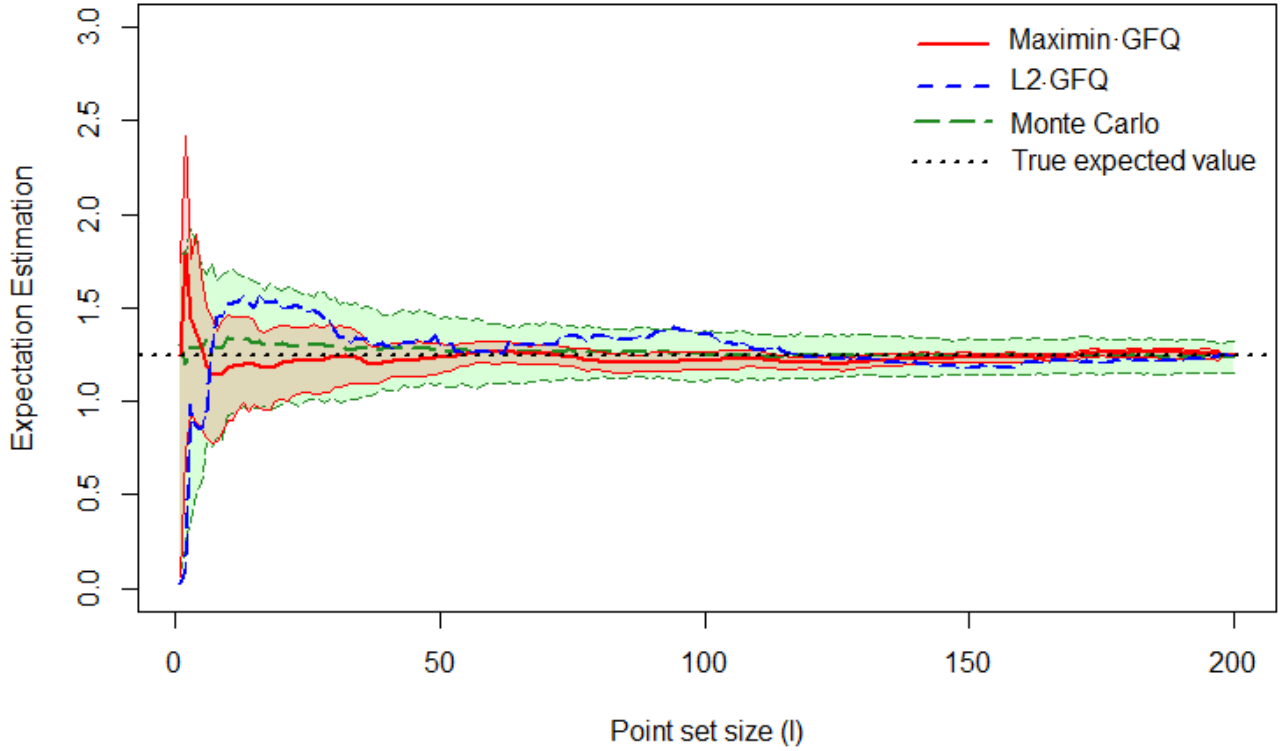
Comparison of the two proposed constructions to the MC method:

$$f : (\mathbf{x}, v) \mapsto \max_t v_t.|0.1\cos(x_1 \max_t v_t)\sin(x_2).(x_1+x_2 \min_t v_t)^2|. \int_0^T (30+v_t)^{\frac{x_1 \cdot x_2}{20}} \mathrm{dt},$$

- $\mathbf{x} = (x_1, x_2) = (2.95, 3.97)$
- $m = 2$
- $\mathtt{card}(\Xi) = 200$ (Brownian motion)



Algorithm Data-driven stochastic inversion via functional quantization

Create an initial DoE $\mathcal{X}_n$ of n points in the control space $\mathbb{X}$
Gaussian process regression from $\mathcal{X}_n$ and $\mathbf{g}_{\mathcal{X}_n}$
Vorob'ev estimation of $\Gamma^* : Q_n$
while Stopping criterion not met (SUR) **do**
    Choose $\mathbf{x}_{n+1}$ through a Stepwise Uncertainty Reduction (SUR) strategy
    while Stopping criterion not met **do**
        $\Theta_{l+1} = \{\hat{v}_1, \ldots, \hat{v}_l\} \cup \{\hat{v}_{l+1}\}$ through GFQ
        Approximate $\mathbf{g}(\mathbf{x}^{n+1}) = \mathbb{E}\left(f(\mathbf{x}^{n+1}, \mathbf{V})\right)$ by $\mathbb{E}[f(\mathbf{x}_{n+1}, \hat{\mathbf{V}}_{l+1})]$
        Set $l = l + 1$
    end while
    Update DoE, Meta-model and estimation of Γ^*
    Set $n = n + 1$
end while
end

Analytical function with the Brownian motion

$$f : (\mathbf{x}, v) \mapsto \max_t v_t.|0.1\cos(x_1 \max_t v_t)\sin(x_2).(x_1+x_2 \min_t v_t)^2|. \int_0^T (30+v_t)^{\frac{x_1 . x_2}{20}} \mathrm{dt},$$

$$\Gamma^* := \{\mathbf{x} \in [1.5, 5] \times [3.5, 5] \ : \ g(\mathbf{x}) = \mathbb{E}[f(\mathbf{x}, \mathbf{V})] \leq c\}.$$

- Significant enhancement in term of precision in comparison with MC,
- more efficient in term of number of calls to the simulator (MC $\sim$ 4000),
- less sensitive to the truncation argument m.

