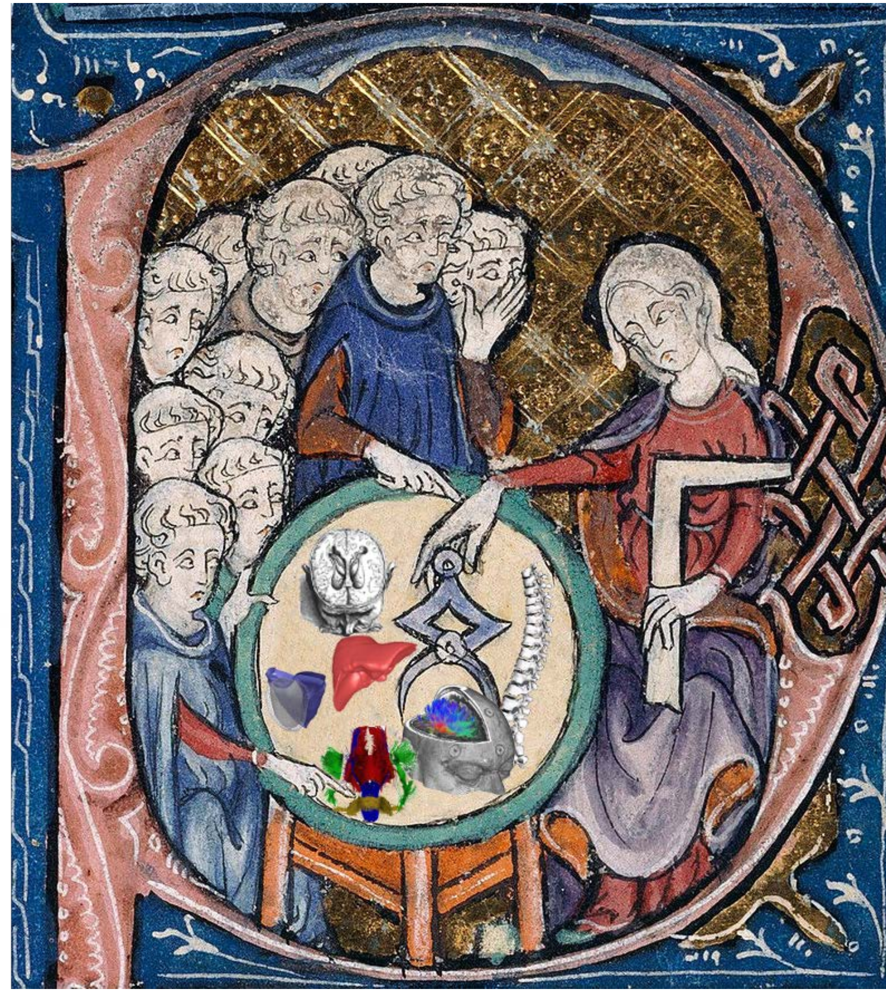
Xavier Pennec

Univ. Côte d'Azur and Inria, France

Geometric Statistics

Mathematical foundations and applications in computational anatomy



Freely adapted from "Women teaching geometry", in Adelard of Bath translation of Euclid's elements, 1310.

2/ Metric and Affine Geometric Settings for Lie Groups

Geometric Statistics workshop 09/2019

Epione e-patient / e-medicine

UNIVERSITÉ CÔTE D'AZUR

Inria

Geometric Statistics: Mathematical foundations and applications in computational anatomy

Intrinsic Statistics on Riemannian Manifolds

Metric and Affine Geometric Settings for Lie Groups

- Riemannian frameworks on Lie groups
- Lie groups as affine connection spaces
- The SVF framework for diffeomorphisms

Advances Statistics: CLT & PCA

Natural Riemannian Metrics on Transformations

Transformation are Lie groups: Smooth manifold G compatible with group structure

- Composition g o h and inversion g^{-1} are smooth
- Left and Right translation $L_g(f) = g \circ f \quad R_g(f) = f \circ g$
- Conjugation $Conj_g(f) = g \circ f \circ g^{-1}$
- Symmetry: $S_g(f) = g \circ f^{-1} \circ g$

Natural Riemannian metric choices

- Chose a metric at Id: $<x,y>_{Id}$
- Propagate at each point g using left (or right) translation
  $<x,y>_g = < DL_{g^{(-1)}} .x , DL_{g^{(-1)}} .y >_{Id}$

Implementation

- Practical computations using left (or right) translations

$$\mathrm{Exp}_f(x) = f \circ \mathrm{Exp}_{Id}(DL_{f^{(-1)}}.x) \qquad \overrightarrow{fg} = \mathrm{Log}_f(g) = DL_f.\mathrm{Log}_{Id}(f^{(-1)} \circ g)$$

General Non-Compact and Non-Commutative case

No Bi-invariant Mean for 2D Rigid Body Transformations

- Metric at Identity: $dist(Id, (\theta; t_1; t_2))^2 = \theta^2 + t_1^2 + t_2^2$
- $T_1 = \left(\frac{\pi}{4}; -\frac{\sqrt{2}}{2}; \frac{\sqrt{2}}{2}\right) \quad T_2 = \left(0; \sqrt{2}; 0\right) \quad T_3 = \left(-\frac{\pi}{4}; -\frac{\sqrt{2}}{2}; -\frac{\sqrt{2}}{2}\right)$
- Left-invariant Fréchet mean: $(0; 0; 0)$
- Right-invariant Fréchet mean: $\left(0; \frac{\sqrt{2}}{3}; 0\right) \simeq (0; 0.4714; 0)$

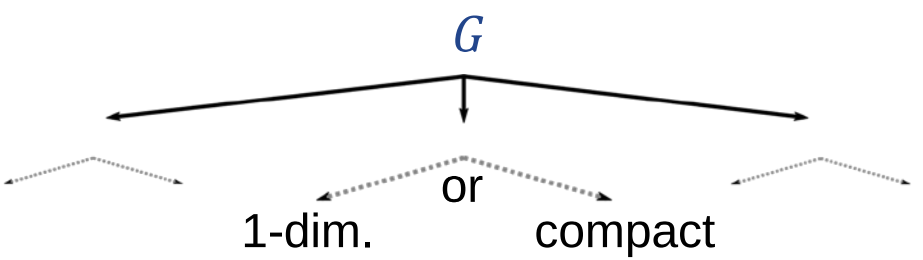
Questions for this talk:

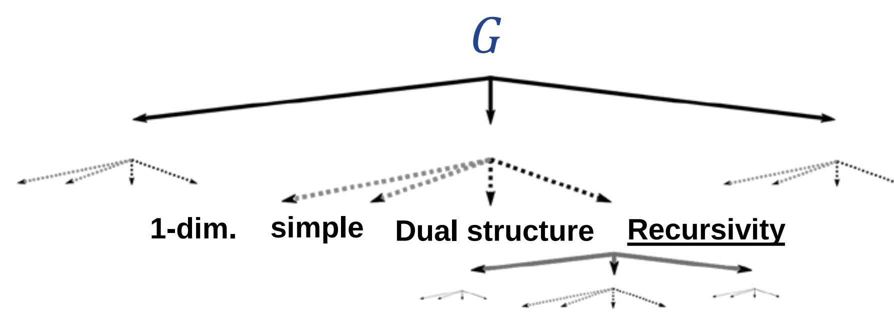
- **Can we design a mean compatible with the group operations?**
- **Is there a more convenient structure for statistics on Lie groups?**

Existence of bi-invariant (pseudo) metrics

[Cartan 50's]:
Bi-invariant metric on G

[Medina, Revoy 80's]:
Bi-invariant pseudo-metric on G





[Miolane, XP, Computing Bi-Invariant Pseudo-Metrics on Lie Groups for Consistent Statistics. Entropy, 17(4):1850-1881, April 2015.]

