Shape Analysis, Moving frames and Infinite-dimensional Geometry

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Outline

Part I : Shape Analysis

1. Shape spaces as **Quotient versus Sections** of fiber bundles
2. 3 different ways of putting a **intrinsic Riemannian metric** on Shape space

Part II : Moving Frames

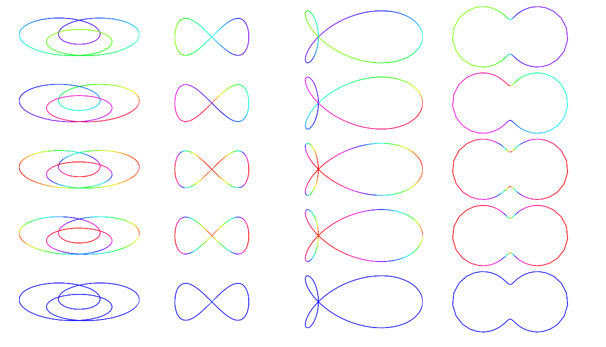
1. Cartan's method of moving frames
2. Resampling using structural invariants of shapes

Part III : Infinite-dimensional Geometry

1. What are the **Model** spaces of infinite-dimensional geometry?
2. What are the **Tools** from Functional Analysis?
3. Which **Geometric structures** can we consider?
4. What are the **Traps** of infinite-dimensional geometry?

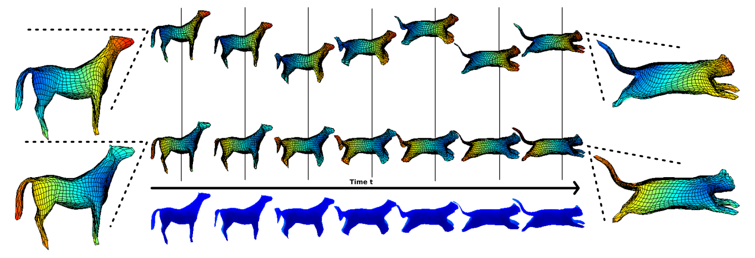
Part I : Shape analysis

Shape spaces



Pre-shape space $\mathscr{F} := \{f \text{ immersion } : \mathbb{S}^1 \to \mathbb{R}^2\} \subset \mathscr{C}^\infty(\mathbb{S}^1, \mathbb{R}^2)$
Shape space $\mathscr{S} :=$ 1-dimensional immersed submanifolds of $\mathbb{R}^2$

Shape spaces



Pre-shape space $\mathscr{F} := \{f \text{ embedding } : \mathbb{S}^2 \to \mathbb{R}^3\} \subset \mathscr{C}^\infty(\mathbb{S}^2, \mathbb{R}^3)$
Shape space $\mathscr{S} :=$ 2-dimensional submanifolds of $\mathbb{R}^3$

Shape spaces are non-linear manifolds

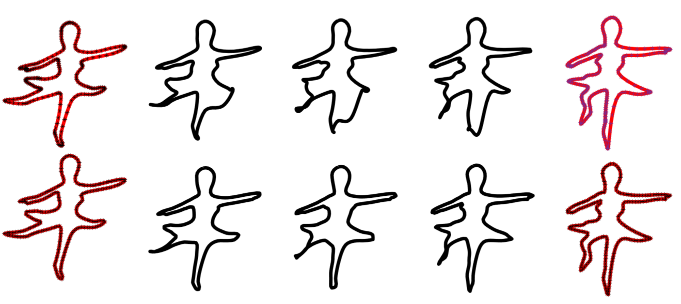
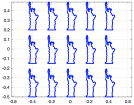
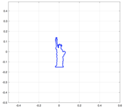
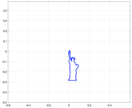
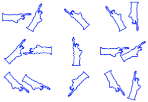
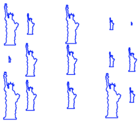
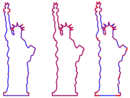


Figure: First line : linear interpolation between some parameterized ballerinas, second line : linear interpolation between arc-length parameterized ballerinas.

