Optimal Riemannian quantization
for air traffic management

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Geometric statistics workshop
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- The Fisher information metric
- Optimal Riemannian quantization
- Implementation with geomstats
- Real data analysis

Shape matching between curves

- Shape analysis framework
- Optimal matching
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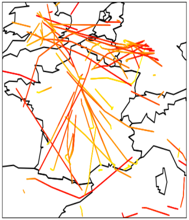
Context

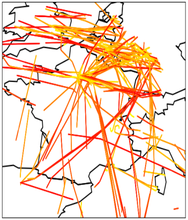
Air traffic control

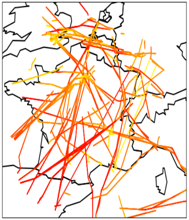
- Air traffic ontrollers act on flying or taxiing aircraft in such a way that separation norms are satisfied at all time.
- The airspace is segmented in elementary cells that can be regrouped or degrouped according to traffic complexity.
- Major concern : automatically evaluate the complexity of an air traffic situation.

What is an air traffic situation ?

- A set of positions and speeds $(x_i, v_i) \in \mathbb{R}^2 \times \mathbb{R}^2$, $i = 1, \ldots, N$ of the aircraft present in the airspace at a given time.





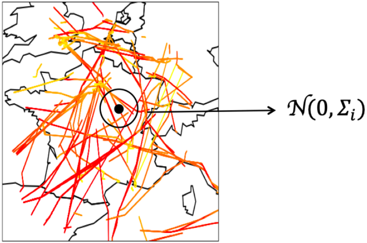


A geometric complexity indicator

- In the neighborhood of each point (x_i, v_i), we assume that the spatial distribution of the speeds is Gaussian.
- We estimate its mean and covariance matrix using a kernel K, $K_h(x) = \frac{1}{h}K(\frac{x}{h})$,

$$m_i = \frac{\sum_{j=1}^{N} v_j K_h(x_i - x_j)}{\sum_{j=1}^{N} K_h(x_i - x_j)}, \quad \Sigma_i = \frac{\sum_{j=1}^{N} (v_j - m_i)(v_j - m_i)^T K_h(x_i - x_j)}{\sum_{j=1}^{N} K_h(x_i - x_j)}.$$

- Σ_i measures the "local disorder" = "local complexity" of the traffic at point x_i
- We neglect the mean and represent complexity at x_i by $\mathcal{N}(0, \Sigma_i)$



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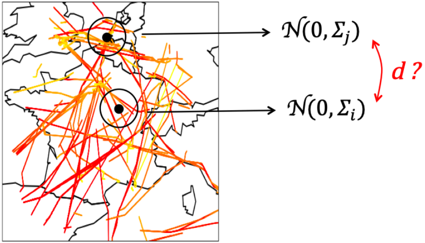

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Information geometry

- Geometric approach to probability and statistics based on the Fisher information
- The Fisher information is defined for a parametric statistical model $\{p_\theta \mu | \theta \in \Theta\}$

$$I(\theta) = \mathbb{E}_\theta[\partial_\theta \ell_\theta(X) \cdot \partial_\theta \ell_\theta(X)^t], \quad \ell_\theta = \log p_\theta.$$

- In parametric estimation, the Fisher information gives a limit to the precision of the estimation given by an unbiased estimator T of θ function of a sample of size n (Cramer-Rao bound)

$$\mathrm{Var}_\theta(T) \geq (nI(\theta))^{-1}$$

- The Fisher information is the curvature of the Kullback-Leibler divergence $K(p,q) = \mathbb{E}_p \log(p/q)$

$$\partial_\theta K(\theta^*, \theta)|_{\theta=\theta^*} = 0, \quad \partial_{\theta_i}\partial_{\theta_j} K(\theta^*, \theta)\Big|_{\theta=\theta^*} = I(\theta^*)_{i,j}$$

- The KL divergence is not symmetric and does not verify the triangular inequality. We use the Fisher information to define a real distance.

The Fisher information metric

- Parametric statistical model $\mathcal{P} = \{P_\theta = p_\theta \mu | \theta \in \Theta\}$ on $\mathcal{X}$, with $\Theta \subset \mathbb{R}^d$ open.

- Θ is a differentiable manifold, and can be equipped with a Riemannian metric using the Fisher information $I(\theta)$

$$g_\theta(u, v) = u^t I(\theta) v, \quad u, v \in T_\theta \Theta \simeq \mathbb{R}^d$$

  g is called the Fisher information metric or Fisher-Rao metric.
  (Θ, g) is a Riemannian manifold.

- The geodesic distance induced on Θ and therefore on $\mathcal{P}$

$$d_F(P_\theta, P_{\theta'}) = d_\Theta(\theta, \theta') = \inf_{\gamma,\, \gamma(0)=\theta, \gamma(1)=\theta'} \int_0^1 \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} dt,$$

  is called the Fisher information distance.

Invariance properties of the Fisher information metric

- The Fisher geometry is invariant with respect to diffeomorphic parameter change $\forall \varphi : \Theta \to \tilde{\Theta}$, $\theta \mapsto \tilde{\theta}$ diffeomorphism,

$$d_\Theta(\theta, \theta') = d_{\tilde{\Theta}}(\varphi(\theta), \varphi(\theta'))$$

$\to$ the geometric structure does not depend on the parameter choice.

- The Fisher metric is the only invariant metric with respect to sufficient statistics (Chentsov's theorem) : $T : \mathcal{X}^n \to \mathbb{R}^d$ sufficient statistic of $\mathcal{P}$, i.e.

$$P_\theta((X_1, \ldots, X_n)|T(X_1, \ldots, X_n)) \text{ is independant of } \theta$$

T transforms the sampling model $(\{P_\theta^n\}_{\theta\in\Theta}, d_F^n)$ on $\mathcal{X}$ into an isometric sampling model $(\{T_*(P_\theta^n)\}_{\theta\in\Theta}, d_F^n)$ on $\mathbb{R}^d$

$$d_F^n(P_\theta^n, P_{\theta'}^n) = d_F^n(T_*(P_\theta^n), T_*(P_{\theta'}^n))$$

$\to$ the geometry of a parametric model is preserved through transformation by a sufficient statistic.

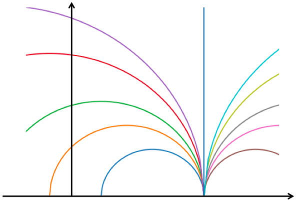
Example : univariate normal distributions

$X \sim \mathcal{N}(m, \sigma^2)$ has probability density function

$$p_\theta(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}, \qquad \theta = (m, \sigma) \in \mathbb{R} \times \mathbb{R}_+^*, \quad x \in \mathbb{R}.$$

The Fisher information is

$$I(\theta) = \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 2/\sigma^2 \end{bmatrix}, \quad \|d\theta\|^2 = \frac{dm^2 + 2d\sigma^2}{\sigma^2}.$$



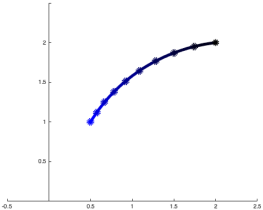
The change of variables $(m, \sigma) \mapsto (m/\sqrt{2}, \sigma)$ yields the Poincaré half-plane, i.e. hyperbolic geometry.

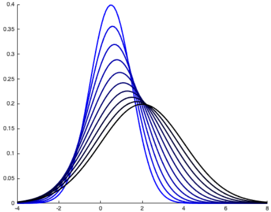
The Wasserstein distance yields Euclidean geometry

$$\|d\theta\|^2 = dm^2 + d\sigma^2.$$

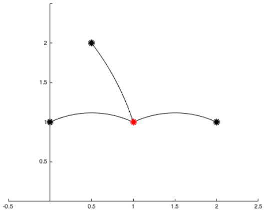
Example : univariate normal distributions

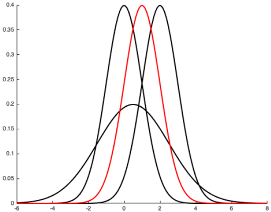
The geodesics yield optimal interpolations between probability distributions.





Since the curvature is negative, the Riemannian center of mass is well defined.





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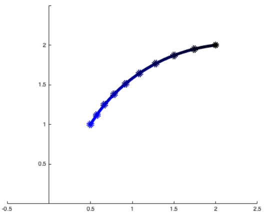

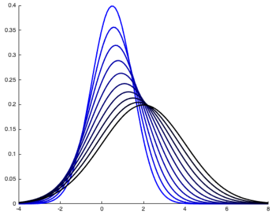

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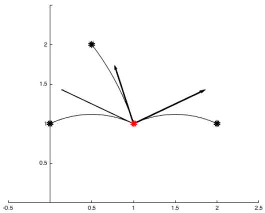

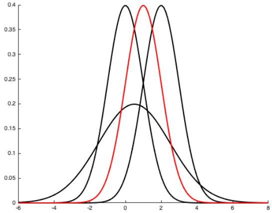

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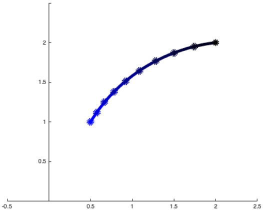

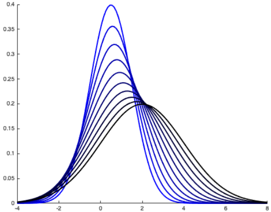

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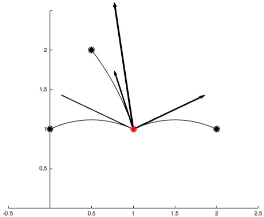

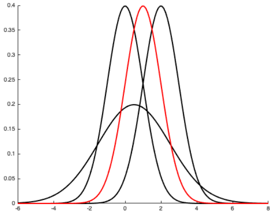

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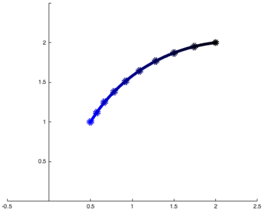

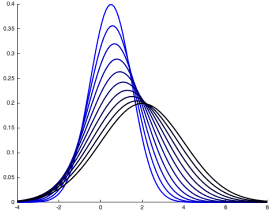

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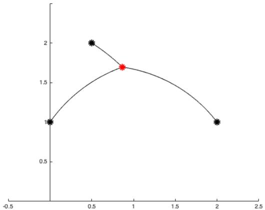

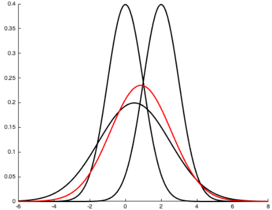

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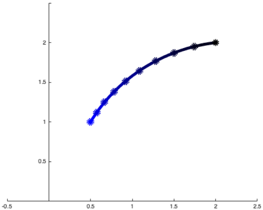

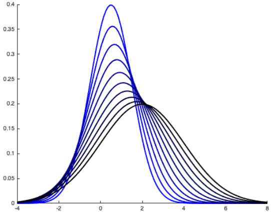

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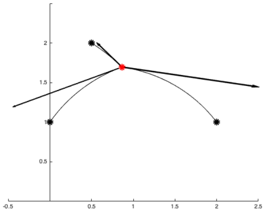

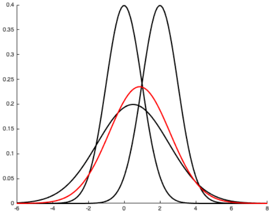

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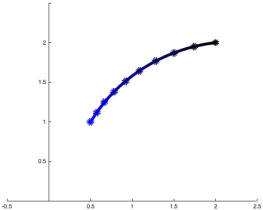

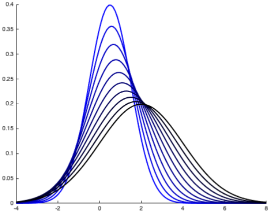

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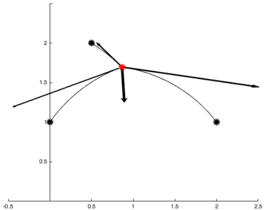

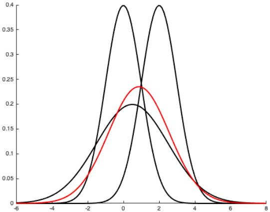

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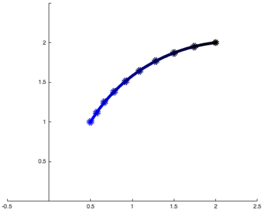

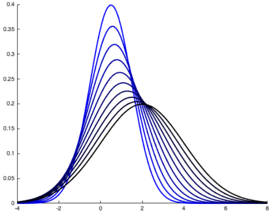

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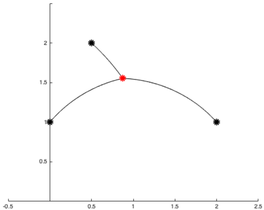
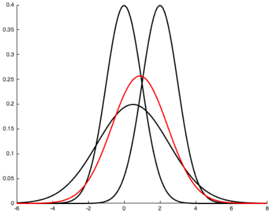




$$\bar{P} = \underset{P}{\operatorname{argmin}} \sum_{i=1}^{n} d^2(P, P_i)$$

Example : centered multivariate normal distributions

$X \sim \mathcal{N}(0, \Sigma)$, $\theta = \Sigma \in S_n^+$ symmetric positive definite matrix.