- Algorithm: decompose the Lie algebra and find a bi-inv. pseudo-metric
- Test on rigid transformations SE(n): bi-inv. ps-metric for n=1 or 3 only

Geometric Statistics: Mathematical foundations and applications in computational anatomy

Intrinsic Statistics on Riemannian Manifolds

Metric and Affine Geometric Settings for Lie Groups

- Riemannian frameworks on Lie groups
- Lie groups as affine connection spaces
- The SVF framework for diffeomorphisms

Advances Statistics: CLT & PCA

Basics of Lie groups

Flow of a left invariant vector field $\tilde{X} = DL.x$ from identity

- $\gamma_x(t)$ exists for all time
- One parameter subgroup: $\gamma_x(s+t) = \gamma_x(s).\,\gamma_x(t)$

Lie group exponential

- Definition: $x \in \mathfrak{g} \rightarrow Exp(x) = \gamma_x(1) \epsilon\, G$
- Diffeomorphism from a neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of e in G (not true in general for inf. dim)

3 curves parameterized by the same tangent vector

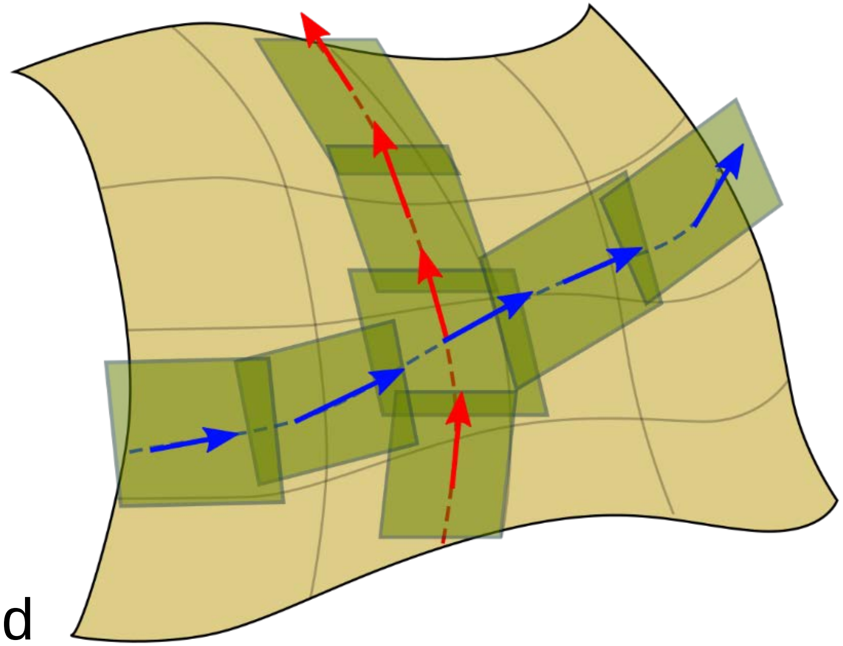
- Left / Right-invariant geodesics, one-parameter subgroups

Question: Can one-parameter subgroups be geodesics?

Affine connection spaces:
Drop the metric, use connection to define geodesics

Affine Connection (infinitesimal parallel transport)

- Acceleration = derivative of the tangent vector along a curve
- Projection of a tangent space on a neighboring tangent space



Geodesics = straight lines

- Null acceleration: $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$
- 2nd order differential equation: Normal coordinate system
- Local exp and log maps, well defined in a convex neighborhood

[Lorenzi, Pennec. Geodesics, Parallel Transport & One-parameter Subgroups for Diffeomorphic Image Registration. Int. J. of Computer Vision, 105(2):111-127, 2013.]

Canonical Affine Connections on Lie Groups

A unique Cartan-Schouten connection

- Bi-invariant and symmetric (no torsion)
- Geodesics through Id are one-parameter subgroups (group exponential)
  - Matrices : M(t) = A exp(t.V)
  - Diffeos : **translations of Stationary Velocity Fields (SVFs)**

Levi-Civita connection of a bi-invariant metric (if it exists)

- Continues to exists in the absence of such a metric (e.g. for rigid or affine transformations)

Symmetric space with central symmetry $S_\psi(\phi) = \psi\phi^{-1}\psi$

- Matrix geodesic symmetry: $S_A(M(t)) = A\exp(-tV)A^{-1}A = M(-t)$

[Lorenzi, Pennec. Geodesics, Parallel Transport & One-parameter Subgroups for Diffeomorphic Image Registration. Int. J. of Computer Vision, 105(2):111-127, 2013.]

Statistics on an affine connection space

~~Fréchet mean~~: exponential barycenters

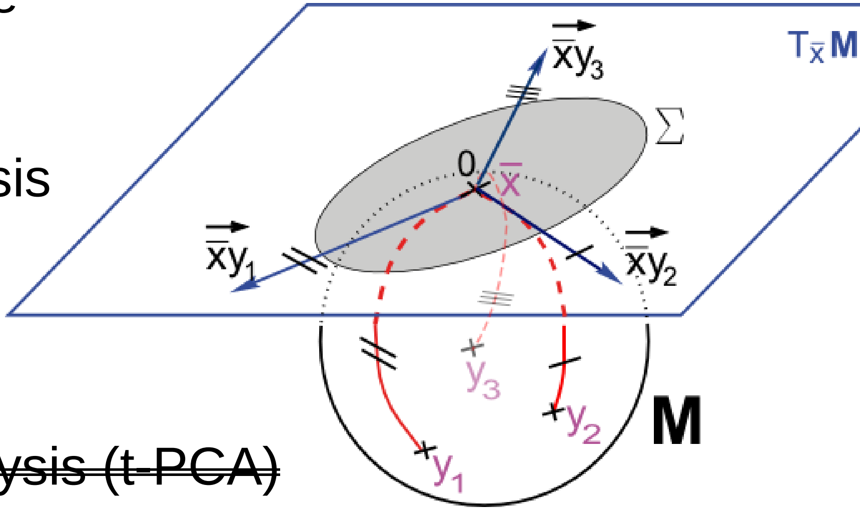
- $\sum_i Log_x(y_i) = 0$ [Emery, Mokobodzki 91, Corcuera, Kendall 99]
- Existence **local uniqueness** if local convexity [Arnaudon & Li, 2005]

Covariance matrix & higher order moments

- Defined as tensors in tangent space

$$\Sigma = \int Log_x(y) \otimes Log_x(y)\, \mu(dy)$$

- Matrix expression changes with basis



Other statistical tools

- Mahalanobis distance, chi^2 test
- ~~Tangent Principal Component Analysis (t-PCA)~~
- Independent Component Analysis (ICA)?