Group G	Some elements of one orbit under the group G	a preferred element in the orbit	another choice of preferred element in the orbit
$\mathbb{R}^3$ acting by translation		 centered curve : $\int_0^1 \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} \|f'(s)\| ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.	 curve starting at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
SO(3) acting by rotation		 axes of approximating ellipse aligned	 tangent vector at starting point horizontal

Group G	Some elements of one orbit under the group G	a preferred element in the orbit	another choice of preferred element in the orbit
$\mathbb{R}^+$ acting by scaling		 length = 1	 enclosed area =1
$\mathrm{Diff}^+([0,1])$ acting by reparameterization		 arc-length parameterization	 curvature proportional parameterization

For $I = [0,1]$ or $I = \mathbb{Z}/\mathbb{R} \simeq \mathbb{S}^1$, the space of smooth immersions

$$\mathcal{C}(I) = \bigcap_{k=1}^{\infty} \mathcal{C}^k(I) = \{\gamma \in \mathscr{C}^\infty(I, \mathbb{R}^2)/\mathbb{R}^2, \gamma'(s) \neq 0, \forall s \in I\}.$$

is an open set of $\mathscr{C}^\infty(I, \mathbb{R}^2)/\mathbb{R}^2$ for the topology induced by the family of norms $\|\cdot\|_{\mathscr{C}^k}$, hence a Fréchet manifold.

$$\mathcal{C}_1(I) = \{\gamma \in \mathcal{C}(I) \ : \int_0^1 |\gamma'(s)| ds = 1\}.$$

$$\mathscr{A}_1(I) = \{\gamma \in \mathcal{C}(I) \ : |\gamma'(s)| = 1, \ \forall s \in I\} \subset \mathcal{C}_1(I).$$

Theorem (A.B.T, S.Preston)

The subset $\mathcal{C}_1(I)$ is a tame $\mathscr{C}^\infty$-submanifold of $\mathcal{C}(I)$ and $\mathscr{A}_1(I)$ is a tame $\mathscr{C}^\infty$-submanifold of $\mathcal{C}(I)$, and thus also of $\mathcal{C}_1(I)$. Its tangent space at a curve γ is

$$T_\gamma \mathscr{A}_1 = \{w \in \mathscr{C}^\infty(\mathbb{S}^1, \mathbb{R}^2), w'(s) \cdot \gamma'(s) = 0, \quad \forall s \in \mathbb{S}^1\}.$$

Proof : Uses the implicit function theorem of Nash-Moser.

$\mathscr{G}(I) = \mathrm{Diff}^+([0,1])$ or $\mathrm{Diff}^+(\mathbb{S}^1)$ is a tame Fréchet Lie group [Hamilton].

Theorem (A.B.T, S.Preston)

The right action $\Gamma : \mathcal{C}(I) \times \mathscr{G}(I) \to \mathcal{C}(I)$, $\Gamma(\gamma, \psi) = \gamma \circ \psi$ of the group of reparameterizations $\mathscr{G}(I)$ on the tame Fréchet manifold $\mathcal{C}(I)$ is smooth and tame, and preserves $\mathcal{C}_1(I)$.

Theorem (A.B.T, S.Preston)

Given a curve $\gamma \in \mathcal{C}_1(I)$, let $p(\gamma) \in \mathscr{A}_1(I)$ denote its arc-length-reparameterization, so that $p(\gamma) = \gamma \circ \psi$ where

$$\psi'(s) = \frac{1}{|\gamma'(\psi(s))|}, \qquad \psi(0) = 0. \tag{1}$$

Then p is a smooth retraction of $\mathcal{C}_1(I)$ onto $\mathscr{A}_1(I)$.

Theorem (A.B.T, S.Preston)

$\mathscr{A}_1([0,1])$ is diffeomorphic to the quotient Fréchet manifold $\mathcal{C}_1([0,1])/\mathscr{G}([0,1])$.

Riemannian metrics on Shape space

We will consider the 2-parameter family of elastic metrics on $\mathcal{C}_1(I)$ introduced by Mio et al. :

$$G^{a,b}(w,w) = \int_0^1 \left(a\left(D_s w \cdot \mathrm{v}\right)^2 + b\left(D_s w \cdot \mathrm{n}\right)^2\right) |\gamma'(t)|\, dt, \qquad (2)$$

where a and b are positive constants, γ is any parameterized curve in $\mathcal{C}_1(I)$, w is any element of the tangent space $T_\gamma \mathcal{C}_1(I)$, with $D_s w = \frac{w'}{|\gamma'|}$ denoting the arc-length derivative of w, $\mathrm{v} = \gamma'/|\gamma'|$ and $\mathrm{n} = \mathrm{v}^\perp$.

