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# Numerical analysis of a bisection-exclusion method to find zeros of univariate analytic functions

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**Abstract**

We state precise results on the complexity of a classical bisection-exclusion method to locate zeros of univariate analytic functions contained in a square. The output of this algorithm is a list of squares containing all the zeros. It is also a robust method to locate clusters of zeros. We show that the global complexity depends on the following quantities: the size of the square, the desired precision, the number of clusters of zeros in the square, the distance between the clusters and the global behavior of the analytic function and its derivatives. We also prove that, closed to a cluster of zeros, the complexity depends only on the number of zeros inside the cluster. In particular, for a polynomial which has $d$ simple roots separated by a distance greater than *sep*, we will prove the bisection-exclusion algorithm needs $O(d^3 \log(d/sep))$ tests to isolate the $d$ roots and the number of squares suspected to contain a zero is bounded by $4d$. Moreover, always in the polynomial case, we will see the arithmetic complexity can be reduced to $O(d^2(\log d)^2 \log(d/sep))$ using $\lceil \log d \rceil$ steps of the Graeffe iteration.


*MSC:* 65H05

*Keywords:* $\alpha$-Theory; Bisection; Exclusion; Clusters of zeros

## Contents



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## 1. Bisection-exclusion method and main results

The goal of the bisection-exclusion method which is studied in this paper is to locate and to approximate the zeros of an analytic function $f$ in a specified bounded domain. In all the paper, $f$ will be an analytic function defined on $\mathbb{C}$ and the domain will be the square $S_0$ introduced below. The set of zeros of $f$ inside $S_0$ is denoted by $Z$. The principle of this method is to remove from this domain subsets which do not contain any zero and to return arbitrary small subsets containing the zeros. Such a method mainly depends on two ingredients: the choice of an exclusion test and a strategy to remove subsets of a initial domain. The subsets here considered will be squares. We will denote by $S(x, s)$ the closed square centered at $x \in \mathbb{C}$ with side length $2s$. The set $\mathcal{S}$ will be the set of closed squares contained in the square $S_0 := S(x_0, s_0)$.

*The exclusion tests*: Let us consider a function $E$ defined from $\mathcal{S}$ into $\{True, \ False\}$ satisfying the following property: $E(S) = True$ implies the square $S \subset S_0$ does not contain any zero of $f$. Such a function $E$ is an exclusion test associated to $f$. When $E(S) = False$ nothing can be deduced and the square $S$ may contain a zero.

*The exclusion test used here*: Let us consider the following function $M$ defined over $\mathbb{C} \times \mathbb{R}$:

$$M(x,t) = |f(x)| - \sum_{k \geqslant 1} \frac{|f^{(k)}(x)|}{k!} t^k.$$

We will prove in Section 3 that the function *Exclusion* defined by

$$Exclusion(S(x,s)) = True \Leftrightarrow M(x, s\sqrt{2}) > 0,$$

is an exclusion test.

**The Algorithm.** We start with the initial square $S_0$, the analytic function $f$ and a precision $\varepsilon$. The result of the algorithm is a set of squares $Z_\varepsilon$ containing $Z \cap S_0$. Each square of the output has a size less than or equal to $\varepsilon$. Let us describe the first step of the algorithm. We consider a set of squares $Z_\varepsilon$ initialized to $Z_\varepsilon = \{S_0\}$. If $Exclusion(S_0) = True$ then we stop and $Z_\varepsilon = \emptyset$. In the contrary case, $Exclusion(S_0) = False$, we divide $S_0$ into four closed squares with size $s_0/2$ and we replace the square $S_0$ by these four new squares in the set $Z_\varepsilon$. At step $k \geqslant 0$ of the algorithm, the set $Z_\varepsilon$ is constituted of squares with the same size $s_0/2^k$. Then we compute $Exclusion(S)$ for each square $S$ of $Z_\varepsilon$. If $Exclusion(S) = True$, we remove this square of the set $Z_\varepsilon$. In the contrary case if $Exclusion(S)$ is *False* and the size of the square $S$ is greater than $\varepsilon$, we divide $S$ in four squares with size $s_0/2^{k+1}$ and we replace the square $S$ by these four new squares into the set $Z_\varepsilon$. The algorithm stops when $Z_\varepsilon = \emptyset$ or if the size of each square of $Z_\varepsilon$ is less than or equal to $\varepsilon$.

We will denote

$$divide(S(x,2s)) := \{S(x - ws, s), S(x + ws, s), S(x - \bar{w}s, s), S(x + \bar{w}s, s)\}$$

with $w = 1 + \sqrt{-1}$. Introducing an intermediate set $Z_{false}$, this algorithm is written in a pseudo-code language like:

**Inputs**: $f$ a polynomial, $S_0 = S(x_0, s_0)$ a square, $\varepsilon > 0$ a precision.

$Z_\varepsilon = \{S_0\}$

**Repeat**

  $Z_{false} = \emptyset$

  **For** each square $S(x,s) \in Z_\varepsilon$ **do**

    **If** $Exclusion(S(x,s)) = False$ **then**

      $Z_{false} := Z_{false} \cup divide(S(x,s))$.

    **end if**

  **end for**

  $Z_\varepsilon = Z_{false}$

**Until** $Z_\varepsilon = \emptyset$ or the size of each square of $Z_\varepsilon$ is less than or equal to $\varepsilon$.

**Output:** The set of squares $Z_\varepsilon$.

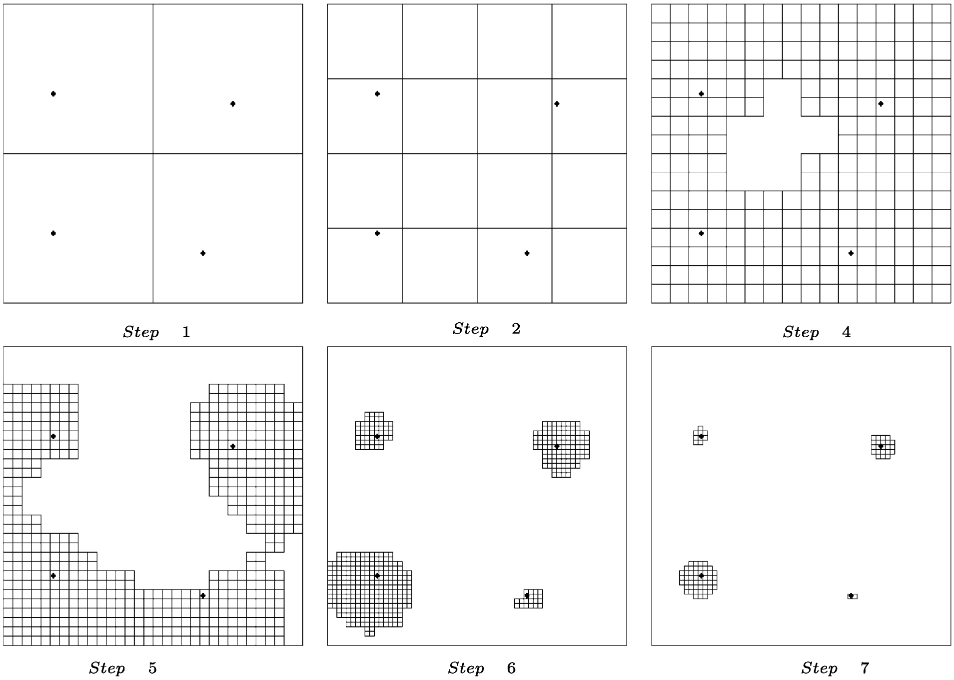


Fig. 1.

Fig. 1 illustrates how the algorithm works with functions like $f(x) = g_1(x)e^{ix} + g_2(x)e^{(-1+2i)x}$, where $g_1(x)$ and $g_2(x)$ are univariate complex polynomials given in Section 10.2. This function has four clusters of zeros: a simple zero $0.5 - i$, a cluster of two zeros in the disk $D(-1 + 0.6i, 10^{-3})$, a cluster of three zeros in $D(0.8 + 0.5i, 10^{-4})$, a cluster of four zeros in $D(-1 - 0.8i, 10^{-4})$. The algorithm is initialized with the initial square $S(0, 1.5)$ and the precision $\varepsilon = 0.03$. Fig. 1 shows the steps from 1 to 7 skipping the step 3: at steps 1–3 all the squares are retained. Some squares begin to be excluded at steps 4 and 5. The four clusters of zeros are separated at step 6. At this step the radius of squares is equal to $\varepsilon$. Hence, the step 7 is the last and the squares not excluded after the exclusion test are in the output set $Z_\varepsilon$. We see the clusters which appear with a black dot on the figures are contained in the set $Z_\varepsilon$.