[XP & Arsigny, 2012, XP & Lorenzi, Beyond Riemannian Geometry, 2019]

Statistics on an affine connection space

For Cartan-Schouten connections [Pennec & Arsigny, 2012]

- Locus of points *x* such that $\sum Log(x^{-1}.y_i) = 0$
- Algorithm: fixed point iteration (**local convergence**)

$$x_{t+1} = x_t \circ Exp\left(\frac{1}{n}\sum Log(x_t^{-1}.y_i)\right)$$

- **Mean stable by left / right composition and inversion**

Matrix groups with no bi-invariant metric

- Heisenberg group: bi-invariant mean is unique (conj. ok for solvable)
- Rigid-body transformations: uniqueness if unique mean rotation
- SU(n) and GL(n): log does not always exist (need 2 exp to cover)

[XP and V. Arsigny. Exponential Barycenters of the Canonical Cartan Connection and Invariant Means on Lie Groups. In Matrix Information Geometry. 2012]

Example mean of 2D rigid-body transformation

$$T_1 = \left(\frac{\pi}{4};\ -\frac{\sqrt{2}}{2};\frac{\sqrt{2}}{2}\right) \quad T_2 = (0;\ \sqrt{2};0\ ) \quad T_3 = \left(-\frac{\pi}{4};\ -\frac{\sqrt{2}}{2};-\frac{\sqrt{2}}{2}\right)$$

- Metric at Identity: $dist(Id,(\theta;t_1;t_2))^2 = \theta^2 + t_1^2 + t_2^2$
- Left-invariant Fréchet mean: $(0;0;0\ )$
- Log-Euclidean mean: $\left(0;\frac{\sqrt{2}-\pi/4}{3};0\right) \simeq (0;0.2096;0)$
- Bi-invariant mean: $\left(0;\frac{\sqrt{2}-\pi/4}{1+\pi/4(\sqrt{2}+1)};0\right) \simeq (0;0.2171;0)$
- Right-invariant Fréchet mean: $\left(0;\frac{\sqrt{2}}{3};0\right) \simeq (0;0.4714;0)$

Cartan Connections vs Riemannian

What is similar

- Standard differentiable geometric structure [curved space without torsion]
- Normal coordinate system with Exp_x et Log_x [finite dimension]

Limitations of the affine framework

- No metric (but no choice of metric to justify)
- The exponential does always not cover the full group
  - Pathological examples close to identity in finite dimension
  - In practice, similar limitations for the discrete Riemannian framework

What we gain with Cartan-Schouten connection

- A globally invariant structure invariant by composition & inversion
- Simple geodesics, efficient computations (stationarity, group exponential)
- Consistency with any bi-invariant (pseudo)-metric
- The simplest linearization of transformations for statistics on Lie groups?

Geometric Statistics: Mathematical foundations and applications in computational anatomy

Intrinsic Statistics on Riemannian Manifolds

Metric and Affine Geometric Settings for Lie Groups

- Riemannian frameworks on Lie groups
- Lie groups as affine connection spaces
- The SVF framework for diffeomorphisms

Advances Statistics: CLT & PCA

Riemannian Metrics on diffeomorphisms

Space of deformations

- Transformation y=ϕ(x)
- Curves in transformation spaces: ϕ(x,t)
- Tangent vector = speed vector field

$$v_t(x) = \frac{d\phi(x,t)}{dt}$$

Right invariant metric

$$\|v_t\|_{\phi_t} = \|v_t \circ \phi_t^{-1}\|_{Id}$$

- Eulerian scheme
- Sobolev Norm H_k or H_∞ (RKHS) in LDDMM → diffeomorphisms [Miller, Trouve, Younes, Holm, Dupuis, Beg… 1998 – 2009]

Geodesics determined by optimization of a time-varying vector field

- Distance

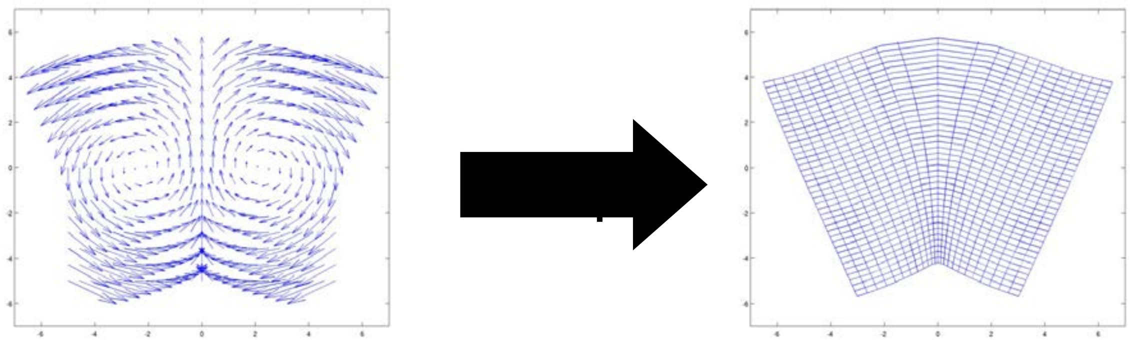
$$d^2(\phi_0, \phi_1) = \arg\min_{v_t}\left(\int_0^1 \|v_t\|_{\phi_t}^2 dt\right)$$

- Geodesics characterized by initial velocity / momentum
- Optimization for images is quite tricky (and lenghty)

The SVF framework for Diffeomorphisms

Idea: [Arsigny MICCAI 2006, Bossa MICCAI 2007, Ashburner Neuroimage 2007]

- Exponential of a smooth vector field is a diffeomorphism
- Parameterize deformation by ~~time-varying~~ Stationary Velocity Fields



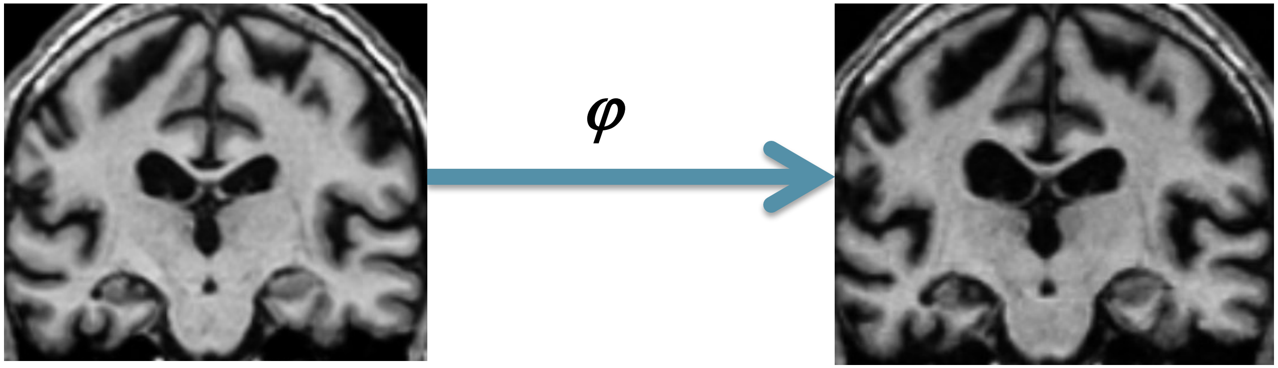
Stationary velocity field   Diffeomorphism