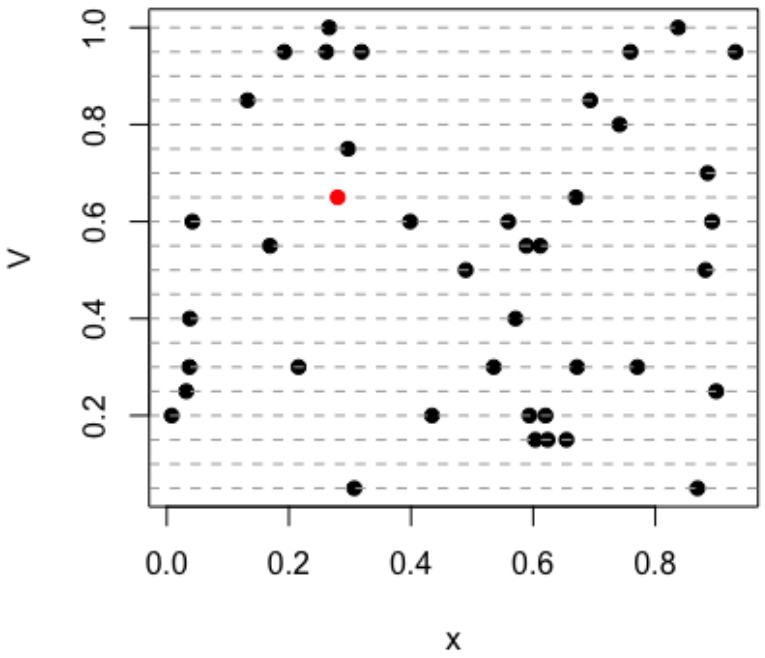
$\mathbb{L}^2$-GFQ	$\mu(\Gamma^* \triangle Q_{n_{\mathtt{last}}, \alpha^*_{n_{\mathtt{last}}}})/\mu(\Gamma^*)$	Cumulative number of calls to f
$m = 2$	**8.90 %** (3.71)	**1286** (26)
$m = 3$	**7.72 %** (3.38)	**1139** (21)
$m = 4$	**7.40 %** (3.13)	**1236** (20)
$m = 5$	**7.05 %** (5.09)	**1214** (25)
$m = 6$	**6.96 %** (3.32)	**1142** (21)

m	2	3	4	5	6
Explained variance	90.2 %	93.4 %	95.1 %	96 %	96.7%

Outline

1. Data-driven stochastic inversion via functional quantization
2. Stochastic inversion via meta-modelling in the joint space
3. IFPEN test case
4. Conclusion and outlook

The framework



Gaussian Process meta-modeling for f

We assume that $f(\mathbf{x}, \mathbf{v})$ is a realization of a Gaussian Process $Z_{(\mathbf{x},\mathbf{u})}$ defined on $\mathbb{X} \times \mathbb{R}^m$, where $\mathbf{u} = (< \mathbf{v}, \psi_1 >, \ldots, < \mathbf{v}, \psi_m >)^\top$.

Let m_Z be the mean function of $Z_{(\mathbf{x},\mathbf{u})}$ and k_Z its covariance function.

Let us denote Z^n, the GP Z conditioned on the set of n observations (simulations) $\mathbf{Z}_n = \{f(\mathbf{x}_1, \mathbf{v}_1), \ldots, f(\mathbf{x}_n, \mathbf{v}_n)\}$ of Z at $\mathcal{X}_n \times \mathcal{U}_n = \{(\mathbf{x}_1, \mathbf{u}_1), \ldots, (\mathbf{x}_n, \mathbf{u}_n)\}$ where $\mathbf{u}_i = (< \mathbf{v}_i, \psi_1 >, \ldots, < \mathbf{v}_i, \psi_m >)^\top$

$$Z^n_{(\mathbf{x},\mathbf{u})} = [Z_{(\mathbf{x},\mathbf{u})} | Z_{\mathcal{X}_n \times \mathcal{U}_n} = \mathbf{Z}_n].$$

The conditional expectation and the conditional covariance are

$$\mathbb{E}[Z^n_{(\mathbf{x},\mathbf{u})}] = m_Z(\mathbf{x}, \mathbf{u}) + k_Z(\mathbf{x}, \mathbf{u}; \mathcal{X}_n \times \mathcal{U}_n) k_Z(\mathcal{X}_n \times \mathcal{U}_n; \mathcal{X}_n \times \mathcal{U}_n)^{-1} (Z - m_Z(\mathcal{X}_n \times \mathcal{U}_n)).$$

$$\begin{aligned}\mathrm{Cov}(Z^n_{(\mathbf{x},\mathbf{u})}, Z^n_{(\mathbf{x}',\mathbf{u}')}) &= k_Z(\mathbf{x}, \mathbf{u}; \mathbf{x}', \mathbf{u}') \\ &\quad - k_Z(\mathbf{x}, \mathbf{u}; \mathcal{X}_n \times \mathcal{U}_n) k_Z(\mathcal{X}_n \times \mathcal{U}_n; \mathcal{X}_n \times \mathcal{U}_n)^{-1} k_Z(\mathcal{X}_n \times \mathcal{U}_n; \mathbf{x}', \mathbf{u}').\end{aligned}$$

Meta-modeling for g : integrated process

Recall that $\Gamma^* = \{\mathbf{x} \in \mathbb{X}\ ,\ g(\mathbf{x}) = \mathbb{E}[f(\mathbf{x}, \mathbf{V})] \leq c\}$. Therefore, to model the function g, we introduce the integrated process

$$Y_{\mathbf{x}}^n = \mathbb{E}_{\mathbf{U}}[Z_{(\mathbf{x},\mathbf{U})}^n] = \int_{\mathbb{R}^m} Z_{(\mathbf{x},\mathbf{u})}^n d\rho(\mathbf{u}),$$

where $d\rho(u)$ is the probability distribution of $\mathbf{U} = (U_1, \ldots, U_m)^T$.