The tangent vectors U, V in Σ are symmetric matrices

$$g_\Sigma(U, V) = \text{tr}(\Sigma^{-1} U \Sigma^{-1} V).$$

The geodesics and geodesic distance have closed forms

$$\Gamma(t) = \Sigma^{1/2} \exp\left(t\Sigma^{-1/2} U \Sigma^{-1/2}\right) \Sigma^{1/2}, \quad U \in T_\Sigma S_n^+$$

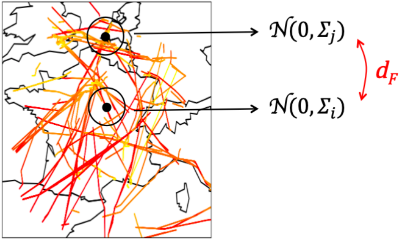
$$d(\Sigma_1, \Sigma_2) = \sqrt{\sum_{i=1}^{n} \log \lambda_i \left(\Sigma_1^{-1/2} \Sigma_2 \Sigma_1^{-1/2}\right)}, \quad \lambda_i(A) = i^{\text{th}} \text{ eigenvalue of } A.$$

This distance on S_n^+ is also called affine-invariant for its invariance w.r.t. GL_n

$$d(A^T \Sigma_1 A, A^T \Sigma_2 A) = d(\Sigma_1, \Sigma_2).$$

Summarizing the complexity information

We can now compare the complexity level of different zones in an image.



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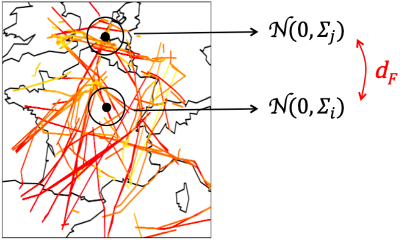

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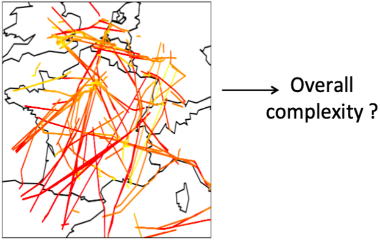

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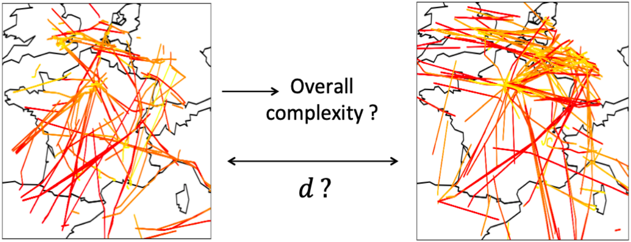

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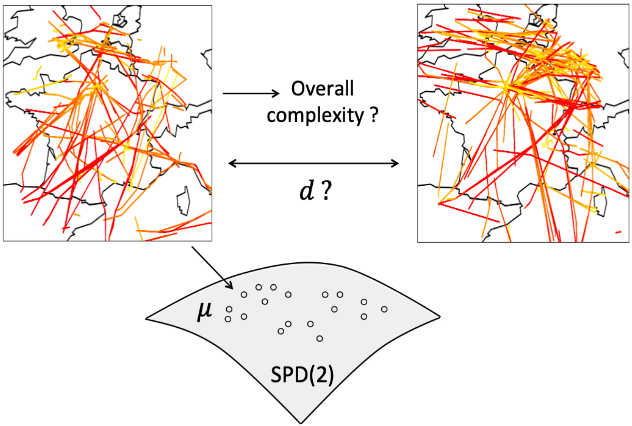

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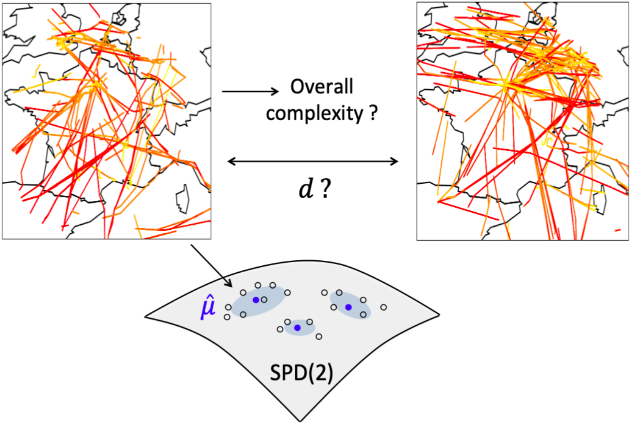

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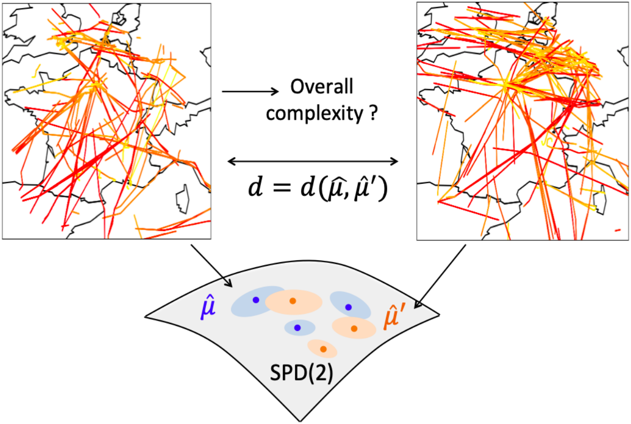

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- Goal : approximate $X \sim \mu$ by a quantized version $q(X)$ where

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- Optimal quantization is an optimal transport problem (Graf, Luschgy, 2000)

$$\inf_{q \in Q_n} \mathbb{E}_\mu \left[d(X, q(X))^p \right] = \inf_{\nu \in \mathcal{P}_n(M)} W_p(\mu, \nu)^p,$$

where $\mathcal{P}_n(M) = \{ \nu \text{ measure on } M, |\operatorname{supp} \nu| \leq n \}$ and W_p is the p^{th} order Wasserstein distance, i.e.,

$$W_p(\mu, \nu) = \inf_{P \in \Pi(\mu, \nu)} \int_{M \times M} d(y, z)^p \, dP(y, z),$$

where $\Pi(\mu, \nu) = \{ P \in \mathcal{P}(M \times M) \text{ has marginals } \mu \text{ and } \nu \}$.

Optimal quantization II

- The search for an optimal quantizer q can be restricted to nearest neighbor projections in a set $\alpha = \{a_1, \ldots, a_n\}$ of size n

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$$q_\alpha(x) = \sum_{i=1}^{n} a_i \mathbf{1}_{Vi}(x), \quad x \in M,$$

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where

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- The minimizers $q = q_\alpha$, $\alpha = \{a_1, \dots, a_n\}$ and $\hat{\mu}$ are related by :

$$\boxed{\hat{\mu} = (q_\alpha)_* \mu = \sum_{i=1}^n \mu(V_i) \delta_{a_i}.}$$

Finding the optimal quantized measure I

- We choose to optimize over n-tuples $\alpha = \{a_1, \ldots, a_n\}$. We set

$$F_{n,p}(a_1, \ldots, a_n) = \mathbb{E}_\mu \left[\min_{1 \le i \le n} d(X, a_i)^p \right] = \int_M \min_{1 \le i \le n} d(x, a_i)^p d\mu(x).$$

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- The minimizer is in general not unique, e.g. in case of symmetries of μ.
- Gradient of the cost function (LB, Puechmorel, 2019) Let $\alpha = (a_1, \dots, a_n) \in M^n$ be a n-tuple of pairwise distinct components. Then $F_{n,2}$ is differentiable and its gradient in α is
$$\operatorname{grad}_\alpha F_{n,2} = \left(-2 \int_{\mathring{V}_i} \overrightarrow{a_i x} \mu(\mathrm{d}x) \right)_{1 \le i \le n} = -2 \left(\mathbb{E}_\mu \mathbf{1}_{\{X \in \mathring{V}_i\}} \overrightarrow{a_i X} \right)_{1 \le i \le n}, \qquad (1)$$
where $\overrightarrow{xy} := \log_x(y)$.

Finding the optimal quantized measure II

- The average opposite direction of the gradient is given by

$$\begin{bmatrix} \mathbf{1}_{\{X \in \mathring{V}_1\}} \overrightarrow{a_1 X} \\ \vdots \\ \mathbf{1}_{\{X \in \mathring{V}_n\}} \overrightarrow{a_n X} \end{bmatrix}.$$

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- Algorithm (Competitive Learning Riemannian Quantization)
  Initialization : $\alpha(0) = (a_1(0), \ldots, a_n(0))$, discrete steps $\sum \gamma_k = \infty$, $\sum \gamma_k^2 < \infty$
  For each new observation X_k, repeat until convergence :
  1. find $i^* = \operatorname{argmin}_i d(X_k, a_i(k))$,
  2. update

$$a_{i^*}(k+1) = \exp_{a_{i^*}(k)}\left(\gamma_k \overrightarrow{a_{i^*}(k) X_k}\right),$$

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- Theorem (LB, Puechmorel 2019, Bonnabel 2013) If the injectivity radius of M is uniformly bounded from below by $I > 0$, and if $(\alpha(k))_{k \geq 0}$ is computed using the above algorithm and a sample of a compactly supported distribution μ, then $F_{n,2}(\alpha(k))$ converges a.s. and $\text{grad}_{\alpha(k)} F_{n,2} \to 0$ when $k \to \infty$ a.s.

Link with k-means clustering

- Let $X_1, \dots, X_N$ be an i.i.d. sample of empirical distribution

$$\mu = \frac{1}{N} \sum_{k=1}^{N} \delta_{X_k},$$

The associated optimal quantized distribution is

$$\hat{\mu}_n = \sum_{i=1}^{n} \frac{|V_i|}{N} \delta_{a_i},$$

where $a_1, \dots, a_n$ minimizes the sum of intra-class variance of each Voronoi cell

$$F_{n,2}(a_1, \dots, a_n) = \sum_{i=1}^{n} \sum_{x_k \in V_i} d^2(x_k, a_i).$$

This is the cost function of the k-means algorithm. The clusters are given by the Voronoi cells.

- Competitive Learning Quantization is an online version of the k-means algorithm $\rightarrow$ adapted to large datasets.