Direct generalization of numerical matrix algorithms

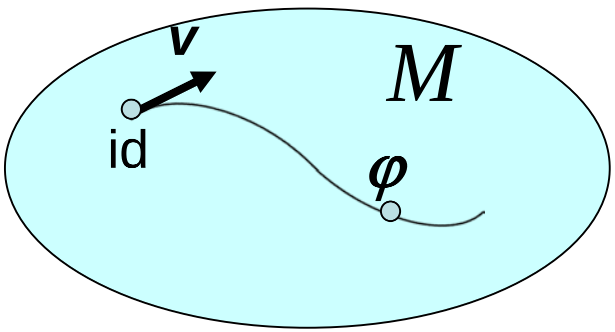
- Computing the deformation: **Scaling and squaring** **[Arsigny MICCAI 2006]**
  recursive use of exp(***v***)=exp(***v***/2) o exp(***v***/2)
- Computing the Jacobian: Dexp(v) = Dexp(v/2) o exp(v/2) . Dexp(v/2)
- Updating the deformation parameters: **BCH formula** **[Bossa MICCAI 2007]**

  $\exp(\boldsymbol{v}) \circ \exp(\varepsilon \boldsymbol{u}) = \exp(\boldsymbol{v} + \varepsilon \boldsymbol{u} + [\boldsymbol{v},\varepsilon \boldsymbol{u}]/2 + [\boldsymbol{v},[\boldsymbol{v},\varepsilon \boldsymbol{u}]]/12 + \ldots)$
  - Lie bracket $[\boldsymbol{v},\boldsymbol{u}](p) = Jac(\boldsymbol{v})(p).\boldsymbol{u}(p) - Jac(\boldsymbol{u})(p).\boldsymbol{v}(p)$

Parallel transport of deformation trajectories



SVF setting
- ***v stationary*** *velocity field*
- *Lie group Exp(v) non-metric geodesic wrt Cartan connections*



LDDMM setting
- ***v time-varying*** *velocity field*
- *Riemannian $exp_{id}(v)$ metric geodesic wrt Levi-Civita connection*
- *Defined by* ***intial momentum***

Transporting trajectories:

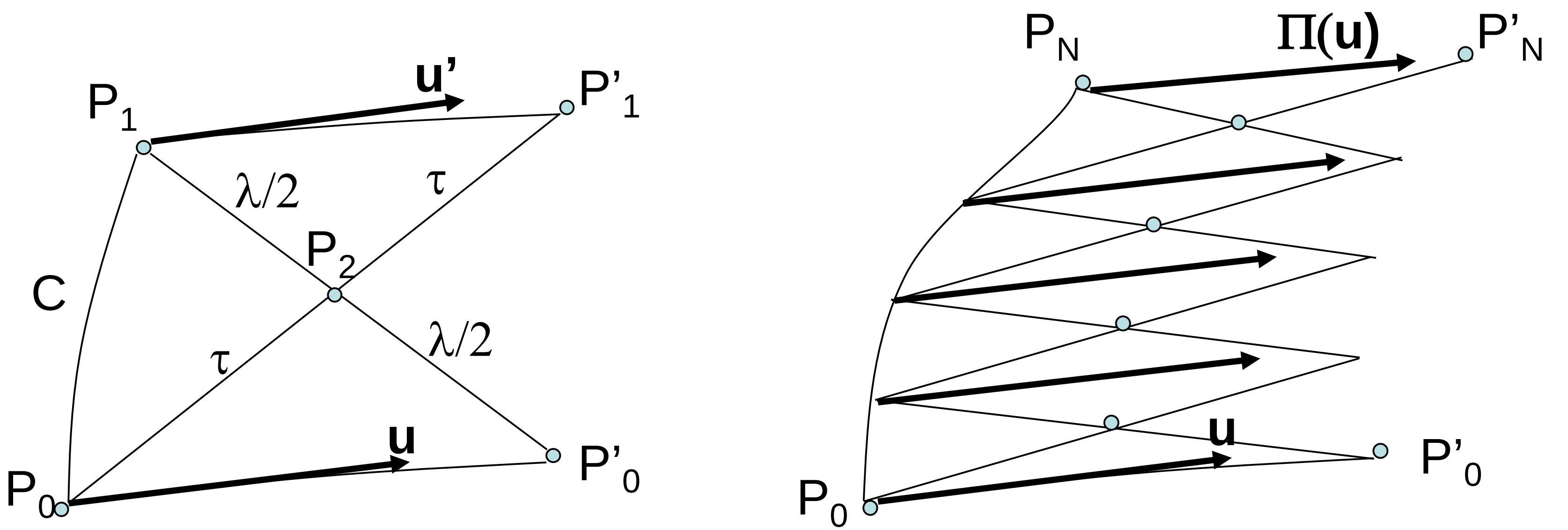
Parallel transport of initial tangent vectors

LDDMM: parallel transport along geodesics using Jacobi fields [Younes et al. 2008]

Parallel transport along arbitrary curves

A numerical scheme to integrate for symmetric connections: Schild's Ladder [Elhers et al, 1972]

- Build geodesic parallelogrammoid
- Iterate along the curve



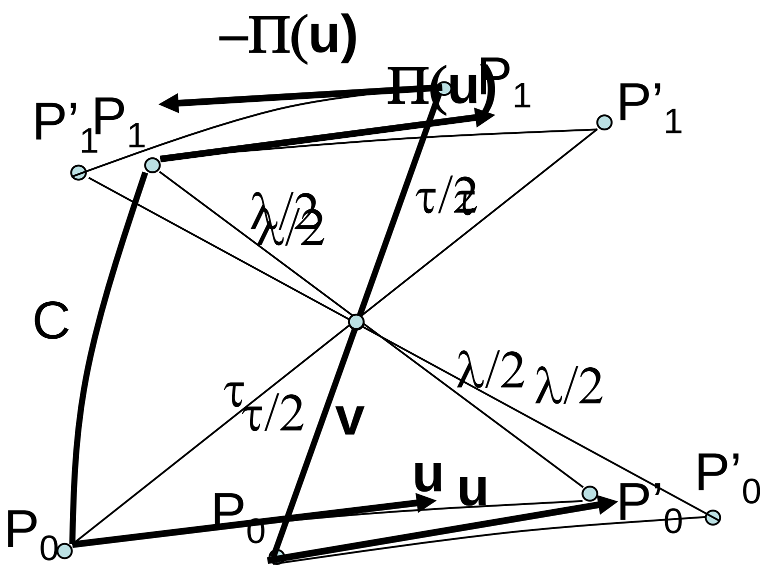
[Lorenzi, Pennec: Efficient Parallel Transport of Deformations in Time Series of Images: from Schild's to pole Ladder, JMIV 50(1-2):5-17, 2013]

