NUMERICAL APPROXIMATION OF MULTIPLE ISOLATED ROOTS OF ANALYTIC SYSTEMS

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Abstract. We propose a numerical analysis of a simplified version of the previous paper *Multiplicity hunting and approximating multiple roots of polynomial systems* written by the two authors.

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1. Equivalent systems and multiplicity

The paper **Multiplicity hunting and approximating multiple roots of polynomial systems** [13] was written in a heuristic way. We achieve its numerical analysis in the present paper, by the way simplifying the procedure given previously.

Definition 1. *A root ζ of an analytic system $f = 0$ (defined in a neighbourhood of ζ) is isolated and singular if*

1– *there exists a neighbourhood of ζ where ζ is the only root of $f = 0$.*
2– *the Jacobian matrix $Df(\zeta)$ is not full rank.*

Remark that the first assumption implies that the number of equations is larger or equal than the number of variables. Note also that this frame includes the important particular case of an analytic system obtained by localizing a polynomial system.
We shall use equally the words singular or multiple for such a root. We have explained in [13] how to derive a regular system (i.e admitting ζ as regular root) from a singular system at a multiple isolated root, provided the assumption that ζ is exactly known. We formalized this transformation by the notion of equivalent systems at a point ζ. More precisely let ζ be a multiple isolated root of an analytic system $f(x) = (f_1(x), \ldots, f_s(x))$ with x in a neighbourhood of ζ in $\mathbf{C}^n$ (note that $s \geq n$). Our method computed a regular system admitting the same root ζ, and that we called *equivalent*. Note that this is obtained **without adding new variables** (important feature we underline).

The *multiplicity* of a root is an important numerical invariant. In the case where there is only one variable and one equation, the multiplicity of a root is exactly the number of derivatives which vanishes at the root, which is unfortunately no longer true in the multivariate situation. We have to introduce a more complicated machinery.
Let us call

1– $\mathbf{C}\{x - \zeta\}$ the ring of the germs of analytic functions at ζ, i.e. the ring of convergent power series in a neighbourhood of ζ, with maximal ideal generated by $x_1 - \zeta_1, \ldots, x_n - \zeta_n$.
2– $I\mathbf{C}\{x - \zeta\}$ the ideal induced generated by the ideal $I = I(f) :=< f_1, \ldots, f_s >$ in $\mathbf{C}\{x - \zeta\}$.

Definition 2. *The multiplicity $\mu(\zeta)$ of an isolated root ζ is defined as the dimension of the quotient space $\mathbf{C}\{x - \zeta\}/I\mathbf{C}\{x - \zeta\}$.*

Relatively to $<$ a admissible local order in $\mathbf{C}\{x - \zeta\}$, we denote by $LT(IC(x - \zeta))$ the ideal generated by the leading terms of all elements of $I\mathbf{C}\{x - \zeta\}$.

Definition 3. *A (minimal) standard basis of $I\mathbf{C}\{x-\zeta\}$ is a finite set of series of $I\mathbf{C}\{x-\zeta\}$ whose leading terms generate minimally $LT(IC(x - \zeta))$.*

We can prove that there is only a finite number of monomials, named standard monomials, which are not in I. The following theorem is classical in the literature about standard bases.

Theorem 1. *The following are equivalent:*

1– *The root ζ is isolated.*

2– *dim* $\mathbf{C}\{x-\zeta\}/IC(x-\zeta)$ *is finite.*
3– *dim* $\mathbf{C}\{x-\zeta\}/LT(IC(x-\zeta))$ *is finite.*
4– *There are only finitely many standard monomials.*

Furthermore, when any of these conditions is satisfied, we have

$$\mu(\zeta) = dim\, C\{x-\zeta\}/LT(IC(x-\zeta)) = \textit{number of standard monomials.}$$

In the particular case of a localized polynomial system, whose equation have a total degree upper bounded by some integer d, the multiplicity is upper bounded by d^n.

2. Overview of this study

To approximate a multiple isolated root is difficult because the root can be a repulsive point for a fixed point method like the classical Newton's method (see the example given by Griewank and Osborne in [18], p. 752). From a point of view of the theoretical analysis, the technical background used when the derivative has constant rank is not possible. This case is well understood and there are many papers on this subject, see for instance [41], [1] and references within. To overcome this drawback, the goal is to define an operator named singular Newton operator generalizing the classical Newton operator defined in the regular case. To do so we construct a finite sequence of equivalent systems named *deflation sequence*, where the multiplicity of the root drops strictly between two successive elements of the sequence. Hence the root is a regular root for the last system. Then we extract from it a regular square system we named deflated system. The singular Newton operator is defined as the classical Newton operator associated to this deflated system.

We now explain the main idea of the construction of the deflation sequence. Since the Jacobian matrix is rank deficient at the root, it means that there exists relations between the lines (respectively columns) of this Jacobian matrix. These relations are given by the Schur complement of the Jacobian matrix. When adding the elements of the Schur complement to the initial system (we call this operation *kerneling*), we obtain an equivalent system where the multiplicity of the root has dropped. In this way, a sequence of equivalent system can be defined iteratively. This will be explained in section 7.

Then we perform a local α-theory of Smale of this singular Newton operator. We first state a γ-theorem, i.e., a result which gives the radius of quadratic convergence of this singular Newton operator and next we give a condition using Rouché's theorem to prove the existence of a singular root.

The context of this study is that of square integrable analytic functions. In this way, it is possible to represent an analytic function and its derivatives thanks to an efficient kernel : the Bergman kernel. Moreover, our study is free of ε (the measure of the numerical approximation) in the following sense:

Definition 4. *We said that a numerical algorithm is free of* ε *if the input of the algorithm does not contain the variable* ε*.*

The determination of a deflation sequence presented in the table 2 is free of ε under the assumption that the norm defined in section 5 (or an upper bound) is given. To do that we present new results to determine by algorithms free of ε:

1– The numerical rank of a matrix : this is achieved in section 4.

2– How close to zero is the evaluation map, see the section 6.

We will see that the two previous problems are applications of the α-theory.

The analysis we present here generalizes what was done by Lecerf, Salvy and the authors of the present work [14]. Under the hypothesis of a square system ($s = n$) and a multiple root of embedding dimension one, i.e., where the the rank of Jacobian matrix drops numerically by one, we treated the case of cluster of zeroes using numerically the implicit function theorem. More precisely, there exists an analytic function $\varphi(x_1, \ldots, x_{n-1})$ such that $\zeta_n = \varphi(\zeta_1, \ldots, \zeta_{n-1})$ and hence ζ_n is a root of the univariate function $h(x_n) = f_n(\varphi(x_1, \ldots, x_{n-1}), x_n)$. Applying the results established in [15] on the function $h(x_n)$, we can deduce both the multiplicity of ζ_n and a way to approximate quickly the root ζ_n. Note that this work extends the case of "simple double zeroes" previously studied by Dedieu and Shub [9].

3. Related works

The case of one variable and one equation was hugely studied in the literature and the generalization of the classical Newton operator is the Schröder operator defined in page 324 of [38]. Moreover, the α-theory of this operator is done in [15] with main references on this subject.

The multivariate case has been studied from purely symbolic and/or numerical points of view. We will not discuss here the works with only a symbolic treatment, see for instance [3]. One of numerical pioneers is Rall [34]. He treats the particular case where the singular root satisfies the following assumption: there exists an index m, defined as the multiplicity of ζ, such that $N_m = \{0\}$ where

$$N_1 = \mathsf{Ker}\, Df(\zeta), \quad N_{k+1} = N_k \cup \mathsf{Ker}\, Df^{k+1}(\zeta), \quad k = 1 : m-1.$$

Then it is possible to construct iteratively an operator to retrieve the local quadratic convergence of the classical Newton operator. The idea of the construction of this operator consists to project iteratively the error $x_0 - \zeta$ on the kernels N_k and its orthogonal $N_k^{\perp}$.

At the same time, the idea to use a variant of a Gauss-Newton's method to approximate a singular isolated root has been investigated by Shamanskii in [39]. But this method converges quadratically towards the singular root under very particular assumptions.

Another techniques are bordered techniques, where some assumption is done on the root. For instance, if the operator induced by the projection from $\mathsf{Ker}\, Df(\zeta)$ into $\mathsf{Ker}\,(Df(\zeta)^*)^{\perp}$:

$$\pi_{(\,\mathsf{Ker}\, Df(\zeta)^*)^{\perp}} D^2 f(\zeta)(z, \pi_{\,\mathsf{Ker}\, Df(\zeta)})$$

is invertible, then the $(\zeta, 0)$ is a regular root of a system, called bordered system, having $2n - r$ variables. The bordered system is constructed from the initial system and from the singular value decomposition of the Jacobian matrix. This way has been developed by Shen and Ypma in [40] and extends this bordered technique used by Griewank [16] in the case of deficient rank one. At the beginning of the eighties a collection of papers addresses the problem of the approximation of the singular roots with similar techniques [35], [36], [7], [8], [17], [6] [21], [42]. These methods previously cited are purely numerical methods and neither the geometry of the problem nor the notion of multiplicity are mentioned.

Ojika in [32] proposes a similar method called deflation method to compute a regular system from the singular initial one, by mixing both symbolic and numerical calculations. This paper is an extension of an algorithm previously developed in [33]. The search of a regular system deals with Gauss forward elimination but there is no analysis of this procedure, especially no numerical determination of the rank. Note also that the attempt to classify the singular roots suffers from not being related to the concept of multiplicity. Moreover, there is no study of complexity, in the case where we study a localized polynomial system. This approach was echoed by Lecerf in [23]. He was able to give a deflation algorithm which outputs a regular triangular system at a root ζ. Moreover he studied precisely the complexity of his deflation algorithm, which is in:

$$\mathcal{O}\left(n^3(nL + n^{\Omega})\mu(\zeta)^2 \log(n\, \mu(\zeta))\right)$$

where n is the number of variables, $\mu(\zeta)$ the multiplicity, $3 \leq \Omega < 4$ and L is the length of the straight line program describing the system.