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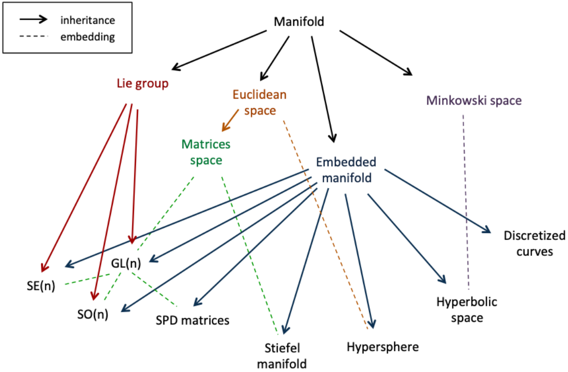
geomstats

- Created by Nina Miolane and Xavier Pennec
- Python package that factorizes code for geometric statistics into a shared unit-test library, with several backends : numpy, tensorflow and pytorch.
- Riemannian geometry is implemented in geomstats.geometry with 4 base classes
  - Manifold and EmbeddedManifold
  - RiemannianMetric and InvariantMetric
- The other manifold classes inherit from these 4 base classes



Quantization in geomstats

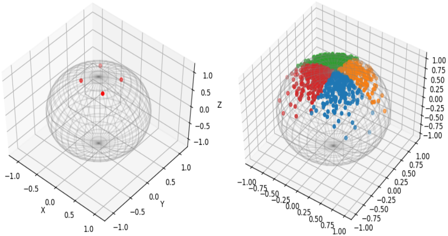
- Machine Learning is implemented in geomstats.learning, using scikit-learn classes
  - BaseEstimator
  - ClassifierMixin , RegressorMixin, TransformMixin, ClusterMixin and others.

```
sphere = Hypersphere(dimension=2)

data = sphere.random_von_mises_fisher(kappa=10, n_samples=1000)

clustering = Quantization(metric=sphere.metric, n_clusters=4)
clustering = clustering.fit(data)

cluster_centers = clustering.cluster_centers_
labels = clustering.labels_
```



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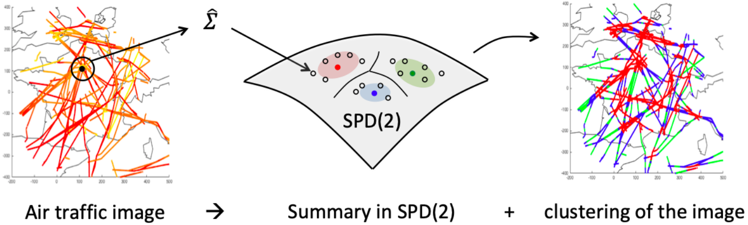
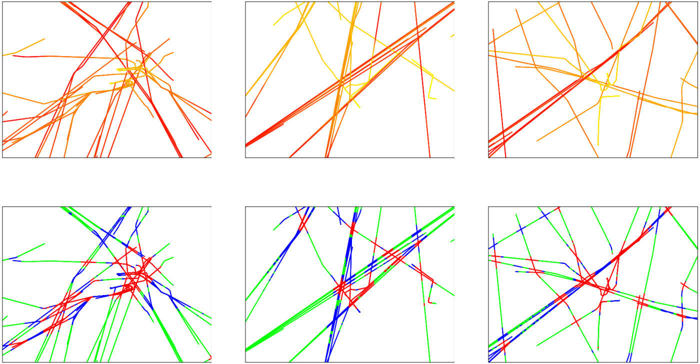
Real data analysis

- Given an air traffic image, we extract N SPD matrices $\Sigma_1, \ldots, \Sigma_N$, with empirical distribution
$$\mu = \frac{1}{N}\sum_{i=1}^{N} \delta_{\Sigma_i}$$
- We use optimal quantization to find a summary
$$\hat{\mu} = \sum_{i=1}^{n} w_i \delta_{A_i}, \quad \text{where} \quad w_i = |V_i|/N.$$
- In practice, we choose $n = 3$ because the centers can then be ordered (Loewner order : $A \geq B \Leftrightarrow A - B$ positive definite).
- Mapping back the labels to the image, this yields a clustering of the image in zones of homogeneous complexity.

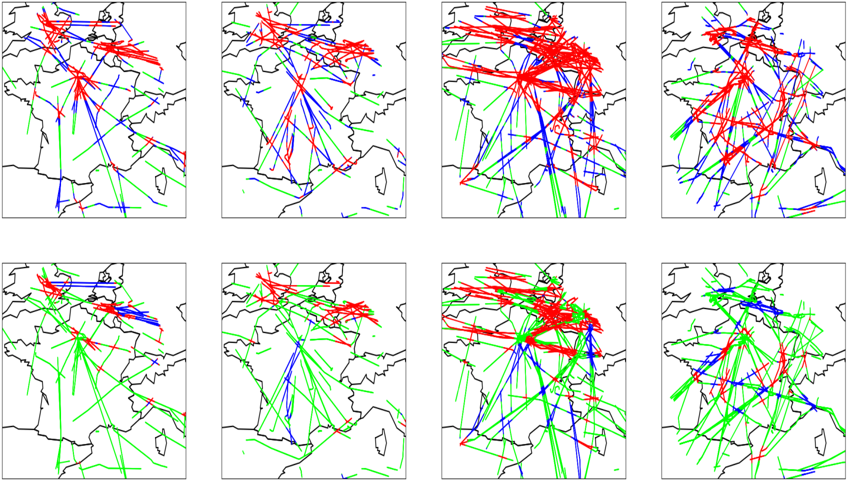


Three levels of complexity



Clustering of the airspace above Paris (left), Toulouse (middle) and Lyon (right).

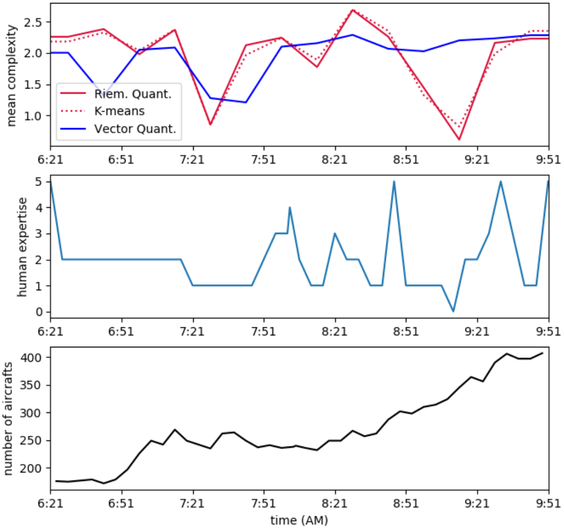
Comparison to Euclidean geometry



Clustering of the French airspace with Fisher-Rao (up) vs Euclidean (down) geometry.

Comparison to human perception

mean complexity index $= \lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3$



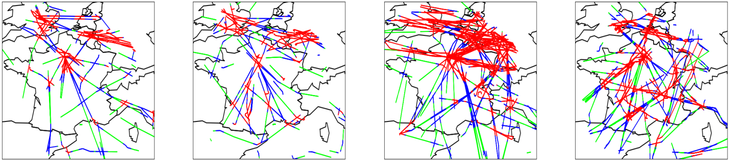
Comparison of summaries

To compare summaries $\mu = \mu_1 \delta_{A_1} + \mu_2 \delta_{A_2} + \mu_3 \delta_{A_3}$ and $\nu = \nu_1 \delta_{B_1} + \nu_1 \delta_{B_1} + \nu_1 \delta_{B_1}$, it suffices to find the transport plan $\pi = (\pi_{ij})_{i,j}$

π_{11}	π_{12}	π_{13}	μ_1
π_{21}	π_{22}	π_{23}	μ_2
π_{31}	π_{32}	π_{33}	μ_3
ν_1	ν_2	ν_3	

solution of

$$\min_{\pi} \sum_{i=1}^{3} \sum_{j=1}^{3} \pi_{ij} d(A_i, B_j)^2.$$



Distances matrix between the summaries :

0.00	1.92	6.74	4.55
1.92	0.00	8.31	6.07
6.74	8.31	0.00	1.22
4.55	6.07	1.22	0.00

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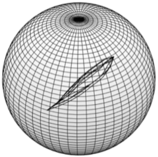
Shape analysis

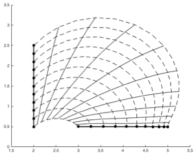
Some interesting questions :

- how can we compare two shapes ?
- how can we interpolate between two shapes ?
- how can we compute a mean shape ?
- how can we perform clustering on shapes ?

shapes in $\mathbb{R}^3$



shapes in $\mathbb{S}^2$



interpolation between shapes in $\mathbb{H}^2$

$\rightarrow$ Riemannian geometry : convenient framework to generalize

- usual statistical notions (mean, covariance, Gaussian distribution...)
- data processing algorithms (clustering, PCA...)

Model of a curve

- We consider smooth curves in a space M ($\mathbb{R}^n$ or manifold) with non zero speed

$$\mathcal{M} = \{c : [0,1] \to M \quad C^\infty, \quad c'(t) \neq 0 \, \forall t\}$$

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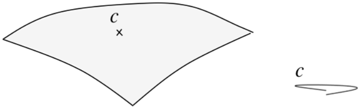
- The space of curves $\mathcal{M}$ can be seen as an (∞-dim) differentiable manifold



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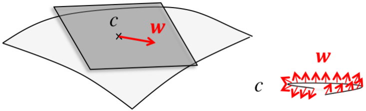

A tangent vector $w \in T_c\mathcal{M}$ is an infinitesimal vector field along c.

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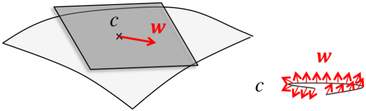

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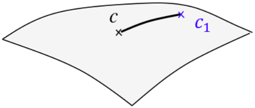

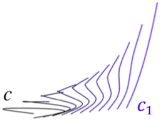

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$$G_c(v,w), \qquad c \in \mathcal{M}, \quad v,w \in T_c\mathcal{M}, \qquad \text{then}$$

  $\rightarrow$ a geodesic in $\mathcal{M}$ is an interpolation between two curves

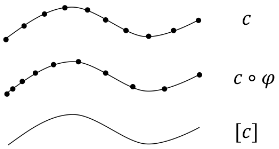
  $\rightarrow$ $\text{dist}(c, c_1) = L(\text{geodesic between } c \text{ à } c_1)$

Model of a shape

- Curves are reparameterized by the action of increasing diffeomorphisms

$$c \mapsto c \circ \varphi, \qquad \varphi \in \Gamma := \text{Diff}_+([0,1])$$

- A shape is an element of the quotient space $\mathcal{M}/\Gamma$



- If the Riemannian metric on $\mathcal{M}$ is invariant w.r.t. the action of Γ

$$G_c(v,w) = G_{c\circ\varphi}(v\circ\varphi, w\circ\varphi), \quad \forall \varphi \in \Gamma$$

it induces a Riemannian metric on $\mathcal{M}/\Gamma$ for which the distance is

$$\text{dist}([c_0],[c_1]) = \inf_{\varphi\in\Gamma} \text{dist}(c_0, c_1\circ\varphi).$$

How to compare two shapes ?

- To compare two shapes in $\mathcal{M}/\Gamma$:
  1. define a reparameterization invariant metric on $\mathcal{M}$
  2. find its geodesics (solve geodesic equations)
  3. solve the optimal matching problem φ between two curves c_0 et c_1

$$\text{dist}([c_0],[c_1]) = \inf_{\varphi\in\Gamma} \text{dist}(c_0, c_1 \circ \varphi)$$

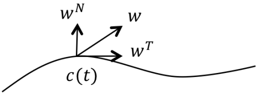
- (Michor, Mumford, 2005) The reparameterization invariant L^2 metric yields a vanishing distance on the quotient space

$$G_c(w,z) = \int_0^1 \langle w(t), z(t)\rangle |c'(t)| dt$$

- Need to include higher order derivatives, e.g. elastic metrics

$$G_c^{a,b}(w,z) = \int a^2 \langle D_\ell w^N, D_\ell z^N \rangle + b^2 \langle D_\ell w^T, D_\ell z^T \rangle \, \mathrm{d}\ell$$

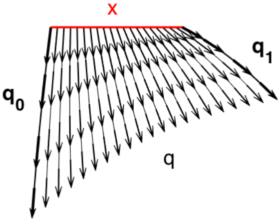
where $D_\ell w = w'/|c'|$, $d\ell = |c'(t)|\mathrm{d}t$.