For smallest values of $\varepsilon$, $0.02 \leqslant \varepsilon \leqslant 0.0002$, the numerical results show that the number of retained squares around of each zero does not change. For these values of $\varepsilon$ the figures representing $Z_\varepsilon$ are similar that of the figure of step 7. If we continue this process in the square $S(-1 - 0.8i,\ 10^{-4})$ with the precision $4 \times 10^{-6}$, Fig. 2 below shows the three last steps where the four zeros of the cluster are located. These results have been obtained with a precision of 30 digits under the Maple software.

These numerical experiments illustrate a property of the bisection-exclusion algorithm: the number of retained squares around each zero mainly depends on the multiplicity of the

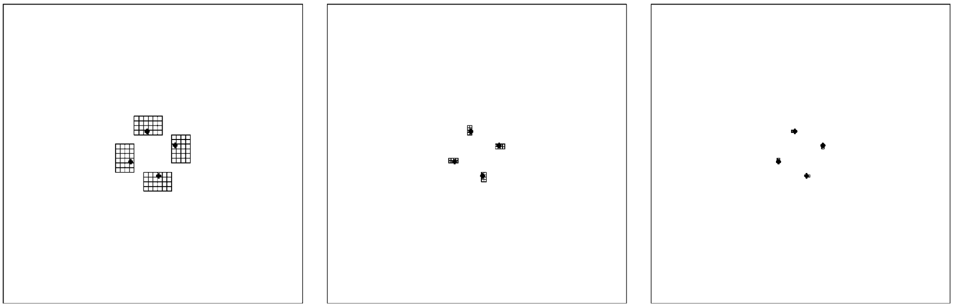

Fig. 2.

zero. This paper will prove this fact. In particular we will show that the number of retained squares around a simple zero is bounded by 4.

The analysis of this algorithm depends on three quantities: the number $q_\varepsilon$ of squares of the output set $Z_\varepsilon$, the total number $Q_\varepsilon$ of exclusion tests, and finally the numerical quality of the obtained approximation.

Before stating a theoretical result which explains the experiments above, we need to introduce some notations and to precise the context. We will suppose that the analytic function $f$ has $d$ zeros $z_1, \ldots, z_d$ in the square $S_0$. Let $g(z) = \prod_{k=1}^{d}(z - z_k)$ and $h(z)$ be the analytic function, such that $f(z) = g(z)h(z)$. The global behavior of $h(z)$ and its derivatives in the square $S_0$ is described by the quantities $\lambda$ and $\tau$ defined by

$$\forall x \in S_0, \quad \frac{|h^{(k)}(x)|}{k!|h(x)|} \leqslant \lambda\tau^{k-1}, \quad k \geqslant 1. \tag{1}$$

In all the paper, $\lambda$ and $\tau$ are chosen in order to verify

$$2\tau s_0\sqrt{2} \leqslant \tfrac{1}{2}. \tag{2}$$

The background of the analysis is done with respect to the following quantities, see [37]:

$$\beta_m(f;\zeta) = \max_{0\leqslant k\leqslant m-1}\left|\frac{m!f^{(k)}(\zeta)}{k!f^{(m)}(\zeta)}\right|^{\frac{1}{m-k}},$$
$$\gamma_m(f;\zeta) = \max_{k\geqslant m+1}\left|\frac{m!f^{(k)}(\zeta)}{k!f^{(m)}(\zeta)}\right|^{\frac{1}{k-m}},$$
$$\alpha_m(f;\zeta) = \beta_m(f;\zeta)\gamma_m(f;\zeta).$$

These quantities have been introduced in the case $m = 1$ by Smale [4] and we will denote $\beta(f,\zeta)$, $\gamma(f,\zeta)$ and $\alpha(f,\zeta)$. We also need several auxiliary functions. First, we let for

$u \in [0, 1/2[$ and $\delta \in \{0, 1\}$:

$$L_{m,\delta}(u) = \frac{2^{m-1}\delta u}{1-u} + \frac{(2-u)^m}{(1-u)^{m+1}} - 2^m + \frac{u}{(1-u)^{m+1}(1-2u)}.$$

In the case $m = 1$, $\delta = 0$ we have $L_1(u) := L_{1,0}(u) = \frac{4u}{1-2u}$. Next

$$b_m(u) = \begin{cases} 1 + \dfrac{1}{(2 - L_{m,1}(u))^{\frac{1}{m}} - 1} & \text{if } m > 1, \\ 1 + \dfrac{1}{1 - L_1(u)} = \dfrac{2(1-4u)}{1-6u} & \text{if } m = 1. \end{cases}$$

Let $\mu_m$ be such that

$$\forall \mu,\ \mu \geqslant 0,\ \mu \leqslant \mu_m,\ q(b_m(0) + \mu) = q(b_m(0)),$$

where $q(b)$ is the number of squares of size $r$ included in a disk of radius $br\sqrt{2}$ ( see Lemma 4.2). We then define $u_m(\mu)$ as the first positive zero of the equation

$$\begin{cases} L_{m,1}(u) = 2 - \left(1 + \frac{1}{\mu + b_m(0) - 1}\right)^m & \text{if } m > 1, \\ L_1(u) = \frac{\mu}{\mu+1} & \text{if } m = 1. \end{cases}$$

where $\mu \geqslant 0$. It is equivalent to

$$b_m(u_m(\mu)) = b_m(0) + \mu.$$

By cluster of $m$ zeros of $f$ around $\zeta \in S_0$ and of radius $\rho$, we mean a closed disk of radius $r$ centered in a zero $\zeta$ of $f$. We will suppose that the clusters of zeros centered in a zero of $f$ to simplify the technical computations. This does not remove anything with the generality of the results obtained. We gather the zeros of $f$ in $p$ clusters of zeros denoted by $\bar{D}_i := \bar{D}_{m_i}(\zeta_i, \rho_i)$, $1 \leqslant i \leqslant p$, such that $f(\zeta_i) = 0$ and the two following requirements:

$$\rho_i = \left(\frac{\beta_{m_i}(f;\zeta_i)}{\gamma_{m_i}(f;\zeta_i)}\right)^{1/2}, \tag{3}$$

$$4\sqrt{2}b\rho < r = \min_{1 \leqslant i \leqslant p} \frac{u_{m_i}(\mu_{m_i})}{\gamma_{m_i}(f;\zeta_i)} \tag{4}$$

hold with $\rho = \max_i \rho_i$, $\bar{b} = \max_{1 \leqslant i \leqslant p} b_{m_i}(0) + \mu_{m_i}$ and $b = \max\left(b_d(0) + \frac{3^d\lambda}{\tau}, \bar{b}\right)$. Evidently a regrouping of the zeros according to the criteria above is always possible. For example, we can consider the regrouping of distinct zeros of $f$. In this case, all the $\rho_i$'s are equal to zero. We will also consider $D$ which satisfies $b \leqslant 1 + \frac{1}{2^{1/D} - 1}$. In this paper log will be the logarithm to the base 2. We then can state

**Theorem 1.1.** *Let us consider an analytic function f defined on $\mathbb{C}$ which has p clusters of zeros in a square $S_0$ described as previous. Let us suppose that the requirements* (1) *and* (2), (3) *and* (4) *are satisfied. Let* $j_0 = \left\lceil \log \frac{\sqrt{2}bs_0}{r} \right\rceil$ *and* $j_1 = \left\lfloor \log \frac{s_0}{\rho} \right\rfloor$. *Then we have* $j_0 < j_1$.

*Let $\varepsilon$ verifying $\frac{s_0}{2^j} \leqslant \varepsilon < \frac{s_0}{2^{j-1}}$. Then the output set $Z_\varepsilon$ of the bisection-exclusion algorithm is a union of p pairwise disjoint sets $Z_{\varepsilon,i}$ such that $\bar{D}_i \subset Z_{\varepsilon,i} \subset D(\zeta_i, r)$. Let $q_{\varepsilon,i}$ be the number of retained squares in $Z_{\varepsilon,i}$. We then have*

1. *For all $x \in Z_{\varepsilon,i}$,* $\quad d(x, \zeta_i) \leqslant \left(2m_i + \mu_{m_i}\right)\sqrt{2}\varepsilon, \quad 1 \leqslant i \leqslant p.$
2. $q_{\varepsilon,i} \leqslant 4m_i^2, \quad 1 \leqslant i \leqslant p.$
3. $Q_\varepsilon \leqslant 1 + 16\left(j_0 p D^2 + (j - j_0)\displaystyle\sum_{i=1}^{p} m_i^2\right).$

Let us comment the two terms which contribute to the upper bound of the number of tests $Q_\varepsilon$. We will see the first count the number of steps to isolate the cluster of roots while the second gives the number of tests when the algorithm works closed to the clusters of roots.

