Statistical structures — completeness of metrics and connections

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Statistical structures

M – connected manifold, g – metric tensor field on M (positive definite)
∇ – affine torsion-free connection (always torsion-free in this lecture)
(g, ∇) – statistical structure if the cubic form ∇g is symmetric

$$\nabla g(X, Y, Z) = (\nabla_X g)(Y, Z)$$

∇ – statistical connection
$\overline{\nabla}$ – dual (conjugate) connection, that is,

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \overline{\nabla}_X Z)$$

$\hat{\nabla}$ – Levi-Civita connection for g
K – difference tensor: $K_X Y = \nabla_X Y - \hat{\nabla}_X Y$, $K(X, Y) = K_X Y$ symmetric for X, Y
A – cubic form: $A(X, Y, Z) = g(K(X, Y), Z)$ – symmetric for X, Y, Z

$$\nabla g = -2A$$

(g, ∇) – **trivial** $\Leftrightarrow \nabla = \hat{\nabla} \Leftrightarrow K = 0 \Leftrightarrow C = 0 \Leftrightarrow A = 0$

Geodesics

By a geodesic we mean a parametrized curve $\gamma(t)$, where $t \in (a, b)$, $a, b \in \mathbb{R}$ or $a = -\infty$ or $b = \infty$, such that

$$\nabla_{\dot\gamma}\dot\gamma = 0.$$

Such a parametrization is called affine. It is unique up to affine changes of the parameter. A geodesic is maximal if it cannot be extended as a geodesic beyond the interval (a, b). A geodesic is complete if the affine parameter runs from $-\infty$ to ∞.
By a pregeodesic we'll mean either the image of a geodesic or any of its parametrizations. For a pregeodesic (parametrized) we have

$$\nabla_{\dot\gamma}\dot\gamma = \rho\dot\gamma,$$

where ρ is a function.

Lemma 1

Let M be equipped with a connection. If $\gamma : (a, b) \to M$ is a geodesic and there is the limit of $\gamma(t)$ in M for $t \to b$ then γ can be extended as a geodesic beyond b.

Lemma 2

Let M be equipped with a connection. If $\gamma : (a, \infty) \to M$ is a geodesic then there is no limit of $\gamma(t)$ in M for $t \to \infty$.

Lemma 3

Let M be equipped with a connection. If $\gamma(t)$ for $t \in (a, b)$, where $b \in \mathbb{R}$ or $b = \infty$ is a pregeodesic and there is $p = \lim_{t \to b} \gamma(t)$ in M then the pregeodesic can be extended as a pregeodesic beyond p.

Geodesics of affine connections on Riemannian manifolds

Assume now that we also have a metric tensor g on M but our connection ∇ is not related with the metric g in any sense. Again we can define the difference tensor $K = \nabla - \hat{\nabla}$ and the cubic form A (non- symmetric, in general):

$$A(X, Y, Z) = g(K(X, Y), Z)$$

If ∇ is torsion-free, $A(X, Y, Z)$ is symmetric for X, Y. We also have the cubic form ∇g and the relation

$$\nabla g(X, Y, Z) = -A(X, Y, Z) - A(X, Z, Y).$$

In particular,

$$\nabla g(X, X, X) = -2A(X, X, X).$$

For a connection ∇ on a Riemannian manifold one can consider the arc-length parametrization $r(s)$ of a ∇-geodesic and the scalar speed $l(t) = \|\dot\gamma(t)\|$ of the affine paramatrization of a ∇-geodesic. The following formulas hold

$$\nabla_{\dot r}\dot r = A(\dot r, \dot r, \dot r)\dot r,$$

$$\left(\frac{1}{l}\right)' = A(u, u, u),$$

where $u = \frac{\dot\gamma}{l}$,

$$A(\dot r, \dot r, \dot r) = -\frac{d}{ds}\ln\|\dot\gamma\circ\varphi\|,$$

where $\varphi(s) = t(s)$ is the change between the arc-length parameter s and the affine parameter t of the geodesic.