The SRV framework

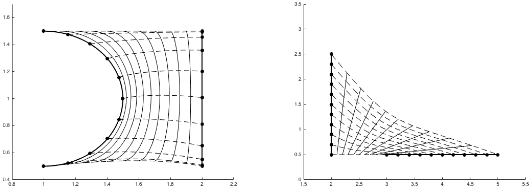
- For the special case $a = 1$, $b = 1/2$, the elastic metric can be mapped to an L^2-metric through the *square root velocity transform* $q = c'/\sqrt{|c'|}$ (Srivastava et al. 2011)

$$d^2_{G^{1,\frac{1}{2}}}(c_0, c_2) = d^2_{L^2}(q_0, q_1) = \int_0^1 |q_1(t) - q_0(t)|^2 \, \mathrm{d}t.$$

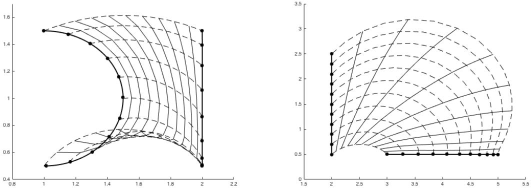


- Many extensions
  - curves in a manifold (J.Su et al. 2014, LB 2017, Zhang et al. 2018)
  - curves in a Lie group (Celledoni et al. 2016)
  - curves in homogeneous spaces (Z.Su et al. 2017, Celledoni et al. 2017)
  - surfaces (square root normal field, Jermyn et al. 2012)

Examples of geodesics between curves



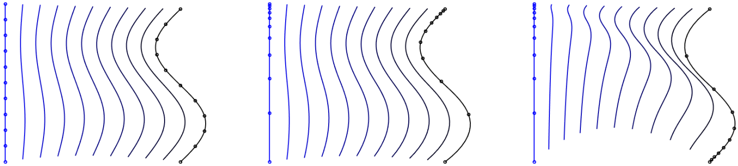
Geodesics between curves in the plane $\mathbb{R}^2$



Geodesics between curves in the Poincaré upper half-plane $\mathbb{H}^2$

Are we really comparing shapes ?

- At this stage, the distance between two curves does not change if we reparameterize them the same way, but it does change if we reparameterize them in different ways !



(c_0, c_1) $\quad$ $(c_0 \circ \varphi, c_1 \circ \varphi)$ $\quad$ $(c_0 \circ \varphi, c_1 \circ \psi)$

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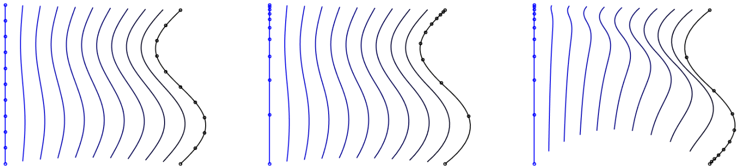

(c_0, c_1) $\quad$ $(c_0 \circ \varphi, c_1 \circ \varphi)$ $\quad$ $(c_0 \circ \varphi, c_1 \circ \psi)$

- We need to solve the optimal matching problem

$$\text{dist}([c_0],[c_1]) = \inf_{\varphi \in \Gamma} \text{dist}(c_0, c_1 \circ \varphi).$$

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- Optimal Riemannian quantization
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- Principal bundle structure $\pi : \mathcal{M} \to \mathcal{M}/\Gamma \Rightarrow$ Decomposition of the tangent space

$$T_c\mathcal{M} = V_c\mathcal{M} \oplus H_c\mathcal{M}$$

Tangent vector $=$ Vertical part $+$ Horizontal part

Vertical part : reparametrizes the curve without changing its shape

Horizontal part : changes the shape, and is orthogonal to the vertical part (w.r.t. G).

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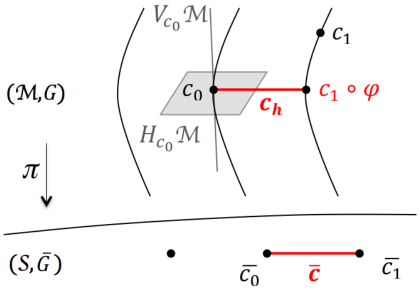

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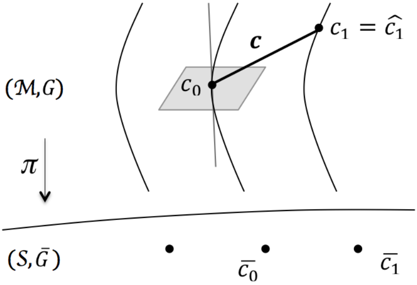

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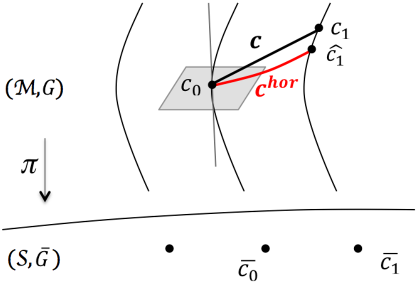

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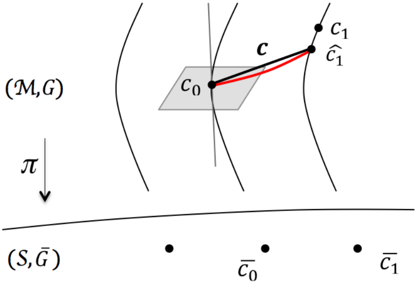

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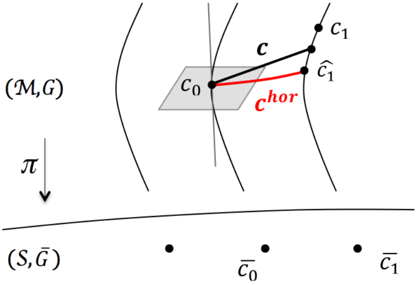

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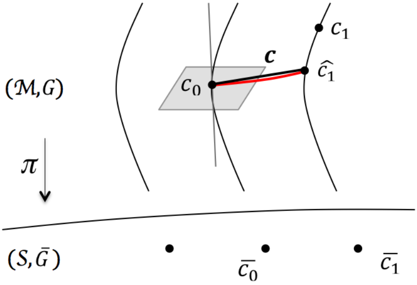

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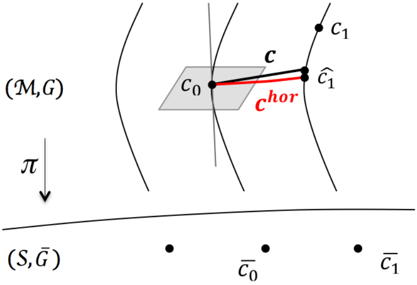

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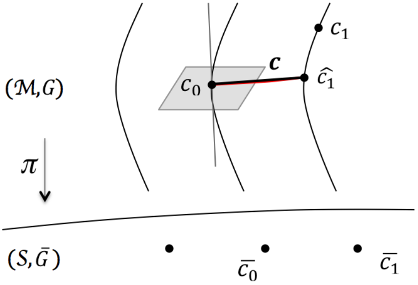

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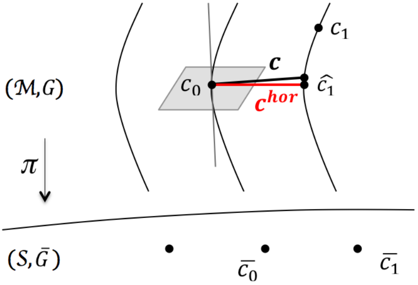

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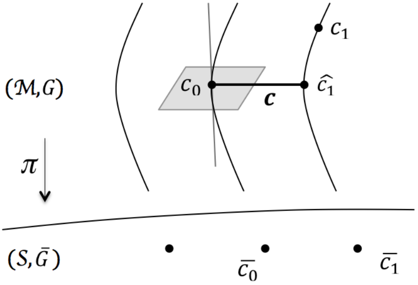

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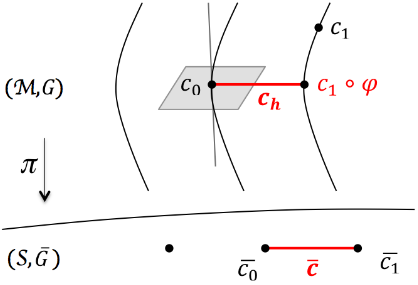

More formally

- We decompose any path of curves $s \mapsto c(s,\cdot) \in \mathcal{M}$ into

$$c(s,t) = c^{hor}(s, \varphi(s,t)), \qquad \begin{array}{l} s \mapsto c^{hor}(s,\cdot) \text{ horizontal path} \\ s \mapsto \varphi(s,\cdot) \text{ path in } \mathrm{Diff}^+([0,1]) \end{array}$$

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- Assuming that we know $\partial_s c(s,t)^{ver} = m(s,t)v(s,t)$, we can show (LB 2019) : The path of diffeomorphisms is solution of the PDE

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$$= \langle \nabla_t \nabla_t w, v\rangle - \left((a/b)^2 - 1\right) \langle \nabla_t w, \nabla_t v\rangle - \langle \nabla_t c'/|c'|, v\rangle \langle \nabla_t w, v\rangle.$$

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$\rightarrow$ From a path of curves $c(s,t)$, find $m(s,t)$ for $w(s,t) = \partial_s c(s,t)$, then $\varphi(s,t)$ and then

$$c^{hor}(s,t) = c(s, \varphi(s)^{-1}(t)).$$

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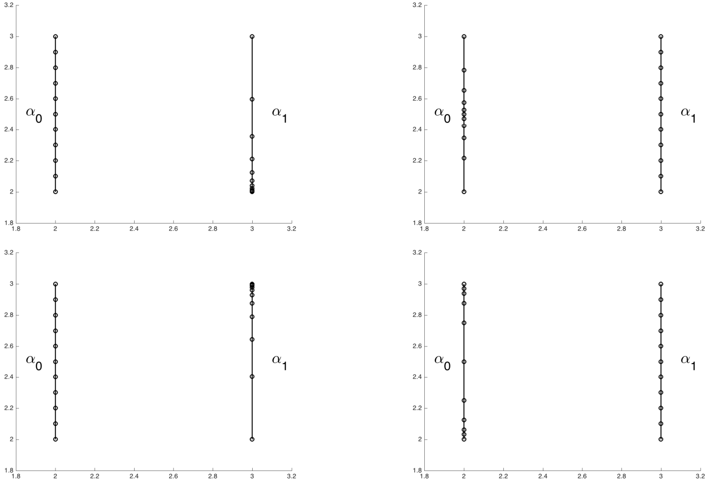

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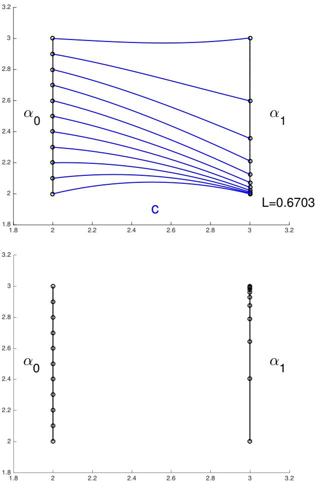

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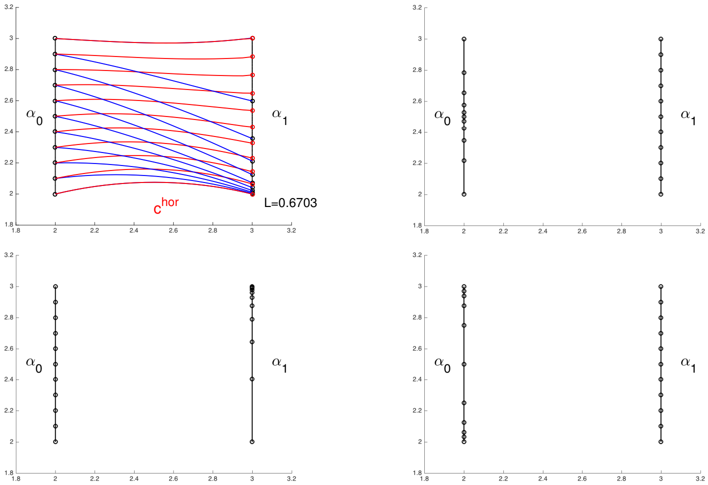