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The process $Y_{\mathbf{x}}^n$ is a Gaussian Process fully characterized by its mean and covariance functions which are given by

$$\mathbb{E}[Y_{\mathbf{x}}^n] = \int_{\mathbb{R}^m} m_Z(\mathbf{x}, \mathbf{u}) + \\ k_Z((\mathbf{x}, \mathbf{u}); \mathcal{X}_n \times \mathcal{U}_n) k_Z(\mathcal{X}_n \times \mathcal{U}_n; \mathcal{X}_n \times \mathcal{U}_n)^{-1} (\mathbf{Z} - m_Z(\mathcal{X}_n \times \mathcal{U}_n)) d\rho(\mathbf{u}),$$

and

$$\mathrm{Cov}(Y_{\mathbf{x}}^n, Y_{\mathbf{x}'}^n) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} k_Z((\mathbf{x}, \mathbf{u}); (\mathbf{x}', \mathbf{u}')) \\ - k_Z((\mathbf{x}, \mathbf{u}); \mathcal{X}_n \times \mathcal{U}_n) k_Z(\mathcal{X}_n \times \mathcal{U}_n; \mathcal{X}_n \times \mathcal{U}_n)^{-1} k_Z(\mathcal{X}_n \times \mathcal{U}_n; (\mathbf{x}', \mathbf{u}')) d\rho(\mathbf{u}) d\rho(\mathbf{u}').$$

Infill strategy in the joint space

Estimate $\Gamma^* = \{\mathbf{x} \in \mathbb{X} \;\; s.t. \;\; g(\mathbf{x}) = \mathbb{E}[f(\mathbf{x}, \mathbf{V})] \leq c\}$.

Step I: choose the `control` variable site

(Stepwise Uncertainty Reduction) $\mathbf{x}_{(n+1)} = \arg\min_{\mathbf{x} \in \mathbb{X}} \; \mathbb{E}_{n,x}[\mathcal{H}_{n+1}^{\text{uncert}}(\mathbf{x})]$.

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Step II: choose the `uncertain` variable site

Choose the point in the uncertain space that reduces the most the uncertainty of the estimated $\mathbb{E}_{\mathbf{U}}[g(\mathbf{x}, \mathbf{U})]$ at the point $\mathbf{x}_{(n+1)}$.

$$\mathbf{u}_{n+1} = \text{argmin}_{\tilde{\mathbf{u}}\in\mathbb{R}^m} \mathbb{VAR}(Y_{\mathbf{x}_{n+1}}^{n+1}),$$

with

$$\mathbb{VAR}(Y_{\mathbf{x}_{n+1}}^{n+1}) = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} k_Z((\mathbf{x}_{n+1}, \mathbf{u}); (\mathbf{x}_{n+1}, \mathbf{u}')) - k_Z((\mathbf{x}_{n+1}, \mathbf{u}); \mathcal{X}_{n+1} \times \mathcal{U}_{n+1})$$
$$k_Z(\mathcal{X}_{n+1} \times \mathcal{U}_{n+1}; \mathcal{X}_{n+1} \times \mathcal{U}_{n+1})^{-1} k_Z(\mathcal{X}_{n+1} \times \mathcal{U}_{n+1}; (\mathbf{x}_{n+1}, \mathbf{u}'))d\rho(\mathbf{u})d\rho(\mathbf{u}'),$$

where $\mathcal{X}_{n+1} \times \mathcal{U}_{n+1} = \mathcal{X}_n \times \mathcal{U}_n \cup \{(\mathbf{x}_{n+1}, \tilde{\mathbf{u}})\}$.

Algorithm Stochastic inversion via joint space modelling

Create an initial DoE of n points in the joint space $(\mathbb{X}, \Xi)$
Calculate simulator responses at the design points
while $n \leq$ budget **do**
    Fit a GP model $Z^n_{(\mathbf{x},\mathbf{u})}$ of the simulator f
    Induce a GP model $Y^n_{\mathbf{x}}$ for the unobservable simulator g
    $\mathbf{x}_{(n+1)} \leftarrow$ *SUR strategy*
    $\mathbf{u}_{(n+1)} \leftarrow \arg\min_{\tilde{\mathbf{u}} \in \mathcal{G}} \mathbb{VAR}(Y^{n+1}_{\mathbf{x}_{(n+1)}})$
    Response at $(\mathbf{x}_{(n+1)}, \mathbf{v}_{(n+1)})$; $\mathbf{v}_{(n+1)} \in \Xi$ is the associated curve to $\mathbf{u}_{(n+1)}$
    Update DoE
    Set $n = n + 1$
end while
end

Application of approach II

$$f : (\mathbf{x}, v) \mapsto \max_t v_t.|0.1\cos(x_1 \max_t v_t)\sin(x_2).(x_1+x_2 \min_t v_t)^2|. \int_0^T (30+v_t)^{\frac{x_1 . x_2}{20}} \mathrm{dt},$$

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- Constant mean function m,
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- $n = 20$ calls to f,
- threshold $c = 1.2$,
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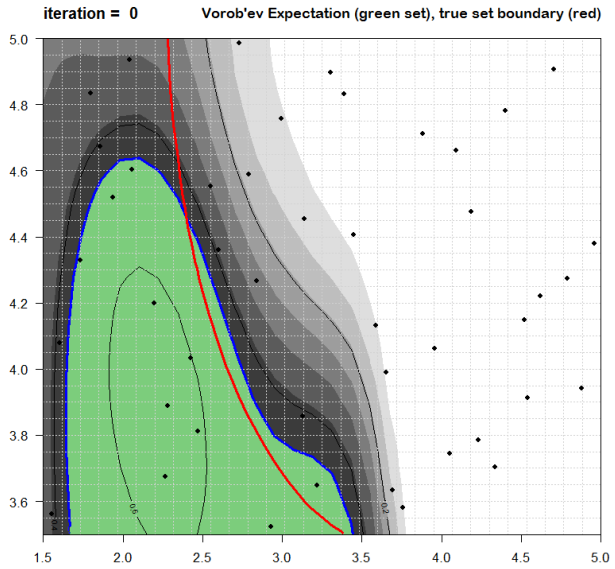