Lemma 4

Let (M, g) be a complete Riemannian manifold and ∇ be a connection on M. If $r(s)$ is an arc-length parametrization of a maximal ∇-geodesic then the parameter s runs from $-\infty$ to ∞. In particular, every maximal ∇-geodesic has infinite length.

Example

$M = \mathbb{R} \times (-\frac{\sqrt{\pi}}{2}, \frac{\sqrt{\pi}}{2}) \subset \mathbb{R}^2$, g - standard metric
$G'(t) = e^{-t^2}$ with $G(0) = 0$
(U, V) – canonical frame on M. Each point of M has the coordinates $(x, G(t))$ for some $t \in \mathbb{R}$. Define K (symmetric) as follows

$$K_{(x,G(t))}(V, V) = 2te^{t^2} U, \qquad K(U, U) = 0, \qquad K(U, V) = 2te^{t^2} V.$$

$\nabla := \hat{\nabla} + K$. The piece of the straight line $\gamma(t) = (0, G(t))$ for $t \in (-\infty, \infty)$ is a complete ∇-geodesic. It has finite length $\sqrt{\pi}$.

Lemma 5

Let (M, g) be a complete Riemannian manifold and $r(s)$, $s \in \mathbb{R}$, be an arc-length parametrization of a maximal geodesic of some affine connection ∇ on M. Let $\nabla_{\dot r}\dot r = \Lambda' \dot r$ where Λ is a function bounded from below on the whole $\mathbb{R}$. Then the geodesic is complete.

Lemma 6

Let (M, g) be a complete Riemannian manifold and ∇ a connection on M. If a maximal ∇-geodesic has scalar speed bounded from above then the geodesic is complete.

statistical structures – continuation

If (g, ∇) is statistical, so is $(g, \overline{\nabla})$.
For $(g, \overline{\nabla})$ the difference tensor equals to $-K$.
More generally, one can define ∇^α by using the difference tensor αK, where $\alpha \in \mathbb{R}$.
∇^α - α-connection
We have the family of statistical structures (g, ∇^α) – α-family of statistical structures
In particular, $\nabla^0 = \hat{\nabla}$, $\nabla^{-1} = \overline{\nabla}$

$$\nu_g - \text{the volume form determined by} g$$

A statistical structure is called **trace-free** if $\nabla \nu_g = 0$. This condition is equivalent to the condition $\operatorname{tr}_g K = 0$ or, equivalently, $\operatorname{tr} K_X = 0$ for every X.

More generally, we set

$$\tau(X) = \operatorname{tr} K_X .$$

A statistical connection ∇ is Ricci-symmetric if and only if $d\tau = 0$. Hence ∇ and $\overline{\nabla}$ (in general ∇^α)are simultaneously Ricci-symmetric.

For a statistical structure $(g, \nabla, \overline{\nabla})$ we set
R, $\overline{R}$, $\hat{R}$ – curvature tensors for ∇, $\overline{\nabla}$, $\hat{\nabla}$. The corresponding Ricci tensors: Ric, $\overline{\mathrm{Ric}}$, $\widehat{\mathrm{Ric}}$.
The function

$$\rho = \mathrm{tr}_g \mathrm{Ric} \tag{1}$$

is the **scalar curvature** of ∇. Similarly one can define the scalar curvature $\overline{\rho}$ for $\overline{\nabla}$, but $\overline{\rho} = \rho$. We also have the scalar curvature $\hat{\rho}$ of g. For a trace-free statistical structure the following formula holds (called "theorema egregium")

$$\hat{\rho} = \rho + g(A, A). \tag{2}$$

Conjugate symmetric statistical structures

The $(0,4)$-tensor field defined by $g(R(X,Y)Z,W)$ is not, in general, skew-symmetric for Z,W, but we have

$$g(R(X,Y)Z,W) = -g(\overline{R}(X,Y)W,Z) \tag{3}$$

Lemma 7

Let (g,∇) be a statistical structure. The following conditions are equivalent:
1) $R = \overline{R}$,
2) $\hat{\nabla} K$ is symmetric as a $(1,3)$-tensor field (equiv. $\hat{\nabla} A$ is symmetric as a $(0,4)$-tensor field),
3) $g(R(X,Y)Z,W)$ is skew-symmetric for Z,W.