In the simple roots case the radius $\rho_i$'s are zero. We can state

**Theorem 1.2.** *Let us consider an analytic function f defined on $\mathbb{C}$ which has only simple zeros $z_1, \ldots, z_d$ in $S_0$. We denote by $\gamma(f) = \max_i \, \gamma(f, z_i)$. Let us suppose* (1) *and* (2). *Let* $j_0 = \left\lceil \log\left(23\sqrt{2}(2d + \lambda\frac{3^d}{\tau})\gamma(f)s_0\right)\right\rceil$ *and a precision $\varepsilon$ satisfying* $\frac{s_0}{2^j} \leqslant \varepsilon < \frac{s_0}{2^{j-1}}$ *with $j_0 < j$. Then the output set $Z_\varepsilon$ of the bisection-exclusion algorithm is a union of d pairwise disjoint sets $Z_{\varepsilon,i}$ with $\zeta_i \in Z_{\varepsilon,i}$. We then have*

1. *For all $x \in Z_{\varepsilon,i}$,* $\quad d(x, \zeta_i) \leqslant 3\sqrt{2}\varepsilon, \quad 1 \leqslant i \leqslant d.$
2. *Each set $Z_{\varepsilon,i}$ contains at most four squares.*
3. $Q_\varepsilon \leqslant 1 + 16dD^2 j_0 + 16d(j - j_0).$

This paper is organized as follows: in Section 3, we introduce the notion of exclusion function on which is based the complexity of the bisection-exclusion algorithm. In Section 4, we put this problem in a more general setting to understand the notions on which this analysis is founded. To do that we develop a theoretical way to study the complexity of the bisection-exclusion method in the general case where the zeros of the analytic function are gathered in clusters. The main Theorem 4.1 of this section shows that the complexity mainly depends on the distance between the clusters of zeros and on the behavior of the exclusion function in the square $S_0$. Always in this section we show how this exclusion function is related to the number of squares that is possible to include in a disk: Lemma 4.2 states a precise result in this way. In Section 5, we introduce the notion of separation number and give a lower bound of the minimal distance between the clusters of zeros. In Section 6, we study the behavior of this exclusion function. This section is the technical background of our paper. Two new results will be given. The first concerns the local behavior of the exclusion function. The second generalizes in the analytic case a classical result concerning the global behavior of the exclusion function associated to a polynomial. The proofs of Theorems 1.1 and 1.2 are done in Section 7. To do that we verify the assumptions of Theorem 4.1 combining the results obtained in Sections 5 and 6. In Section 8, we will specialize the previous results of complexity to only find the nearly real zeros in a given interval. We will also discuss the localization of real roots of a polynomial. Section 9 is devoted to the polynomial case. We will give a synthesis of the previous results. We also discuss the question of rounding error for the computation of the exclusion polynomial. Moreover, in the case of simple roots we will give a result of bit complexity. Finally, we will also show that a number of $O(d^2(\log\, d)^2 \log(d\gamma(f)s_0))$ of arithmetic operations is sufficient to isolate the roots of a polynomial using $\lceil \log d \rceil$ steps of the Graeffe iteration. This bound of arithmetic complexity is closed to that of Pan [28] which is $O(d^2 \log\, d \log(ds_0/r))$ with our notations. Finally, Section 10 is devoted to practical comments and numerical experiments.

## 2. Context and links with related works

This type of bisection-exclusion algorithm appears for the first time in a paper of Weyl [33] without study of the cost of this algorithm. This task is realized by Gargantini and Henrici in [15], where the authors study four different exclusion tests only in the polynomial case. We focus on their tests $T_2$ and $T_1$. The test $T_2$ corresponds to the test studied here. In our context of notations, the test $T_1$ asserts that if $|f(x)| > d(1+\sqrt{2})^{d-1}s_0\sqrt{2}s$ then the square $S(x, s)$ included in $S_0$ does not contain any root of the polynomial *f*. The test $T_1$ requires no more than $16d\pi \left(\dfrac{2^{5/2}d}{sep}\right)^{2d-2}$ tests to isolate the *p* roots contained in a disk of radius one, [15, p. 92, formula (3)–(11)], where *sep* is the minimal separation distance of zeros. Concerning the test $T_2$, these authors show that the test $T_2$ is at least as effective that test $T_1$, [15, p. 95]: "Although the convergence estimates do not show it, the test $T_2$ is asymptotically likely to be much more effective than $T_1$ ...". Although, the case of exact multiple roots is considered for the test $T_1$, the global behavior of tests $T_1$ and $T_2$ is only studied without estimates of the local behavior of these tests.

Thereafter, several authors gave modifications and improvements by combining it with other method like Newton method or other exclusion tests like Schur–Cohn test and Turan's test: see [27,28] for a precise review on this subject. In this vein, the report of Schönhage [30] is certainly the first significant paper which deals with the splitting circle method. The previous papers are devoted to polynomials. In [38,39], the authors propose to count the number of zeros of an analytic function thanks to a reliable test based on the argument principle, see also [34,35]. But the algorithms are given without precise study of complexity.

From a point of view of some practitioners in the scientific and engineering communities, these bisection-exclusion-type methods are frequently used when the number of variables is small. For example to draw implicit curves or surfaces, these methods are easy to implement, see [32]. They are also used in many areas: in dynamical systems [13,14], in the localization of solutions of systems of equations [9,19–21] and in optimization [1,22].

Our aim in this paper was to study more precisely the complexity of the bisection-exclusion algorithm using an exclusion test based on the Taylor formula without seeking to optimize or to link with other methods. The analysis we propose uses $\alpha$-theory of Smale [31] and its generalization for multiple roots [36]. This technical background permits to obtain precise results in an efficient way since the complexity is described with respect to invariant quantities which depend only on the zeros. Indeed, we have focused our study on the link between this algorithm and the geometry of zeros and this paper is the theoretical answer to a unpublished report [11], see also [10]. A more recent study to fast compute clusters of zeros has been done by the author in collaboration with others [16]. The results of this previous paper can be used to link in a robust manner a method of global localization of zeros like bisection-exclusion with Newton generalized method.

## 3. The exclusion function

The study of the complexity depends on the existence of an exclusion function defined in the following statement (see [9]).

**Theorem and Definition 3.1.** *The following implicit function* $x \in \mathbb{C} \to m(x) \in \mathbb{R}_+$ *defined by*

$$M(x, m(x)) = 0$$

*exists. Moreover* $m(x)$ *is a continuous function. If* $M(x, \sqrt{2}\,s) > 0$ *then f has no zero in the square* $S(x, s)$. *Moreover* $M(x, \sqrt{2}\,s) > 0 \Leftrightarrow \sqrt{2}\,s < m(x)$. *It is why we will say* $m(x)$ *is the exclusion function associated to f at x.*

**Proof.** Let $d$ be an integer. Then we have $M(x, t) \leqslant |f(x)| - \sum_{k=1}^{d} \frac{|f^{(k)}(x)|}{k!} t^k$. Since the analytic function $f$ is defined on $\mathbb{C}$ it follows that $\lim_{t \to \infty} M(x, t) = -\infty$. The real function $t \in \mathbb{R}_+ \to M(x, t) \in ]-\infty, |f(x)|]$ is strictly decreasing. There is only one positive zero and the existence of $m(x)$ is established. The continuity of $m(x)$ can be proved in the following way (see [9]). For $\varepsilon > 0$ and $x, y \in \mathbb{C}$, the decreasing of $M(x, t)$ with respect $t$ implies: $M(x, m(x) + \varepsilon) < M(x, m(x)) = M(y, m(y)) = 0 < M(x, m(x) - \varepsilon)$. From the continuity of $M(x, t)$ with respect $x$, there exits a neighborhood of $x$ such that for all $y$ lying in this neighborhood we have $M(y, m(x) + \varepsilon) < M(x, m(x)) = M(y, m(y)) = 0 < M(y, m(x) - \varepsilon)$. Always from the decreasing of $M(x, t)$ with respect $t$ it follows $m(x) - \varepsilon < m(y) < m(x) + \varepsilon$. The continuity of $m(x)$ is established. Let $z \in S(x, s)$. From Taylor's formula and the triangle inequality we get $|f(z)| \geqslant M(x, |z-x|)$. Since the function $M(x, t)$ decreases and $|z - x| \leqslant \sqrt{2}\,s$ we have also $|f(z)| \geqslant M(x, |z - x|) \geqslant M(x, \sqrt{2}\,s)$. Hence if $M(x, \sqrt{2}\,s) > 0$ then $f$ has no zero in the square $S(x, s)$.