Since the reparameterization group preserves the elastic metric $G^{a,b}$, it defines a quotient elastic metric on the quotient space $\mathcal{C}_1([0,1])/\mathscr{G}([0,1])$, which we will denote by $\overline{G}^{a,b}$.

$$\overline{G}^{a,b}([w],[w]) = \inf_{u \in T_\gamma \mathscr{O}} G^{a,b}(w+u, w+u)$$

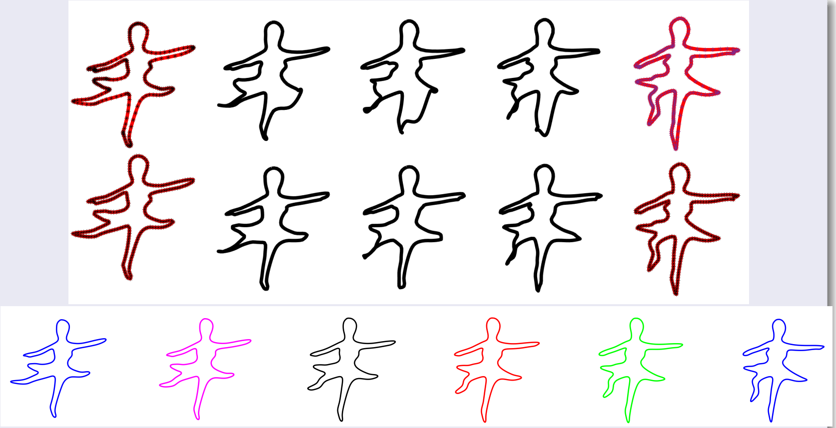


Figure: First line : linear interpolation between some parameterized ballerinas, second line : linear interpolation between arc-length parameterized ballerinas. Geodesic between some parameterized ballerinas with 200 points using Qmap : execution time = 350 s.

Since $\mathscr{A}_1([0,1])$ is diffeomorphic to the quotient Fréchet manifold $\mathcal{C}_1([0,1])/\mathscr{G}([0,1])$, we can pull-back the quotient elastic metric $\overline{G}^{a,b}$ to the space of arc-length parameterized curves $\mathscr{A}_1([0,1])$ and define

$$\widetilde{G}^{a,b}(w,w) = G^{a,b}([w],[w]) = \inf_{u \in T_\gamma \mathscr{O}} G^{a,b}(w+u, w+u)$$

where w is tangent to $\mathscr{A}_1([0,1])$.
If $T_\gamma \mathcal{C}_1([0,1])$ decomposes as $T_\gamma \mathcal{C}_1([0,1]) = T_\gamma \mathscr{O} \oplus Hor_\gamma$, this minimum is achieved by the unique vector $P_h(w) \in [w]$ belonging to the horizontal space Hor_γ at γ. In this case:

$$\widetilde{G}^{a,b}(w,w) = G^{a,b}(P_h(w), P_h(w)), \tag{3}$$

where $P_h(w) \in T_\gamma \mathcal{C}_1([0,1])$ is the projection of w onto the horizontal space.

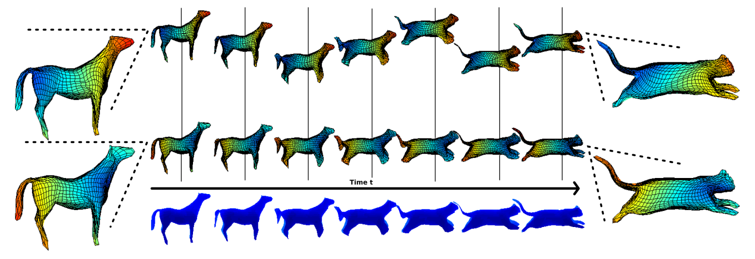
Theorem (A.B.T- S. Preston)

Let w be a tangent vector to the manifold $\mathscr{A}_1([0,1])$ at γ and write $w' = \Phi\, \mathrm{n}$, where Φ is a real function in $\mathscr{C}^\infty([0,1],\mathbb{R})$. Then the projection $P_h(w)$ of $w \in T_\gamma\mathscr{A}_1([0,1])$ onto the horizontal space Hor_γ reads $P_h(w) = w - m\,\mathrm{v}$ where $m \in \mathscr{C}^\infty([0,1],\mathbb{R})$ is the unique solution of

$$-\frac{a}{b}m'' + \kappa^2 m = \kappa\Phi, \qquad m(0) = 0, \quad m(1) = 0 \tag{4}$$

where κ is the curvature function of γ.