Leykin, Verschelde and Zhao proposed in [24] a similar modified deflation method, based on the following observation: if the numerical rank of the system is r, there exists an isolated solution $(\zeta, \delta) \in \mathbf{C}^n \times \mathbf{C}^{r+1}$ of the system

$$Df(x)B\delta = 0, \quad \delta^* h - 1 = 0, \tag{1}$$

where $B \in \mathbf{C}^{n\times(r+1)}$ and $h \in \mathbf{C}^{r+1}$ are randomly chosen. The multiplicity of the root (ζ, δ) of the deflated system is lower than the multiplicity $\mu(\zeta)$ of the root ζ of the initial system. Then a step of deflation consists to add the equations (1) to the initial system. The theorem is then that it is enough to perform $\mu(\zeta) - 1$ steps of deflation to get a regular system. This implies that the numbers of variables and equations can double in the worst case. And unfortunately the determination of the numerical rank, based on the work of [12], is not free of ε.

In the same same vein we have the papers of Dayton and Zeng [5] which treats the polynomial case, Dayton, Li and Zeng in the analytic case [4] and Nan Li, Lihong Zhi [27]. Particular cases were studied by Nan Li and Lihong Zhi in several papers [26], [25]. But all these papers furnish a superficial numerical analysis of their algorithms.

The duality and the relationship with the Macaulay matrices constitute the theoretical background of Mourrain [31], Mantzaflais and Mourrain [28] or more recently Hausenstein, Mourrain, Szanto in [19]. Actually, the relations between the columns (respectively the lines) represent those of the space (respectively, columns). As we shall point out, a classical fact show that all these relations can be found through the Schur complement.

4. Tracking the rank of a matrix

Let $s \geq n$ be two integers, M a $s \times n$-matrix with complex coefficients, $U\Sigma V^*$ a singular value decomposition of M, and $\sigma_1 \geq \ldots \geq \sigma_n$ its singular values.

We consider the elementary symmetric sums of the σ_i's, i.e.:

$$s_k = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \sigma_{i_1} \ldots \sigma_{i_k}, \quad k = 1 : n.$$

In other words, the singular values are the roots of the polynomial $s(\lambda)$ of degree n

$$s(\lambda) = \prod_{i=1}^{n} (\lambda - \sigma_i) = \lambda^n + \sum_{1 \leq i \leq n} (-1)^{(n-i)} s_{n-i} \lambda^i.$$

By convention $s_0 = 1$. Let us remark that this convention is natural: it allows to treat the case where all the singular values are zero, which means that the matrix M is null and its rank is zero.

More generally if the rank of M is r, the s_i's are non-zero up to the rank ($i = 0 : r$), and zero after. Then for $k = n - r : n$, the quantities s_{n-k} are non-zero and we can introduce:

1– $b_k(M) := \max_{0 \leq i \leq k-1} \left(\dfrac{s_{n-i}}{s_{n-k}} \right)^{\frac{1}{k-i}}$.

2– $g_k(M) := \max_{k+1 \leq i \leq n} \left(\dfrac{s_{n-i}}{s_{n-k}} \right)^{\frac{1}{i-k}}$.

3– $a_k(M) := b_k(M)\, g_k(M)$.

with the convention $g_n(M) = 1$.

We precise the notion of ε-rank used in the sequel.

Definition 5. *Let ε be a nonnegative number. A matrix M has ε-rank equal to r_ε if its singular values verify*

$$\sigma_1 \geq \ldots \geq \sigma_{r_\varepsilon} > \varepsilon \geq \sigma_{r_\varepsilon + 1} \geq \ldots \geq \sigma_n. \tag{2}$$

Observe that an upper bound for the ε-rank is the rank r itself.
Let Σ_ε the matrix obtained from Σ by putting $\sigma_{r+1} = \ldots = \sigma_n = 0$. We define $M_\varepsilon = U\Sigma_\varepsilon V^*$.

Remark 1. *If rank $M \geq r$, we know that M_ε is the nearest matrix of M which is of rank r.*

Remark 2. *The definition 5 is justified by the Eckardt-Young-Mirsky theorem which has a long story in low rank approximation theory: see* [10], [30] *and* [29] *for more recent developments.*

For simplicity let us denote by a_k, b_k, g_k the corresponding values $a_k(M)$, $b_k(M)$, $g_k(M)$.

Theorem 2. *Let a matrix M be such that* $\mathrm{rank}(M) = r$. *Let m an integer be such that $n - r \le m \le n$, and*

$$\varepsilon = \frac{3a_m + 1 - \sqrt{(3a_m+1)^2 - 16a_m}}{4g_m}.$$

If $a_m < 1/9$ then the matrix M has ε-rank equal to $n - m$.

Proof. As $n - r \le m$, the quantity s_{n-m} is not zero since it is positive. Let us consider the polynomials

$$p(\lambda) = \frac{1}{s_{n-m}} s(\lambda) = \frac{1}{s_{n-m}} \prod_{i=1}^{n} (\lambda - \sigma_i) = \sum_{i=0}^{n} (-1)^{n-i} \frac{s_{n-i}}{s_{n-m}} \lambda^i$$

and

$$q(\lambda) = \sum_{i=m}^{n} (-1)^{n-i} \frac{s_{n-i}}{s_{n-m}} \lambda^i.$$

Lemma 1. *Let $\tau := g_m|\lambda|$. Then for all λ such that $|\lambda| < 1/g_m$, hence for all $\tau < 1$:*

$$|q(\lambda)| \ge |\lambda|^m \frac{1-2\tau}{1-\tau}$$

Proof.

$$\begin{aligned}
|q(\lambda)| &= \left| \lambda^m + \sum_{i=m+1}^{n} (-1)^{n-i} \frac{s_{n-i}}{s_{n-m}} \lambda^i \right| \\
&\ge |\lambda|^m - \sum_{i=m+1}^{n} \frac{s_{n-i}}{s_{n-m}} |\lambda|^i \\
&\ge |\lambda|^m \left(1 - \sum_{i=m+1}^{n} \frac{s_{n-i}}{s_{n-m}} |\lambda|^{i-m} \right) \\
&\ge |\lambda|^m \left(1 - \sum_{i \ge m+1} (g_m|\lambda|)^{i-m} \right) \\
&\ge |\lambda|^m \frac{1 - 2g_m|\lambda|}{1 - g_m|\lambda|}. \qquad (3)
\end{aligned}$$

□

We first prove that 0 is the only root of $q(\lambda)$ in the open ball $B\left(0, \frac{1}{2g_m}\right)$. Let ν be a non-zero root of $q(\lambda)$. Then we have by lemma 1

$$0 = q(\nu) = |q(\nu)| \ge |\nu|^m \frac{1 - 2g_m|\nu|}{1 - g_m|\nu|}.$$

Hence $|\nu| \ge \dfrac{1}{2g_m}$.
Now consider the trinomial

$$2\tau^2 - (3a_m + 1)\tau + 2a_m. \qquad (4)$$

If $a_m < 1/9$, then this trinomial has two real roots $\tau_1 < \tau_2$, since its

$$\Delta = (3a_m + 1)^2 - 16a_m = 9{a_m}^2 - 10a_m + 1 = (9a_m - 1)(a_m - 1)$$

is positive. We can check explicitely that τ_1 is positive, since it boils down to a_m being positive.

We prove that for $|\lambda|$ satisfying $\frac{\tau_1}{g_m} \leq |\lambda| < \frac{1}{2g_m}$, $p(\lambda)$ has m roots counting with multiplicities in the open ball $B(0, |\lambda|)$ (note that the range of the interval where $|\lambda|$ is asked to live is positive, since $\tau_1 < 1/2$). To do that, we verify that Rouché's inequality

$$|p(\lambda) - q(\lambda)| < |q(\lambda)| \tag{5}$$

holds on the sphere of radius $|\lambda|$. We have

$$\begin{aligned} |p(\lambda) - q(\lambda)| &\leq \sum_{i=0}^{m-1} \frac{s_{n-i}}{s_{n-m}} |\lambda|^i \\ &\leq \sum_{i=0}^{m-1} b_m^{m-i} |\lambda|^i \\ &\leq |\lambda|^m \frac{b_m/|\lambda|}{1 - b_m/|\lambda|} \\ &\leq \frac{a_m}{g_m|\lambda| - a_m} |\lambda|^m. \end{aligned} \tag{6}$$

We check that $\tau - a_m > \tau_1 - a_m = \dfrac{-a_m + 1 - \sqrt{\Delta}}{4}$ is positive if $a_m < 1/9$.
From (6) and lemma 1, we see that the Rouché's inequality is satisfied if

$$\frac{a_m}{\tau - a_m} |\lambda|^m < \frac{1 - 2\tau}{1 - \tau} |\lambda|^m.$$

Since $|\lambda|$, $1-\tau$ and $\tau - a_m$ are positive, this is equivalent to the trinomial (4) being negative, which is insured by the condition $a_m < 1/9$.
Hence under the condition $a_m < 1/9$ the polynomial $p(\lambda)$ has exactly m roots counting the multiplicities in the open ball $B(0, |\lambda|)$ where

$$\varepsilon := \frac{\tau_1}{g_m} \leq |\lambda| < \frac{1}{2g_m}.$$

Consequently we have

$$\sigma_1 \geq \ldots \geq \sigma_{n-m} > \varepsilon \geq \sigma_{n-m+1} \geq \ldots \geq \sigma_n.$$

We are done. □

Theorem 3. *The algorithm of the table 1 computes the ε-rank of a matrix thanks to the theorem 2.*

Remark 3. *In fact this algorithm is free of ε and we call the computed ε-rank the* numerical *rank of the given matrix.*

numerical rank

Input : a matrix $M \in \mathbf{C}^{s \times n}$, $s \geq n$.
Compute the singular values of M : $\sigma_1 \geq \ldots \geq \sigma_n$.
Let r be the rank of M, i.e. $\sigma_{r+1} > 0, \sigma_r = 0$.
From these σ_i's, compute the quantities a_k, $k = n - r : n$ and g_k defined in the section 4.
if there exists $m \geq n - r$ s.t. $a_m < 1/9$ then

$$\varepsilon := \frac{3a_m + 1 - \sqrt{(3a_m+1)^2 - 16a_m}}{4g_m}$$

  the ε-rank of the matrix M is $n - m$. from the theorem 2
else
  $\varepsilon < \sigma_n$. The ε-rank of the matrix M is n.
end if
Output : the ε-rank of the matrix M.