Finally since $M(x, t)$ decreases, it implies $M(x, \sqrt{2}\,s) > M(x, m(x)) = 0 \Leftrightarrow \sqrt{2}\,s < m(x)$. $\square$

This previous result shows that the complexity of the bisection-exclusion algorithm depends on the behavior of the exclusion function $m(x)$. In the polynomial case this exclusion function is equivalent to the distance function in the following sense, see [17, p. 457]:

$$2^{1/d} - 1 \leqslant \frac{m(x)}{d(x, Z)} \leqslant 1, \tag{5}$$

where $d(x, Z)$ is the distance function from $x$ to $Z$. Hence, the question to know a lower bound of the exclusion function is fundamental to analyze the complexity of the bisection-exclusion algorithm. Indeed this complexity is less than that of the algorithm which uses this lower bound as exclusion test. Our analysis is based on this property.

In the analytic case, it seems that there is not any reference for such a lower bound. This is why in Section 6 we will perform a general analysis on the behavior of $m(x)$.

## 4. Theoretical complexity of the bisection-exclusion algorithm

We will suppose that the analytic function defined on $\mathbb{C}$ has $p$ clusters $\bar{D}_i := D_{m_i}(\zeta_i, \rho_i)$ inside the square $S_0$ with $\rho_i > 0$, $\zeta_i \in S_0$, $1 \leqslant i \leqslant p$. As in the introduction the $\zeta_i$'s are zeros of $f$. Let $\rho = \max_i \rho_i$. Intuitively $\rho_i$'s are small with respect to the precision $\varepsilon$. The results will specify this fact. We recall that $Z$ is the zeros' set of $f$. Evidently, we have always $m(x) \leqslant d(x, Z)$ for all $x \in S_0$. In this section, we will suppose that the exclusion function $m(x)$ associated to $f$ satisfies the four following assumptions H1–H4 below.

The global behavior of $m(x)$ in the initial square is described by the following: there exists $a > 0$, such that

(H1) $\qquad \forall x \in S_0, \quad a d(x, Z) \leqslant m(x).$

The local behavior of exclusion closed to a cluster of zeros $\bar{D}_i$ is described in the following way: we will assume for all $i$, $1 \leqslant i \leqslant p$, there exists $a_i > 0$, $r > \rho_i$, such that

(H2) $\qquad \forall x \in D(\zeta_i, r) \backslash \bar{D}_i, \quad a_i d(x, \zeta_i) \leqslant m(x).$

(H3) $\qquad \forall i \neq k, \quad D(\zeta_i, r) \cap D(\zeta_k, r) = \emptyset.$

It is a natural way to suppose that

(H4) $\qquad a \leqslant a_i, \quad 1 \leqslant i \leqslant p.$

We will denote

1. $b = 1 + \dfrac{1}{a}$.
2. $b_i = 1 + \dfrac{1}{a_i}, \quad 1 \leqslant i \leqslant p.$

From (H4) it follows $b \geqslant b_i$.

The set $Z_\varepsilon$ is a set of squares $S(x, s)$ for which $Exclusion(S) = False$ or equivalently $m(x) \leqslant s\sqrt{2}$. Such squares are called retained squares. We say that an exclusion test has level $k \geqslant 0$ when the size of the square is $s_0/2^k$. We define the integers $p_k$ and $q_k$ as the numbers of *True* and *False*, respectively, at level $k$. We have clearly

1. $p_0 = 0, q_0 = 1.$
2. $p_k + q_k = 4q_{k-1}, \quad k \geqslant 1.$

Finally, we need to introduce $q(b)$ as the number of squares with size $s$ strictly included in a disk of radius $\sqrt{2}bs$. This number $q(b)$ is independent of $s$, see Lemma 4.2.

The bounds on the distance of retained squares to $Z$, the number $q_\varepsilon$ of retained squares and the total number $Q_\varepsilon$ of exclusion tests are given by the following:

**Theorem 4.1.** *Using the previous notations, let us suppose that* (H1)–(H4) *hold. Moreover let us also require* $4\sqrt{2}b\rho < r$. *Let us consider the two integers* $j_0 = \left\lceil \log \dfrac{\sqrt{2}bs_0}{r} \right\rceil$ *and*

*$j_1 = \left\lfloor \log \frac{s_0}{\rho} \right\rfloor$. We then have $j_0 < j_1$. Let $\varepsilon$ be a precision satisfying $\frac{s_0}{2^j} \leqslant \varepsilon < \frac{s_0}{2^{j-1}}$, where $j_0 < j < j_1$. Then $Z_\varepsilon$ is a union of p pairwise disjoint sets $Z_{\varepsilon,i}$, such that $\bar{D}_i \subset Z_{\varepsilon,i} \subset D(\zeta_i, r)$. Let $q_{\varepsilon,i}$ be the number of retained squares in $Z_{\varepsilon,i}$. We then have*

1. *For all $x \in Z_{\varepsilon,i}$, $d(x, \zeta_i) \leqslant \sqrt{2} b_i \varepsilon$, $1 \leqslant i \leqslant p$.*
2. *$q_{\varepsilon,i} \leqslant q(b_i) \leqslant \frac{\pi}{2} b_i^2$, $1 \leqslant i \leqslant p$.*
3. *$Q_\varepsilon \leqslant 1 + 4 j_0 p q(b) + 4(j - j_0) \sum_{i=1}^{p} q(b_i)$.*

**Proof.** Let us recall that log is the logarithm to base 2. From $4\sqrt{2} b\rho < r$, it follows $\log\left(\frac{\sqrt{2} b s_0}{r}\right) + 2 < \log \frac{s_0}{\rho}$. Hence $j_0 < j_1$. Let us show that the output set $Z_\varepsilon$ of the bisection-exclusion algorithm is a union of $p$ pairwise disjoint sets $Z_{\varepsilon,i}$ each one containing a cluster of zeros. We first prove we need $j_0$ steps in the algorithm for that the distance from all point belonging to a retained square at the set $Z$ is less than $r$. Let $s_k = \frac{s_0}{2^k}$ and $S := S(x, s_k)$ be a non-excluded square at a level $k \leqslant j_0$. From (H1), we have

$$ad(x, Z) \leqslant m(x) \leqslant s_k \sqrt{2}.$$

Since $\varepsilon < s_k$ we get for all $z \in S$,

$$d(z, Z) \leqslant d(z, x) + d(x, Z) \leqslant b s_k \sqrt{2}. \tag{6}$$

Let us consider $\bigcup_{i=1}^{p} D\left(\zeta_i, b\sqrt{2} s_k\right)$. We know from Lemma 4.2 below, the number of squares with size $s_k$ in each disk $D(\zeta_i, b\sqrt{2} s_k)$ is bounded by $q(b) \leqslant \frac{\pi}{2} b^2$. Hence, the number $q_k$ of retained squares at level $k$ is bounded by

$$q_k \leqslant p q(b) \leqslant \frac{\pi}{2} p b^2. \tag{7}$$

The index $j_0$ has been selected so that for all $z$ belonging to a retained square $S$ the inequality

$$\frac{\sqrt{2} s_0 b}{2^{j_0}} \leqslant r \tag{8}$$

holds. From (H3) it follows that the $p$ clusters of roots are contained in $p$ pairwise disjoint sets. Let us make $j_0 = k$ in (6). We obtain $d(z, Z) \leqslant r$. Hence, at level $j$, $j_0 < j < j_1$, we have also $d(z, Z) \leqslant r$. Since the $D(\zeta_i, r)'s$ are pairwise disjoint disks, the set $Z_\varepsilon$ will be an union of $p$ pairwise disjoint sets $Z_{\varepsilon,i} \subset D(\zeta_i, r)$, $1 \leqslant i \leqslant p$. Moreover, from construction of the bisection-exclusion algorithm, one has $Z \cap \bar{D}_i \subset Z_{\varepsilon,i}$. From definition of $j_1$ the inequalities $2\rho < \dfrac{s_0}{2^{j_1-1}} \leqslant \varepsilon$ imply $\bar{D}_i \subset Z_{\varepsilon,i}$;

Let us now prove the 1. We bound $d(z, \zeta_i)$ for any $z$ in a retained square $S(x, s_j)$ at level $j$ included in $Z_{\varepsilon,i}$. For that there are two cases. First, if $x$ lies in $D(\zeta_i, r) \backslash \bar{D}_i$ it follows:

$$d(z, \zeta_i) \leqslant d(z, x) + d(x, Z).$$

Since $a_i d(x, \zeta_i) \leqslant m(x) \leqslant s_j \sqrt{2}$ we get

$$d(z, \zeta_i) \leqslant b_i s_j \sqrt{2}.$$

Next, if $x \in \bar{D}_i$ then since $\rho_i < s_j$ it implies $\zeta_i \in S(x, s_j)$. Hence

$$d(z, \zeta_i) \leqslant d(z, x) + d(x, \zeta_i) \leqslant s_j \sqrt{2} + \rho_i.$$

It follows $d(z, \zeta_i) \leqslant \max(b_i s_j \sqrt{2},\ s_j \sqrt{2} + \rho_i)$. But, by definition $a_i \leqslant 1$ and $b_i = 1 + 1/a_i \geqslant 2$. The inequalities $\dfrac{s_j\sqrt{2}}{a_i} > \dfrac{\rho\sqrt{2}}{a_i} \geqslant \dfrac{\rho_i\sqrt{2}}{a_i} > \rho_i$ imply $s_j\sqrt{2} + \rho_i < b_i s_j \sqrt{2}$. Finally $d(x, \zeta_i) \leqslant b_i s_j \sqrt{2}$.