If a statistical structure satisfies one of the above conditions, it is called **conjugate symmetric**.

Sectional ∇-curvature

The sectional curvature is attributed to Riemannian geometry in a strong way, but it can be also defined for statistical structures. First, we can consider the average of the tensors R and $\overline{R}$:

$$\mathcal{R} = \frac{R + \overline{R}}{2} \tag{4}$$

It has all symmetries needed for defining a sectional curvature. In particular, we have the skew-symmetry for the last arguments in $g(\mathcal{R}(X, Y)Z, W)$.
If π is a vector plane in the tangent space T_xM, e_1, e_2 is an orhonormal basis of π, then we set

$$k(\pi) = g(\mathcal{R}(e_1, e_2)e_2, e_1) \tag{5}$$

and we call this sectional curvature **the sectional ∇-curvature**.

Sectional ∇-curvature on conjugate symmetric statistical manifolds

In general, Schur's lemma does not hold for the sectional ∇-curvature. **Schur's lemma holds if a statistical structure is conjugate symmetric.**
If a conjugate symmetric statistical structure has constant sectional ∇-curvature k, then

$$R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y) \tag{6}$$

If dim M=2 and the given statistical structure is conjugate symmetric then the above formula always holds with k being a function. Of course, if the above formula holds then the statistical structure is conjugate symmetric. Moreover, if $n > 2$ and (6) holds, then both connections ∇ and $\overline{\nabla}$ are projectively flat. More generally, we have

Proposition 8

For a conjugate symmetric statistical structure the statistical connection and its dual are simultaneously projectively flat.

Statistical structures on equiaffine hypersurfaces

$f : M \to \mathbb{R}^{n+1}$ - locally strongly convex hypersurface with an equiaffine transversal vector field ξ, that is, $D_X\xi$ is tangent to f for every tangent vector $X \in TM$, where D is the standard flat connection on $\mathbb{R}^{n+1}$. The Weingarten formula

$$D_X\xi = -f_*(S(X))$$

for $X \in TM$, defines the shape operator S. By the Gauss formula

$$D_X f_* Y = f_*(\nabla_X Y) + g(X, Y)\xi$$

we define the induced (torsion-free) connection ∇ and the second fundamental form g (symmetric (0,2)-tensor field). Since our hypersurface is locally strongly convex, g is a definite. We usually choose the sign of a transversal vector field ξ in such a way that the second fundamental form is positive definite.

For a hypersurface equipped with a transversal vector field we have four fundamental equations (Gauss, Codazzi I, Codazzi II, Ricci). If a transversal vector field is equiaffine then the first Codazzi equation says that the cubic form ∇g is symmetric. For a locally strongly convex hypersurface the pair (g, ∇) is a statistical structure. On an equiaffine hypersurface the induced statistical structure is Ricci-symmetric.
Given a locally strongly convex immersion f endowed with a transversal vector field ξ we also have the conormal map

$$\overline{f} : M \to (\mathbb{R}^{n+1})^* \setminus \{0\}$$

defined by the conditions

$$\overline{f}(x)(\xi_x) = 1, \qquad (\overline{f}(x))_{|f_*(T_xM)} \equiv 0$$

If f is locally strongly convex then $\overline{f}$ is an immersion. For each $x \in M$ the conormal vector $\overline{f}_x = \overrightarrow{0\overline{f}_x}$ is transversal to $\overline{f}$. We equip the immersion $\overline{f}$ with this equiaffine transversal vector field $-\overline{f}$. Again, we receive the induced objects on M. In particular, the induced connection turns out to be the dual connection for ∇ relative to g.