Table 1

5. The functional framework

Let $n \geq 2$, $R_\omega \geq 0$ and $\omega \in \mathbf{C}^n$. We consider the set $\mathbf{A}^2(\omega, R_\omega)$ of the square integrable analytic functions in the open ball $B(\omega, R_\omega)$, which is an Hilbert space equipped with the inner product

$$< f, g >= \int_{B(\omega, R_\omega)} f(z)\overline{g(z)} d\nu(z),$$

where ν is the Lebesgue measure on $\mathbf{C}^n$, normalized so that $\nu(B(\omega, R_\omega)) = R_\omega^{2n}$.
Next $(\mathbf{A}^2(\omega, R_\omega))^s$ has an hilbertian structure with the inner product

$$< f, g >= \sum_{i=1}^{s} < f_i, g_i > .$$

We denote by $||f||$ the associated norm.

Observe that this framework includes the case of an analytic system obtained by localizing a polynomial system.

5.1. The Bergman kernel.

in [37] and S.G. Krantz in [22]. Since for each $x \in B(\omega, R_\omega)$ and $f \in \mathbf{A}^2(\omega, R_\omega)$, the evaluation map $f \mapsto f(x)$ is a continuous linear functional $eval_x$ on $\mathbf{A}^2$, there exists from the Riesz representation theorem an element $h_x \in \mathbf{A}^2$ such that

$$f(x) = eval_x(f) =< f, h_x > .$$

Set down the function $\rho := x \mapsto \rho_x = \|x - \omega\|$.

Definition 6. *The function* $(z,x) \mapsto H(z,x) := \overline{h_x(z)}$ *is named the Bergman kernel. It has the reproducing property :*

$$f(x) = \int_{B(\omega,R_\omega)} f(z)\, H(z,x)\, d\nu(z), \quad \forall f \in \mathbf{A}^2(\omega, R_\omega).$$

We say that the Bergman kernel reproduces $\mathbf{A}^2(\omega, R_\omega)$*. We state some classical properties of this reproducing kernel.*

5.2. **Properties.**

Proposition 1.

1– $H(z,x) = \dfrac{R_\omega^2}{(R_\omega^2 - < z-\omega, x-\omega >)^{n+1}}$

2– $H(x,x) = \|H(\bullet,x)\|^2 = \dfrac{R_\omega^2}{(R_\omega^2 - \|x-\omega\|^2)^{n+1}} = \dfrac{R_\omega^2}{(R_\omega^2 - \rho_x^2)^{n+1}}.$

3– *For all* $f \in \mathbf{A}^2(\omega, R_\omega)$ *we have*

$$|f(x) = | \int_{B(\omega,R_\omega)} f(z) H(z,x) d\nu(z)| \le \frac{\|f\|\, R_\omega}{(R_\omega^2 - \rho_x^2)^{\frac{n+1}{2}}}$$

Proof. See Theorem 3.1.3. page 37 in [37]. ☐

The previous proposition generalizes to higher derivatives.

Proposition 2. *Let* $k \ge 0$*,* $\omega \in \mathbf{C}^n$*,* $x \in B(\omega, R_\omega)$ *and* $u_i \in \mathbf{C}^n$*,* $i = 1 : k$*. Let us introduce*

$$H_k(z,x,u_1,\ldots,u_k) = \frac{(n+1)\cdots(n+k) < z-\omega, u_1 > \cdots < z-\omega, u_k >}{(R_\omega^2 - < z-\omega, x-\omega >)^k} H(z,x).$$

We have

1– $D^k f(x)(u_1,\cdots,u_k) = \displaystyle\int_{B(\omega,R_\omega)} f(z)\, H_k(z,x,u_1,\cdots,u_k)\, d\nu(z).$

2– $\|D^k f(x)\| \le ||f|| \, \dfrac{(n+1)\cdots(n+k)\, R_\omega^{1+k}}{(R_\omega^2 - \rho_x^2)^{\frac{n+1}{2}+k}}$

(evidently if $k = 0$ *the range where* i *lives is empty, and the products* $(n+1)\cdots(n+k)$ *and* $< z-\omega, u_1 > \cdots < z-\omega, u_k >$ *are 1.)*

To prove this we need the following

Lemma 2.

$$\|H_k(\bullet, x, u_1, \ldots, u_n)\| \le \frac{(n+1)\ldots(n+k)\, R_\omega^{1+k}}{(R_\omega^2 - \rho_x^2)^{\frac{n+1}{2}+k}} \|u_1\| \ldots \|u_k\|.$$

Proof. We have to compute the integral of $H_k \bar{H}_k$ on the ball $B(\omega, R_\omega)$. This is reduced to estimate

$$I_k = \int_{B(\omega,R_\omega)} \frac{R_\omega^2}{(R_\omega^2 - < z-\omega, x-\omega >)^{n+1+k} (R_\omega^2 - \overline{< z-\omega, x-\omega >})^{n+1+k}} d\nu(z)$$

since

$$\|H_k(z,x,u_1,\ldots,u_n)\| \le (n+1)\ldots(n+k)\|u_1\|\ldots\|u_k\|\, R_\omega^{1+k}\, I_k^{1/2}.$$

We have

$$\begin{aligned} I_k &= \int_{B(\omega,R_\omega)} H(z,x)\frac{1}{(R_\omega^2 - < z-\omega, x-\omega >)^k (R_\omega^2 - \overline{< z-\omega, x-\omega >})^{n+1+k}} d\nu(z) \\ &= \frac{1}{(R_\omega^2-\rho_x^2)^{n+1+2k}} \end{aligned}$$

using the formula for the Bergman kernel (Proposition 1) and its reproducing property applied to the function $\dfrac{1}{(R_\omega^2-\rho_x^2)^{n+1+2k}}$.

The proof of the lemma follows. □

We now prove the proposition 2.

Proof. We proceed by induction. The proposition 1 treats the case $k=0$. Next, we have:

$$\begin{aligned} D^{k+1}f(x)(u_1,\ldots,u_k,u_{k+1}) &= \frac{d}{dt}D^k f(x+tu_{k+1})(u_1,\ldots,u_k)\Big|_{t=0} \\ &= \frac{d}{dt}\int_{B(\omega,R_\omega)} f(z)H_k(z,x+tu_{k+1},u_1,\ldots,u_k)d\nu(z)\Big|_{t=0} \\ &= \int_{B(\omega,R_\omega)} f(z)\frac{H_k(z,x,u_1,\ldots,u_k)(n+1+k) < z-\omega, u_{k+1} >}{(R_\omega^2 - < z-\omega, x-\omega >)} d\nu(z) \\ &= \int_{B(\omega,R_\omega)} f(z)H_{k+1}(z,x,u_1,\ldots,u_{k+1})d\nu(z). \end{aligned}$$

Hence the first assertion holds. For the second assertion, we write

$$\|D^k f(x)(u_1,\ldots,u_k)\| \le \|f\|\,\|H_k(\bullet,x,u_1,\ldots,u_k)\|.$$

Using the lemma 2, we are done. □

From the propositions 1 and 2 we deduce easily the following

Proposition 3. *For all $k\ge 0$, $x\in\mathbf{C}^n$ and $f\in(\mathbf{A}^2(\omega,R_\omega))^s$ we have*

$$\|D^k f(x)\| \le ||f||\,\frac{(n+1)\ldots(n+k)R_\omega^{1+k}}{(R_\omega^2-\rho_x^2)^{\frac{n+1}{2}+k}}.$$

6. Analysis of the evaluation map

The evaluation map is defined by

$$eval\ : (f,x)\mapsto eval_x(f) = f(x)$$

from $(\mathbf{A}^2(\omega,R_\omega))^s\times B(\omega,R_\omega)$ to $\mathbf{C}^s$.

Let $c_0 := \sum_{k\ge 0}(1/2)^{2^k-1}$ ($\sim$ 1.63...), and α_0 ($\sim$ 0.13...) be the first positive root of the trinomial $(1-4u+2u^2)^2-2u$.

We study the question: when the value $f(x)$ can be considered as small? We give a precise meaning of being small without the use of any ε.