Let us prove the 2. We have $b_i s_j \sqrt{2} \leqslant b\sqrt{2} s_{j_0} \leqslant r$. Hence $Z_{\varepsilon,i} \subset D\left(\zeta_i, b_i \sqrt{2}\, s_j\right) \subset D(\zeta_i, r)$. Using Lemma 4.2 below, we then can bound the number $q_{\varepsilon,i}$ of retained squares at level $j$ contained in $D(\zeta_i, b_i \sqrt{2} s_j)$. We obtain

$$q_{\varepsilon,i} \leqslant q(b_i) \leqslant \frac{\pi}{2} b_i^2. \tag{9}$$

To prove the 3, let us remember we have $p_0 + q_0 = 1$ and $p_k + q_k = 4q_{k-1}$ for $k \geqslant 1$. Then using the bounds (7) and (9) on the $q_k$'s, we find a bound for the total number $Q_\varepsilon$ of exclusion tests is

$$\begin{aligned} Q_\varepsilon = \sum_{k=0}^{j} p_k + q_k = 1 + \sum_{k=1}^{j} 4q_{k-1} &\leqslant 1 + \sum_{k=1}^{j_0} 4q_{k-1} + \sum_{k=j_0+1}^{j} 4q_{k-1} \\ &\leqslant 1 + 4 j_0 p q(b) + 4(j - j_0) \sum_{i=1}^{p} q(b_i). \end{aligned}$$

We are done. □

**Remark.** From Lemma 4.2 we also have $Q_\varepsilon \leqslant 1 + 2\pi j_0 p b^2 + 2\pi (j - j_0) \sum_{i=1}^{p} b_i^2$.

We now state the lemma used in the previous result on the number of squares that is possible to include in a closed disk. The bound $q(b) \leqslant \dfrac{\pi b^2}{2}$ is easy to prove. But we need a better bound to theoretically explain the numerical results shown in the introduction.

**Lemma 4.2.** *Let $r > 0$, $b \geqslant 2$, $0 \leqslant s, t \leqslant 2$ be real numbers. Let us introduce the quantities*:

1. $q_k(b,s,t) = \left\lfloor \dfrac{\sqrt{2b^2-(2k+t)^2}-s}{2} \right\rfloor.$
2. $k_1(b,s,t) = \min(q_0(b,t,s), q_0(b,t,2-s)).$
3. $\bar{q}_1(b,s,t) = \displaystyle\sum_{k=1}^{k_1(b,s,t)} q_k(b,s,t) + q_k(b,-s,t).$

*We then have*

1. *The number of squares of size r included in a closed disk of radius $\sqrt{2}br$ is equal to*

$$q(b) = 1 + \max_{0\leqslant s,t\leqslant 2} \left(k_1(b,t,s) + k_1(b,t,2-s) + \bar{q}_1(b,s,t) + \bar{q}_1(b,s,2-t)\right).$$

2. $q(b) \leqslant \dfrac{\pi b^2}{2}.$

**Proof.** The proof is done in the Appendix A.1. $\square$

In the sequel, we will be interested to bound $q(b_m)$ with $b_m = 1 + \dfrac{1}{2^{1/m}-1}$. For that we have

**Lemma 4.3.**
1. *For $m \geqslant 4$ we have $b_m \leqslant 2\sqrt{2/\pi}\, m$.*
2. *For $m \geqslant 1$ we have $b_m \leqslant 2m$.*
3. *For $\mu < \sqrt{5} - 2 \sim 0.23607$ we have $q(b_1 + \mu) = q(b_1) = 4$.*
4. *Let $m \geqslant 1$. Then we have $q(b_m) \leqslant 4m^2$.*

**Proof.** The derivative of the function $m \in [1, +\infty[ \to b_m \in [2, +\infty[$ is $b'_m = \frac{2^{1/m}\log(2)}{(2^{1/m}-1)^2 m^2}$. It is a strictly increasing function from $2\log(2)$ to $1/\log(2)$. Since $2\sqrt{2/\pi} > 1/\log(2)$ the function $m \to b_m - 2\sqrt{2/\pi}\, m$ decreases. Then the inequalities $b_4 - 2\sqrt{2/\pi} \times 4 < 0 < b_3 - 2\sqrt{2/\pi} \times 3$ imply the part 1.

Since $b_1 = 2$ and the function $m \to b_m - 2m$ decreases also, the part 2 follows.

Taking $\mu < \sqrt{5} - b_1$, a straightforward numerical computation from Lemma 4.2, part 1 gives the part 3. The value $\mu = \sqrt{5} - 2$ is not convenient because $q(\sqrt{5}) = 5$.

For the part 4 we first prove $q(b_m) \leqslant 4m^2$ for $m = 1, 2, 3$, thanks to Lemma 4.2, part 1. We find, respectively, $q(b_1) = q(2) = 4$, $q(b_2) = 12 \leqslant 16$, $q(b_3) = 27 \leqslant 36$. Next for $m \geqslant 4$, thanks to $b_m \leqslant 2\sqrt{2/\pi}\, m$ and Lemma 4.2, part 2, we get $q(b_m) \leqslant 4m^2$. We are done. $\square$

## 5. Geometry of zeros

The complexity of the bisection-exclusion algorithm is related to the geometry of zeros of the analytic function *f*. By geometry of zeros we mean mainly the separation number which is the minimum distance between two distinct zeros. Since this algorithm isolates the zeros, it is a natural way to describe the complexity in terms of a lower bound of the separation number. For polynomials, a result established in [7,36] states:

**Theorem 5.1.** *Let $\zeta$ be a simple root of a polynomial f. We have*

$$\min_{f(w)=0,\, \zeta\neq w} |\zeta - w| > \frac{1}{2\gamma(f;\zeta)}.$$

But this result holds in the analytic case. Here, it is more convenient to reformulate the notion of separation number from the point of view of clusters of zeros.

**Definition 5.2.** Let $\bar{D}_i$, $1 \leqslant i \leqslant p$, the clusters of zeros of an analytic function *f* defined on $\mathbb{C}$. We denote by $sep(f, \zeta_i, m_i) = \min\{|\zeta_i - w| \; : \; w \notin \bar{D}_i, \; f(w) = 0\}$. The separation number is defined by $sep(f) = \min\limits_{1 \leqslant i \leqslant p} sep(f, \zeta_i, m_i)$.

Evidently, we need a lower bound of this separation number to quantify the step of the bisection-exclusion algorithm from which all the clusters are contained in pairwise disjoint subsets of squares. Such a bound has been given in [37] in the polynomial case. We will give the proof of this result in the analytic case.

**Theorem 5.3.** *Let $D(\zeta, \rho)$ be an open disk. We note $\beta_m := \beta_m(f;\zeta)$, $\gamma_m := \gamma_m(f;\zeta)$ and $\alpha_m := \alpha_m(f;\zeta)$. Let us suppose $\rho = 3\beta_m$ and $9\alpha_m \leqslant 1$. Then*

1. *The analytic function f has m zeros* (*counting multiplicities*) *in $D(\zeta, \rho)$.*
2. $sep(f, \zeta, m) > \dfrac{1}{2\gamma_m} - \dfrac{3}{2}\beta_m.$

**Proof.** The proof is done in the Appendix A.2. □

## 6. Behavior of the exclusion function

We now describe the behavior of the exclusion function in the square $S_0$. As we can see it on the figures of the introduction we will distinguish a global behavior and a local behavior of the exclusion function.

### *6.1. Local behavior*

The result is the exclusion function closed to a cluster of *m* zeros has the same behavior of the exclusion function associated to the polynomial $x^m$.

**Proposition 6.1.** *We have* $M_{x^m}(x,t) = 2|x|^m - (t+|x|)^m$ *and the exclusion function associated to* $x^m$ *is equal to* $(2^{1/m}-1)\,|x|$.