A.B.T., S. Preston, *Quotient elastic metrics on the manifold of arc-length parameterized plane curves*, Journal of Geometric Mechanics.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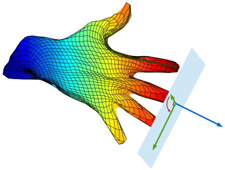
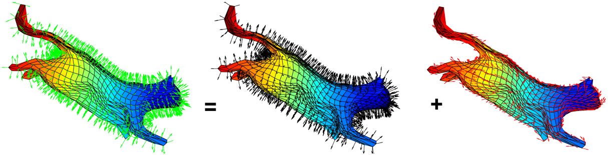
Canonical parameterizations of surfaces

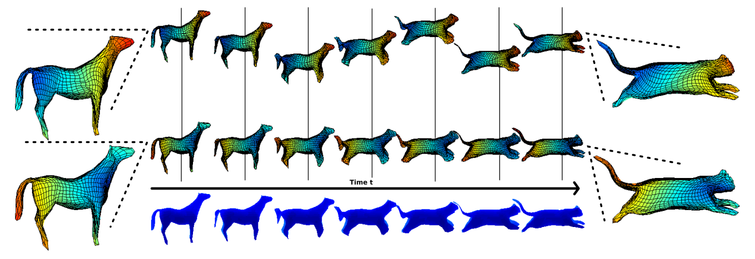


Genus-0 surfaces of $\mathbb{R}^3$ are *Riemann surfaces*. Since they are compact and simply connected, the Uniformization Theorem says that they are conformally equivalent to the unit sphere. This means that, given a spherical surface, there exists a homeomorphism, called the *uniformization map*, which preserves the angles and transforms the unit sphere into the surface.

$\Rightarrow$ This gives a canonical parameterization of the surface modulo the choice of 3 points. (or unique modulo the action of $PSL(2, \mathbb{C})$).

Gauge invariante degenerate Riemannian metrics





A.B.Tumpach, H. Drira, M. Daoudi, A. Srivastava, *Gauge invariant Framework for shape analysis of surfaces*, IEEE TPAMI.
A.B.Tumpach, *Gauge invariance of degenerate Riemannian metrics*, Notices of AMS.

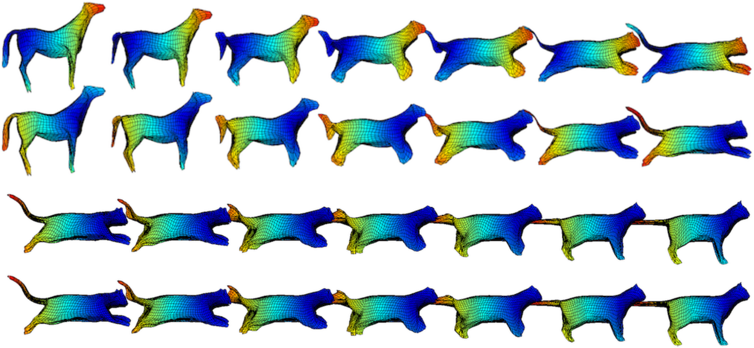
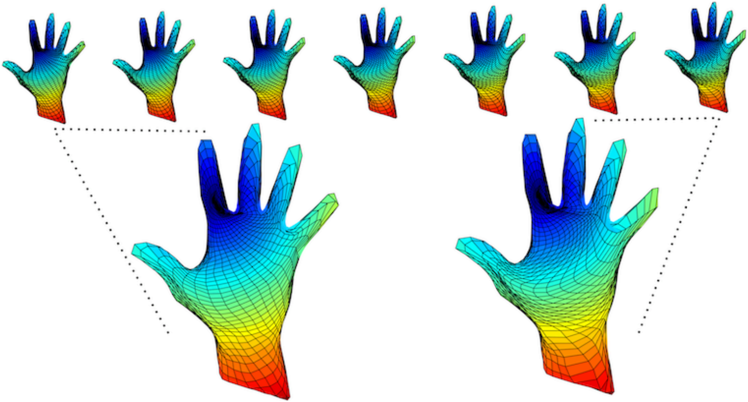


Figure: Pairs of paths projecting to the same path in Shape space, but with different parametrizations. The energies of these paths, as computed by our program, are respectively (from the upper row to the lower row): $E_\Delta = 225.3565$, $E_\Delta = 225.3216$, $E_\Delta = 180.8444$, $E_\Delta = 176.8673$.