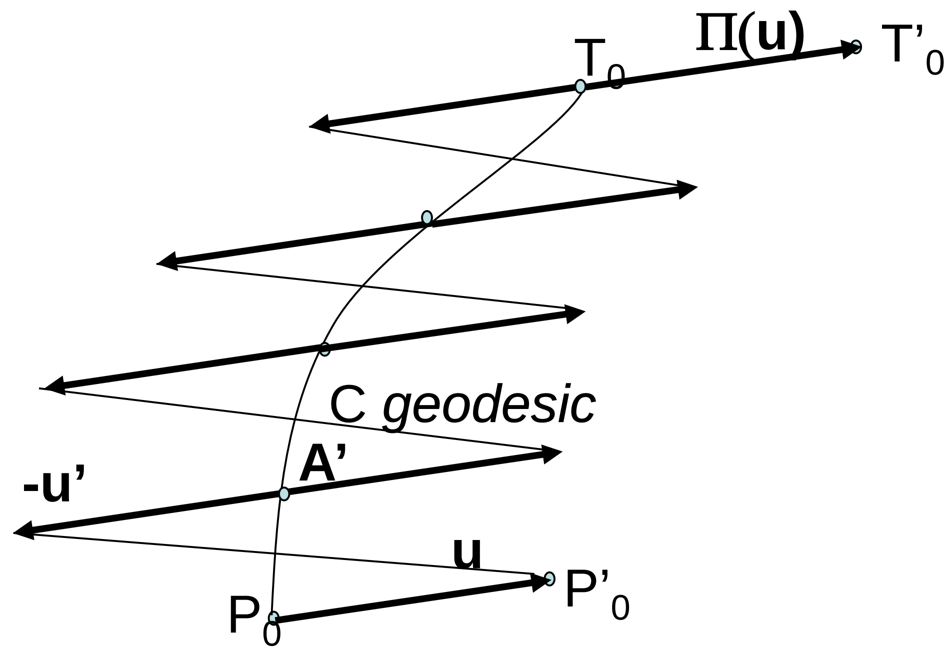
Parallel transport along geodesics

Simpler scheme along geodesics: Pole Ladder

$$\mathrm{Exp}(\Pi(u)) = \mathrm{Exp}(v/2) \circ \mathrm{Exp}(u) \circ \mathrm{Exp}(-v/2)$$

$$\Pi_{BCH}(u) = u + [v,u] + \frac{1}{2}[v[v,u]]$$

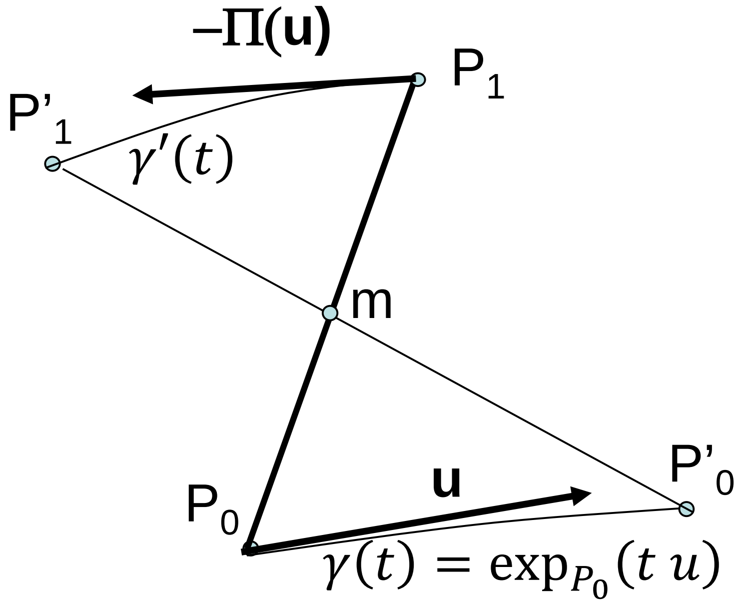




[Lorenzi, Pennec: Efficient Parallel Transport of Deformations in Time Series of Images: from Schild's to pole Ladder, JMIV 50(1-2):5-17, 2013]

Parallel transport along geodesics

Simpler scheme along geodesics: Pole Ladder



Pole ladder is exact in 1 step in symmetric space

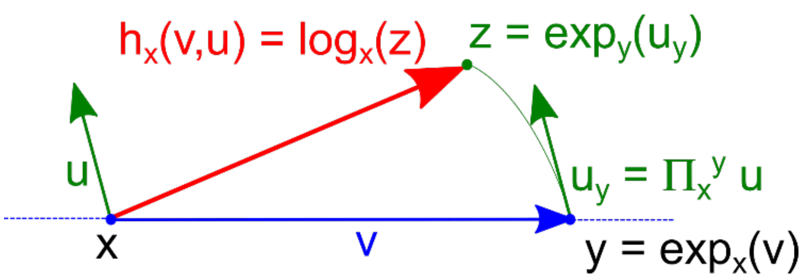
- Symmetry preserves geodesics:
$$S_m\big(\gamma(t)\big) = \gamma'(t)$$
- Parallel transport is differential of symmetry
$$\gamma'(t) = \exp_{P_1}(-\Pi(u))$$

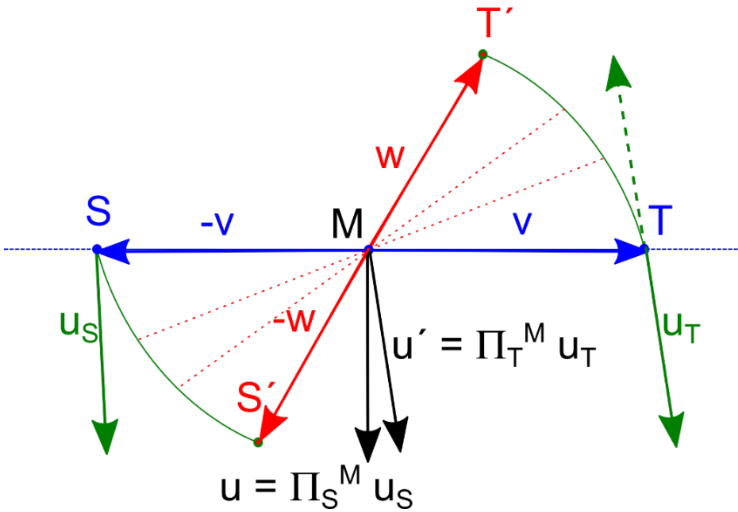
[XP. Parallel Transport with Pole Ladder: a Third Order Scheme in Affine Connection Spaces which is Exact in Affine Symmetric Spaces. Arxiv 1805.11436]

Accuracy of pole ladder

Gavrilov's double exponential series (2006):

$$h_x(v,u) = \log_x(\Pi_x^{\exp_x(v)}\, u)$$
$$= \; v + u + \frac{1}{6}R(u,v)v + \frac{1}{3}R(u,v)u + \frac{1}{24}\nabla_v R(u,v)(2v+5u) + \frac{1}{24}\nabla_u R(u,v)(v+2u) + O(5)$$





Find u' that satisfies:

$$h_M(v,-u') + h_M(-v,u) = 0$$

$$u' = u + \frac{1}{12}\nabla_v R(u,v)(5u-2v) + \frac{1}{12}\nabla_u R(u,v)(v-2u) + O(5)$$