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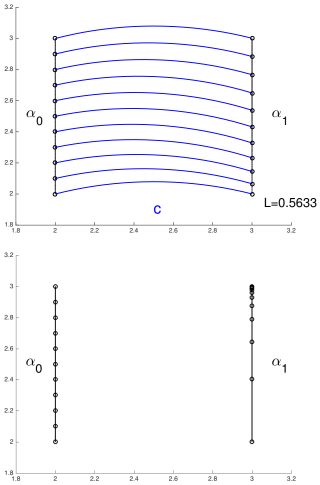

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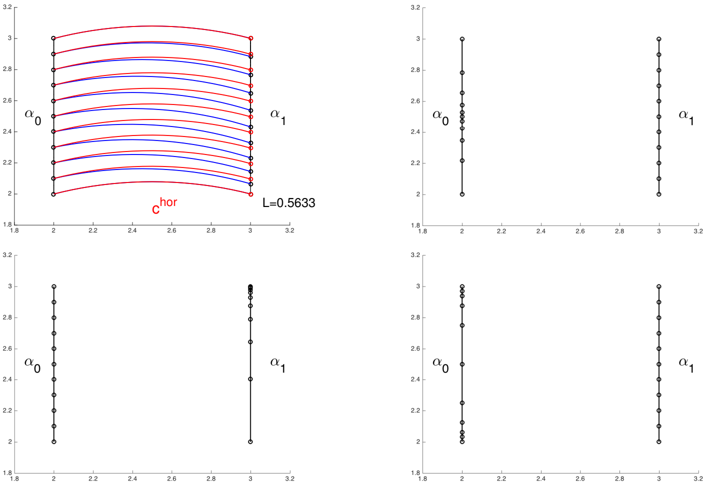

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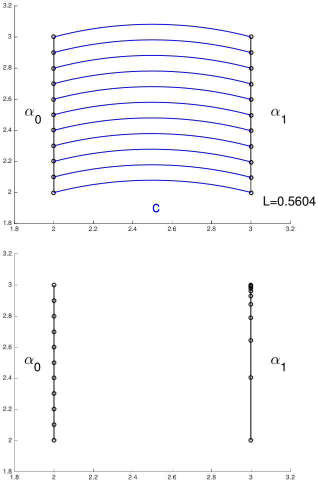

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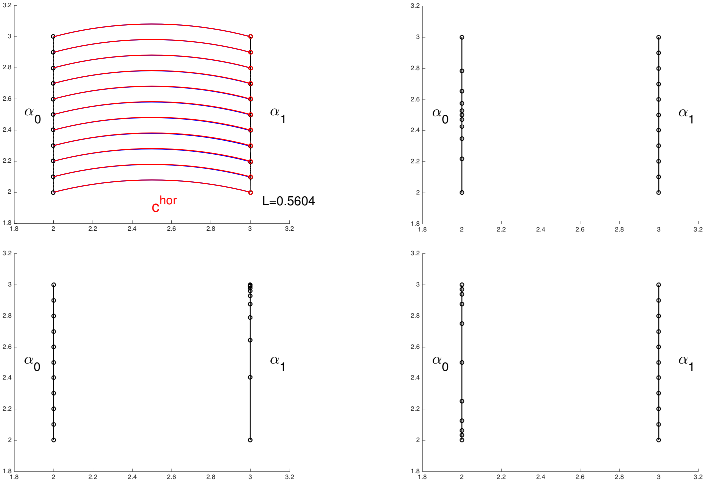

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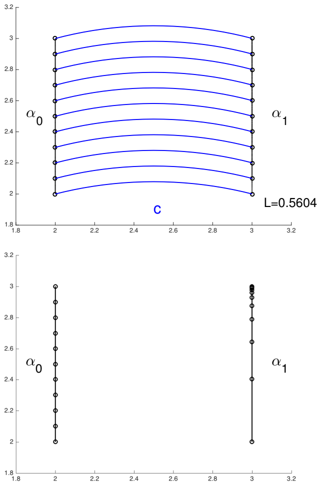

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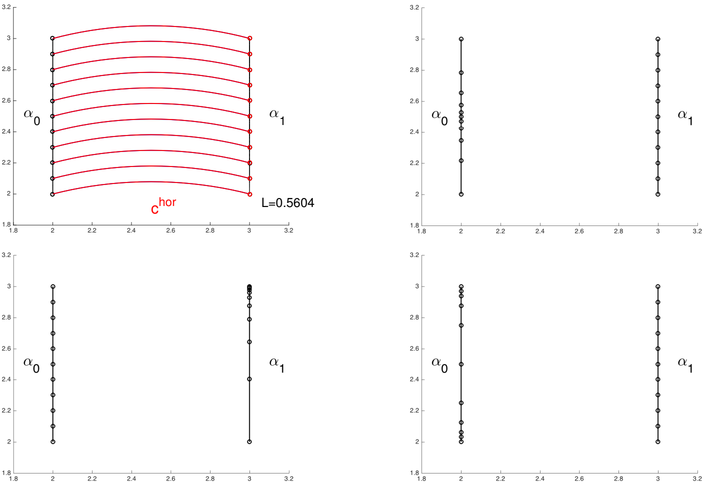

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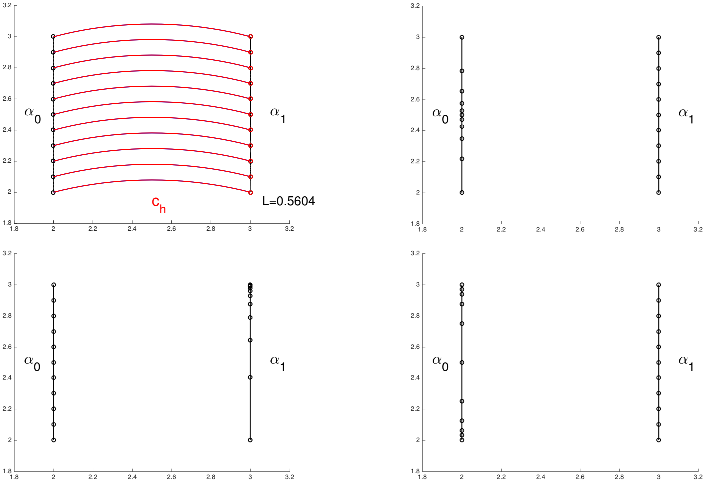

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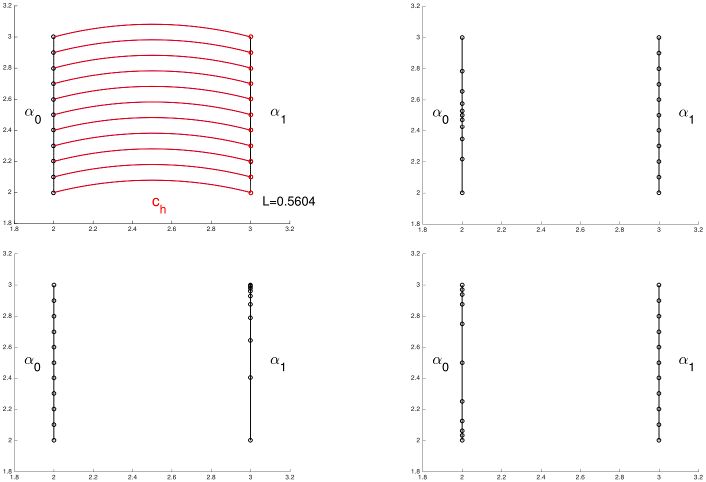

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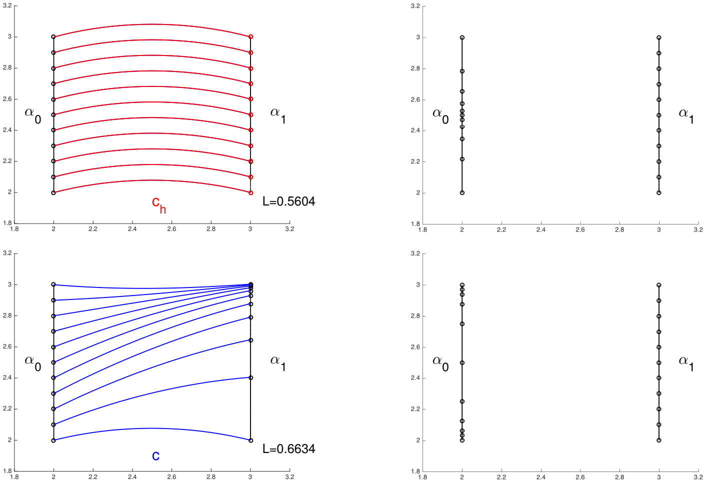

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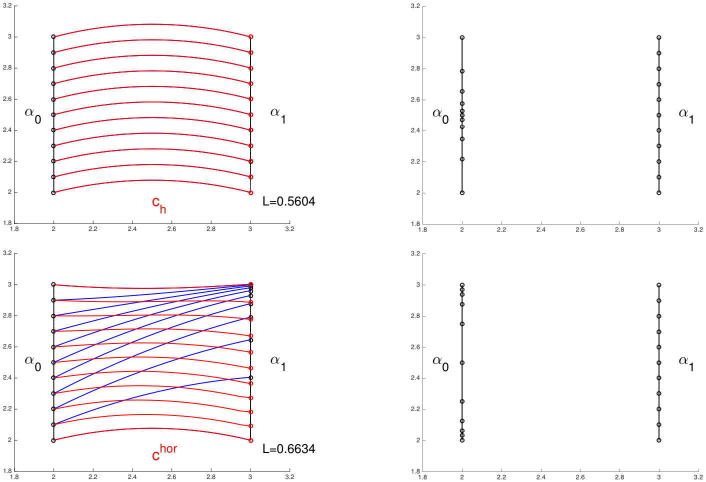

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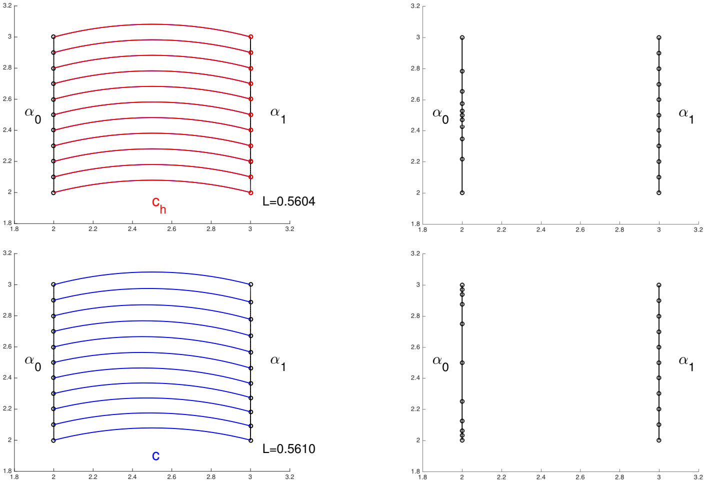