Theorem 4. *Let $f = (f_1, \dots, f_s) \in \mathbf{A}^2(\omega, R_\omega)^s$. Let $x \in B(\omega, R_\omega)$ and $\|x - \omega\| = \rho_x$. If*

$$\frac{c_0}{R_\omega}\,(R_\omega^2 - \rho_x^2)^{\frac{n+1}{2}}\,\|f(x)\| + \rho_x < R_\omega$$

and

$$\frac{(n+1)(n+2)}{2}\,(R_\omega^2 - \rho_x^2)^{(n-3)/2}\left(\|f\|\,R_\omega + (R_\omega^2 - \rho_x^2)\right)\,\|f(x)\| \le \alpha_0$$

then $f(x)$ is small at the following sense : the Newton sequence defined by

$$(f^0, x_0) = (f, x), \quad (f^{k+1}, x_{k+1}) = ((f^k, x_k) - D\,eval\,(f^k, x_k)^\dagger\,eval\,(f^k, x_k)), \quad k \ge 0,$$

converges quadratically towards a certain $(g, y) \in (\mathbf{A}^2(\omega, R_\omega))^s \times B(\omega, R_\omega)$ satisfying $g(y) = 0$. More precisely we have

$$(\|f - g\| + \|x - y\|^2)^{1/2} \le \frac{c_0}{R_\omega}\,(R_\omega^2 - \rho_x^2)^{\frac{n+1}{2}}\,\|f(x)\|.$$

In a straightforward way, we get the corollary

Corollary 1. *Let us consider $x = \omega$ in the theorem 4. If*

$$c_0 R_\omega^{n-1}\|f(x)\| < 1$$

and

$$\frac{(n+1)(n+2)}{2}\,R_\omega^{n-2}\,(\|f\| + R_\omega)\,\|f(x)\| \le \alpha_0.$$

then $f(x)$ is small. More precisely there exists $(g, y) \in (\mathbf{A}^2(x, R_\omega))^s \times B(x, R_\omega)$ such that $g(y) = 0$ and

$$(\|f - g\| + \|x - y\|^2)^{1/2} \le c_0\,R_\omega^n\|f(x)\|.$$

6.1. **Estimates about the derivatives of the evaluation map.**

Proposition 4.

$$\|D\,eval\,(f, x)^\dagger\| \le \frac{1}{R_\omega}(R_\omega^2 - \rho_x^2)^{\frac{n+1}{2}}.$$

Proof. The derivative of the evaluation map is given by

$$D\,eval\,(f, x)(g, y) = g(x) + Df(x)y.$$

Hence $(g, y) \in \ker D\,eval\,(f, x)$ iff $g(x) + Df(x)y = 0$. That is

$$< g_i, H(\bullet, x) > + < y, Df_i(x)^* >= 0, \quad i = 1 : s.$$

In term of inner product in $(\mathbf{A}^2)^s \times \mathbf{C}^n$ we have

$$< g, (0, \dots, 0, H(\bullet, x), 0, \dots, 0) > + < y, Df_i(x)^* >= 0, \quad i = 1 : s.$$

This shows that the vector space $(\ker D\,eval\,(f, x))^\perp$ is generated by the set of

$$(H(\bullet, x)v, Df(x)^* v)$$

where $v \in \mathbf{C}^n$. The condition

$$D\,eval\,(f, x)(H(\bullet, x),\ Df(x)^* v) = u$$

becomes

$$(H(x,x)I_s + Df(x)Df(x)^*)\, v = u.$$

The matrix $\mathcal{E} = H(x,x)I_s + Df(x)Df(x)^*$ is the sum of a diagonal positive matrix and an hermitian matrix. By Weyl theorem (page 203 in G.W. Stewart, J.Q. Sun, Matrix Perturbation Theory, Academic Press, 1990) the eigenvalues of the matrix $\mathcal{E}$ are greater than those of $H(x,x)I_s > 0$. Hence the norm of the inverse matrix $\mathcal{E}^{-1}$ satisfies

$$\|\mathcal{E}^{-1}\| \le \frac{1}{H(x,x)}.$$

This permits to calculate $\|D\, eval\,(f,x)^\dagger\|$. In fact, let $u, v \in \mathbf{C}^n$ be such that $\mathcal{E}v = u$. We have

$$\begin{aligned} \|D\, eval\,(f,x)^\dagger u\|^2 &= \|H(\bullet, x)\|^2\, \|v\|^2 + \|Df(x)v\|^2 \\ &= H(x,x)\, \|v\|^2 + \|Df(x)^* v\|^2. \end{aligned}$$

Since the matrix $\mathcal{E}^{-1}$ is hermitian, we can write

$$\begin{aligned} \|D\, eval\,(f,x)^\dagger u\|^2 &= v^* \mathcal{E} v \\ &= u^* \mathcal{E}^{-1} u \\ &\le \|\mathcal{E}^{-1}\|\, \|u\|^2. \end{aligned}$$

Finally

$$\begin{aligned} \|D\, eval\,(f,x)^\dagger\|^2 &\le \|\mathcal{E}^{-1}\| \\ &\le \frac{1}{H(x,x)} \\ &\le \frac{1}{R_\omega^2} \left(R_\omega^2 - \rho_x^2\right)^{n+1}. \end{aligned}$$

This proves the proposition. $\square$

Proposition 5.

$$\|D^k\, eval\,(f,x)\| \le \frac{(n+1)\dots(n+k)\, \|f\|\, R_\omega^{1+k}}{(R_\omega^2 - \rho_x^2)^{\frac{n+1}{2}+k}} + \frac{k(n+1)\dots(n+k-1)\, R_\omega^k}{(R_\omega^2 - \rho_x^2)^{\frac{n+1}{2}+k-1}}.$$

Proof. We have

$$\begin{aligned} &D^k\, eval\,(f,x)(g^{(1)}, y^{(1)}, \dots, g^{(k)}, y^{(k)}) \\ &\qquad = D^k f(x)(y^{(1)}, \dots, y^{(k)}) + \sum_{j=1}^{k} D^{k-1} g^{(j)}(x)(y^{(1)}, \dots, \widehat{y^{(j)}}, \dots, y^{(k)}), \end{aligned}$$

where $\widehat{y^{(j)}}$ signifies that this term does not appear. Then using the proposition 2 we find that

$$\begin{aligned}
&\|D^k\, eval\,(f,x)(g^{(1)},y^{(1)},\ldots,g^{(k)},y^{(k)})\| \\
&\quad\le \|D^k f(x)(y^{(1)},\ldots,y^{(k)})\| + \sum_{j=1}^{k} \|D^{k-1}g^{(j)}(x)(y^{(1)},\ldots,\widehat{y^{(j)}},\ldots,y^{(k)})\| \\
&\quad\le \frac{(n+1)\ldots(n+k)\,\|f\|\,R_\omega^{1+k}}{(R_\omega^2-\rho_x^2)^{\frac{n+1}{2}+k}}\|y^{(1)}\|\ldots\|y^{(k)}\| \\
&\qquad + \sum_{j=1}^{k} \frac{(n+1)\ldots(n+k-1)\,\|g^{(j)}\|\,R_\omega^{k}}{(R_\omega^2-\rho_x^2)^{\frac{n+1}{2}+k-1}}\|y^{(1)}\|\ldots\widehat{\|y^{(j)}\|}\ldots\|y^{(k)}\|.
\end{aligned}$$

We bound $\|y^{(j)}\|$ and $\|g^{(j)}\|$ by $\|(g^{(j)},y^{(j)})\|$. We obtain

$$\begin{aligned}
&\|D^k\, eval\,(f,x)(g^{(1)},y^{(1)},\ldots,g^{(k)},y^{(k)})\| \\
&\quad\le \left(\frac{(n+1)\ldots(n+k)\,\|f\|\,R_\omega^{1+k}}{(R_\omega^2-\rho_x^2)^{\frac{n+1}{2}+k}} + \frac{k(n+1)\ldots(n+k-1)\,\|g^{(j)}\|\,R_\omega^{k}}{(R_\omega^2-\rho_x^2)^{\frac{n+1}{2}+k-1}}\right) \\
&\qquad\qquad ||(g^{(1)},y^{(1)})\|\ldots\|(g^{(k)},y^{(k)})\|.
\end{aligned}$$

Finally

$$\|D^k\, eval\,(f,x)\| \le \frac{(n+1)\ldots(n+k)\,\|f\|\,R_\omega^{1+k}}{(R_\omega^2-\rho_x^2)^{\frac{n+1}{2}+k}} + \frac{k(n+1)\ldots(n+k-1)\,R_\omega^{k}}{(R_\omega^2-\rho_x^2)^{\frac{n+1}{2}+k-1}}.$$

□

6.2. **Proof of the theorem 4.** The proof uses the theorem 128 page 121 in J.-P. Dedieu, Points fixes, zéros et la méthode de Newton. Springer, 2006.

Theorem 5. *Let f an analytic map from $\mathbf{E}$ to $\mathbf{F}$ two Hilbert spaces be given. Let $x \in \mathbf{C}^n$. We suppose that $Df(x)$ is surjective. We introduce the quantities*

1– $\beta(f,x) = \|Df(x)^\dagger f(x)\|$.

2– $\gamma(f,x) = \sup_{k\ge 2} \|\frac{1}{k!}Df(x)^\dagger D^k f(x)\|^{\frac{1}{k-1}}$.

3– $\alpha(f,x) = \beta(f,x)\gamma(f,x)$.

Let α_0 and c_0 be the constants introduced in this section.
If $\alpha(f,x) \le \alpha_0$ then there exists a zero ζ of f in the ball $B(x_0, c_0\beta(f,x_0))$ and the Newton sequence

$$x_0 = x, \quad x_{k+1} = x_k - Df(x_k)^\dagger f(x_k), \quad k \ge 0,$$

converges quadratically towards ζ.

We are now ready to prove the theorem 4.