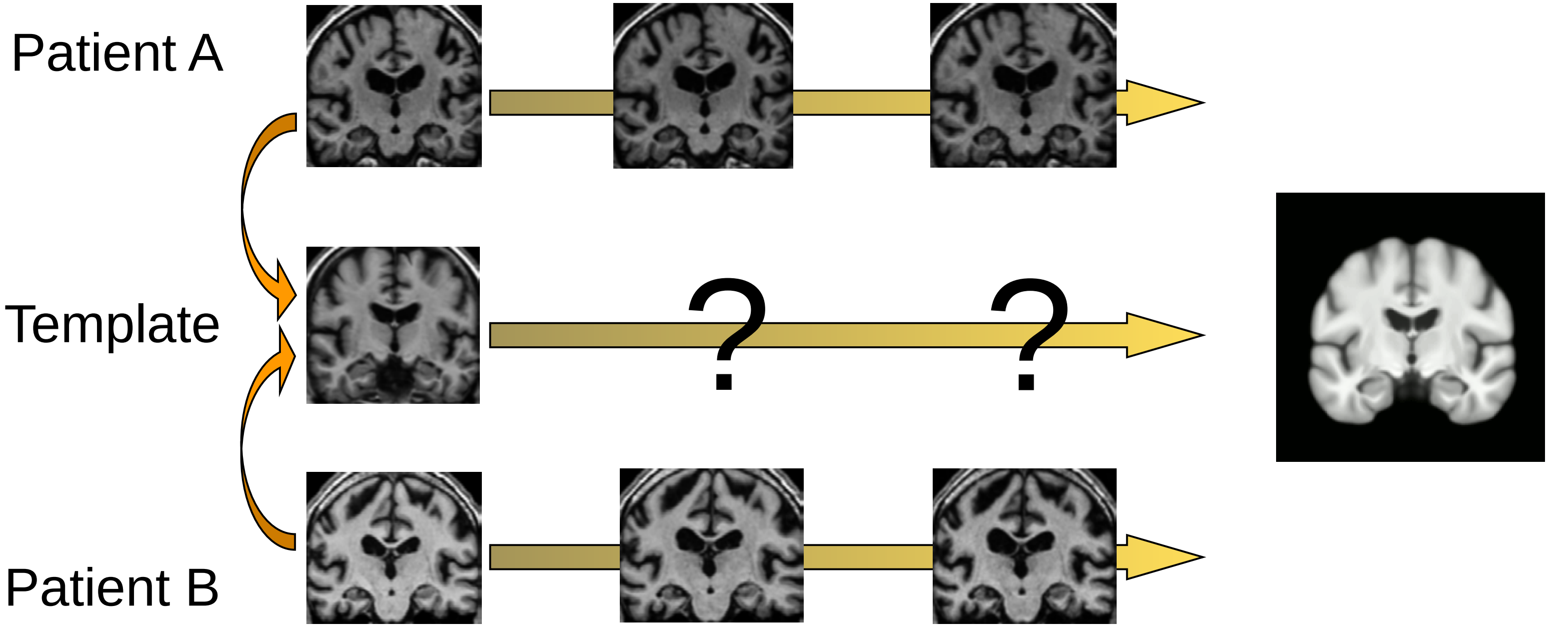
- Error term is of order 4 in general affine manifolds
- Error is even zero for symmetric spaces: pole ladder is exact in one step!

[XP. Parallel Transport with Pole Ladder: a Third Order Scheme in Affine Connection Spaces which is Exact in Affine Symmetric Spaces. Arxiv 1805.11436]

The Stationnary Velocity Fields (SVF) framework for diffeomorphisms

- SVF framework for diffeomorphisms is algorithmically simple
- Compatible with “inverse-consistency” **[Lorenzi, XP. IJCV, 2013]**
- Vector statistics directly generalized to diffeomorphisms.
- **Exact parallel transport** using one step of pole ladder **[XP arxiv 1805.11436 2018]**

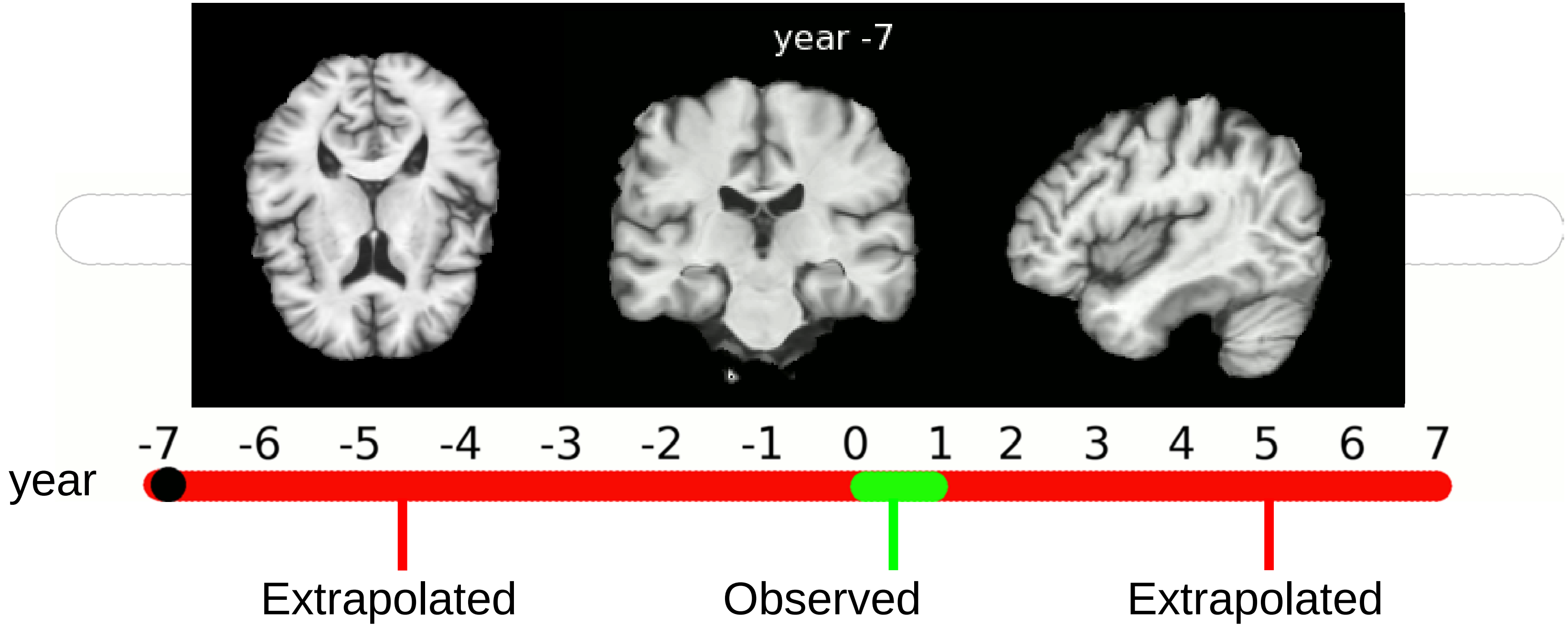
Longitudinal modeling of AD: 70 subjects extrapolated from 1 to 15 years



