Bregman superquantiles. Estimation methods and applications

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Joint work with F. Gamboa, A. Garivier (IMT) and B. Iooss (EDF R&D).

1. Coherent measure of risk
   - Quantile and subadditivity
   - Coherent measure of risk
2. Bregman superquantile and Coherent measure of risk
   - Bregman divergence, mean and superquantile
   - Coherence of Bregman superquantile
3. Estimation and asymptotics of the Bregman superquantile
   - Plug-in estimator
   - Asymptotics of the plug-in estimator
   - Examples
4. Applications
5. Conclusion

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Let X be a real-valued random variable and let F_X be its cumulative distribution function. We denote for $u \in]0,1[$, the quantile function

$$F_X^{-1}(u) := \inf\{x : F_X(x) \geq u\}.$$

A usual way to quantify the risk associated with X is to consider, for a given number $\alpha \in]0,1[$ close to 1, its lower quantile

$$q_\alpha := F_X^{-1}(\alpha).$$

The quantile is not subadditive
(in some examples $q_\alpha^{X+Y} > q_\alpha^X + q_\alpha^Y$)

- Subadditivity is interesting in finance.
- Rockafellar introduces a new quantity which is subadditive, the superquantile.

Definition of the superquantile

The superquantile Q_α of a law X is defined in this way

$$Q_\alpha := \mathbb{E}(X|X \geq q_\alpha) = \mathbb{E}(X|X \geq F_X^{-1}(\alpha)) = \mathbb{E}\left(\frac{X\mathbf{1}_{X \geq F_X^{-1}(\alpha)}}{1-\alpha}\right)$$

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Subadditivity is not the sole interesting property for a measure of risk.

Definition of a coherent measure of risk

Let $\mathcal{R}$ be a measure of risk and X and X' be two real-valued random variables. We say that $\mathcal{R}$ is coherent if, and only if, it satisfies the five following properties :

i) **Constant invariant** : let $C \in \mathbb{R}$, If $X = C$ (a.s.) then $\mathcal{R}(C) = C$.

ii) **Homogeneity** : $\forall \lambda > 0$, $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$.

iii) **Subaddidivity** : $\mathcal{R}(X + X') \leq \mathcal{R}(X) + \mathcal{R}(X')$.

iv) **Non decreasing** : If $X \leq X'$ (a.s.) then $\mathcal{R}(X) \leq \mathcal{R}(X')$.

v) **Closeness** : Let $(X_h)_{h\in\mathbb{R}}$ be a collection of random variables. If $\mathcal{R}(X_h) \leq 0$ and $\lim\limits_{h\to 0} ||X_h - X||_2 = 0$ then $\mathcal{R}(X) \leq 0$.

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Bregman divergence

Let γ be a **strictly convex function**, $\overline{\mathbb{R}}$-valued on $\mathbb{R}$. We assume that domγ is a non empty open set and that γ is closed proper and differentiable on the interior of domγ.
The **Bregman divergence** d_γ associated to γ is a function defined on dom$\gamma \times$ domγ by

$$d_\gamma(x,x') := \gamma(x) - \gamma(x') - \gamma'(x')(x-x') \ \ ;(x,x' \in \text{dom}\gamma).$$

Definition of the Bregman mean

Let μ be a probability measure whose support is included in domγ such that $\mu(\overline{\text{dom}\gamma}\backslash\text{dom}\gamma) = 0$ and γ' is integrable with respect to μ. **The Bregman mean is the unique point b in the support of μ satisfying**

$$\int d_\gamma(b,x)\mu(dx) = \min_{m\in\text{dom}\gamma} \int d_\gamma(m,x)\mu(dx). \qquad (1)$$

By differentiating it's easy to see that

$$b = {\gamma'}^{-1}\left[\int \gamma'(x)\mu(dx)\right].$$

Examples

- Euclidean. $\gamma(x) = x^2$ on $\mathbb{R}$, we obviously obtain, for $x, x' \in \mathbb{R}$,
$$d_\gamma(x, x') = (x - x')^2$$
and b is the **classical mean**.
- Geometric. $\gamma(x) = x \ln(x) - x + 1$ on $\mathbb{R}_+^*$ we obtain, for $x, x' \in \mathbb{R}_+^*$,
$$d_\gamma(x, x') = x \ln \frac{x}{x'} + x' - x$$
and b is the **geometric mean**.
- Harmonic. $\gamma(x) = -\ln(x) + x - 1$ on $\mathbb{R}_+^*$ we obtain, for $x, x' \in \mathbb{R}_+^*$,
$$d_\gamma(x, x') = -\ln \frac{x}{x'} + \frac{x}{x'} - 1$$
and b is the **harmonic mean**.