Theorem 9 (Berwald)

The induced statistical structure on a locally strongly convex equiaffine hypersurface is trivial if and only if the hypersurface is a locally strongly convex quadric.

Centroaffine normalizations, proper equiaffine spheres

$f : M \to \mathbb{R}^{n+1}$ l.s.c. hypersurface, $p \in \mathbb{R}^{n+1}$, $\overrightarrow{pf(x)}$ is transversal to f for every $x \in M$. Equip f with the transversal vector field $\xi = -\overrightarrow{pf}$. This vector field ξ is equiaffine and the corresponding shape operator $S = id$.
p – the center of the centroaffine normalization.
We usually can assume that $p = 0 \in \mathbb{R}^{n+1}$.
In particular, the conormal map has the natural centroaffine normalization.
Hypersurfaces with centroaffine normalization are also called proper equiaffine spheres.
Improper equiaffine spheres are hypersurfaces equipped with a constant transversal vector field.

Trace-free statistical structures play a crucial role in the classical affine differential geometry. Namely, we have the following basic theorem

Theorem 10

Let $f : M \to \mathbb{R}^{n+1}$ be a locally strongly convex hypersurface. There is a unique (up to a constant) equiaffine transversal vector field ξ such that the induced statistical structure is trace-free.

– Blaschke hypersurface, ξ - Blaschke affine normal, g - Blaschke metric.

A fundamental theorem

F. Dillen, K. Nomizu, L. Vrancken, [4]

Theorem 11

Let M be a simply connected manifold and (g, ∇) be a statistical structure on M such that ∇ is Ricci-symmetric and $\overline{\nabla}$ is projectively flat. Then there is a locally strongly convex immersion $f : M \to \mathbb{R}^{n+1}$ and its equiaffine transversal vector field ξ such that ∇ is the induced connection and g the second fundamental form for f, ξ. The immersion is unique up to an affine transformation of $\mathbb{R}^{n+1}$.
If, moreover, the given statistical structure is conjugate symmetric, the immersion f is an equiaffine sphere.
If the given statistical structure is trace-free, ξ is the Blaschke affine normal (up to a constant) for f.

Affine spheres

In the cathegory of Blaschke hypersurfaces there is a very important and rich class of the so called affine spheres. Namely, if a Blaschke hypersurface is an equiaffine sphere it is called an affine sphere. For an equiaffine sphere the shape operator S is a multiple of the identity, i.e. $S = \lambda \mathrm{id}$ (λ is constant on a connected domain). For a locally strongly convex affine sphere we always choose the sign of the affine normal in such a way that the induced second fundamental form is positive definite. The following names of spheres are in use:

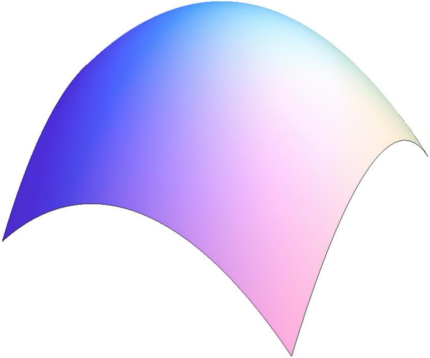
if $\lambda > 0$, the sphere is called elliptic;

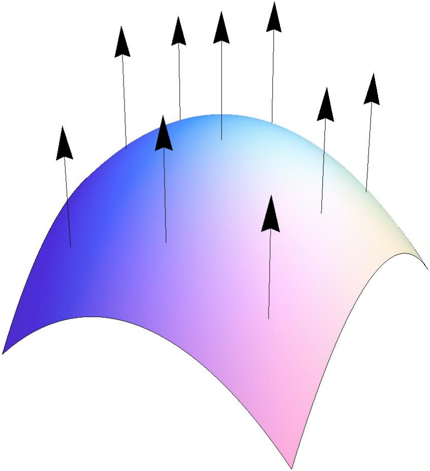
if $\lambda < 0$, the sphere is called hyperbolic.