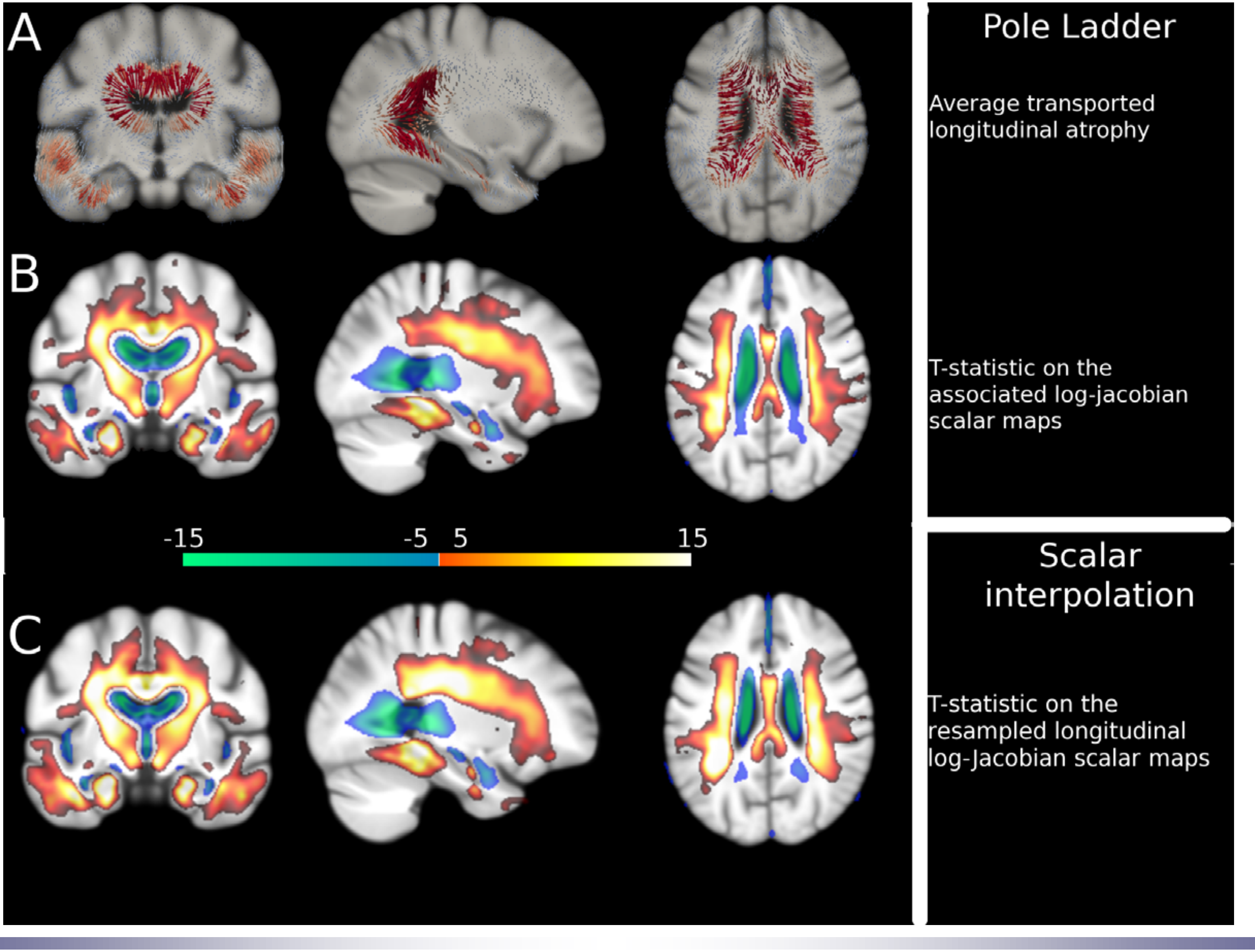
The Stationnary Velocity Fields (SVF) framework for diffeomorphisms

- SVF framework for diffeomorphisms is algorithmically simple
- Compatible with "inverse-consistency" **[Lorenzi, XP. IJCV, 2013]**
- Vector statistics directly generalized to diffeomorphisms.
- **Exact parallel transport** using one step of pole ladder **[XP arxiv 1805.11436 2018]**