Definition of the Bregman superquantile

Let $\alpha \in]0,1[$, the Bregman superquantile $Q_\alpha^{d_\gamma}$ is defined by

$$Q_\alpha^{d_\gamma} := \gamma'^{-1}\left(\mathbb{E}(\gamma'(X)|X \geq F_X^{-1}(\alpha))\right) = \gamma'^{-1}\left[\mathbb{E}\left(\frac{\gamma'(X)\mathbf{1}_{X\geq F_X^{-1}(\alpha)}}{1-\alpha}\right)\right].$$

In words $Q_\alpha^{d_\gamma}$ satisfies (1) taking for μ the distribution of X conditionally to $X \geq F_X^{-1}(\alpha)$.

Examples

- Geometric. $\gamma(x) = x\ln(x) - x + 1$ on $\mathbb{R}_+^*$ we obtain,

$$\exp\left[\mathbb{E}\left(\frac{\ln(X)\mathbf{1}_{X\geq F_X^{-1}(\alpha)}}{1-\alpha}\right)\right]$$

- Harmonic. $\gamma(x) = -\ln(x) + x - 1$ on $\mathbb{R}_+^*$ we obtain,

$$\frac{1}{1-\left[\mathbb{E}\left(\frac{\left(-\frac{1}{X}+1\right)\mathbf{1}_{X\geq F_X^{-1}(\alpha)}}{1-\alpha}\right)\right]}$$

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Proposition

Fix α in $]0,1[$.

i) Any Bregman superquantile always satisfies the properties of constant invariance and non decreasing.

ii) The Bregman superquantile associated to γ is homogeneous if and only if
$\gamma''(x) = \beta x^{\delta}$ for some real numbers β and δ.

iii) If γ' is concave and sub-additive, then subadditivity and closeness axioms both hold.

Examples

- **Bregman geometric function** : $\gamma'(x) = x \mapsto \ln(x)$.
  - $\rightarrow$ i) and ii) satisfied.
  - $\rightarrow$ γ' is concave but subadditive only on $[1, +\infty[$. Then it satisfies iii) only for couples (X, X') such that,

$$\min\left(q_\alpha^X(\alpha), q_\alpha^{X'}(\alpha), q_\alpha^{X+X'}(\alpha)\right) > 1$$

- **The subadditivity and the homogeneity are not true in the general case**. For $\gamma(x) = \exp(x)$ and $X \sim \mathcal{U}([0,1])$:

$$\mathcal{R}(2X) - 2\mathcal{R}(X) = 0.000107 > 0,$$

and

$$\frac{\mathcal{R}(4X)}{4\mathcal{R}(X)} = 1,000321 > 1$$

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Plug-in estimator

Assume that we have at hand $(X_1, \dots, X_n)$ an i.i.d sample with same distribution as X. If we wish to estimate $Q_\alpha^{d_\gamma}$, we may use the following empirical estimator :

$$\hat{Q_\alpha^{d_\gamma}} = \gamma'^{-1}\left[\frac{1}{1-\alpha}\left(\frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor+1}^{n}\gamma'(X_{(i)})\right)\right]$$

where $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ is the re-ordered sample built with $(X_1, \dots, X_n)$.

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Theorem : consistency

Let $\alpha \in]0,1[$ be close to 1 and X be a real-valued random variable. Let $(X_1, \dots X_n)$ be an independant sample with the same distribution as X.

We assume that

H1) γ is twice differentiable.

H2) F_X is absolutely continuous and $f_X > 0$ on its support.

H3) The derivative of $\gamma' \circ F_X^{-1}$ that we denote l_γ is non-decreasing and $o\left((1-t)^{-2}\right)$ in the neighborhood of 1.

Then the plug-in estimator is consistent.

Theorem : asymptotic normality

Strongly, we assume that

H1) γ is three times differentiable.