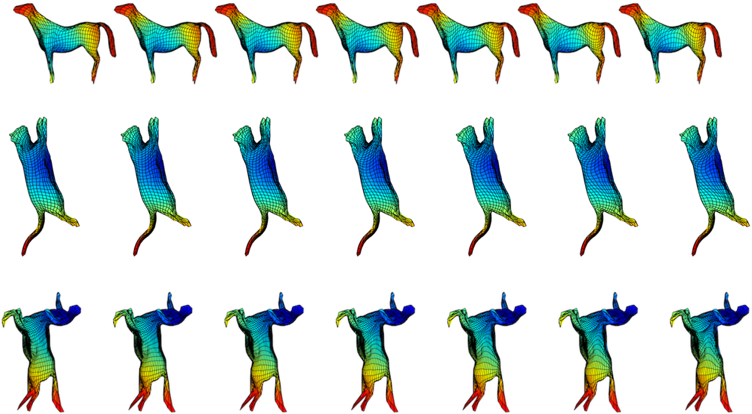


Figure: Four Paths connecting the same shape but with a parametrization depending smoothly on time. The energy computed by our program is respectively $E_\Delta = 0$ for the path of hands, $E_\Delta = 0.1113$ for the path of horses, $E_\Delta = 0$ for the path of cats, and $E_\Delta = 0.0014$ for the path of Centaurs.

Part II

Cartan's method of Moving frames

f = curve in an homogeneous space G/H,
$\hat{f}$ = curve in G projecting to f.
Suppose that we have a natural procedure to associate $\hat{f}$ to f. Then:
$c = \hat{f}^{-1}\frac{d}{ds}\hat{f}$ is a curve in the Lie algebra of G such that

1. c remains the same if one replace f by any $g \cdot f$ with $g \in G$.
2. from c one can recover the initial curve f, uniquely modulo the action of G.

$\Rightarrow$ The Lie-algebra valued curve is characteristic of the orbit of f under G , and is a geometric invariant of the G/H-valued curve.

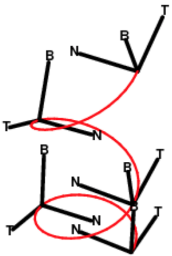
Curves in $\mathbb{R}^3$: Frenet frame

$f \; : I \to \mathbb{R}^3$ parameterized by arc-length.

Unit tangent vector : $\vec{v}(s) = f'(s)$,

Unit normal vector : $\vec{n}(s) = \frac{f''(s)}{\|f''(s)\|}$,

Unit bi-normal : $\vec{b}(s) = \vec{v}(s) \wedge \vec{n}(s)$.



Curves in $\mathbb{R}^3$: Frenet-Serret equations

Frenet-Serret equations with $\kappa =$ *curvature* and $\tau =$ *torsion* :

$$\begin{cases} D_s\vec{v} = \kappa\vec{n} \\ D_s\vec{n} = -\kappa\vec{v} + \tau\vec{b} \\ D_s\vec{b} = -\tau\vec{n}, \end{cases}$$

$$\Leftrightarrow O(s)^{-1}\frac{d}{ds}O(s) = \begin{pmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \in \mathfrak{so}(3),$$

$$\text{for } O(s) = \begin{pmatrix} \vec{v}(s) & \vec{n}(s) & \vec{b}(s) \end{pmatrix}.$$

Curves in $\mathbb{R}^3$: Reparameterization taking curvature and torsion into account

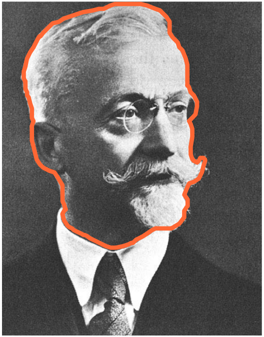
Endowing the space $\mathscr{C}^\infty([0,1], \mathfrak{so}(3))$ with the L^2 metric given by

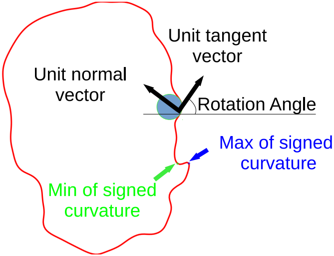
$$\langle\!\langle A, B \rangle\!\rangle = -\frac{1}{2}\int_0^1 \mathrm{Tr}(A(s)B(s))ds.$$