Analytical function with Brownian motion

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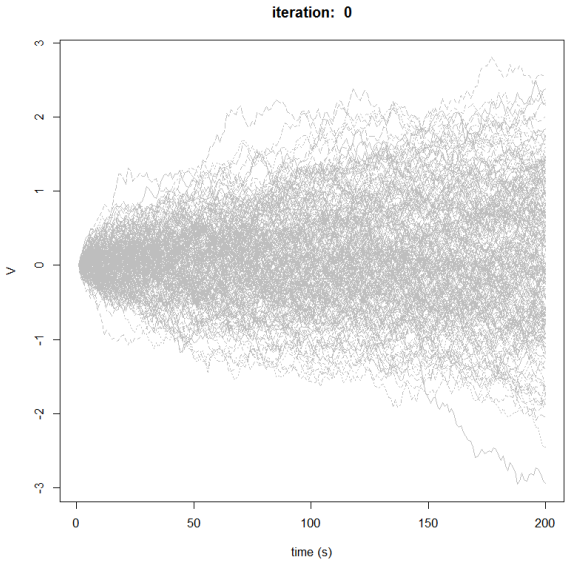
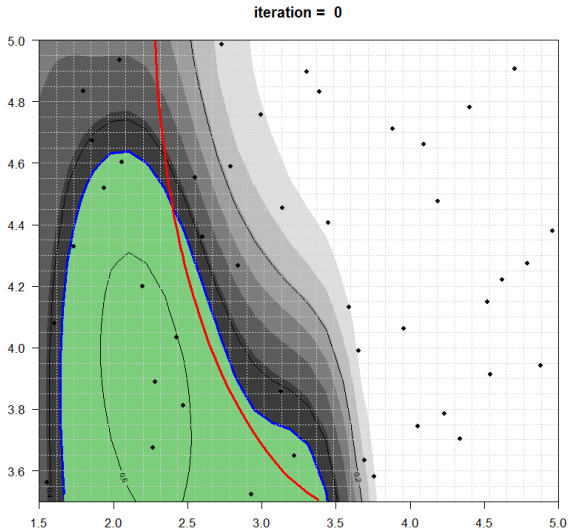
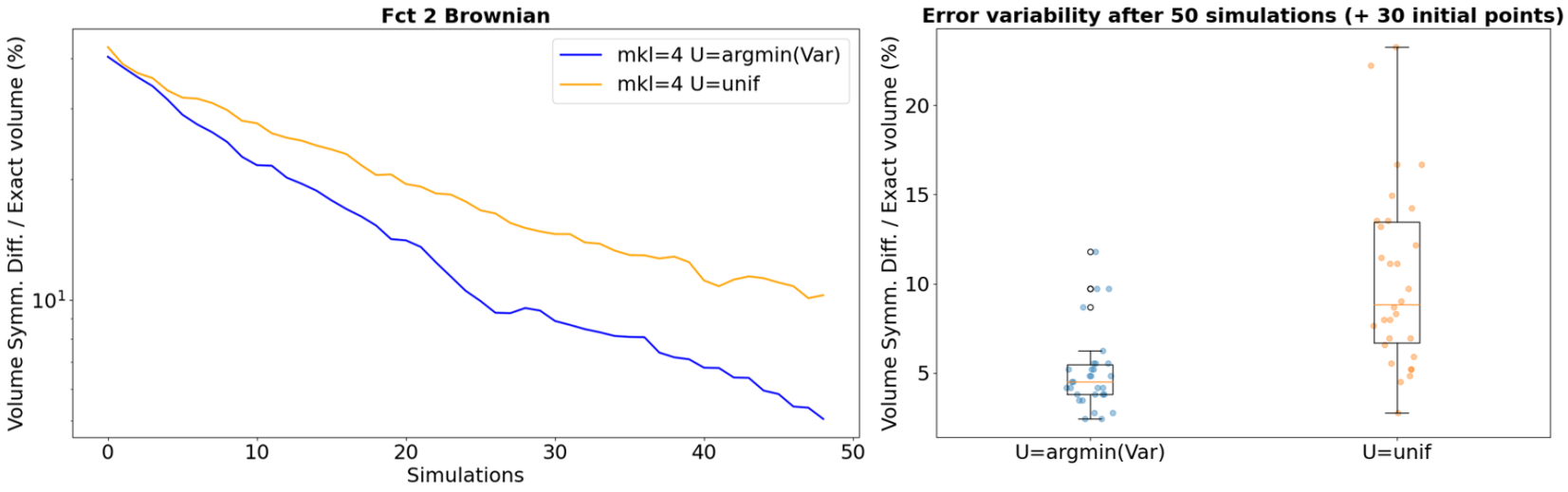
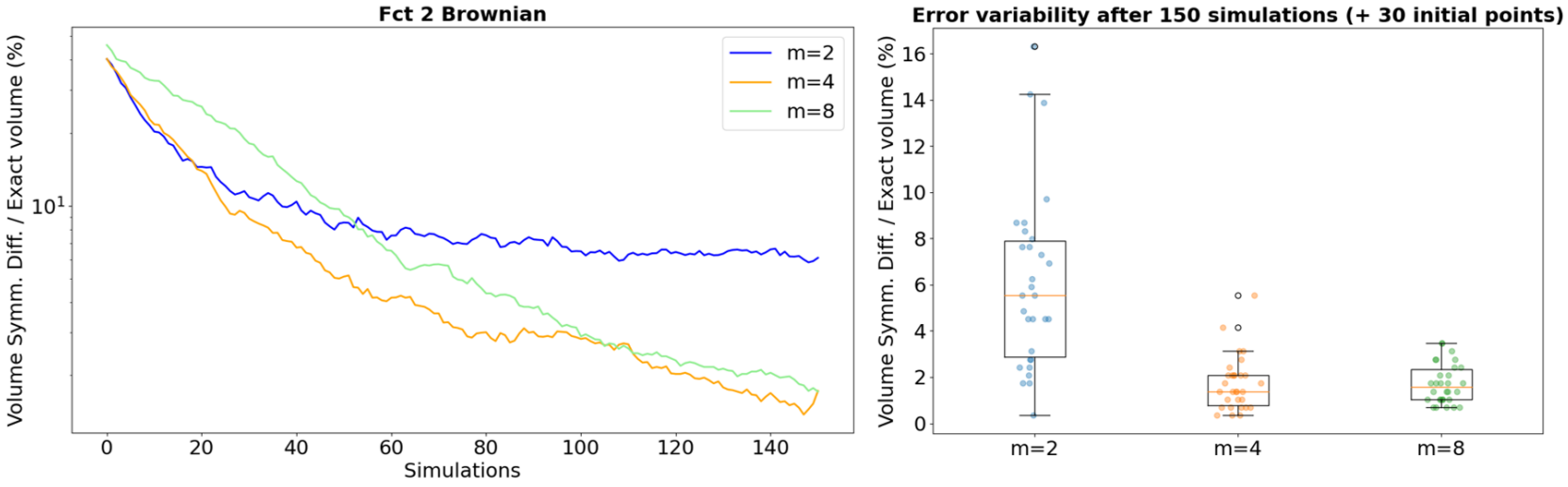