H2) F_X is absolutely continuous, the density f_X is $\mathcal{C}^1$ and $f_X > 0$ on its support.

H3) The second derivative of $\gamma' \circ F_X^{-1}$ that we denote L_γ is non decreasing and $O\left((1-t)^{-m_L}\right)$ for an $1 < m_L < \frac{5}{2}$, in the neighborhood of 1.

Then the estimator is asymptotically normal.

$$\sqrt{n}\left(\gamma'^{-1}\left(\frac{1}{n(1-\alpha)}\sum_{i=\lfloor n\alpha \rfloor+1}^{n}\gamma'(X_{(i)})\right) - Q_\alpha^{d\gamma}(X)\right) \Longrightarrow \mathcal{N}\left(0, \frac{\sigma_\gamma^2}{\left(\gamma''\circ\gamma'^{-1}(Q_\alpha(\gamma'(X)))\right)^2(1-\alpha)^2}\right)$$

where

$$\sigma_\gamma^2 := \int_\alpha^1\int_\alpha^1 \frac{(\min(x,y)-xy)}{f_Z(F_Z^{-1}(x))f_Z(F_Z^{-1}(y))} \quad \textit{and } Z := \gamma'(X)$$

Remark

The key point of the proof is to notice that we have the fundamental link between these two quantities

$$Q_\alpha^{d\gamma}(X) = \gamma'^{-1}\left(Q_\alpha(\gamma'(X))\right).$$

Indeed, as $\gamma'\left(F^{-1}_{(\gamma')^{-1}(Z)}(\alpha)\right) = F_Z^{-1}(\alpha)$, we have

$$\mathbb{E}\left(\gamma'(X)\mathbf{1}_{X>F_X^{-1}(\alpha)}\right) = \mathbb{E}\left(Z\mathbf{1}_{Z>F_Z^{-1}(\alpha)}\right).$$

Then we have to study the asymptotic behaviour of the superquantile.

Plug-in estimator of the superquantile

For $\gamma' = id$, the estimator becomes $(n(1-\alpha))^{-1} \sum_{i=\lfloor n\alpha \rfloor}^{n} X_{(i)}$.

Proposition : consistency of the plug-in estimator of the superquantile

Let $\alpha \in]0,1[$ be close to 1 and X be a real-valued random variable. Let $(X_1, \ldots X_n)$ be an independant sample with the same distribution as X. We assume that

H1) F_X is absolutely continuous and $f_X > 0$ on its support.

H2) The derivative of the quantile function F_X^{-1} denoting l is non-decreasing and $o\left((1-t)^{-2}\right)$ in the neighborhood of 1.

Then, the plug-in estimator is **consistent**.

Proposition : asymptotic normality of the plug-in estimator of the superquantile

Strongly, we assume that

H1) F_X is absolutely continuous, the density f_X is $\mathcal{C}^1$ and $f_X > 0$ on its support.

H2) The second derivative of the quantile function that we denote L is non decreasing and $O\left((1-t)^{-m_L}\right)$ for an $1 < m_L < \frac{5}{2}$, in the neighborhood of 1.

Then the estimator is **asymptotically normal.**

$$\sqrt{n}\left(\frac{1}{n(1-\alpha)}\sum_{i=\lfloor n\alpha\rfloor+1}^{n} X_{(i)} - Q_\alpha\right) \Longrightarrow \mathcal{N}\left(0, \frac{\sigma^2}{(1-\alpha)^2}\right)$$

where $\sigma^2 := \int_\alpha^1 \int_\alpha^1 \frac{(\min(x,y)-xy)}{f(F^{-1}(x))f(F^{-1}(y))}$.

Sketch of proof

- Step 1 : Using properties on ordered statistics, we show that our proposition is equivalent to show the convergence in law of

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor}^{n} X_{(i)} - \frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor}^{n} F^{-1}\left(\frac{i}{n+1}\right)\right].$$

Then we use Taylor-Lagrange formula

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor+1}^{n}\left[X_{(i)} - F^{-1}\left(\frac{i}{n+1}\right)\right]\right) \underset{\mathcal{L}}{=} \sqrt{n}\left[\frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor+1}^{n}\left(U_{(i)} - \frac{i}{n+1}\right)\frac{1}{f\left(F^{-1}\left(\frac{i}{n+1}\right)\right)}\right]$$
$$+\frac{1}{\sqrt{n}}\sum_{i=\lfloor n\alpha\rfloor+1}^{n}\left[\int_{\frac{i}{n+1}}^{U_{(i)}}\frac{f'(F^{-1}(t))}{\left(f(F^{-1}(t))\right)^{3}}\left(U_{(i)} - t\right)dt\right].$$

- Step 2 : Convergence of the second order term to 0 in probability (Markov's inequality).
- Step 3 : Identification of the limit in law of the first order using a corollary of the Lindenberg-Feller theorem.