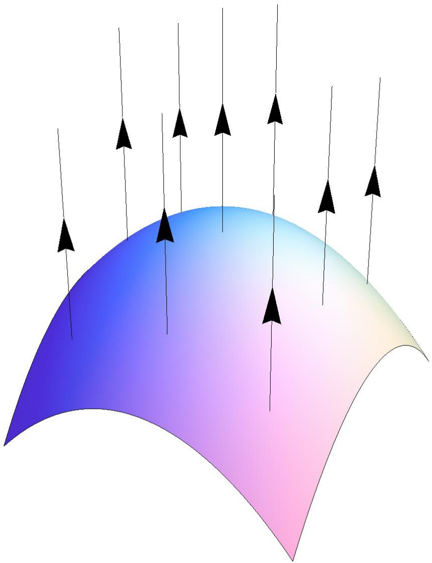
For improper affine spheres $\lambda \equiv 0$.

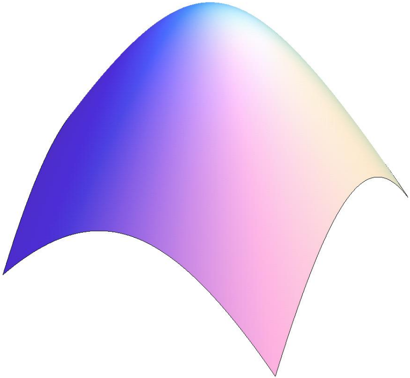
If $\lambda \equiv 0$, the sphere is called parabolic.

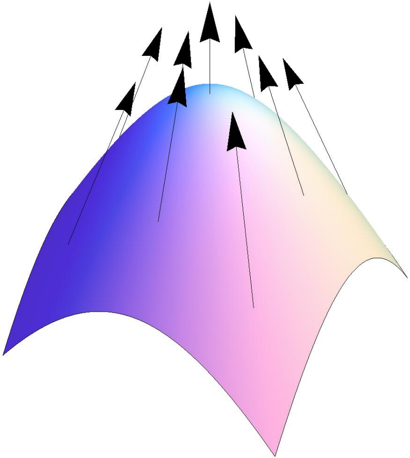
The only compact affine spheres are ellipsoids. An ellipsoid is a locally strongly convex proper affine sphere (elliptic) whose center lies at the center of the ellipsoid. Hyperboloids are proper affine spheres (hyperbolic). Elliptic paraboloids are improper locally strongly convex affine spheres.

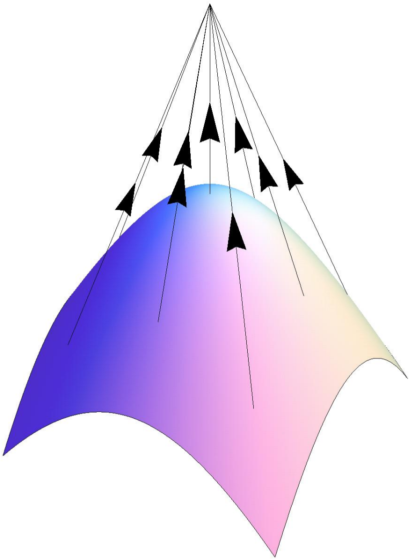












Completness

For a l.s.c. hypersurface we can consider:
1) the completeness relative to the second fundamental form (for Blaschke hypersurfaces – the Blaschke metric) (**affine completeness**)
2) the completeness of the Riemannian metric induced from the Euclidean space $\mathbb{R}^{n+1}$ (**Euclidean completeness**)
3) the completeness relative to the induced connection
4) the completeness relative to the dual connection

For a statistical structure (g, ∇) we can consider
1) the completeness relative to the metric tensor g
2) the completeness relative to the statistical connection ∇
3) the completeness relative to the dual statistical connection $\overline{\nabla}$

Completeness of the Blaschke metric and the induced metric

W. Blaschke, A. Deicke; 1917, 1918

Theorem 12

Every locally strongly convex elliptic affine sphere that is complete relative to the Blaschke metric must be an ellipsoid with the trivial affine structure.