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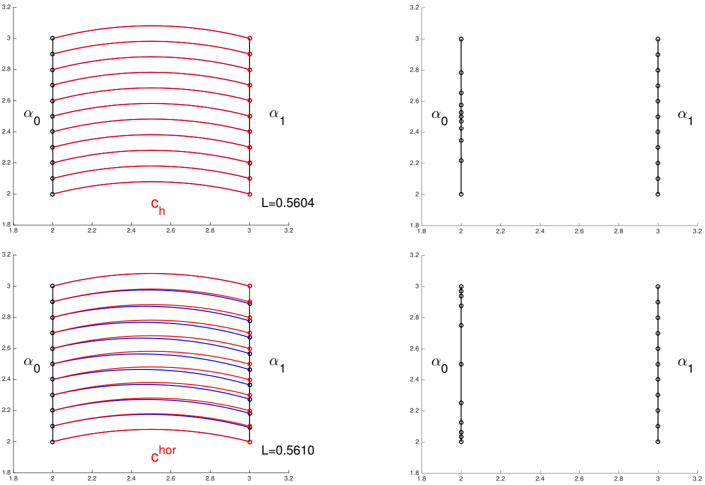

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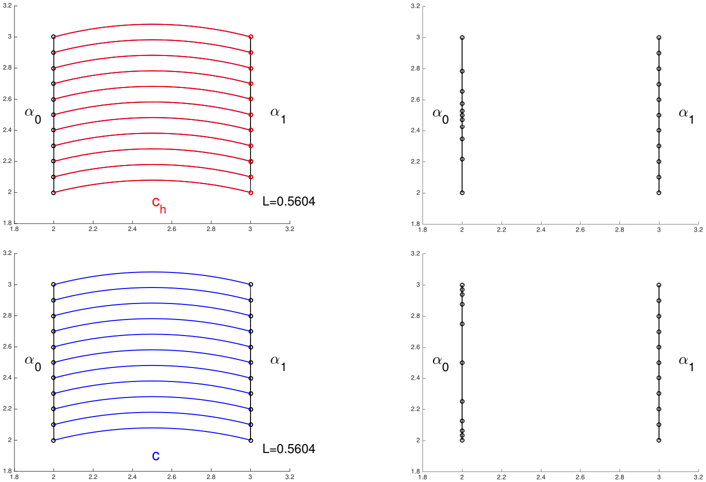

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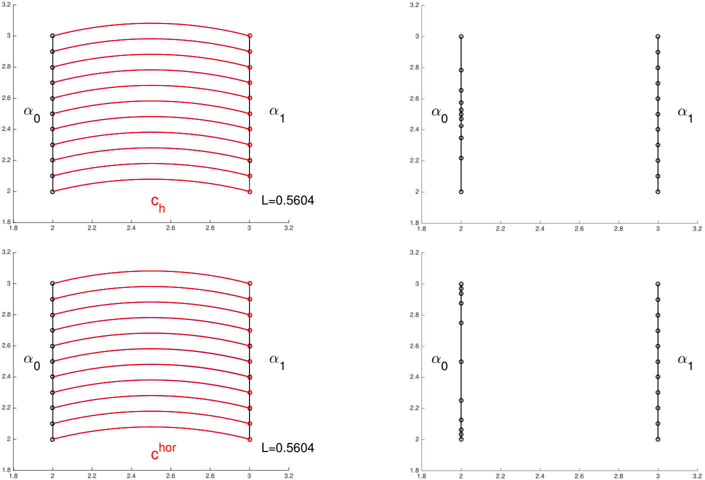

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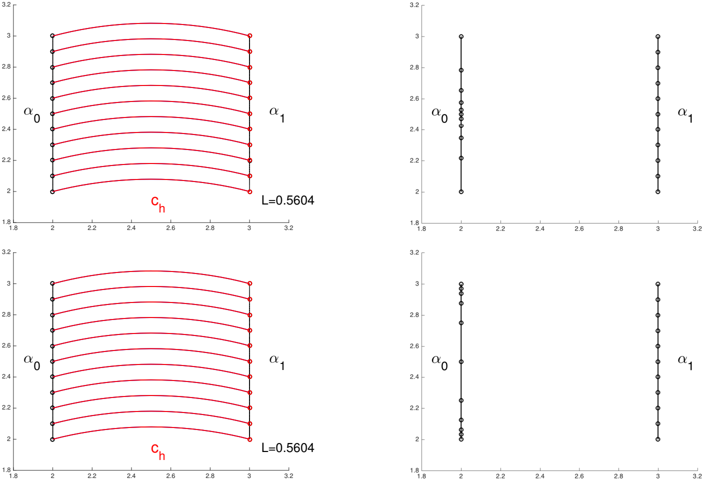

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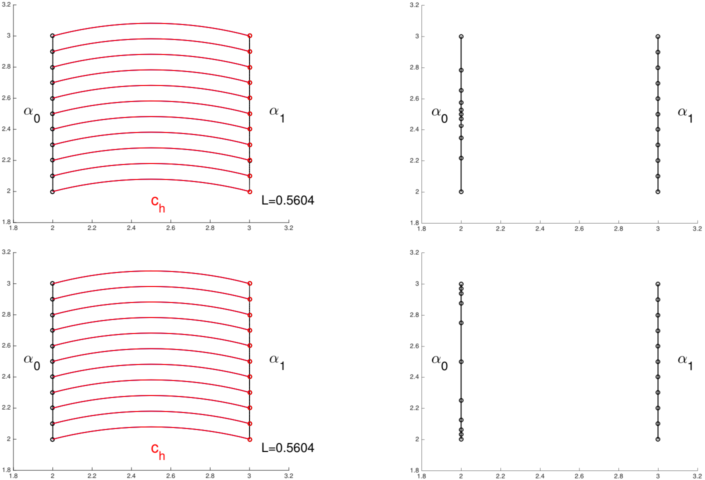

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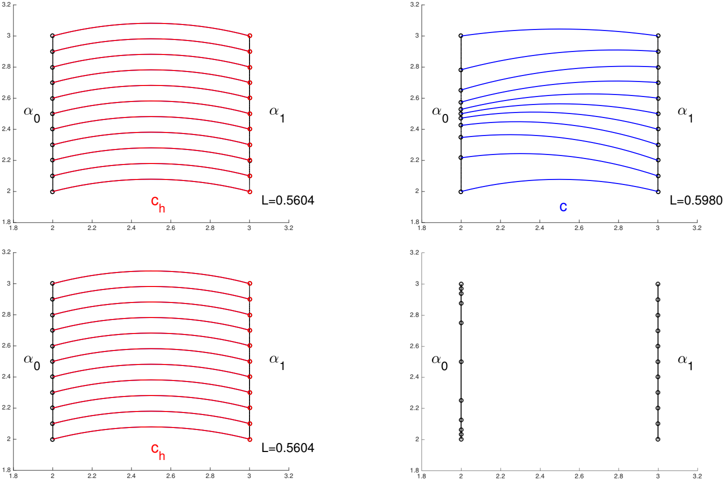

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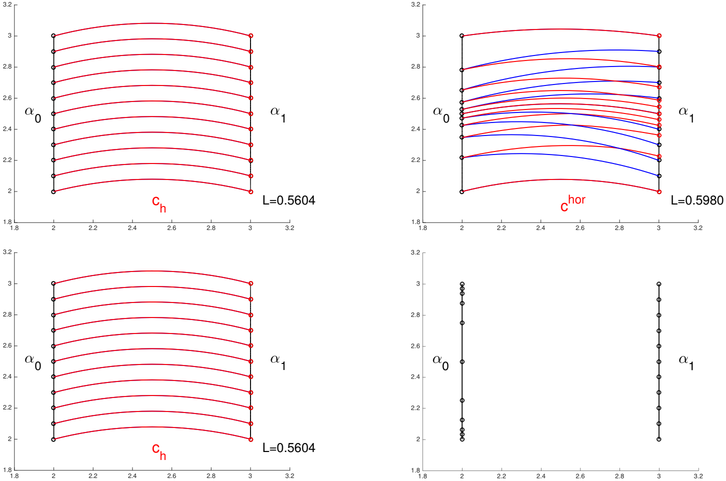

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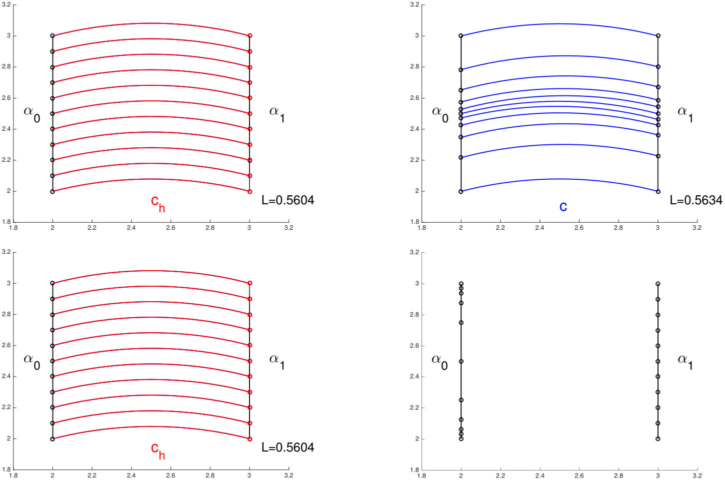

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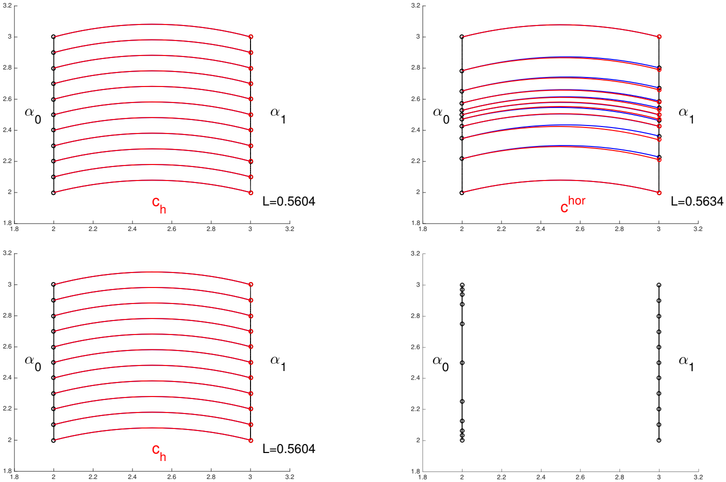

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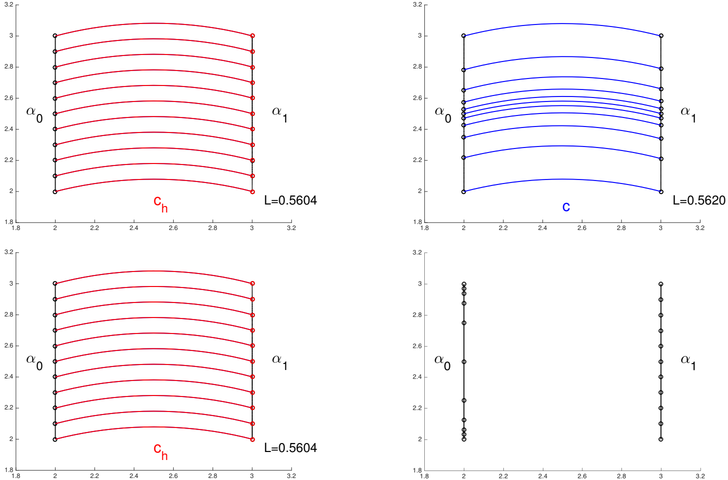

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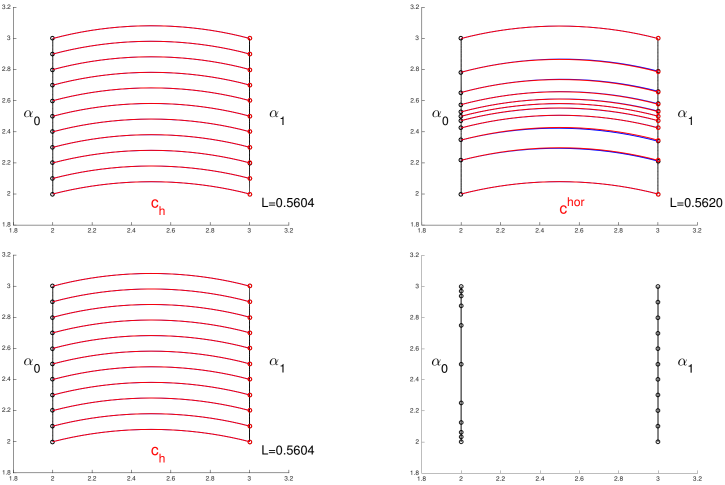