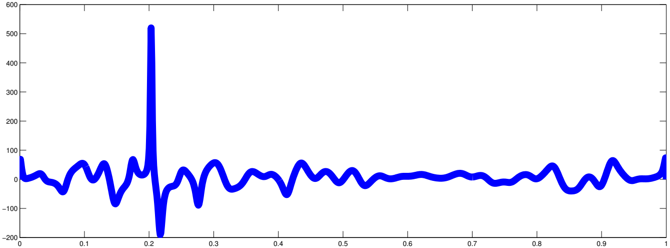
Given a curve in $\mathbb{R}^3$ parameterized proportionally to arc-length, the speed of the corresponding moving frame $s \mapsto O(s)$ with respect to the scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is $\sqrt{\kappa(s)^2 + \tau(s)^2}$. Now the parameterization of the 3D curve proportional to curvature-length corresponds to parameterization proportional to arc-length of the corresponding moving frame. The corresponding parameter is

$$r(s) = \frac{\int_0^s \sqrt{\kappa(s)^2 + \tau(s)^2}ds}{\int_0^1 \sqrt{\kappa(s)^2 + \tau(s)^2}ds}.$$

Curves in $\mathbb{R}^2$






600
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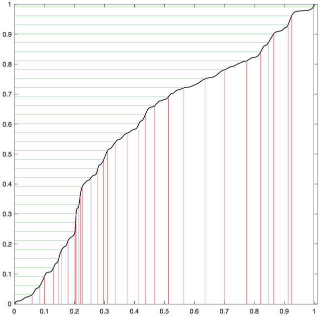
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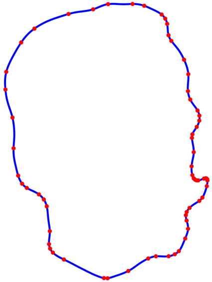

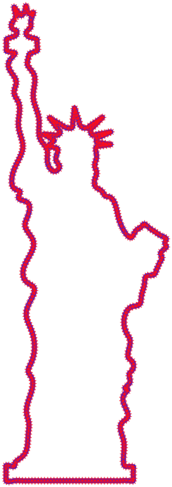

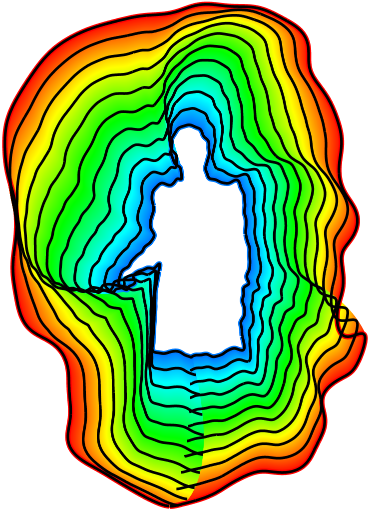


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American Mathematical Society
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What are the Model spaces of infinite-dimensional geometry?

Hilbert $\subset$ Banach $\subset$ Fréchet $\subset$ Locally Convex spaces

Hilbert space H = complete vector space for the distance given by an inner product = $\langle \cdot, \cdot \rangle \ : H \times H \to \mathbb{R}^+$

- symmetric :$\langle x, y \rangle = \langle y, x \rangle$
- bilinear : $\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle$
- non-negative : $\langle x, x \rangle \geq 0$
- definite : $\langle x, x \rangle = 0 \Rightarrow x = 0$

$H^* = H$ (Riesz Theorem).

What are the Model spaces of infinite-dimensional geometry?

Hilbert $\subset$ **Banach** $\subset$ Fréchet $\subset$ Locally Convex spaces

Banach space B = complete vector space for the distance given by a norm $= \|\cdot\| \ : B \to \mathbb{R}^+$

- triangle inequality : $\|x+y\| \leq \|x\| + \|y\|$
- absolute homogeneity : $\|\lambda x\| = |\lambda| \|x\|$.
- non-negative : $\|x\| \geq 0$
- definite : $\|x\| = 0 \Rightarrow x = 0$.

B^* = Banach space.

What are the Model spaces of infinite-dimensional geometry?

Hilbert $\subset$ Banach $\subset$ **Fréchet** $\subset$ Locally Convex spaces

Fréchet space F = complete Hausdorff vector space for the distance $d \; : F \times F \rightarrow \mathbb{R}^+$ given by a countable family of semi-norms $\| \cdot \|_n$:

$$d(x, y) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

$F^* \neq$ Fréchet space in general, but locally convex
$F^{**} =$ Fréchet space.

What are the Model spaces of infinite-dimensional geometry?

Hilbert $\subset$ Banach $\subset$ Fréchet $\subset$ **Locally Convex spaces**

Locally Convex spaces = Hausdorff topological vector space whose topology is given by a (possibly not countable) family of semi-norms.