Proof. The proof consists to verify the condition $\alpha(\mathit{eval}\,,(f,x)) \le \alpha_0$. Using the propositions 4 and 5, we are able to bound the quantity $\gamma(\mathit{eval}\,,(f,x))$. We obtain

$$\begin{aligned}\gamma(\mathit{eval}\,,(f,x)) &\le \sup_{k\ge 2}\left(\frac{1}{k!}\,\|D\,\mathit{eval}\,(f,x)^\dagger\|\,\|D^k\,\mathit{eval}\,(f,x)\|\right)^{\frac{1}{k-1}}\\ &\le \sup_{k\ge 2}\left(\binom{n+k}{k}\frac{\|f\|\,R_\omega^k}{(R_\omega^2-\rho_x^2)^k}+\binom{n+k-1}{k-1}\frac{R_\omega^{k-1}}{(R_\omega^2-\rho_x^2)^{k-1}}\right)^{\frac{1}{k-1}}.\end{aligned}$$

We know that $\binom{n+k}{k}=\frac{n+k}{k}\binom{n+k-1}{k-1}$. Moreover the function $k\mapsto\binom{n+k}{k}^{\frac{1}{k-1}}$ decreases. Hence $\binom{n+k}{k}^{\frac{1}{k-1}}\le\frac{(n+1)(n+2)}{2}$. Then we get the following point estimate

$$\gamma(\mathit{eval}\,,(f,x)) \le \frac{(n+1)(n+2)R_\omega}{2(R_\omega^2-\rho_x^2)}\left(\frac{\|f\|\,R_\omega}{(R_\omega^2-\rho_x^2)}+1\right). \tag{7}$$

In the same way the quantity $\alpha(\mathit{eval}\,,(f,x))$ can be bounded by

$$\begin{aligned}\alpha(\mathit{eval}\,,(f,x)) &\le \gamma(\mathit{eval}\,,(f,x))\,\beta(\mathit{eval}\,,(f,x))\\ &\le \gamma(\mathit{eval}\,,(f,x))\,\|D\,\mathit{eval}\,(f,x)^\dagger\|\,\|f(x)\|\end{aligned}$$

Using the inequalities of propositions 4 and (7) we get

$$\alpha(\mathit{eval}\,,(f,x)) \le \frac{(n+1)(n+2)}{2}\,(R_\omega^2-\rho_x^2)^{(n-3)/2}\left(\|f\|\,R_\omega+(R_\omega^2-\rho_x^2)\right)\,\|f(x)\|. \tag{8}$$

The condition

$$\frac{(n+1)(n+2)}{2}\,(R_\omega^2-\rho_x^2)^{(n-3)/2}\left(\|f\|R_\omega+(R_\omega^2-\rho_x^2)\right)\,\|f(x)\|\le\alpha_0$$

implies evidently $\alpha(\mathit{eval}\,(f,x))\le\alpha_0$.
Hence the theorem 5 applies. The Newton sequence

$$(f^0,x_0)=(f,x),\quad (f^{k+1},x_{k+1})=((f^k,x_k)-D\,\mathit{eval}\,(f^k,x_k)^\dagger\,\mathit{eval}\,(f^k,x_k),\quad k\ge 0,$$

is convergent towards a certain $(g,y)\in B((f,x),c_0\beta(\mathit{eval}\,,(f,x))\subset(\mathbf{A}^2(\omega,R_\omega)^s\times\mathbf{C}^n$. That is to say

$$\begin{aligned}(\|f-g\|^2+\|x-y\|^2)^{\frac12} &\le c_0\beta(\mathit{eval}\,,(f,x))\\ &\le c_0\|D\,\mathit{eval}\,(f,x)^\dagger\|\,\|f(x)\|\\ &\le \frac{c_0}{R_\omega}\,(R_\omega^2-\rho_x^2)^{\frac{n+1}{2}}\|f(x)\|.\end{aligned}$$

This implies that $y\in B(\omega,R_\omega)$ since we have

$$\begin{aligned}\|y-\omega\| &\le \|y-x\|+\rho_x\\ &\le \frac{c_0}{R_\omega}\,(R_\omega^2-\rho_x^2)^{\frac{n+1}{2}}\|f(x)\|+\rho_x\\ &< R_\omega. \qquad \text{from assumption.}\end{aligned}$$

We are done. □

7. Kerneling and singular Newton operator

It consists to prepare the system by dividing the generators into two families. The invariant leading to this partition is the rank r of the Jacobian matrix $Df(\zeta)$ which is not maximal since ζ is singular. Without loss of generality we can assume that the first r generators have linearly independent affine parts.
Since the notion of Schur complement is intensively used in the sequel, we remember its definition.

Definition 7. *The Schur complement of a matrix* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *of rank* $r > 0$ *associated to an invertible submatrix* A *of rank* r *is by definition* $\mathit{Schur}\,(M) := D - CA^{-1}B$. *If* $r = 0$ *we define* $\mathit{Schur}\,(M) := M$.

We also note by $\mathit{vec}\,(\bullet)$ the operator which transforms a matrix into a line vector by concatenating its lines.

Definition 8. *Let* $\varepsilon \geq 0$, $0 \leq r < n$ *and* $f = (f_1, \ldots, f_s) \in \mathbf{C}\{x - x_0\}^s$. *Let us suppose* $D_{1:r}f_{1:r}(x_0)$ *has an* ε*-rank equal to* r. *We define the kerneling operator*

$$K \,:\, f \mapsto (f_1, \ldots, f_r, \mathit{vec}(\,\mathit{Schur}\,(Df(x)))) \in \mathbf{C}\{x - x_0\}^{r+(n-r)\,(s-r)}.$$

We say that $K(f)$ *is an* ε*-kerneling of* f *if we have*

$$\|K(f)\| \leq \varepsilon. \tag{9}$$

We say that the kerneling is exact when $\varepsilon = 0$.

Definition 9. (**Deflation sequence**). *Let* $\varepsilon \geq 0$, $x_0 \in \mathbf{C}^n$ *and* $f = (f_1, \ldots, f_s) \in \mathbf{C}\{x - x_0\}^s$. *The sequence*

$$\begin{aligned} F_0 &= f \\ F_{k+1} &= K(F_k), \quad k \geq 0. \end{aligned}$$

is named the deflation sequence.
The thickness is the index ℓ *where the* ε*-rank of* $DF_\ell(x_0)$ *is equal to* n, *and not before.*
We name deflation system $\mathit{dfl}(f)$ *of* f *a system of rank* n *extracted from* F_ℓ.

We adopt the term *thickness* which is the translation of the french word *épaisseur* introduced by Ensalem in [11] rather than the term *depth* more recently used by Mourrain, Matzaflaris in [28] or Dayton, Li, Zeng [5], [4]. We shall see in section 8 that the thickness is finite. $\circ$

Theorem 6. *Let* $x_0 \in \mathbf{C}^n$ *and* $f \in \mathbf{A}^2(x_0, R_\omega)$. *Then the algorithm described in the table 2 proves the existence of a deflation sequence where the tests verifying the inequalities 5 and 8 are performed respectively thanks to the theorem 2 and the corollary 1.*

Definition 10. *The classical Newton operator associated to the deflation system* $dfl(f)$ *of* ε*-rank* n *is named the singular Newton operator of the initial system* f.

Rather than to compute the deflation sequence introduced in the definition 9, it is sufficient to start from a truncated deflation sequence. To do that we need the following definition.

deflation sequence and deflated system

1- Input : $x_0 \in \mathbf{C}^n$, $f \in \mathbf{A}^2(x_0, R_{x_0})$
2- $\mathsf{dfl}\,(f) = \{\emptyset\}$
3- $F := f$.
4- $\eta := \dfrac{2\alpha_0}{(n+1)(n+2)(R_{x_0} + \|F\|)R_{x_0}^{n-2}}$
5- if $\|F(x_0\| \leq \eta$ then   test justified by corollary 1
6- $r :=$ **numerical rank** $(DF(x_0))$
7-   if $r < n$ then
8- $F := K(F)$
9-   go to 2
10-   else
11- $\mathsf{dfl}\,(f)$ a deflated system of numerical rank n extracted from F
12-   end if
13- end if
14- Output : $\mathsf{dfl}\,(f)$.

TABLE 2

Definition 11. *Let $p \geq 1$. We note by $Tr_{x_0,p}(F)$ the truncated series at the order p of the analytic function F at x_0.*
We name the truncated deflation sequence at the order p at x_0 the sequence :

$$\begin{aligned} T_0 &= Tr_{x_0,p}(f) \\ T_{k+1} &= Tr_{x_0,p-k-1}\left(K\left(T_k\right)\right), \quad 0 \leq k \leq p. \end{aligned}$$

To define the singular Newton operator it is sufficient to know the thickness of the deflation sequence of the definition 9. From this knowledge the determination of the singular Newton operator will use the truncated deflation sequence at the order of the thickness, say ℓ, i.e. that is to say that the rank of F_ℓ is full.

Proposition 6. *Let $\eta > 0$ and $x_0 \in \mathbf{C}^n$. Let ℓ the thickness of the deflation sequence of the definition 9. Let us consider the truncated deflation sequence $(T_k)_{k\geq 0}$ at the order $\ell+1$ at x_0 of the definition 11. Then the singular Newton operator associated to f is equal to the Newton operator associated to T_ℓ.*

Proof. Since T_0 is the truncated series at the order ℓ of F_0, from construction it is easy to see that for all $k = 0 : \ell$, T_k is the truncated series of F_k at the order $p-k$. The conclusion of the proposition follows. $\square$

8. THE MULTIPLICITY DROPS THROUGH KERNELING

This section is devoted to prove that the deflation sequence remains constant after a finite index. This will be achieved trough the following proposition :

	singular Newton
1-	Input : $x_0 \in \mathbf{C}^n$, $f \in \mathbf{A}^2(x_0, R_{x_0})$
2-	$\mathsf{dfl}\,(f) = \mathsf{deflated\ system}(f)$.
3-	Output : If $\mathsf{dfl}\,(f) \neq \emptyset$ then $N_{\mathsf{dfl}\,(f)}(x_0)$ else x_0.