Influence of the choice of $\hat{v}_l$: $\mathcal{U}(\Xi)$ versus $\mathbb{VAR}(Y^{n+1}_{\mathbf{x}_{(n+1)}})$



Influence of m



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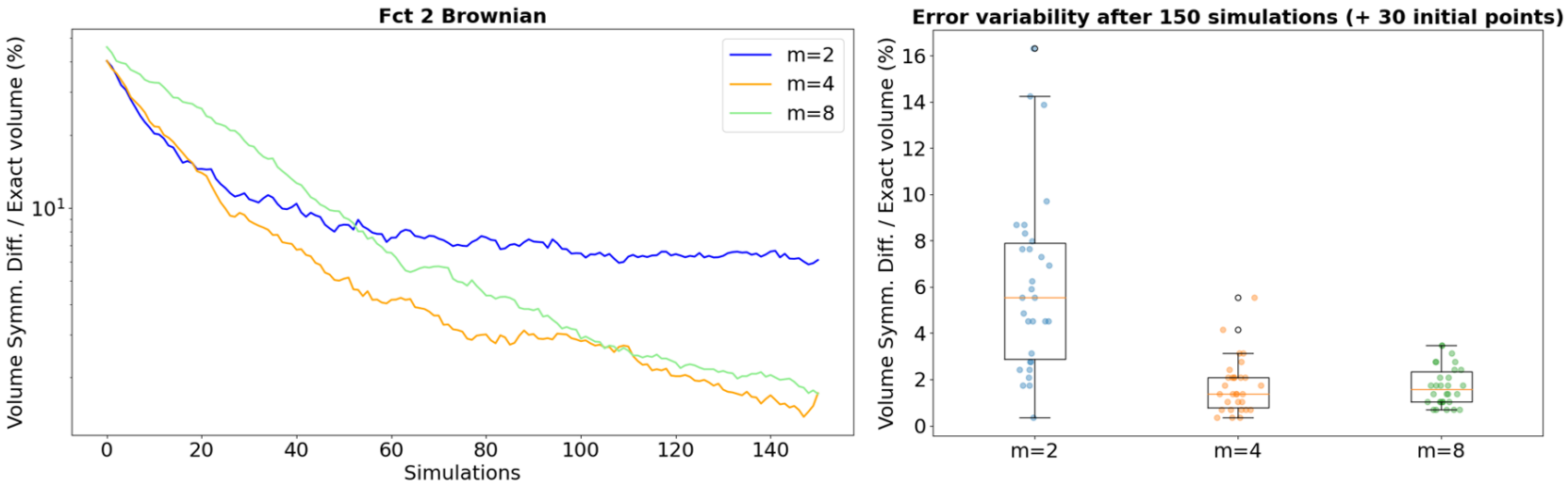
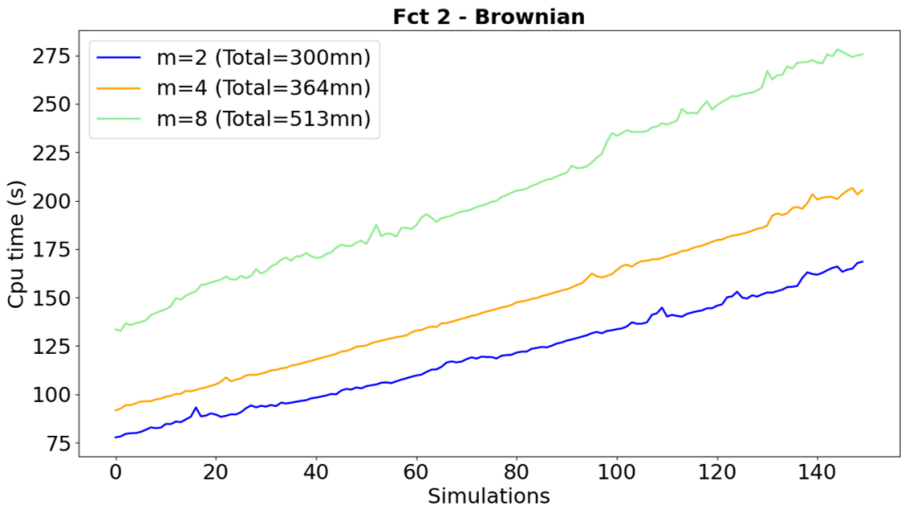
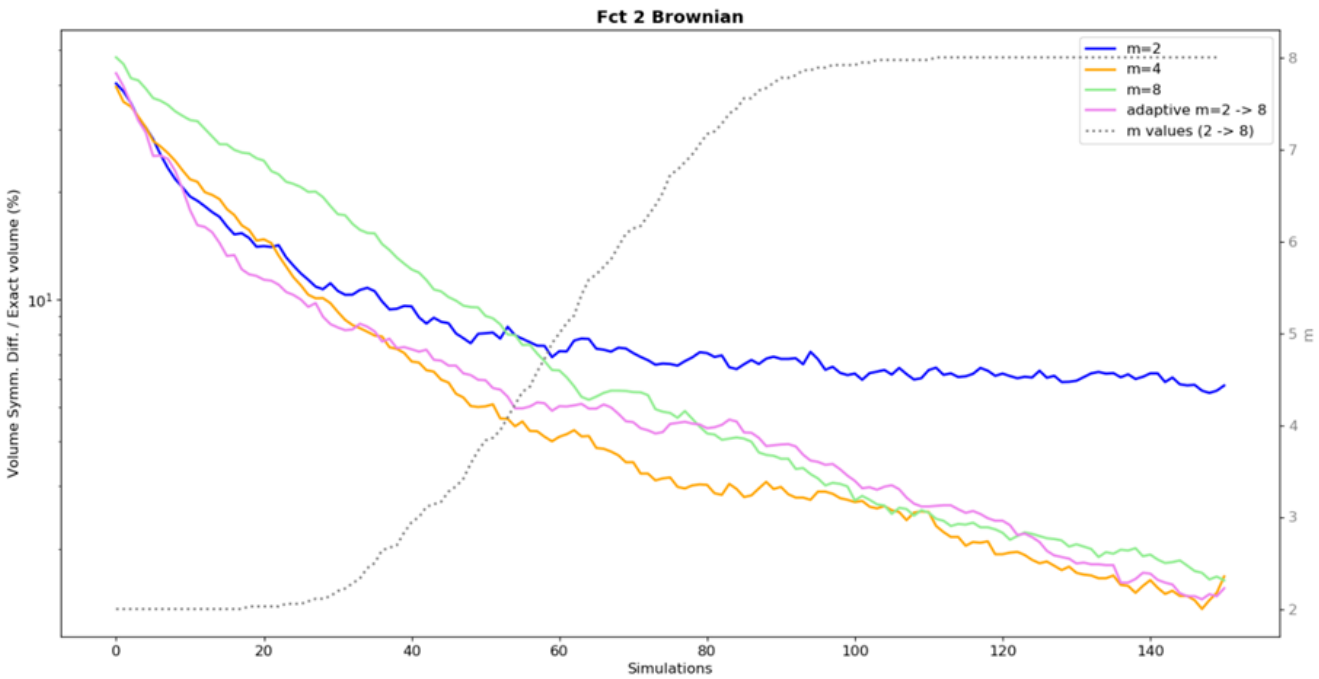