E. Calabi; 1971

Theorem 13

Every locally strongly convex parabolic affine sphere whose Blaschke metric is complete must be an elliptic paraboloid with the trivial affine structure. For a locally strongly convex hyperbolic or parabolic affine sphere whose Blaschke metric is complete the Ricci tensor of the metric is negative semi-definite.

affine completeness $\Rightarrow$ Euclidean completeness

S.Y. Cheng, S.T. Yau; 1986, A-M Li; 1990

Theorem 14

Every locally strongly convex hyperbolic affine sphere that is complete relative to the Blaschke metric is complete relative to the induced Riemannian metric.

N.S. Trudinger, X.-J. Wang; 2002

Theorem 15

Every locally strongly convex hypersurface that is complete relative to the Blaschke metric is complete relative to the induced Riemannian structure.

Euclidean completeness $\Rightarrow$ affine completeness

S.Y. Cheng, S.T. Yau; 1986

Theorem 16

Every locally strongly convex affine sphere that is complete relative to the induced Riemannian metric is complete relative to the Blaschke metric.

A-M Li; 1990

Theorem 17

Every locally strongly convex hypersurface that is complete relative to the induced Riemannian metric and whose eigenvalues of the affine shape operator are bounded from above is also complete relative to the Blaschke metric.

Versions of Theorems 6 and 7 (Blaschke, Deicke, Calabi) for statistical manifolds.

Theorem 18

Let (g, ∇) be a conjugate symmetric trace-free statistical structure on M. Assume that g is complete and the sectional ∇-curvature is constant and non-negative. Then the statistical structure on M is trivial, that is $\nabla = \hat{\nabla}$. Consequently, by Myers' theorem, if the sectional ∇-curvature is positive, M is compact and its first fundamental group is finite.

Theorem 19

Let (g, ∇) be a conjugate symmetric trace-free statistical structure on M. Assume that g is complete and the sectional ∇-curvature is constant and non-positive. Then the Ricci tensor of the metric is negative semi-definite.

Since the structures from the above theorems are conjugate symmetric with constant sectional ∇-curvature both statistical connections ∇ and $\overline{\nabla}$ are projectively flat and therefore the statistical structures are locally realizable on affine spheres. In general, the realizations cannot be global.

Curvature bounded conjugate symmetric trace-free statistical structures with complete metric

Theorem 20

Let (g, ∇) be a trace-free conjugate symmetric statistical structure on a manifold M. Assume that g is complete on M. If the sectional ∇-curvature is non-negative everywhere then the statistical structure is trivial, that is, $\nabla = \hat{\nabla}$. If the sectional ∇-curvature is bounded from 0 by a positive constant then, additionally, M is compact and its first fundamental group is finite.

Theorem 21

Let (g, ∇) be a trace-free conjugate symmetric statistical structure on a manifold M. Assume that g is complete on M. If the sectional ∇-curvature is bounded from below and above on M then the Ricci tensor of g is bounded from below and above on M. If $H - \varepsilon < k(\pi) < H$ then $\hat{\rho} \leqslant \frac{n^2(n-1)}{2}\varepsilon$.

Completeness of the induced connections on equiaffine hypersurfaces and statistical connections

In general, statistical connections are not necessarily complete on compact manifolds.