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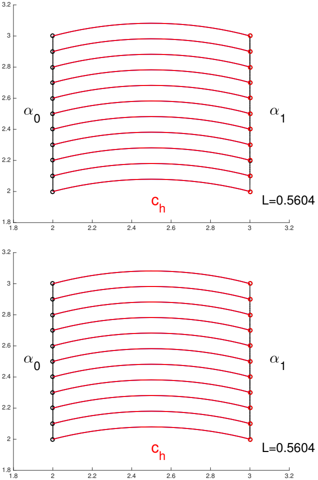

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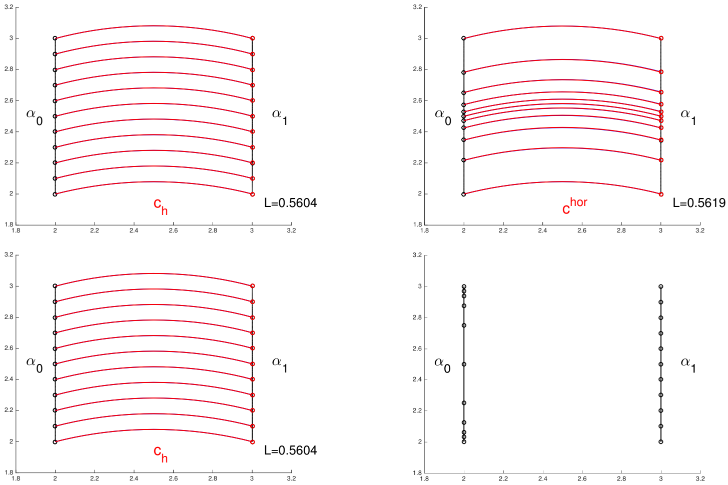
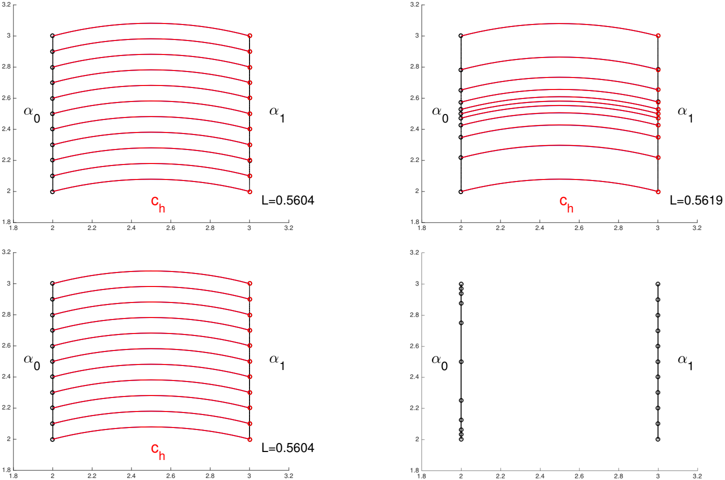
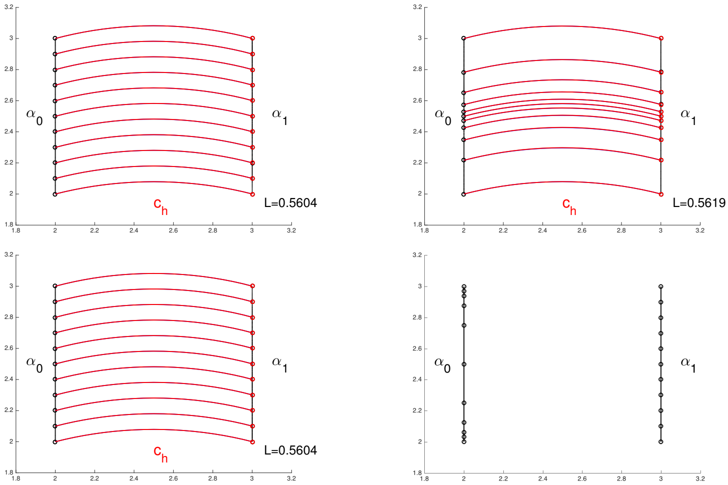