Table 3

Theorem 7. *Let us suppose that the rank of $Df(\zeta)$ is equal to r and that*

$$Df(x) := \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}$$

where $A(\zeta) \in \mathbf{C}^{r\times r}$ is invertible. Then the multiplicity of ζ as root of $K(f)$ is strictly lower than the multiplicity of ζ as root of f.

Proof. If $r = 0$ then the system $K(f)$ consists of all partial derivatives

$$\nabla f(x) := \left(\frac{\partial f_i(x)}{\partial x_j}, \quad 1 \le j \le n, \quad 1 \le i \le s \right).$$

Then, the conclusion follows from the lemma 3.

If $r > 0$ the system $K(f)$ consists of $f_1, \ldots, f_r$ augmented by the elements of the schur complement $D(x) - C(x)\ A(x)^{-1}\ B(x)$. From the proposition 7, the relations between the lines are

$$(C(x), D(x)) - C(x)A(x)^{-1}(A(x), B(x)) = 0.$$

It is easy to see that the system $K(F) = 0$ is equivalent to the following

$$\left(f_1, \ldots, f_r, \nabla f_i(x) - \sum_{j=1}^{r} \lambda_{ij}(x) \nabla f_j(x) = 0, \quad i = r+1 : s \right) = 0, \tag{10}$$

with $(\lambda_{ij}(x)) := \left(C(x)A(x)^{-1}\right)^T$.

From the implicit function theorem, we know that there exists a local isomorphism Φ such that

$$x_{1:r} - \zeta_{1:r} = f_{1:r} \circ \Phi.$$

By substitution of $x_{1:r} - \zeta_{1:r}$ in $f = 0$ we obtain the system

$$(x_1 - \zeta_1, \ldots, x_r - \zeta_r,\ f_{r+1:s} \circ \Phi) = 0. \tag{11}$$

We remark that the multiplicity of the root ζ has not changed. The ideal generated by $f_{r+1:s} \circ \Phi$ only contains the monomials $x_i - \zeta_i$, $i = r+1 : n$. On the another hand the multiplicity of ζ as root of system (11) has not changed : it is also the multiplicity of $\zeta_{r+1:n}$ as root of system $f_{r+1:s} \circ \Phi$. Moreover, the multiplicity of ζ as root of the system (10) is equal to the multiplicity of $\zeta_{r+1:n}$ as root of the system $\nabla(f_{r+1:s} \circ \Phi)$. We now apply the lemma 3 to the system $f_{r+1:s} \circ \Phi$ to deduce that the multiplicity drops. We are done.

□

Proposition 7. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{C}^{s\times n}$ *of rank* r *where* $A \in \mathbf{C}^{r\times r}$ *is invertible. Then the relations between the lines (respectively the columns) of* M *are given by*

$$(C,D) - CA^{-1}(A,B) = 0, \quad (\textit{respectively} \begin{pmatrix} B \\ D \end{pmatrix} - \begin{pmatrix} A \\ C \end{pmatrix} A^{-1}B = 0).$$

Proof. The proposition follows from the equivalence:
$(C,D) - CA^{-1}(A,B) = 0$ and $\begin{pmatrix} B \\ D \end{pmatrix} - \begin{pmatrix} A \\ C \end{pmatrix} A^{-1}B = 0$ iff $D - CA^{-1}B = 0$. Since the rank of matrix M is equal to r, this is classically equivalent to $Schur\,(M) = 0$. □

Definition 12. *The valuation of an analytic system* $f = (f_1, \ldots, f_s)$ *at* ζ *is the minimum of the valuation of* f_i*'s at* ζ.

Remark 4. *A generator of* $I\mathbf{C}\{x-\zeta\}$ *of minimal valuation among others generators can always be taken as one of the generator of a (minimal) standard basis.*

This is a consequence of a fundamental property of local orderings: the valuation of a sum is larger than the valuation of any of the summands.

In the case where the construction of a standard basis of $I\mathbf{C}\{x-\zeta\}$ starts from a given set of *polynomial* generators, the goal can be achieved e.g. through the original Mora's tangent cone algorithm, by successive S-polynomials (and reductions which are particular cases of them). The valuation can only increase through these operations, which forbids to reduce $S(f,g)$ by f (or g by the way).

Lemma 3. *Let* $\nabla f(x) := \left(\frac{\partial f_i(x)}{\partial x_j}, \quad 1 \le j \le n, \quad 1 \le i \le s\right)$. *Let us suppose that* ζ *is an isolated root of* f *and* ∇f. *Then the multiplicity of* ζ *as root of* ∇f *is strictly lower than the multiplicity of* ζ *as root of* $f = 0$.

Proof. Let us take one of the f_k's, say f_i, of minimal valuation at ζ. This valuation is greater than 2. There exists an index j such that the leading term $\frac{\partial f_i(x)}{\partial x_j}$ is not in the ideal generated by f. The conclusion follows. □

Lemma 4. *Let* p *the valuation of* f *at* ζ. *Let us consider the following system*

$$D^{p-1}f(x) \ := \ \left(\frac{\partial^{|\alpha|} f_i(x)}{\partial x^{\alpha}},\ |\alpha| = p-1,\ 1 \le i \le s\right).$$

Let us assume that $p \ge 2$ *and that the rank of* $D^p f(\zeta)$ *is equal to* r. *Then the multiplicity of* ζ *as root of* $D^{p-1}f(x) = 0$ *is strictly lower than the multiplicity of* ζ *as root of* $f = 0$. *More precisely the multiplicity of the root* ζ *drops by at least* p^r.

Proof. Since the valuation $p \ge 2$ then $f(x) = \sum_{k\ge p} \frac{1}{k!} D^k f(\zeta)(x-\zeta)^k$ with $D^p f(\zeta) \ne 0$. The monomials of $LT(f)$ are of type $(x-\zeta)^{\alpha}$ with $|\alpha| \ge p \ge 2$. Hence the number of standard monomials of $\mathbf{C}\{x-\zeta\}/LT(f)$ is bounded below by p^n. Since the rank of the derivative of $D^{p-1}f(x)$ at ζ is $r > 0$, we can suppose without loss in generality that $x_1 - \zeta_1, \ldots, x_r - \zeta_r$

are in the ideal $LT(D^{p-1}f(x))$. Consequently the number of standard monomials dropped by at least p^r. $\square$

9. Quantitative version of Rouchés theorem in the regular case

In this section we consider as previously $\omega \in \mathbf{C}^n$ and the set $\mathbf{A}^2(\omega, R_\omega)$. For $x \in B(\omega, R_\omega)$ we introduce the quantities

$$\beta(f,x) = \|Df(x)^{-1}f(x)\| \tag{12}$$

$$\kappa_x = \max\left(1, \frac{R_\omega(n+1)}{R_\omega^2 - \rho_x^2}\right) \tag{13}$$

$$\gamma(f,x) = \max\left(1, \frac{\|f\|\,\|Df(x)^{-1}\|\,R_\omega\,\kappa_x}{(R_\omega^2-\rho_x^2)^{\frac{n+1}{2}}}\right) \tag{14}$$

$$\alpha(f,x) = \beta(f,x)\,\kappa_x \tag{15}$$

Theorem 8. *(α-Theorem). Let $R_\omega > 0$, $x_0 \in B(\omega, R_\omega)$, and $f = (f_1, \dots, f_n) \in (\mathbf{A}^2(\omega, R_\omega))^n$. Let us note α, β, γ, κ for $\alpha(f, x_0)$, $\beta(f, x_0)$, $\gamma(f, x_0)$, κ_{x_0} respectively defined in (15), (12), (14) and (13).*
Let us suppose that

$$\alpha < 2\gamma + 1 - \sqrt{(2\gamma+1)^2 - 1}.$$

Then for all $\theta > 0$ such that $B(x_0, \theta) \subset B(\omega, R_\omega)$ and

$$\frac{\alpha + 1 - \sqrt{(\alpha+1)^2 - 4\alpha(\gamma+1)}}{2(\gamma+1)} < u := \kappa\theta < \frac{1}{\gamma+1}$$

f has only one root in the ball $B(x_0, \theta)$.

Before proving this theorem we need the following proposition.

Proposition 8. *For all $f \in \mathbf{A}^2(\zeta, R_\omega)^s$ we have*

$$\forall k \geq 0, \quad \frac{1}{k!}\|D^k f(x_0)\| \leq ||f|| \frac{(n+1)^k R_\omega^{1+k}}{\left(R_\omega^2 - \rho_{x_0}^2\right)^{\frac{n+1}{2}+k}}.$$

Proof. It is enough to use the inequality

$$\frac{(n+1)\dots(n+k)}{k!} \leq (n+1)^k$$

in the proposition 3. $\square$

We are now ready to begin the proof of the theorem.