Longitudinal modeling of AD: 70 subjects extrapolated from 1 to 15 years

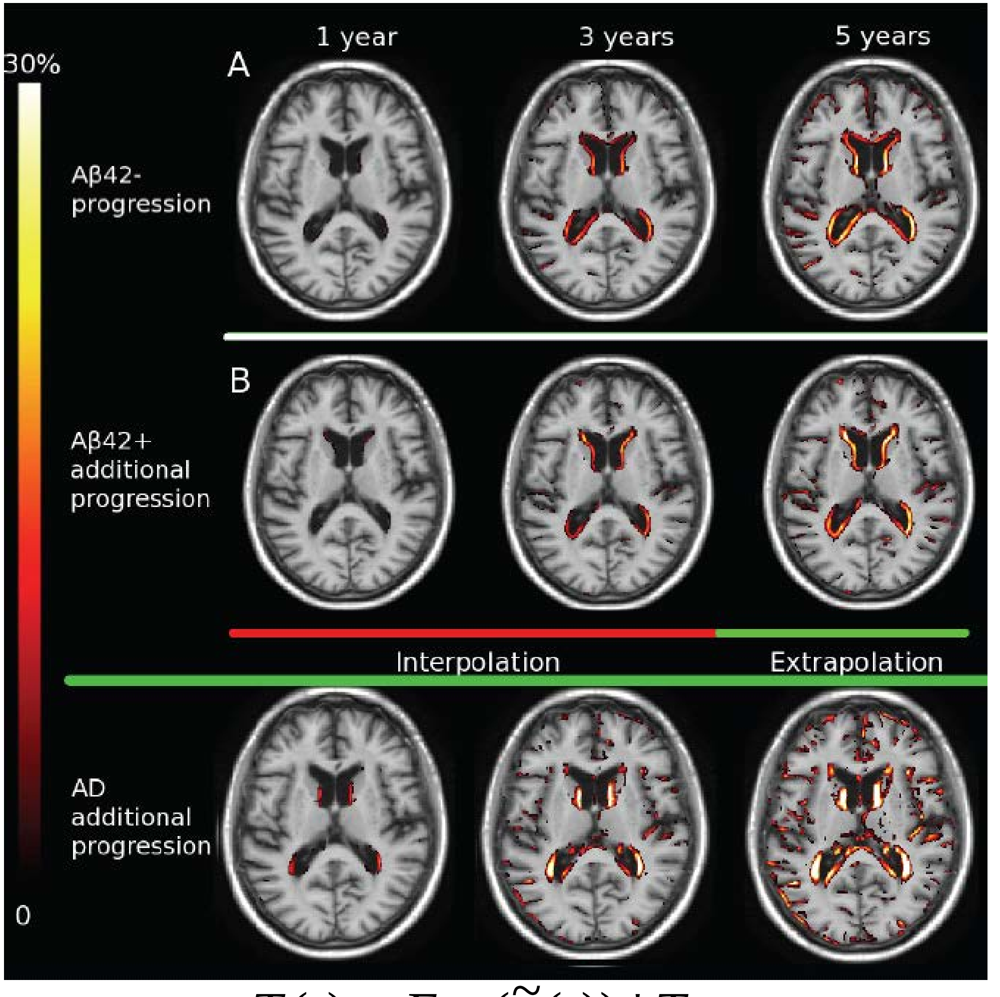


Modeling longitudinal atrophy in AD from images

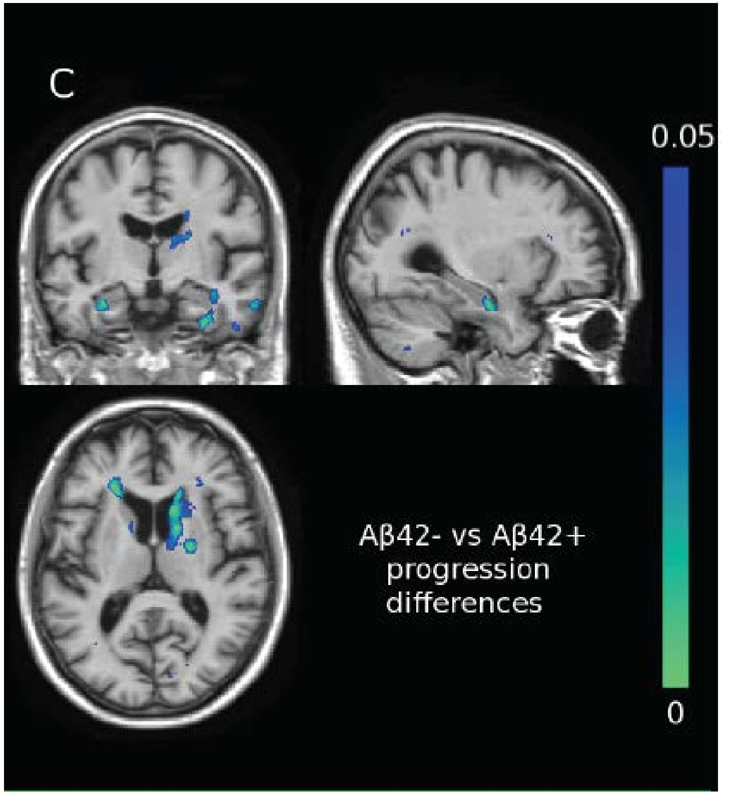


Study of prodromal Alzheimer's disease

Linear regression of the SVF over time: interpolation + prediction



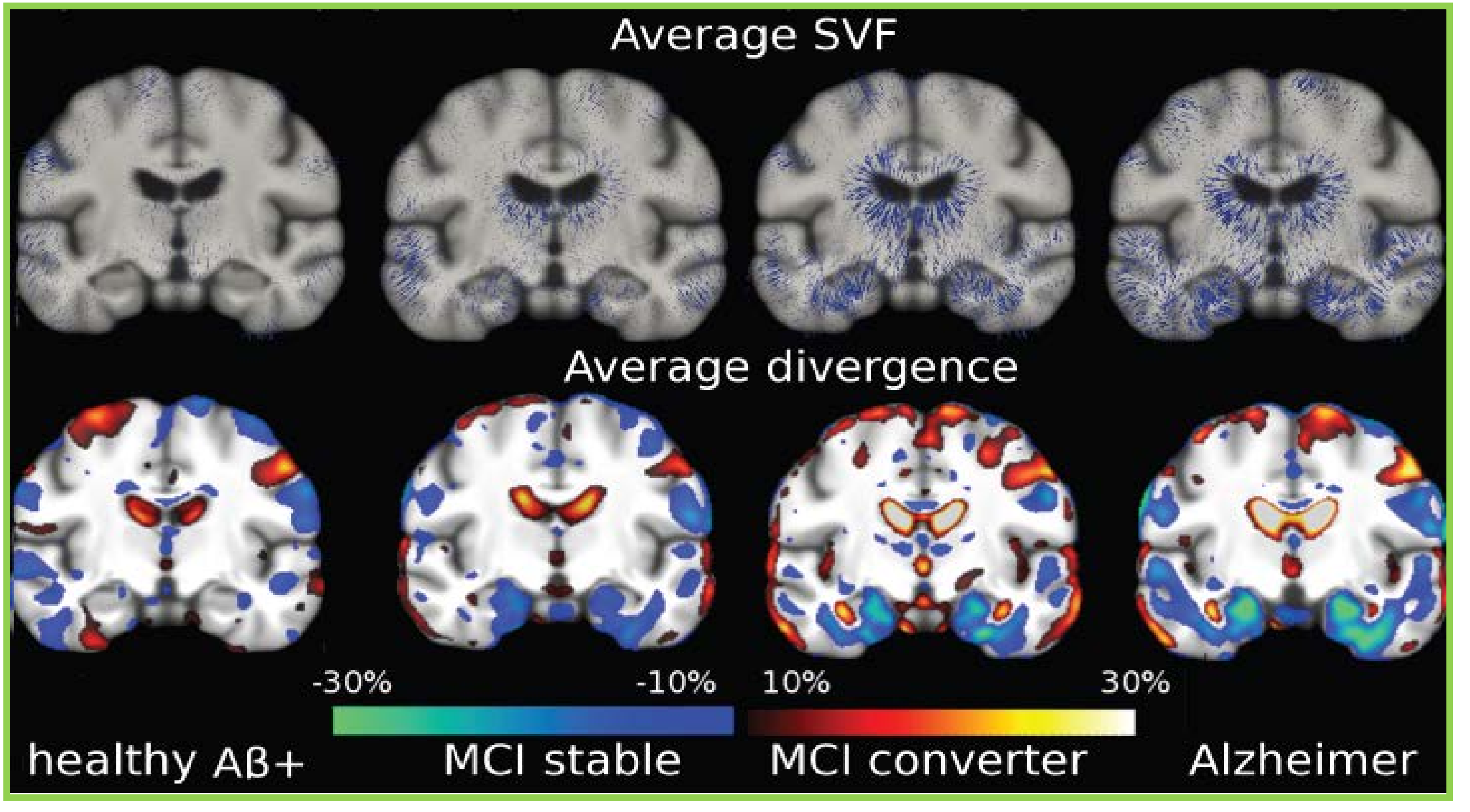
$$T(t) = Exp(\tilde{v}(t)) * T_0$$



Multivariate group-wise comparison of the transported SVFs shows statistically significant differences (nothing significant on log(det))

[Lorenzi, Ayache, Frisoni, Pennec, in Proc. of MICCAI 2011]

Mean deformation / atrophy per group



M Lorenzi, N Ayache, X Pennec G B. Frisoni, for ADNI. Disentangling the normal aging from the pathological Alzheimer's disease progression on structural MR images. 5th Clinical Trials in Alzheimer's Disease (CTAD'12), Monte Carlo, October 2012. (see also MICCAI 2012)

References for Statistics on Manifolds and Lie Groups

Statistics on Riemannnian manifolds

- Xavier Pennec. Intrinsic Statistics on Riemannian Manifolds: Basic Tools for Geometric Measurements. Journal of Mathematical Imaging and Vision, 25(1):127-154, July 2006. http://www.inria.fr/sophia/asclepios/Publications/Xavier.Pennec/Pennec.JMIV06.pdf

Invariant metric on SPD matrices and of Frechet mean to define manifold-valued image processing algorithms

- Xavier Pennec, Pierre Fillard, and Nicholas Ayache. A Riemannian Framework for Tensor Computing. International Journal of Computer Vision, 66(1):41-66, Jan. 2006. http://www.inria.fr/sophia/asclepios/Publications/Xavier.Pennec/Pennec.IJCV05.pdf

Bi-invariant means with Cartan connections on Lie groups

- Xavier Pennec and Vincent Arsigny. Exponential Barycenters of the Canonical Cartan Connection and Invariant Means on Lie Groups. In Frederic Barbaresco, Amit Mishra, and Frank Nielsen, editors, Matrix Information Geometry, pages 123-166. Springer, May 2012. http://hal.inria.fr/hal-00699361/PDF/Bi-Invar-Means.pdf

Cartan connexion for diffeomorphisms:

- Marco Lorenzi and Xavier Pennec. Geodesics, Parallel Transport & One-parameter Subgroups for Diffeomorphic Image Registration. International Journal of Computer Vision, 105(2), November 2013 https://hal.inria.fr/hal-00813835/document