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Examples : Exponential distribution

On $\mathbb{R}_*^+$, $f(t) = \exp(-t)$ and $F^{-1}(t) = -\ln(1-t)$.

- **Superquantile**.

→ **Consistency**.

$$l(t) = (1-t)^{-1} = o\left((1-t)^{-2}\right)$$

Then **the estimator of the superquantile is consistent**.

→ **Asymptotic normality**.

$$L(t) = (1-t)^{-2} = O\left((1-t)^{-m_L}\right)$$

for $2 < m_L < \frac{5}{2}$. **The estimator is asymptotically gaussian.**

Examples : Exponential distribution

- **Bregman superquantile with Bregman geometric function.**

For $\gamma(x) = x\ln(x) - x + 1$, $F_Z^{-1}(t) = 1 + \frac{1}{\ln(1-t)}$.

$\rightarrow$ **Consistency.**

$$l_\gamma(t) = \frac{1}{(1-t)(\ln(1-t))^2} = o\left((1-t)^{-2}\right),$$

for $2 < m_L < \frac{5}{2}$. **The estimator is consistent.**

$\rightarrow$ **Asymptotic normality.**

$$L_\gamma(t) = \frac{(\ln(1-t))^2 + 2\ln(1-t)}{(1-t)^2(\ln(1-t))^4} = O\left((1-t)^{-m_L}\right),$$

for $2 < m_L < \frac{5}{2}$. **Our estimator is asymptotically gaussian.**

Examples : Pareto law of parameter $a > 0$

On $\mathbb{R}_*^+$, $f(t) = ax^{-a-1}$ and $F^{-1}(t) = (1-t)^{\frac{-1}{a}}$.

- **Superquantile**

→ **Consistency.**

$$I(t) = \left(a(1-t)^{-1-\frac{1}{a}}\right) = o\left((1-t)^{-2}\right)$$

as soon as $a > 1$. **The consistency is true when $a > 1$.**

→ **Asymptotic normality.**

$$L(t) = C(a)(1-x)^{-\frac{1}{a}-2} = O\left(\frac{1}{(1-t)^{m_L}}\right)$$

for $\frac{3}{2} < m_L < \frac{5}{2}$ as soon as $a > 2$. **The asymptotic normality is true if and only if $a > 2$.**

Examples : Pareto law

- **Bregman superquantile with the Bregman harmonic function**

For $\gamma(x) = -\ln(x) + x - 1$, $F_Z^{-1}(t) = -\frac{1}{a}\ln(1-t)$.

→ **Consistency**.

$$l_\gamma(t) = \frac{1}{a}\frac{1}{1-t} = o\left(\frac{1}{(1-t)^2}\right).$$

The estimator is consistent for every $a > 0$.

→ **Asymptotic normality.**

$$L_\gamma(t) = \frac{1}{a}\frac{1}{(1-t)^2} = O\left((1-t)^{-m_L}\right),$$

for $2 < m_L < \frac{5}{2}$. **The estimator is normally asymptotic for every $a > 0$.**

Numerical simulations for the Pareto law

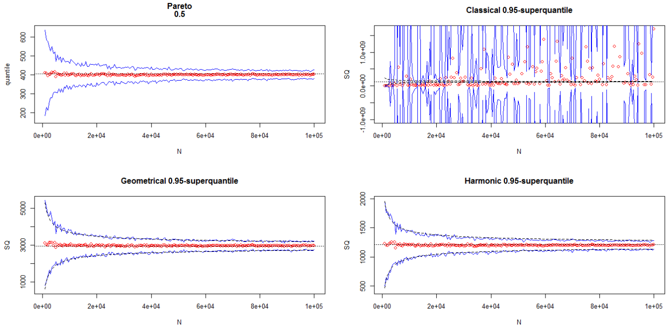


FIGURE: Numerical convergence test for the Pareto distribution ($a = 0.5$).