Proof. We let $Df(x_0)^{-1}f(x) = Df(x_0)^{-1}f(x_0) + g(x)$ with

$$g(x) = x - x_0 + \sum_{k\geq 2} \frac{1}{k!} Df(x_0)^{-1} D^k f(x_0)(x-x_0)^k.$$

We first remark that for all $x \in \mathbf{C}^n$ such that $\|x - x_0\| = \theta$ we have

$$\begin{aligned} \|g(x)\| &\geq \|x - x_0\| - \sum_{k\geq 2} \frac{1}{k!}\|Df(x_0)^{-1}D^k f(x_0)\| \, \|x - x_0\|^k \\ &\geq \theta - \frac{\|f\| \, \|Df(x_0)^{-1} \, R_\omega}{(R_\omega^2 - \rho_{x_0}^2)^{\frac{n+1}{2}}} \sum_{k\geq 2} \left(\frac{(n+1)R_\omega \, \theta}{R_\omega^2 - \rho_{x_0}^2} \right)^k \quad \text{from proposition 8} \\ &\geq \frac{u}{\kappa} - \frac{\gamma}{\kappa} \sum_{k\geq 2} u^k \\ &\geq \frac{1}{\kappa}\left(u - \gamma \frac{u^2}{1-u}\right). \end{aligned} \tag{16}$$

The Rouché's theorem states that the analytic functions $Df(x_0)^{-1}f(x)$ and $g(x)$ have the same number of roots, each one counting with the respective multiplicity, in the ball $B(x_0, \theta)$ if the inequality

$$\|Df(x_0)^{-1}f(x) - g(x)\| < \|g(x)\|$$

holds for all $x \in \partial B(x_0, \theta)$. Let us first prove that x_0 is the only root of $g(x)$ in the ball $B\left(x_0, \dfrac{1}{\kappa(\gamma+1)}\right)$. In fact let $y \neq x_0$ be a root of $g(x)$ in the ball $B(\omega, R_\omega)$. Let $v = \kappa\|y - x_0\|$. If $v \geq 1$ then $\|y - x_0\| \geq 1/\kappa > \dfrac{1}{\kappa(\gamma+1)}$. From the assumption we know that $\dfrac{1}{\kappa(\gamma+1)} \geq \theta$. In this case we conclude that $y \notin B(x_0, \theta)$. Otherwise $v < 1$. We deduce from the inequality (16) that

$$\|g(y)\| = 0 \geq \frac{1}{\kappa}\left(v - \frac{\gamma v^2}{1-v}\right).$$

Hence $\dfrac{1}{\gamma+1} \leq v$. From the assumption on θ, we then deduce that the distance between two distinct roots is bounded from below by

$$\|y - x_0\| \geq \frac{1}{\kappa(\gamma+1)} > \theta.$$

We then have proved that x_0 is the only one root of $g(x)$ in the ball $B\left(x_0, \dfrac{1}{\kappa(\gamma+1)}\right)$.

Now, let $x \in B(\omega, R_\omega)$ be such that $\|x - x_0\| = \theta = \dfrac{u}{\kappa}$. Then $B(x_0, \theta) \subset B(\omega, R_\omega)$. Always from the inequality (16) we deduce that the inequality

$$\beta := \|Df(x_0)^{-1}f(x_0)\| < \frac{1}{\kappa}\left(u - \frac{\gamma u^2}{1-u}\right) \tag{17}$$

implies $\|Df(x_0)^{-1}f(x) - g(x)\| < \|g(x)\|$ on the boundary of the ball $B(x_0, \theta)$. Since $\alpha = \beta\kappa$, this is satisfied if the numerator

$$(\gamma+1)u^2 - (\alpha+1)u + \alpha$$

of the previous expression (17) is strictly negative. Then it is easy to see that under the condition

$$\alpha := \beta\kappa < 2\gamma + 1 - \sqrt{(2\gamma+1)^2 - 1}$$

the trinomial $(\gamma+1)u^2 - (\alpha+1)u + \alpha$ has two roots equal to $\dfrac{\alpha + 1 \pm \sqrt{(\alpha+1)^2 - 4\alpha(\gamma+1)}}{2(\gamma+1)}$. Hence for all θ such that

$$\frac{\alpha + 1 - \sqrt{(\alpha+1)^2 - 4\alpha(\gamma+1)}}{2(\gamma+1)} < u := \kappa\theta < \frac{1}{\gamma+1}$$

we have $(\gamma+1)u^2 - (\alpha+1)u + \alpha < 0$. Then the inequality (17) is satisfied and the system f has only one root in the ball $B(x_0, \theta)$. The theorem follows. □

10. A new γ-theorem

Let $f = (f_1, \ldots, f_n)$ be an analytic system which is regular at a root ζ . The radius of the ball in which the Newton sequence converges quadratically towards a regular root ζ is controlled by the following quantity

$$\gamma(f,\zeta) = \sup_{k\geq 2} \left(\frac{1}{k!} \| Df(\zeta)^{-1} D^k f(\zeta) \| \right)^{\frac{1}{k-1}} .$$

More precisely we have the following result named γ-theorem.

Theorem 9. *(γ-Theorem of [2]). Let $f(x)$ an analytic system and ζ a regular root of $f(x)$. Let $R_\omega = \dfrac{3-\sqrt{7}}{2\gamma(f,\zeta)}$. Then for all $x_0 \in B(\zeta, R_\omega)$ the Newton sequence*

$$x_{k+1} = x_k - Df(x_k)^{-1} f(x_k), \quad k \geq 0,$$

converges quadratically towards ζ.

Taking in account the Bergman kernel to reproduce the analytic functions we are going to prove a new version of γ-theorem for analytical regular systems.

Theorem 10. *(γ-Theorem). Let ζ a regular root of an analytic system $f = (f_1, \ldots, f_n) \in \mathbf{A}^2(\omega, R_\omega)^n$. Let us note γ and κ for $\gamma(f,\zeta)$ and κ_ζ respectively defined in (14), (13). Then, for all x be such that*

$$u := \kappa \, \|x - \zeta\| < \frac{2\gamma + 1 - \sqrt{4\gamma^2 + 3\gamma}}{\gamma+1}$$

the Newton sequence

$$x_0 = x, \quad x_{k+1} = N_f(x_k), \quad k \geq 0,$$

converges quadratically towards ζ. More precisely

$$\|x_k - \zeta| \leq \left(\frac{1}{2}\right)^{2^k - 1} \|x - \zeta\|, \quad k \geq 0.$$

Proof. We use the proposition 9 below to prove by induction the result. The scheme of the proof is classical and can be found for instance in [2] page 158. The assumption $u < \dfrac{2\gamma+1-\sqrt{4\gamma^2+3\gamma}}{\gamma+1}$ implies that $\dfrac{\gamma u}{(1+\gamma)(1-u)^2-\gamma} \leq \dfrac{1}{2}$, that is a sufficient condition for the quadratic convergence of the Newton sequence with ratio $\dfrac{1}{2}$. □

Proposition 9. *With the notations of the theorem 10 we have:*

1– *For all x satisfying $u < 1-\sqrt{\dfrac{\gamma}{1+\gamma}}$, $Df(x)$ is invertible. Moreover we have*

$$\|Df(x)^{-1}Df(\zeta)\| \leq \frac{(1-u)^2}{(1+\gamma)\,(1-u)^2-\gamma}.$$

2– $\|Df(\zeta)^{-1}\left(Df(x)(x-\zeta)-f(x)\right)\| \leq \dfrac{\gamma u^2}{(1-u)^2}.$

3– $\|N_f(x)-\zeta\| \leq \dfrac{\gamma u^2}{(1+\gamma)\,(1-u)^2-\gamma}.$

Proof.

1– We write

$$Df(\zeta)^{-1}Df(x)-I = \sum_{k\geq 1}\binom{k+1}{k} Df(\zeta)^{-1}\frac{D^{k+1}f(\zeta)}{(k+1)!}(x-\zeta)^k.$$

Using proposition 8,

$$\begin{aligned}
\frac{1}{(k+1)!}\|D^{k+1}f(\zeta)\|\,\|Df(\zeta)^{-1}\| &\leq \frac{||f||\,\|Df(\zeta)^{-1}\|\,(n+1)^{k+1}\,R_\omega^{2+k}}{\left(R_\omega^2-\rho_{x_0}^2\right)^{\frac{n+1}{2}+k+1}} \\
&\leq \frac{\|f\|\,\|Df(\zeta)^{-1}\|\,R_\omega\,\kappa^{k+1}}{\left(R_\omega^2-\rho_\zeta^2\right)^{\frac{n+1}{2}}} \\
&\leq \gamma\kappa^k,
\end{aligned}$$

we have

$$\begin{aligned}
\|Df(\zeta)^{-1}Df(x)-I\| &\leq \gamma\sum_{k\geq 1}\binom{k+1}{k}\,(\kappa\,||x-\zeta||)^k \\
&\leq \gamma\left(\frac{1}{(1-u)^2}-1\right)
\end{aligned}$$

with $u=\kappa\|x-\zeta\|$. From this point estimate and thanks the classical Von Neumann lemma, see for instance [20] page 30, the item 1 follows easily.

2– We have $Df(x)(x-\zeta)-f(x)=\sum_{k\geq 2}(k-1)\frac{1}{k!}D^kf(\zeta)(x-\zeta)^k$. Hence, using more the proposition 8 we get from a straightforward calculation

$$\begin{aligned}\|Df(\zeta)^{-1}\left(Df(x)(x-\zeta)-f(x)\right)\| &\leq \gamma\sum_{k\geq 2}(k-1)\left(\kappa\|x-\zeta\|\right)^k\\ &\leq \frac{\gamma u^2}{(1-u)^2}.\end{aligned}$$

This proves the item 2.

3– We write

$$N_f(x)-\zeta = Df(x)^{-1}Df(\zeta)\,Df(\zeta)^{-1}(Df(x)(x-\zeta)-f(x)).$$

Using the items 1 and 2, we get the result.