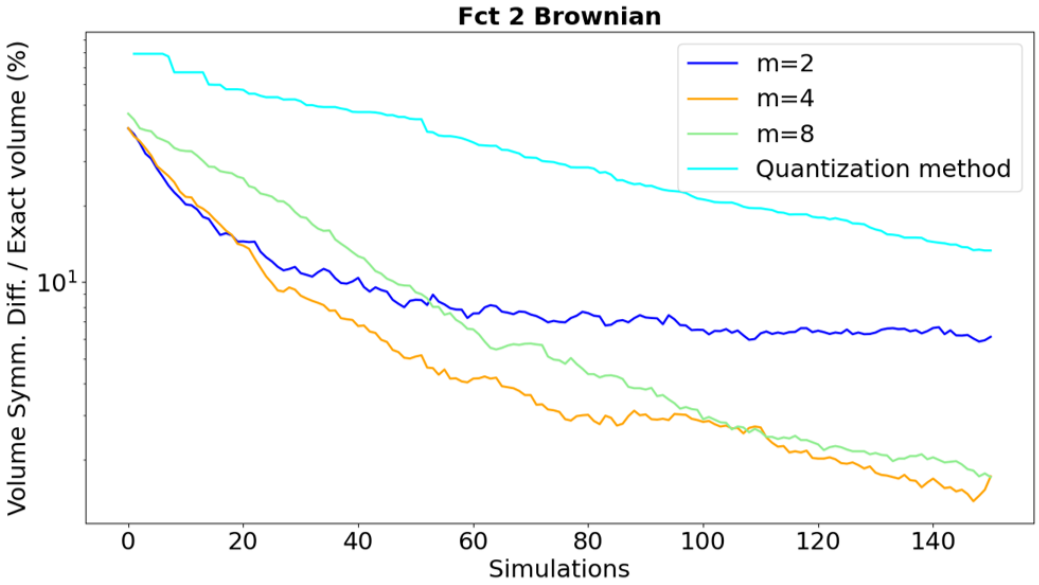
Comparison of the computing time



Sequential incrementation of m



Comparison with approach I



Summary of the approach II

- Extend the concept of Gaussian Process modelling to the case where the inputs contain a functional variable,
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	# calls to the simulator	criterion calculation costs	estimation error
I	-	+	+
II	+	-	-