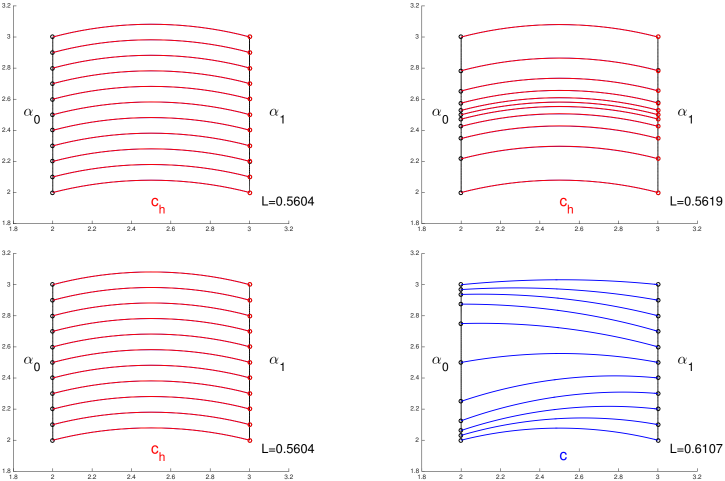
Algorithm in $\mathbb{H}^2$



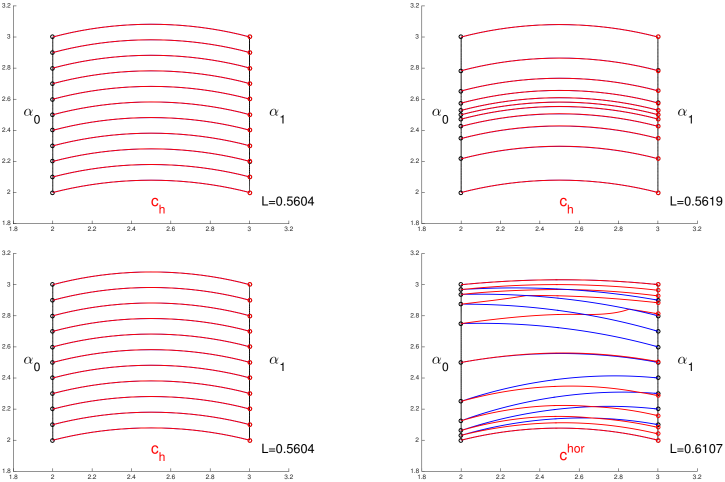
Algorithm in $\mathbb{H}^2$



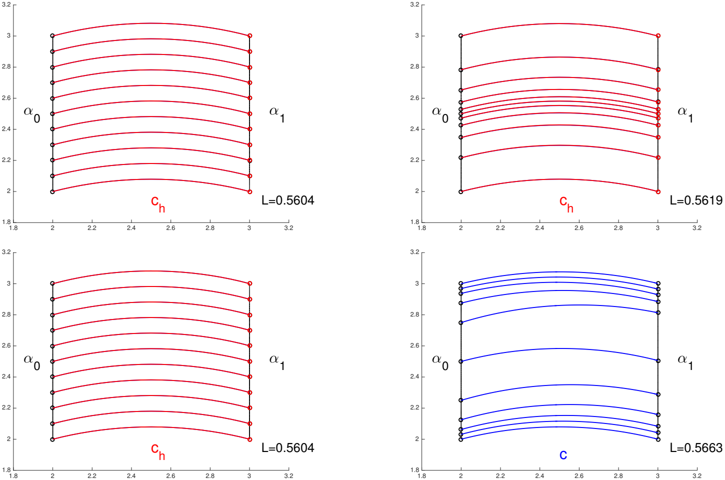
Algorithm in $\mathbb{H}^2$



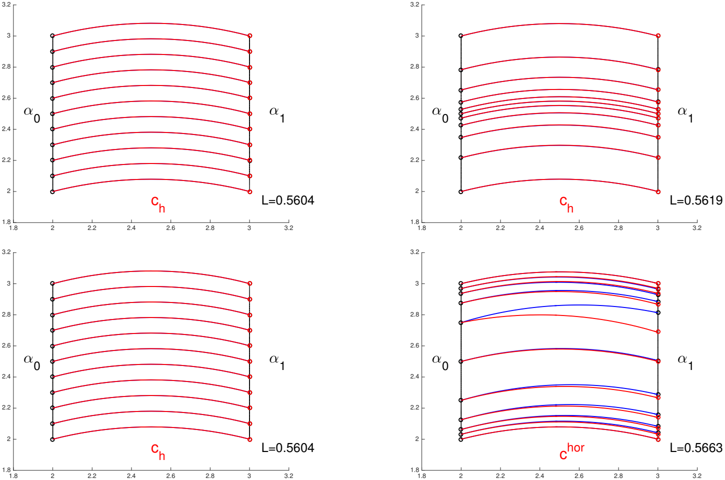
Algorithm in $\mathbb{H}^2$



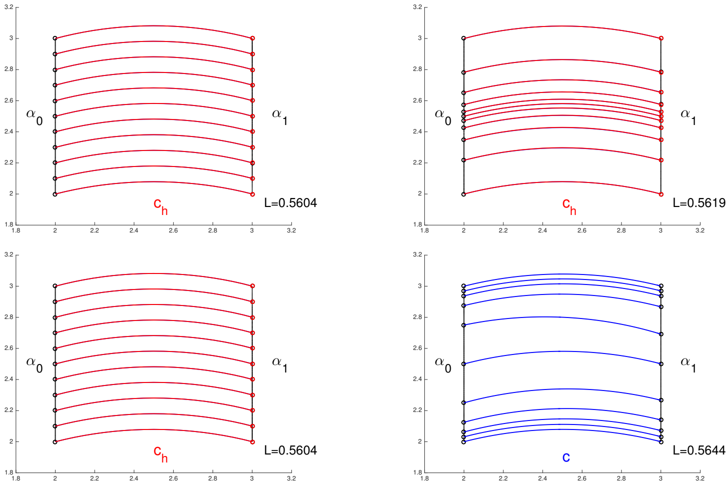
Algorithm in $\mathbb{H}^2$



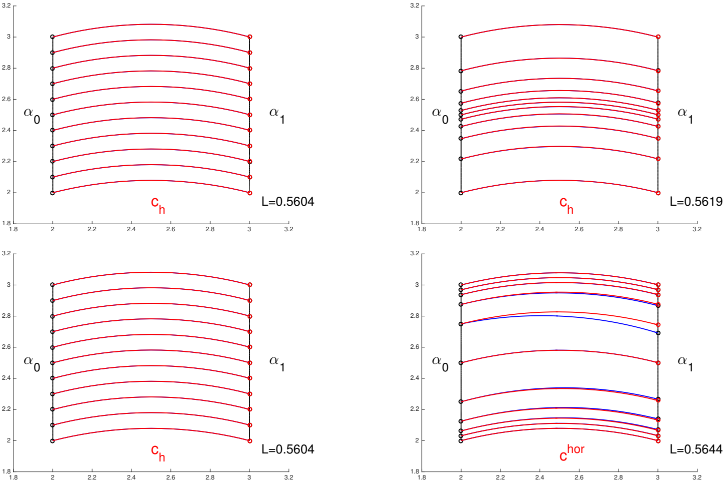
Algorithm in $\mathbb{H}^2$



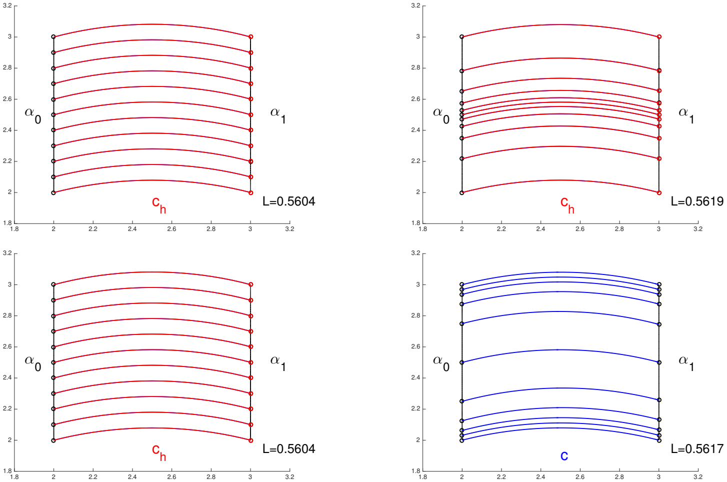
Algorithm in $\mathbb{H}^2$



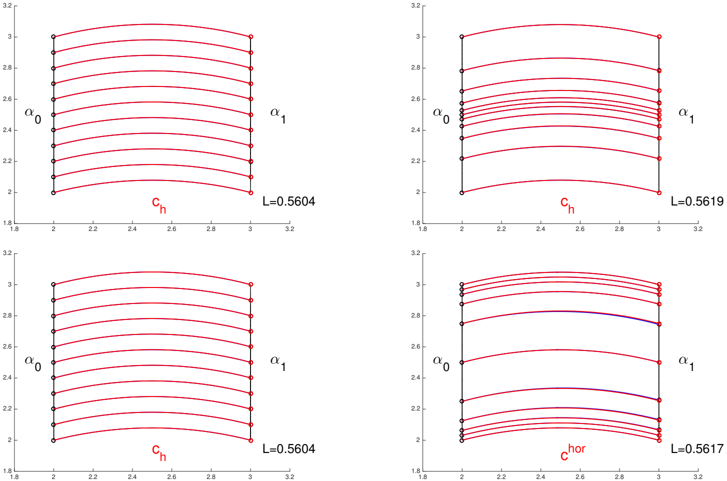
Algorithm in $\mathbb{H}^2$



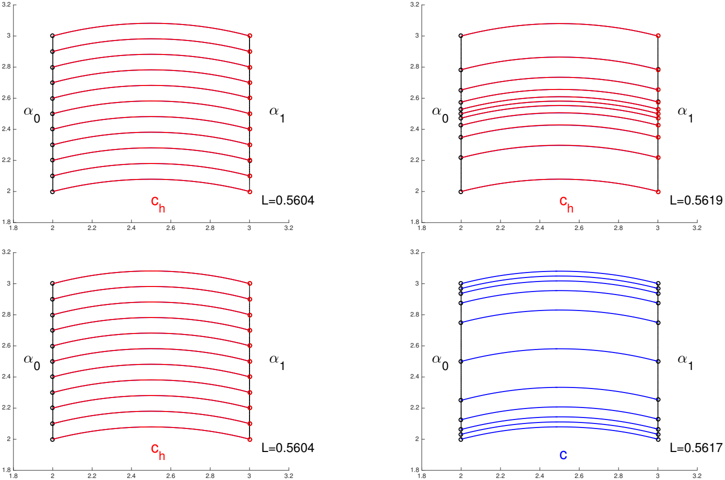
Algorithm in $\mathbb{H}^2$



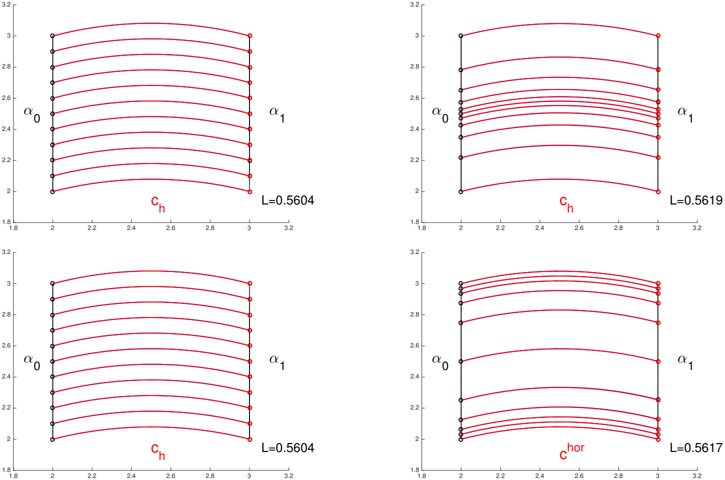
Algorithm in $\mathbb{H}^2$



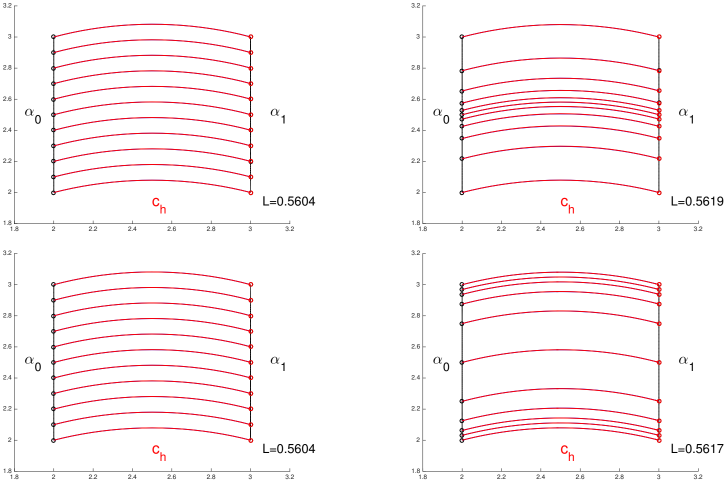
Algorithm in $\mathbb{H}^2$



Algorithm in $\mathbb{H}^2$

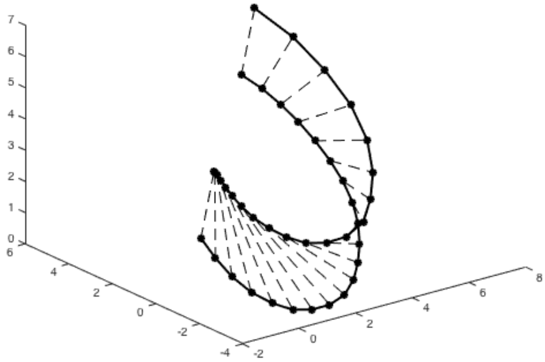


Algorithm in $\mathbb{H}^2$



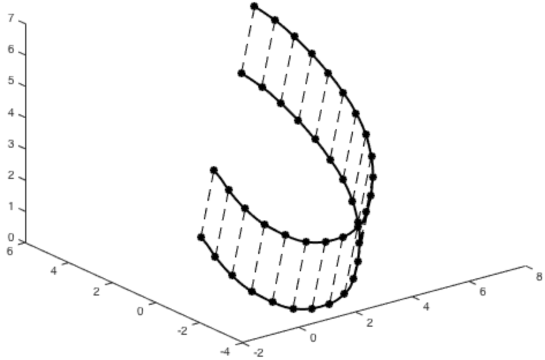
Examples of matchings

Sub-optimal matching



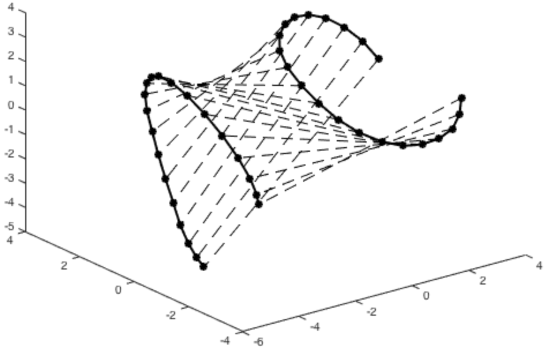
Examples of matchings

Optimal matching



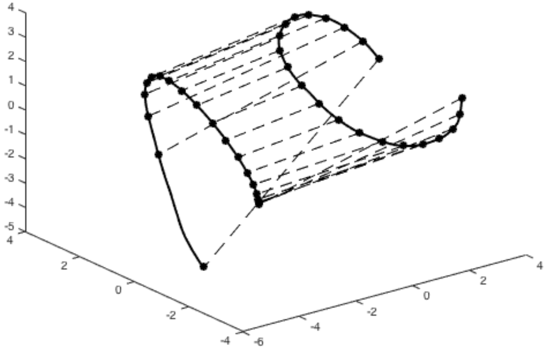
Examples of matchings

Sub-optimal matching



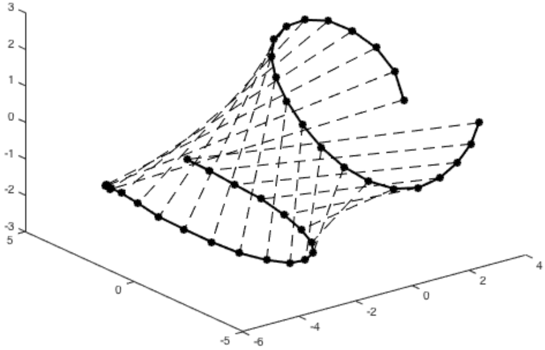
Examples of matchings

Optimal matching



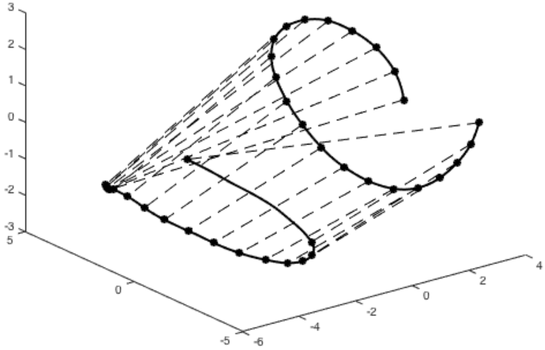
Examples of matchings

Sub-optimal matching

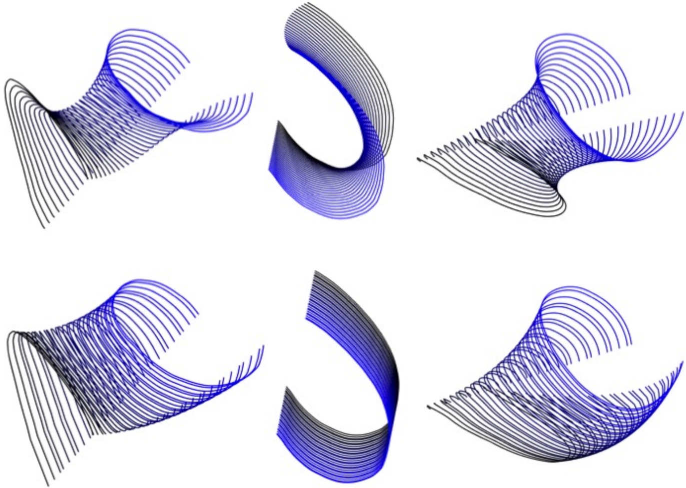


Examples of matchings

Optimal matching

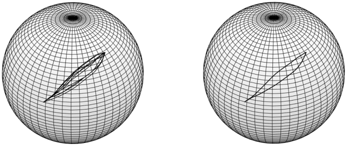


Geodesics between curves vs between shapes

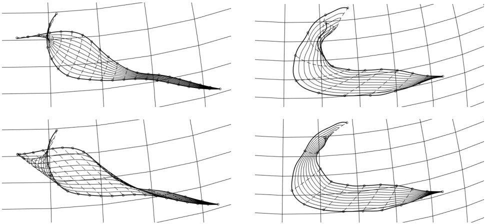


Real data applications

- Trajectory analysis



Clustering of plane trajectories



Comparison of hurricane tracks

Real data applications

- Mean shape of the internal ear (J. M. Loubes)

Thank you for your attention !