References :

- Klingenberg : *Riemannian Geometry*
- Lang : *Differential and Riemannian manifolds*
  Fondamentals of Differential Geometry
- Hamilton : *The inverse function theorem of Nash-Moser*
- A. Kriegl and P. Michor : *Convenient setting of Global Analysis*

What are the Tools from Functional Analysis?

Theorems :	Hilbert	Banach	Fréchet	Locally Convex
Banach-Picard	$\surd$	$\surd$	$\surd$	X
Open Mapping	$\surd$	$\surd$	$\surd$	F webbed G limit of Baire
Hahn-Banach	$\surd$	$\surd$	$\surd$	$\surd$
Inverse function	$\surd$	$\surd$	Nash-Moser	X

Which **Geometric structures** can we consider?

Riemannian $\subset$ Symplectic $\subset$ Poisson Geometry

Riemannian metric = smoothly varying inner product on a manifold M

$$\begin{array}{cccc} g_x & : & T_xM \times T_xM & \rightarrow & \mathbb{R} \\ & & (U, V) & \mapsto & g_x(U, V) \end{array}$$

strong Riemannian metric = for every $x \in M$, $g_x \ : T_xM \rightarrow (T_xM)^*$ is an isomorphism

weak Riemannian metric = for every $x \in M$, $g_x \ : T_xM \rightarrow (T_xM)^*$ is just injective

Levi-Cevita connection may not exist for a weak Riemannian metric.

Which **Geometric structures** can we consider?

Riemannian $\subset$ **Symplectic** $\subset$ Poisson Geometry

Symplectic form = smoothly varying skew-symmetric bilinear form

$$\begin{array}{cccc} \omega_x & : & T_xM \times T_xM & \rightarrow & \mathbb{R} \\ & & (U,V) & \mapsto & \omega_x(U,V) \end{array}$$

with $dw = 0$ and $(T_xM)^{\perp_w} = \{0\}$

strong symplectic form = for every $x \in M$, $\omega_x \ : T_xM \rightarrow (T_xM)^*$ is an isomorphism

weak symplectic form = for every $x \in M$, $\omega_x \ : T_xM \rightarrow (T_xM)^*$ is just injective

Darboux Theorem does not hold for a weak symplectic form

Which **Geometric structures** can we consider?

Riemannian $\subset$ **Symplectic** $\subset$ Poisson Geometry

Hamiltonian Mechanics

(M, g) **strong Riemannian manifold**

- $$\begin{array}{rcccl} \flat & : & T_xM & \simeq & T_x^*M \\ & & U & \mapsto & g_x(U, \cdot) \end{array} \qquad \flat^{-1} = \sharp$$
- **Kinetic energy = Hamiltonian**
  $$\begin{array}{rcccl} H & : & T^*M & \to & \mathbb{R} \\ & & \eta_x & \mapsto & g_x(\eta_x^\sharp, \eta_x^\sharp) \end{array}$$

(T^*M, ω) **strong symplectic manifold**

- $\pi \ : T^*M \to M$
- $\omega = d\theta$
- $$\begin{array}{rcccl} \theta_{(x,\eta)} & : & T_{x,\eta}T^*M & \to & \mathbb{R} \\ & & X & \mapsto & \eta(\pi_*(X)) \end{array}$$ **Liouville 1-form**

geodesic flow = flow of Hamiltonian vector field $X_H : dH = \omega(X_H, \cdot)$

Which **Geometric structures** can we consider?

Riemannian $\subset$ **Symplectic** $\subset$ **Poisson Geometry**

Poisson bracket = family of bilinear maps
$\{\cdot,\cdot\}_U : \mathscr{C}^\infty(U) \times \mathscr{C}^\infty(U) \to \mathscr{C}^\infty(U)$, U open in M with

- skew-symmetry $\{f,g\}_U = -\{g,f\}_U$
- Jacobi identity $\{f,\{g,h\}_U\}_U + \{g,\{h,f\}_U\}_U + \{h,\{f,g\}_U\}_U = 0$
- Leibniz rule $\{f,gh\}_U = \{f,g\}_U h + g\{f,h\}_U$

A strong symplectic form defines a Poisson bracket by
$\{f,g\} = \omega(X_f, X_g)$ where $df = \omega(X_f,\cdot)$ and $dg = \omega(X_g,\cdot)$

A Poisson bracket may not be given by a bivector field

D. Beltita, T. Golinski, A.B.T., *Queer Poisson Brackets*, Journal of Geometry and Physics.

Which **Geometric structures** can we consider?

$$\left.\begin{array}{l}\text{Riemannian}\\ \text{Symplectic}\\ \text{Complex}\end{array}\right\} \subset \text{ Kähler} \subset \text{hyperkähler Geometry}$$

Complex structure $=$ smoothly varying endomorphism J
of the tangent space s.t. $J^2 = -1$.