Outline

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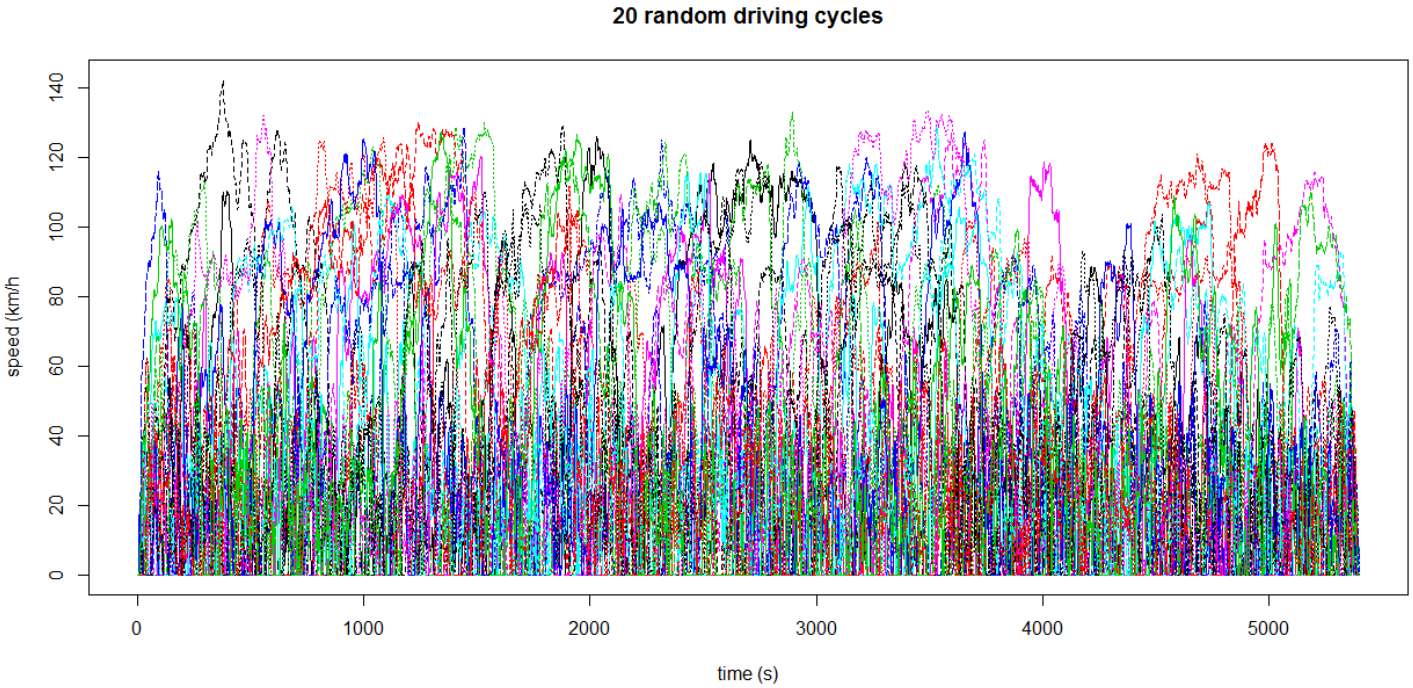
4. Conclusion and outlook

IFPEN test case: control strategy for an automotive NO_x depollution system

- $\mathrm{NH}_3^{\mathrm{out}} \leq 30$ *ppm*
- 2 control parameters in $\mathbb{X} = [0, 0.6] \times [0, 0.6]$,
- **V** represented by 100 driving cycles,

- Approach 1 : initial DOE 8 points in $\mathbb{X} \subset \mathbb{R}^2$ (a mean of 23 calls to f at each point)
- Approach 2 : initial DOE 5*22 in $\mathbb{X} \subset \mathbb{R}^{2+20}$ (20 eigen functions for 80% explained variance)



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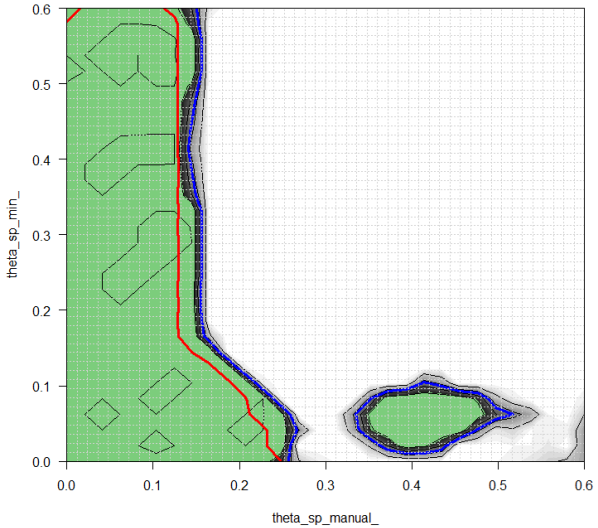
Approach I versus Approach II



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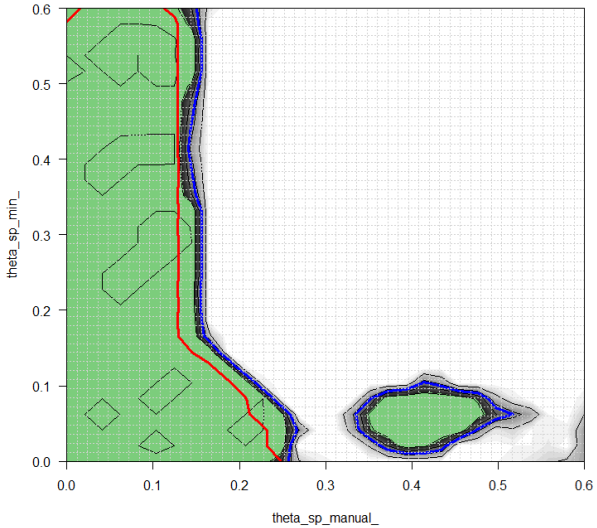
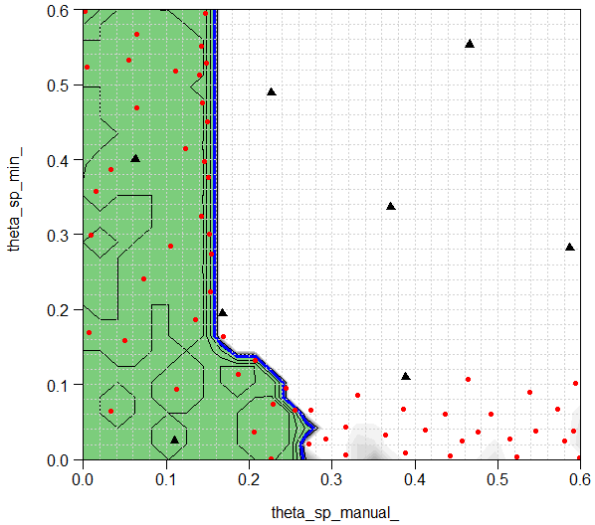


$n =$ **1575** calls to f with approach I
Reference excursion set



Conclusion

- Dealing with functional uncertainties:
  - two greedy expectation estimation via **functional quantization** coupled with a SUR strategy in control variable space,
  - sequential strategy to wisely choose the points in the **joint space**.
- IFPEN test case:
  - application of the approach I to the case with control variables $\mathbf{x} \in \mathbb{X} \subset \mathbb{R}^9$ (not presented),
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Thanks for your attention !