□

From the theorem 10 we can state :

Theorem 11. *(γ-theorem). Let $f\in \mathbf{A}^2(\omega,R_\omega)^s$ and $\zeta\in B(\omega,R_\omega)$ such that $f(\zeta)=0$. Let us suppose there exists a index ℓ be such that*

1– *For all $0\geq k<\ell$ each element $F_k=K(F_{k-1})$ satisfies $F_k(\zeta)=0$ and $\operatorname{rank}(DF_k(\zeta)) < n$.*
2– *The assumptions of γ-theorem 10 hold for the system F_ℓ at ζ.*

Then, for all x be such that

$$u:=\kappa\,\|x-\zeta\| < \frac{2\gamma+1-\sqrt{4\gamma^2+3\gamma}}{\gamma+1}$$

the Newton sequence, computed thanks to the Table 3,

$$x_0=x,\quad x_{k+1}=N_{\mathit{dfl}(f)}(x_k),\quad k\geq 0,$$

converges quadratically towards ζ.

Also an existence result of a singular solution follows from the theorem 8.

Theorem 12. *Let $f\in \mathbf{A}^2(\omega,R_\omega)^s$ and $x_0\in B(\omega,R_\omega)$. Let us suppose that there exists a deflation sequence $(F_k)_{0\leq k\leq \ell}$ of thickness ℓ at x_0. More precisely*

1– *For all $0\geq k<\ell$ each element $F_k=K(F_{k-1})$ satisfies*
   1.1– $\|F_k(x_0)\|\leq \eta_k := \dfrac{2\alpha_0}{(n+1)(n+2)(R_{x_0}+\|F_k\|)R_{x_0}^{n-2}}$.
   1.2– *$DF_k(x_0)$ has a ε_k numerical rank stictly less than n where ε_k is the ε number of the line 6 of the Table 1.*
2– *The assumptions of α-theorem 8 hold for the system F_l at x_0.*

Then f has only one root in the ball $B(x_0,\theta)$ where θ is defined in $\alpha-$theorem 8.

11. Example

Let us give an example to illustrate the exact and numerical algorithm, by considering $f(x,y)=(f_1(x,y),f_2(x,y))$ with

$$f_1(x,y)=x^3/3+y^2x+x^2+2yx+y^2,\quad f_2(x,y)=x^2y-y^2x+x^2+2yx+y^2.$$

The root $(0,0)$ has multiplicity 6.

11.1. **Exact computations.** We have

$$Df(x,y) = \begin{pmatrix} x^2+y^2+2x+2y & 2xy+2x+2y \\ 2xy-y^2+2x+2y & x^2-2xy+2x+2y \end{pmatrix}.$$

The rank of the Jacobian matrix is 0 at $(0,0)$. Hence kerneling consists just to add to the input system the gradients of f_1 and f_2:

$$F_1 = K(f) = (x^2+y^2+2x+2y, 2xy+2x+2y, 2xy-y^2+2x+2y, x^2-2xy+2x+2y).$$

The four last lines of Jacobian matrix of $K(f)$ are:

$$\begin{pmatrix} 2x+2 & 2y+2 \\ 2y+2 & 2x+2 \\ 2y+2 & 2x-2y+2 \\ 2x-2y+2 & -2x+2 \end{pmatrix}.$$

The rank at $0,0$ of the matrix $\begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{pmatrix}$ is one, as the rank of the Jacobian of F_1 at $(0,0)$.

The Schur complement of $DF_1(x,y)$ associated to $2x+2$ is

$$Schur\,(DF_1(x,y)) = \frac{2}{x+1}\begin{pmatrix} 2x-2y+x^2-y^2 \\ 2x-3y+x^2-xy-y^2 \\ -x-x^2-xy+y^2. \end{pmatrix}$$

Then we can easily check that the system $F_2 = (f_1, vec(\,Schur\,(DF_1(x,y))$ is a regular system equivalent at $(0,0)$ to f. Let us remark also the truncated system of F_2 up to the order 1 namely

$$(x+y,\; x-y,\; 2x-3y,\; x)$$

is a regular system equivalent at $(0,0)$ to f.

11.2. **Numerical computations.** We give the behaviour of the deflation sequence.

1– The initial point $(x_0,y_0) = (-0.01, 0.02)$.

2– The system :

$$f = \begin{pmatrix} 1/3\,x^3+y^2x+x^2+2\,xy+y^2 \\ x^2y-y^2x+x^2+2\,xy+y^2 \end{pmatrix}$$

3– The ball $B(x_0,R_{x_0}) := B(x_0,1/4)$

4– Truncated expansion series of the system $F_0 = f(x+x_0, y+y_0)$ up to the order 3.

$$F_0 = \begin{pmatrix} 0.0000957+0.0205\,x+0.0196\,y+0.990\,x^2+2.04\,xy+0.99\,y^2+0.333\,x^3+y^2x \\ 0.000106+0.0205\,y+0.0192\,x+1.94\,xy+1.02\,x^2+1.01y^2+x^2y-y^2x \end{pmatrix}$$

5– Evaluation of F_0 at $(0,0)$: $(0.0000956666667, 0.000106)$.

6– We successively have $\|F_0\| = 8\times 10^{-4}$,

$$\eta = \frac{2\alpha_0}{12(R_{x_0}+\|F_0\|)R_{x_0}^{n-2}} = 0.086 \;>\; \|F_0(0,0)\| \;=\; 0.000106.$$

7– Jacobian of F_0 at $(0,0)$: $DF_0(0,0) = \begin{pmatrix} 0.02050000000 & 0.0196 \\ 0.0196 & 0.0205 \end{pmatrix}$. The singular values of this jacobian are 0.039 and 0.0011. This jacobian has a $\varepsilon_0 = 0.086$-rank equal to 0.

8– Kerneling of F_0 at $(0,0)$:

$$F_1 = K(F_0) = \begin{pmatrix} 0.0205 + 1.98\,x + 2.04\,y + 1.0\,x^2 + y^2 \\ 0.0196 + 2.04\,x + 1.98\,y + 2.0\,xy \\ 0.0192 + 1.94y + 2.04x + 2xy - y^2 \\ 0.0205 + 1.94\,x + 2.02\,y + x^2 - 2xy \end{pmatrix}$$

9– Evaluation of F_1 at $(0,0)$: $F_1(0,0) = (0.0205, 0.0196, 0.0192, 0.0205)$. We have $\|F_1\| = 0.1$ and
$\eta = \dfrac{2\alpha_0}{12(R_{x_0} + \|F_1\|)R_{x_0}^{n-2}} = 0.062 > \|F_1(0,0)\| = 0.034.$

10– Jacobian matrix of F_1 and its evaluation at $(0,0)$:

$$DF_1(x,y) = \begin{pmatrix} 1.98 + 2.0\,x & 2.04 + 2\,y \\ 2.04 + 2\,y & 1.98 + 2\,x \\ 2.04 + 2\,y & 1.94 + 2\,x - 2y \\ 1.94 + 2\,x - 2y & 2.02 - 2x \end{pmatrix} \qquad DF_1(0,0) = \begin{pmatrix} 1.98 & 2.04 \\ 2.04 & 1.98 \\ 2.04 & 1.94 \\ 1.94 & 2.02 \end{pmatrix}$$

The singular values of $DF_1(0,0)$ are 5.6 and 0.1 and its $\varepsilon_1 = 0.21$-rank is one.

11– Kerneling of $DF_1(x,y)$. We compute the truncated series at the order one in $(0,0)$ of each element of the Schur complement of $DF_1(x,y)$ associated to $1.98 + 2.0\,x$. We obtain

$$F_2 := K(F_1) = \begin{pmatrix} 0.0205 + 1.98\,x + 2.04\,y \\ -0.12 + 4.12\,x - 4.12\,y \\ -0.16 + 4.12x - 6.12y \\ 0.021 - 2.04x + 0.1y \end{pmatrix}$$

12– Regular system from F_2 at $(0,0)$. The singular values of $DF_2(0,0)$ are 9.46 and 3.32 and $DF_2(0,0)$ has $\varepsilon_2 = 3.32$-rank equal to 2.

13– If we consider

$$dfl(f) = \begin{pmatrix} 0.0205 + 1.98\,x + 2.04\,y \\ -0.121 + 4.123\,x - 4.121\,y \end{pmatrix}$$

we find that the iterate of $(x_0, y_0) = (-0.01, 0.02)$ is by the singular Newton operator is $(-0.0001017, 0.00034)$. This illustrates the manifestation of a quadratic convergence.

We show below quadratic convergence obtained thanks to the algorithm singular Newton.

$$[-0.01, 0.02]$$

$$[-0.00010175, 0.000343]$$

$$[-1.7 \times 10^{-8}, 8.1 \times 10^{-8}]$$

$$[-7.15 \times 10^{-16}, 4.2 \times 10^{-15}]$$

$$[-1.5 \times 10^{-30}, 1.06 \times 10^{-29}]$$

$$[-7.9 \times 10^{-60}, 6.55 \times 10^{-59}]$$

$$[-2.6 \times 10^{-118}, 2.4 \times 10^{-117}]$$

11.3. **Illustration of the theorems 10 and 8.** This is given by the table below

	β	κ	γ	α	$2\gamma+1-\sqrt{(2\gamma+1)^2-1}$	$\rho_{x_0} = \frac{2\gamma+1-\sqrt{(2\gamma+1)^2-1}}{2\kappa(\gamma+1)}$	$\frac{2\gamma+1-\sqrt{4\gamma^2+3\gamma}}{\gamma+1}$
$(-0.001, 0.002)$	0.00019	6	13.06	0.0119	0.018	0.00051	
$(0, 0)$	0	6	13.012	0			0.0031

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