Integrable complex structure : s. t. there exists an holomorphic atlas
Formally integrable complex structure : with Nijenhuis tensor $= 0$

Newlander-Nirenberg Theorem is not true in general :
formal integrability does not imply integrability.

What are the traps of infinite-dimensional geometry?

"Never believe anything you have not proved yourself!"

- The distance function associated to a Riemannian metric may be 0
- Levi-Cevita connection may not exist for weak Riemannian metrics
- Hopf-Rinow Theorem does not hold in general
- Darboux Theorem does not apply to weak symplectic forms
- A formally integrable complex structure does not imply the existence of a holomorphic atlas
- the tangent space differs from the space of derivations (even on a Hilbert space)
- a Poisson bracket may not be given by a bivector field (even on a Hilbert space)
- Lie algebras may not integrate to Lie groups
- partitions of unity may not exist
- at least 14 differents ways to define tensor products

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Poisson bracket not given by a Poisson tensor

$\mathscr{H}$ separable Hilbert space

Kinetic tangent vector $X \in T_x\mathscr{H}$ equivalence classes of curves $c(t)$, $c(0) = x$, where $c_1 \sim c_2$ if they have the same derivative at 0 in a chart.

Operational tangent vector $x \in \mathscr{H}$ is a linear map $D : C_x^\infty(\mathscr{H}) \to \mathbb{R}$ satisfying Leibniz rule :

$$D(fg)(x) = Df\ g(x) + f(x)\ Dg$$

Poisson bracket not given by a Poisson tensor

Ingredients :

- Riesz Theorem
- Hahn-Banach Theorem
- compact operators $\mathscr{K}(\mathscr{H}) \subsetneq \mathscr{B}(\mathscr{H})$ bounded operators

  $\Rightarrow \exists \ell \in \mathscr{B}(\mathscr{H})^*$ such that $\ell(\mathrm{id}) = 1$ and $\ell_{|\mathscr{K}(\mathscr{H})} = 0$.

Queer tangent vector [Kriegl-Michor]

Define $D_x : C_x^\infty(\mathscr{H}) \to \mathbb{R}$, $D_x(f) = \ell(d^2 f(x))$, where the bilinear map $d^2 f(x)$ is identified with an operator $A \in \mathscr{B}(\mathscr{H})$ by Riesz Theorem

$$d^2 f(x)(X, Y) = \langle X, AY \rangle$$

Then D_x is an operational tangent vector at $x \in \mathscr{H}$ of order 2

Poisson bracket not given by a Poisson tensor

Queer tangent vector [Kriegl-Michor]

$$
\begin{aligned}
d^2(fg)(x) &= d^2 f(x).g(x) + df(x) \otimes dg(x) \\
&\quad + dg(x) \otimes df(x) + f(x) d^2 g(x)(X, Y)
\end{aligned}
$$

$$
\begin{aligned}
D_x(fg) &= \ell(d^2(fg)(x)) \\
&= \ell(d^2 f(x)).g(x) + f(x)\ell(d^2 g(x)) \\
&\quad + \ell(df(x) \otimes dg(x)) + \ell(dg(x) \otimes df(x)) \\
&= D_x f \; g(x) + f(x) \; D_x g
\end{aligned}
$$

Poisson bracket not given by a Poisson tensor

Theorem (Beltita-Golinski-T.)

Consider $\mathscr{M} = \mathscr{H} \times \mathbb{R}$. Denote points of $\mathscr{M}$ as (x, λ). Then $\{\cdot, \cdot\}$ defined by

$$\{f, g\}(x, \lambda) := D_x\left(f(\cdot, \lambda)\right) \frac{\partial g}{\partial \lambda}(x, \lambda) - \frac{\partial f}{\partial \lambda}(x, \lambda) D_x\left(g(\cdot, \lambda)\right)$$

a queer Poisson bracket on $\mathscr{H} \times \mathbb{R}$, in particular it can not be represented by a bivector field $\Pi : T^*\mathscr{M} \times T^*\mathscr{M} \to \mathbb{R}$. The Hamiltonian vector field associated to $h(x, \lambda) = -\lambda$ is the queer operational vector field

$$X_h = \{h, \cdot\} = D_x$$

acting on $f \in C^\infty_x(\mathscr{H})$ by $D_x(f) = \ell(d^2 f(x))$.