**Proof.** It is an easy computation. □

**Theorem 6.2.** *Let* $\bar{D}_m(\zeta,\rho)$ *be a cluster of zeros of* $f$ *with* $f(\zeta) = 0$ *and* $\rho = \left(\frac{\beta_m(f;\zeta)}{\gamma_m(f;\zeta)}\right)^{1/2}$. *Let* $r > \rho$ *be such that the quantity* $u = \gamma_m(f;\zeta)r$ *verifies* $L_{m,\delta}(u) < 1$, *where* $\delta = 1$ *if* $\beta_m(f;\zeta) \neq 0$ *and* $\delta = 0$ *if* $\beta_m(f;\zeta) = 0$. *Then*

$$\forall x \in D(\zeta,r)\backslash\bar{D}_m(\zeta,\rho), \quad (2-L_{m,\delta}(u))^{\frac{1}{m}} - 1 \leqslant \frac{m(x)}{|x-\zeta|} \leqslant 1.$$

*In particular if* $m = 1$ *and* $u \leqslant \frac{1}{6}$, *we have*

$$1 - L_1(u) = \frac{1-6u}{1-2u} \leqslant \frac{m(x)}{|x-\zeta|} \leqslant 1.$$

**Proof.** The proof is done in the Appendix A.3. □

**Proposition 6.3.**
1. *The function* $u \to L_{m,\delta}(u)$ *increases on* $[0,1/2[$ *with* $L_{m,\delta}(0) = 0$ *and* $\lim_{u\to 1/2} L_{m,\delta}(u) = +\infty$.
2. *For* $u \in [0,1/2[$ *the function* $m \to L_{m,\delta}(u)$ *increases.*
3. *Let* $\bar{u}_{m,\delta}$ *the first positive zero of the equation* $L_{m,1}(u) = 1$. *The sequence* $(\bar{u}_{m,\delta})_{m\geqslant 0}$ *decreases to* $0$.
4. $\bar{u}_{1,1} = \frac{1}{2} - \frac{\sqrt{2}}{4} = 0.14$ *and* $\bar{u}_{1,0} = \frac{1}{6}$.

**Proof.** The proof is easy. □

### *6.2. Global behavior*

We now generalize the lower bound of the inequality given in Theorems 6.4(d) and 6.4(i) of [17], in the analytic case.

**Proposition 6.4.** *Let* $f$ *be an analytic function defined on* $\mathbb{C}$ *which has d zeros* $z_1,\ldots,z_d$ *in the square* $S_0$. *Let us consider* $g(z) = \prod_{k=1}^{d}(z-z_k)$ *and* $h(z)$ *the analytic function such that* $f(z) = g(z)h(z)$. *Let us suppose that the requirements* (1) *and* (2) *of the introduction hold. Then, for any* $x \in S_0$ *the exclusion function* $m(x)$ *associated to f satisfies the inequality*

$$\frac{2^{1/d}-1}{\frac{3^d\lambda}{\tau}(2^{1/d}-1)+1} \leqslant \frac{m(x)}{d(x,Z)}.$$

**Proof.** The proof is done in the Appendix A.4. □

## 7. Proofs of the main theorems

### 7.1. Proof of Theorem 1.1

For that it is sufficient to verify the assumptions of Theorem 4.1. From Proposition 6.4 the assumption (H1) is verified with $a = \dfrac{2^{1/d}-1}{\frac{3^d\lambda}{\tau}(2^{1/d}-1)+1}$ and $b \geqslant b_d(0) + \dfrac{3^d\lambda}{\tau}$. We let $\beta_{m_i} := \beta_{m_i}(f;\zeta_i)$, $\gamma_{m_i} = \gamma_{m_i}(f;\zeta_i)$, $\alpha_{m_i} = \alpha_{m_i}(f;\zeta_i)$ and $u_{m_i} = u_{m_i}(\mu_{m_i})$. We then prove the inequality $\frac{1}{2}sep(f,\zeta_i,m_i) > r$ which implies the $D(\zeta_i, r)$'s are pairwise disjoint disks. We remark $u_{m_i}$ is less than $\bar{u}_{1,0} = \frac{1}{6}$ the zero of $L_1(u) = 1$, see Proposition 6.3. It is easy to see the inequality $\rho_i = \left(\dfrac{\beta_{m_i}(f;\zeta_i)}{\gamma_{m_i}(f;\zeta_i)}\right)^{1/2} < r \leqslant \dfrac{u_{m_i}}{\gamma_{m_i}} \leqslant \dfrac{\bar{u}_{1,0}}{\gamma_{m_i}}$ implies $\alpha_{m_i} < \bar{u}_{1,0}^2$. Using both Theorem 5.3 and the 4 of Proposition 6.3, we then get

$$\begin{aligned} sep(f,\zeta_i,m_i) - 2r &\geqslant \frac{1}{2\gamma_{m_i}} - \frac{3}{2}\beta_{m_i} - 2r \\ &\geqslant \left(\frac{1}{2} - \frac{3}{2}\bar{u}_{1,0}^2 - 2\bar{u}_{1,0}\right)\frac{1}{\gamma_{m_i}} \\ &\geqslant \frac{1}{8\gamma_{m_i}} > 0. \end{aligned}$$

Hence the requirement (H3) holds. Let us verify the requirement (H2). For that let us consider $b_i := b_{m_i}(u_{m_i})$. From definition of $u_{m_i}$ we have $L_{m_i,\delta}(u_{m_i}) < 1$. From Theorem 6.2 we know the behavior of the function $m(x)$ in $D(\zeta_i, r)\backslash\bar{D}_i$. From the definition of $r$ it follows $r\gamma_{m_i}(f;\zeta_i) \leqslant u_{m_i}$ and we can write

$$\forall x \in D(\zeta_i, r)\backslash\bar{D}_i, \quad (2 - L_{m_i}(u_{m_i}))^{1/m_i} - 1 \leqslant \frac{m(x)}{d(x,\zeta_i)}, \quad 1 \leqslant i \leqslant p.$$

The requirement H4 holds from the definition of $b$. The requirement 4 implies $j_0 < j_1$. The definitions of $\mu_{m_i}$ and $u_{m_i}$ imply $b_i = b_{m_i}(0) + \mu_{m_i}$ and $q(b_i) = q(b_{m_i}(0))$. Since Lemma 4.3 establishes both $b_{m_i}(0) \leqslant 2m_i$ and $q(b_{m_i}(0)) \leqslant 4m_i^2$, the parts 1 and 2 follow easily.

Finally for the part 3, Theorem 4.1 applies in the right way under these considerations using $q\ (b) \leqslant 4D^2$. We are done. □

**Remark.** The assumption $q(b_i) = q(b_{m_i}(u_{m_i})) = q(b_{m_i}(0))$ permits to understand the local behavior of the bisection-exclusion algorithm closed to a cluster of zero. In fact, for all $\varepsilon$ such that $\rho_i \ll \varepsilon < \frac{s_0}{2^{j_0}}$, the number of retained squares will be constant. On the other hand, if $\varepsilon \leqslant \rho_i$ it is necessary to consider the clusters of zeros inside the initial cluster. Roughly speaking, the algorithm see the cluster of zeros as a multiple zero until a certain scale.

### 7.2. Proof of Theorem 1.2

Now the analytic function $f$ has only simple zeros $z_1, \dots, z_d$. We apply Theorem 1.1 with the following values: $m = 1$, $\rho = 0$, $b_1(0) = 2$, and $q(b_1(0)) = 4$ (see Lemma 4.2). The value of $\mu_1$ is given by a solution of the equation $q(2+\mu) = 4$. From Lemma 4.3 we can choose $\mu_1 = 0.236$. The zero $u_1(\mu_1)$ of the equation $b_1(u) = b_1(0) + \mu_1$ is given by

$$L_1(u) = \frac{4u}{1-2u} = 1 - \frac{1}{\mu_1 + 1}.$$

We find $u_1(\mu_1) \geqslant \frac{1}{23}$. Then we select $r = \frac{1}{23\gamma(f;\zeta_i)}$. Since $b_d(0) \geqslant 1 + \frac{1}{2^{1/2}-1} \geqslant 3.4 > b_1(0) + \mu_1 \sim 2.236$, we choose $b = b_d(0) + \frac{\lambda 3^d}{\tau}$ and $b_i = b_1(0) + \mu_1$. From Lemma 4.3 we have successively $b \leqslant 2d + \frac{\lambda 3^d}{\tau}$ and $b_1(0) = 2$. We also have $sep(f;\zeta_i) - 2r \geqslant \frac{1}{2\gamma(f;\zeta_i)} - \frac{2}{23\gamma(f;\zeta_i)} > 0$.