Numerical simulations for the Pareto law

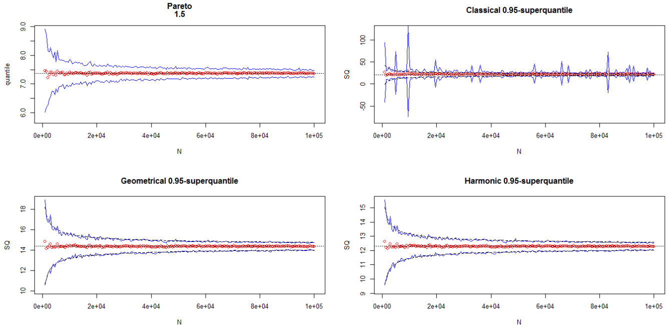


FIGURE: Numerical convergence test for the Pareto distribution ($a = 1.5$).

Numerical simulations for the Pareto law

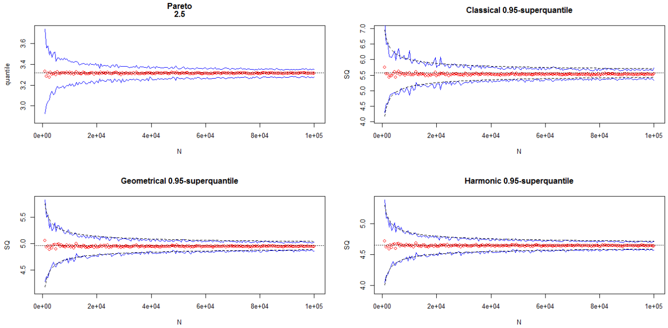


FIGURE: Numerical convergence test for the Pareto distribution ($a = 2.5$).

Gascon

GASCON is a software developed by **CEA** (French Atomic Energy Commission) to study the potential chronological atmospheric releases and dosimetric impact for **nuclear facilities safety assessment**.

It evaluates, from a fictitious radioactive release, the doses received by a population exposed to the cloud of radionuclides and through the food chains.

We focus on one output of GASCON, **the annual effective dose in ^{129}I received in one year by an adult who lives in the neighborhood of a particular nuclear facility**.

As GASCON is relatively **costly in computational time**, a metamodel (of polynomial form and which depends on 10 **input variables**) of GASCON outputs was built in order to perform uncertainty and sensitivity analysis.

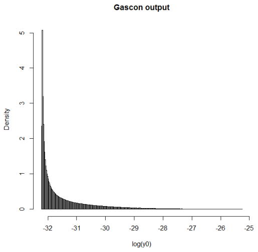
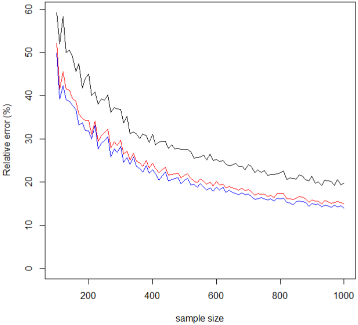


FIGURE: Distribution of the GASCON output variable.

Quantile	Classical superquantile	Geometrical superquantile	Harmonic superquantile
1.304×10^{-13}	4.769×10^{-13}	3.316×10^{-13}	2.637×10^{-13}



Conclusion and perspectives

- The Bregman superquantile is a new coherent measure of risk. This tool is rich thanks to the functions γ because of their diversity.
- The theoretical properties obtained are confirmed on several numerical test cases. More precisely, geometrical and harmonic superquantiles are more robust than the classical superquantile. This robustness is particularly important in risk assessment studies.
- We would like to better understand the choice of the function γ.
- We still work on it in the case of stochastic codes.

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The quantile is not subadditive

A classical counter-example from "Coherent measures of risk", P. Artzner, F. Delbaen, J.M Eber and D. Health. We consider two independent identically distributed random variables X_1 and X_2 having the same density 0.90 on the interval $[0,1]$ and the same density 0.05 on the interval $[-2,0]$. Then

$$q_{X_1}^{10\%} = q_{X_2}^{10\%} = 0 \ \textit{and} \ q_{X_1+X_2}^{10\%} > 0$$