**More on second-order properties of the Moreau regularization-approximation of a convex function**

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"*Optimization is revolutionized by its interactions with Machine Learning and Data Analysis*", S. Wright (2016)

*"Proximal methods are the natural algorithms for solving regularized learning problems"*, F. Iutzeler and J. Malick (2020)

**Abstract**

Go to an Optimization congress, more precisely in sessions devoted to large scale problems such as those found in mathematical imagery, automatic learning or statistics (Machine Learning), and you will hear about Moreau's regularization, proximal (algorithmic) methods, etc. To understand them, you need a minimum of basic theoretical (*i.e.*, mathematical) knowledge. It is to this need that I had to respond by teaching in a Master 2R of Operation Research (course entitled "Contemporary Themes in (continuous) Optimization" during the last six years. The audience of students (all at the graduate level) mainly came from four engineering schools in Toulouse as well as the Paul Sabatier University.

For these purposes, we have chosen to start from a beginner level in the targeted field, hence the title of the text [8], avoiding the temptation to take for granted things that seem simple to us (so much we are "in" by our own practices and work in Optimization).

The text referenced in [8] is divided into 6 parts of very unequal lengths. After the introductory paragraphs of Analysis (§1) and Modern Convex Analysis (§2), we present in §3 the properties of Moreau's regularization (to first order) ; everything is distilled in the form of "facts" (= statements) without proofs. It is, for the student-reader, **the basis for understanding the so-called proximal algorithmic methods.**

Section 4 is dedicated to the second-order properties of Moreau's regularization ; in addition to summarizing the results available in terms of classical differential calculus, we improve some results from the literature. **This is the subject of our talk here.**

In §5, a look is taken at general models of so-called proximal-type algorithms in convex optimization ; indicated references will allow the student-reader to go further and to prepare for the use, even the improvement, of these methods in fields of application. §6 only consists of an inflection towards the world of non-convex optimization.

*"Theory is the first term in the Taylor series of practice"* (Th. M. Cover, 1990 Shannon Lecture).

**References**


1. J.-J. Moreau, *Proximité et dualité dans un espace hilbertien.* Bull. Soc. Math. France 93 (1965), $273 - 279$.

This is the founding paper of the domain. Nearly 60 years after its publication, it has kept all its modernity.

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8. J.-B. Hiriart-Urruty, *La régularisation-approximation de Moreau d'une fonction convexe, l'opérateur proximal, les méthodes de gradient proximal, etc. : une présentation synthétique pour ceux qui n'en ont jamais entendu parler.* Document pédagogique (automne 2021).

9. F. Iutzeler and J. Malick, *Nonsmoothness in Machine Learning : specific structure, proximal identification, and applications.* Journal Set-Valued and Variational Analysis (2020) 28 $(4), 661 - 678$.