Then Theorem 1.1 applies under these considerations. It follows the value of $j_0 = \left\lceil \log\left(23\sqrt{2}(2d + \frac{\lambda 3^d}{\tau})\gamma(f)s_0\right)\right\rceil$ and the bounds given in the parts 1–3. We are done. $\square$

## 8. Bisection-exclusion algorithm for nearly real zeros

In the real case the bisection-exclusion algorithm works in the same way but intervals replace the squares. To study the complexity of the bisection-exclusion algorithm, we must hold into account the complex zeros closed to the real axis. An interval $I(x, s)$ is represented by its center $x$ and its length $2s$. Let $I_0 := I(x_0, s_0)$. We also suppose that there are $p$ clusters of zeros $\bar{D}_{m_i}(\zeta_i, \rho_i)$, $1 \leqslant i \leqslant p$, such that $f(\zeta_i) = 0$ and $\bar{I}_i := \bar{D}_{m_i}(\zeta_i, \rho_i) \cap I_0$. We will say the analytic function has $p$ nearly real clusters of zeros in the interval $I_0$. The exclusion test for an interval $I(x, s)$ becomes

$$Exclusion(I(x,s)) = True \Leftrightarrow M(x,s) > 0.$$

In fact, proving Theorem 3.1 in the real case, it is easy to see the factor $\sqrt{2}$ does not appear. Let us suppose

(H5) $\qquad \forall x \in I_0, \quad ad(x, Z) \leqslant m(x) \leqslant d(x, Z).$

With the same notations as in Section 4 we can state

**Theorem 8.1.** *Let $f$ be an analytic function defined on $\mathbb{C}$ which has $p$ nearly real clusters of zeros in the interval $I_0$. Let us suppose that the assumptions* (H2)–(H5) *hold. Let us also suppose $4b\rho < r$. Let us introduce the two integers $j_0 = \left\lceil \log \frac{bs_0}{r} \right\rceil$ and $j_1 = \left\lfloor \log \frac{s_0}{\rho} \right\rfloor$.*

*We then have* $j_1 < j_0$. *Let also* $\varepsilon$ *be a precision satisfying* $\frac{s_0}{2^j} \leqslant \varepsilon < \frac{s_0}{2^{j-1}}$ *with* $j_0 < j < j_1$. *Then* $Z_\varepsilon$ *is a union of p pairwise disjoint sets* $Z_{\varepsilon,i}$, *such that* $\bar{I}_i \subset Z_{\varepsilon,i} \subset I(\zeta_i, r)$. *Let* $q_{\varepsilon,i}$ *be the number of retained intervals in* $Z_{\varepsilon,i}$. *We then have*

1. *For all* $x \in Z_{\varepsilon,i} \quad d(x,\zeta_i) \leqslant b_i\varepsilon, \quad 1 \leqslant i \leqslant p$.
2. $q_{\varepsilon,i} \leqslant \lfloor b_i \rfloor, \quad 1 \leqslant i \leqslant p$.
3. $Q_\varepsilon \leqslant 1 + 2j_0 p \lfloor b \rfloor + 2(j - j_0) \sum_{i=1}^{p} \lfloor b_i \rfloor$.

**Proof.** See the Appendix A.5. □

To state a more precise result, we proceed as in the introduction. For that we introduce $u_m(\mu_m)$ the first positive zero of

$$b_m(u) = b_m(0) + \mu_m,$$

where $\mu_m$ satisfies

$$\forall \mu \leqslant \mu_m, \quad \lfloor b_m(0) + \mu \rfloor = \lfloor b_m(0) \rfloor .$$

With this new definition of $\mu_m$ and $u_m$, we now suppose that the $p$ clusters of zeros are gathered such that the above requirements are satisfied.

$$\rho_i = \left( \frac{\beta_{m_i}(f;\zeta_i)}{\gamma_{m_i}(f;\zeta_i)} \right)^{1/2}, \tag{10}$$

$$4b\rho < r = \min_{1 \leqslant i \leqslant p} \frac{u_{m_i}(\mu_{m_i})}{\gamma_{m_i}(f;\zeta_i)}, \tag{11}$$

where $\rho = \max_i \rho_i$, $\bar{b} = \max_{1 \leqslant i \leqslant p} b_{m_i}(0) + \mu_{m_i}$ and $b = \max\left( b_d(0) + \frac{3^d \lambda}{\tau}, \bar{b} \right)$.

Let us also suppose that the $\lambda$ and $\tau$ verify (1) and

$$2\tau s_0 \leqslant \tfrac{1}{2}. \tag{12}$$

We have

**Theorem 8.2.** *Let us consider an analytic function f defined on* $\mathbb{C}$ *which has p nearly real clusters of zeros in the interval* $I_0$. *Let us suppose that the requirements* (1), (10), *and* (11), (12) *hold. Let us introduce the two following integers* $j_0 = \left\lceil \log \frac{bs_0}{r} \right\rceil$ *and* $j_1 = \left\lfloor \log \frac{s_0}{\rho} \right\rfloor$. *We then have* $j_0 < j_1$. *Let* $\varepsilon$ *be a precision verifying* $\frac{s_0}{2^j} \leqslant \varepsilon < \frac{s_0}{2^{j-1}}$ *with* $j_0 < j < j_1$. *Then the output set* $Z_\varepsilon$ *of the bisection-exclusion algorithm is a union of p pairwise disjoint sets* $Z_{\varepsilon,i}$ *such that* $\bar{I}_i \subset Z_{\varepsilon,i} \subset I(\zeta_i, r)$. *Let* $q_{\varepsilon,i}$ *the number of intervals in* $Z_{\varepsilon,i}$. *We then have*

1. *For all* $x \in Z_{\varepsilon,i}, \quad d(x,\zeta_i) \leqslant (2m_i + \mu_{m_i})\varepsilon, \quad 1 \leqslant i \leqslant p$.

2. $q_{\varepsilon,i} \leqslant 2m_i, \quad 1 \leqslant i \leqslant p.$
3. $Q_\varepsilon \leqslant 1 + 4j_0pD + 4(j - j_0)p,$

*where D satisfies* $b \leqslant 1 + \dfrac{1}{2^{1/D} - 1}$.

**Proof.** The proof is similar to that of Theorem 1.1. □

In the particular case where the $\zeta_i$'s are simple nearly real zeros we state

**Theorem 8.3.** *Let us consider an analytic function f defined on* $\mathbb{C}$ *which has p simple nearly real zeros* $\zeta_i$, $1 \leqslant i \leqslant p$, *in the interval* $I_0$. *Let* $j_0 = \left\lceil \log_2 \left( 11(2d + \frac{\lambda 3^d}{\tau})\gamma(f)s_0 \right) \right\rceil$ *and* $\varepsilon$ *be a precision verifying* $\frac{s_0}{2^j} \leqslant \varepsilon < \frac{s_0}{2^{j-1}}$ *with* $j_0 < j$. *Then the output set* $Z_\varepsilon$ *of the bisection-exclusion algorithm is a union of p pairwise disjoint sets* $Z_{\varepsilon,i}$ *such that* $\zeta_i \in Z_{\varepsilon,i}$, $1 \leqslant i \leqslant p$. *Moreover, the following estimations hold*:

1. *For all* $x \in Z_{\varepsilon,i}, \quad d(x, \zeta_i) \leqslant 3\varepsilon, \quad 1 \leqslant i \leqslant p.$
2. *Each* $Z_{\varepsilon,i}$ *contains at most two intervals,* $1 \leqslant i \leqslant p$.
3. $Q_\varepsilon \leqslant 1 + 4j_0pD + 4(j - j_0)p,$

*where D satisfies* $b_d(0) + \dfrac{3^d\lambda}{\tau} \leqslant 1 + \dfrac{1}{2^{1/D} - 1}$.

**Proof.** The proof is performed in the same way as the proof of Theorem 1.2. We only explain the factor 11 in the value of $j_0$. We have $\rho = 0$, $m = 1$ and $b_1(0) = 2$. The value $\mu_1$ is bounded by 1 and the zero $u_1(\mu)$ of the equation $b_1(u) = b_1(0) + \mu$ satisfies

$$\frac{4u}{1 - 2u} = 1 - \frac{1}{\mu_1 + 1} < \frac{1}{2}.$$

Hence $u < \dfrac{1}{10}$. We select $r$ so that $u_1(\mu_1) < \dfrac{1}{10}$, i.e; $r = \dfrac{1}{11\gamma(f)}$. □

## 9. The polynomial case

Here $f$ is a polynomial of degree $d$.