Application 1

$$f : (\mathbf{x}, v) \mapsto \max_t v_t.|0.1\cos(x_1 \max_t v_t)\sin(x_2).(x_1+x_2 \min_t v_t)^2|. \int_0^T (30+v_t)^{\frac{x_1 \cdot x_2}{20}} \mathrm{dt},$$

$$\Gamma^* := \{\mathbf{x} \in [1.5, 5] \times [3.5, 5] \ : \ g(\mathbf{x}) = \mathbb{E}[f(\mathbf{x}, \mathbf{V})] \leq c\}.$$

- Constant mean function m,
- Matérn 5/2 covariance kernel,
- $n = 9$ observations,
- threshold $c = 1.2$,
- $m = 2$,
- $\mathtt{card}(\Xi) = 200$.

	$\mu(\Gamma^* \triangle Q_{n_{\mathtt{last}}, \alpha^*_{n_{\mathtt{last}}}})/\mu(\Gamma^*)$		
(l_0, ϵ)	Crude MC	Fpca	$\mathbb{L}^2$-GFQ
(4,1.e-2)	9.53 % (4.12)	9.89 % (4.14)	**7.50 %**
(2,5.e-3)	9.84 % (3.24)	10.86 % (2.82)	**7.93 %**
(3,5.e-3)	9.54 % (3.81)	7.29 % (1.07)	**6.87 %**
(4,5.e-3)	8.98 % (2.62)	7.01 % (1.21)	**6.79 %**

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	Cumulative number of calls to the function f		
(l_0, ϵ)	Crude MC	Fpca	$\mathbb{L}^2$-GFQ
(4,1.e-2)	2849 (27)	2805 (24)	**1225 (42)**
(2,5.e-3)	2393 (26)	2670 (23)	**978 (22)**
(3,5.e-3)	3537 (23)	3661 (20)	**1096 (21)**
(4,5.e-3)	4400 (23)	4278 (20)	**1489 (26)**

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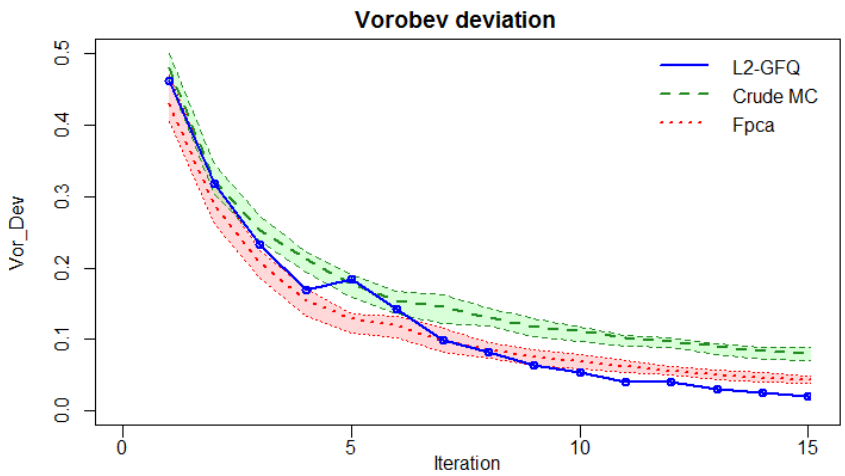

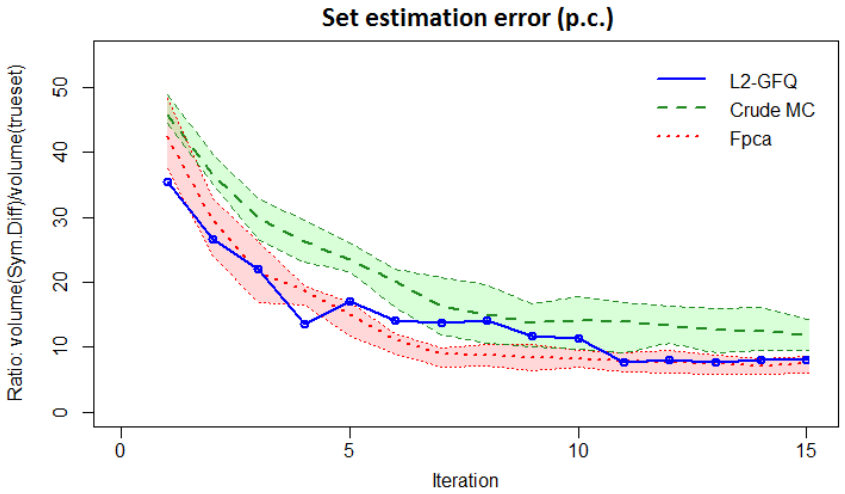

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- $m = \{2, 3, 4, 5, 6\}$,
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	Fpca	$\mathbb{L}^2$-GFQ	Fpca	$\mathbb{L}^2$-GFQ
$m = 2$	11.43 % (3.70)	**8.90 %** (3.71)	3855 (18)	**1286** (26)
$m = 3$	10.70 % (3.38)	**7.72 %** (3.38)	4418 (24)	**1139** (21)
$m = 4$	9.24 % (3.18)	**7.40 %** (3.13)	4438 (21)	**1236** (20)
$m = 5$	8.94 % (2.66)	**7.05 %** (5.09)	4542 (21)	**1214** (25)
$m = 6$	8.27 % (1.67)	**6.96 %** (3.32)	4955 (19)	**1142** (21)