Example

Let g be the standard flat metric on $\mathbb{R}^2$. Let U, V be the canonical frame field on $\mathbb{R}^2$. Define the statistical connection ∇ as follows

$$\nabla_U U = U, \quad \nabla_U V = -V, \quad \nabla_V V = -U. \tag{7}$$

The statistical structure can be projected on the standard torus T^2. Here we have $\hat{\nabla} A = 0$, ∇ is Ricci-symmetric and projectively flat.The curve

$$\gamma(t) = (\ln(1-t), y_0) \tag{8}$$

for $t \in [0, 1)$, is a ∇-geodesic. We have $\|\dot{\gamma}(t)\| = \frac{1}{1-t} \to +\infty$ if $t \to 1$. Hence this geodesic cannot be extended beyond 1.

The above example can be generalized to the following negative result

Theorem 22

Let (g, ∇) be a non-trivial statistical structure such that

$$\hat{\nabla} A(U, U, U, U) \leqslant 0$$

for every $U \in \mathcal{U}M$, where $\mathcal{U}M$ is the unit sphere bundle over M. The statistical connection ∇ is not complete.

In particular, we may have $\hat{\nabla} A = 0$. The fact that $\hat{\nabla} A = 0$ does not trivialize the situation. Even in the theory of affine hypersurfaces one knows examples (non-compact) of non-trivial statistical structures for which $\hat{\nabla} A = 0$. The most famous is the hypersurface of $\mathbb{R}^{n+1}$ given by the equation

$$x_1 \cdot \ldots \cdot x_{n+1} = 1$$

for $x_1 > 0, \ldots, x_{n+1} > 0$. It is a hyperbolic locally strongly convex affine sphere with $\hat{\nabla} A = 0$. It is not a quadric (hence the induced statistical structure is non-trivial), its Blaschke metric is complete, the induced metric is complete, the induced connection is not complete, the dual connection is not complete. Other examples with $\hat{\nabla} A = 0$ can be found in [5] and [7]. For instance, one has the following affine sphere in $\mathbb{R}^4$

$$(y^2 - z^2 - w^2)^3 x^2 = 1$$

and the following affine spheres in $\mathbb{R}^5$

$$(y^2 - z^2 - w^2 - v^2)^2 x = 1,$$

$$(z^2 - w^2 - v^2)^3 (xy)^2 = 1.$$

The following positive result is due to Noguchi (1992)

Theorem 23

Let (M, g) be a complete Riemannian manifold and A be a cubic form given by

$$A = sym(d\sigma \otimes g) \tag{9}$$

for some function σ on M. Assume that the function σ is bounded from below on M. Then the statistical connection of the statistical structure (g, A) is complete.

The theorem is a consequence of Lemma 6 because the scalar speed of any geodesic (relative to the statistical connection determined by A) is here bounded from above.

Remark In a similar way one can produce complete affine connections, not necessarily statistical for the metric g. Namely, one can define the cubic form :

$$A(X, Y, Z) = \beta g(X, Y)d\sigma(Z) + \delta[g(X, Z)d\sigma(Y) + g(Y, Z)d\sigma(X)],$$

where β, δ are real numbers such that $\beta + \delta > 0$.

Corollary 24

Let (M, g) be a compact Riemannian manifold. Each function σ on M gives rise to an α-family of statistical structures whose all statistical connections are complete.

Theorem 25

On a centroaffine ovaloid in $\mathbb{R}^{n+1}$ the induced connection and its dual relative to the second fundamental form of the given centroaffine ovaloid are complete.

Theorem 26

For an ovaloid in $\mathbb{R}^{n+1}$ equipped with any equiaffine transversal vector field, for which the affine Gauss curvature $\det S$ is nowhere zero, the dual connection is complete.

Theorem 27

Let (g, ∇) be a statistical structure on a manifold M diffeomorphic to a Euclidean sphere. If the structure is conjugate symmetric and the connection ∇ is projectively flat then ∇ and its dual connection $\overline{\nabla}$ are complete on M.

In Theorem 27 the assumption that M is diffeomorphic to a Euclidean sphere is important, see the example on page 30.

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