### 9.1. Complexity for the localization of complex roots

Theorem 1.1 holds with $\lambda = 0$ and $D = d$. In the simple roots case, the term $16j_0dD^2 \in O(d^3 \log(d\gamma(f)s_0))$ in $Q_\varepsilon$ gives the number of exclusion tests to isolate the roots. The next term $16d(j - j_0) \in O(d \log d)$ in $Q_\varepsilon$ gives the number of tests when the algorithm works closed to the roots. Then the number of arithmetic operations is bounded by $O(d^5 \log(20\gamma(f)\, ds_0))$ or $O(d^4 \log(d) \log(20\gamma(f)\, ds_0))$ according to the generalized

Horner scheme [17, p. 435] or the fast Fourier transform algorithm [3, p. 36] is used to numerically evaluate all the quantities $f^{(k)}(x)/k!$'s. Note these bounds can be computed only a posteriori since $\gamma(f)$ depends on the roots.

### 9.2. *Complexity for the localization of simple real roots*

Let us consider a polynomial which has $p$ simple real roots in the interval $I_0$. We then have $\lambda = 0$ and $D = d$. The real bisection-exclusion algorithm needs $4pd \log(22\gamma(f)\, ds_0)$ exclusion tests to isolate the roots. Hence, the number of arithmetic operations is bounded by $O(pd^3 \log(22\gamma(f)\, ds_0))$ or $O(pd^2 \log(d) \log(22\gamma(f)\, ds_0))$ according to the generalized Horner scheme [17] or fast Fourier transform algorithm [3] is used to numerically evaluate all the quantities $\dfrac{f^{(k)}(x)}{k!}$'s. Others methods to isolate simple real roots of polynomials are based on the Descartes rule of signs. In [6], the authors obtained an arithmetic complexity when the polynomials are expressed in the monomial basis. Further improvements can be found in [25, Theorem 2.1], where the Bernstein basis is used to represent polynomials. More precisely, the number of arithmetic operations is in $O(d(d+1)r(\log\left(\dfrac{5d}{2sep}\right) - \log(r) + 4)$ where $r$ is the number of sign changes of the Bernstein coefficients' sequence. The gain of a factor $d$ comes from the isolation algorithm does not split the interval when the number of sign changes of the Bernstein sequence does not exceed 1. Consequently, the retained intervals are different sizes contrary to those of the bisection-exclusion algorithm described here. Let us add that in [2] the authors study the bit complexity of these real root isolation algorithms. Moreover a recent report [24], using ideas developed in [29], gives an algorithm which improves this bit complexity.

### 9.3. *Rounding error analysis and bit-complexity*

In this section $f$ is a complex polynomial of degree $d$. We let $f(x) = \sum_{k=0}^{d} f_k x^k$ and $\dfrac{\tilde{f}^{(k)}(x)}{k!} = \sum_{j=0}^{d-k} \binom{d}{k} |f_{k+j}| x^j$, $0 \leqslant k \leqslant d$. We introduce for a complex number $x$ the quantity $|x|_1^2 = 1 + |x|^2$ where $|x|^2 = x\bar{x}$. We defined a norm on the linear space of the complex polynomials of degree $d$ by $\|f\|^2 = \sum_{k=0}^{d} \binom{d}{k}^{-1} |a_k|^2$, see [4, p. 218]. In this section only, we will use the notation $u = \frac{1}{2}\beta^{1-n}$, where $\beta$ and $n$ are, respectively, the base and the precision of the floating point number system. We perform a rigorous rounding error analysis of the evaluation of the exclusion polynomial. To do that, we deal with the standard arithmetic model for the floating point numbers [18, p. 44]. Let us consider the generalized Horner scheme to evaluate the derivatives. Let $f_k$ the floating point number of $|f^{(k)}(x)|/k!$. In this model we know there exists $\delta_k$, such that $\left|\dfrac{f^{(k)}(x)}{k!}\right| = f_k(1+\delta_k)$, $0 \leqslant k \leqslant d$. We then have

$$M(x,t) = f_0 - \sum_{k=1}^{d} f_k t^k - \sum_{k=0}^{d} \delta_k t^k.$$

Consequently if $f_0 - \sum_{k=1}^{d} f_k t^k > \sum_{k=0}^{d} |\delta_k| t^k$ then $M(x,t) > 0$. The question is: what is the precision $n$ in the floating point number system to have $\sum_{k=0}^{d} |\delta_k| t^k \leqslant \varepsilon$ where $\varepsilon$ is a given real number?

**Proposition 9.1.** *Let* $\varepsilon > 0$ *and* $h = \max_{0 \leqslant k \leqslant d} |a_k|$.

*Then for* $n = \left\lceil \log_\beta \left( \frac{3\beta}{8\varepsilon} (2\sqrt{3} h 2^d |x|_1^d + (d-1)\varepsilon) \right) \right\rceil$, *we have* $\sum_{k=0}^{d} |\delta_k| t^k \leqslant \varepsilon$. *Hence a precision of* $\varepsilon$ *on the computation of* $M(x,t)$ *is performed with* $O\left( d \log \left( \frac{h|x|_1}{\varepsilon} \right) \right)$ *bits of precision.*

**Proof.** The proof is done in the Appendix A.6. □

### 9.4. *Bit complexity*

For sake of simplicity, we will suppose that the polynomial $f$ only has simple roots. We introduce $\Sigma$ the variety of polynomials of degree $d$ which have a multiple root and $\Sigma_x$ the variety of polynomials of degree $d$ which have $x$ as multiple root. We note by $d(f, \Sigma)$ (respectively, $d(f, \Sigma_x)$) the distance of $f$ to $\Sigma$ (respectively, $\Sigma_x$) for the norm $\|f\|$ defined above. The goal of this section is to link the number of bit we need to isolate the roots of $f$ with the distance $d(f, \Sigma)$. This question to link the bit complexity with the distance to the ill-posed problems has been studied in a more general setting in [5].

**Proposition 9.2.** *A bound for the number of bits to isolate the roots of the polynomial* $f$ *is given by*

$$\left\lceil \log \left( \frac{23\sqrt{6} \max(1,h) d^{5/2} s_0}{\min(1, d(f,\Sigma))} \right) \right\rceil.$$

**Proof.** The proof is done in the Appendix A.7. □

### 9.5. *Improvement using Graeffe iterates*

In this section, we show how to improve the exclusion test given in the introduction. To do that we use the classical Graeffe process which consists in defining the following polynomial sequence from a given polynomial $g$:

$$g^{<0>}(z) = g(z),$$
$$g^{<N+1>}(z) = g^{<N>}(\sqrt{z})g^{<N>}(-\sqrt{z}), \quad N \geqslant 0.$$

We call $g^{<N>}(z)$ the $N$th Graeffe iterate of $g(z)$. This polynomial is also of degree $d$. Each Graeffe iterate can be computed with $d \log d$ arithmetic operations using the fast Fourier transform algorithm [3]. In many papers the Graeffe iterates are a tool for approximating the distance from a point to the roots, see [26,30,12,27]. In [12] we can find the following:

**Proposition 9.3.** *Let $f$ be a polynomial of degree $d$. For $x \in \mathbb{C}$, let us consider $m^{<N>}(0)$ the exclusion function associated to the $N$th Graeffe iterate of $g(z) = f(x+z)$. Namely*

$$M^{<N>}(0,t) = |g^{<N>}(0)| - \sum_{k \geqslant 1}^{d} \frac{|g^{<N>}(0)|}{k!} t^k.$$

1. *If $M^{<N>}\left(0, (\sqrt{2}r)^{2^N}\right) > 0$ then $Z \cap S(x,r) = \emptyset$.*
2. *We have*

$$\left(2^{1/d} - 1\right)^{2^{-N}} \leqslant \frac{m^{<N>}(0)}{d(x,Z)} \leqslant 1.$$

**Corollary 9.4.**

$$0.638 \cdots = (2^{1/3} - 1)^{1/3} \leqslant \frac{m^{<\lceil \log d \rceil>}(0)}{d(x,Z)} \leqslant 1.$$

**Proof.** If we take $N = \lceil \log d \rceil$, we have $\dfrac{m^{\lceil \log d \rceil}(0)}{d(x,Z)} \geqslant \left(2^{1/d} - 1\right)^{1/d}$. An easy study of the function $d \in \mathbb{N} \to \left(2^{1/d} - 1\right)^{1/d}$ shows that the minimum of this function is reached for $d = 3$. Since $(2^{1/3} - 1)^{1/3} \geqslant 0.638$ the corollary follows. We are done. □

The quantity $b$ of Section 4 is bounded by $1 + (2^{1/3} - 1)^{-1/3} \sim 2.57 < 3$. Then a straightforward computation shows that $q(b) \leqslant 6$. From Theorem 4.1, the number of tests of the bisection-exclusion algorithm which use the exclusion polynomial associated to the $d$th Graeffe iterate of $g(z) = f(x+z)$ will be bounded by $O\left(24d \log\left(\dfrac{3\sqrt{2}s_0}{r}\right)\right)$. Moreover, each step needs $O(d(\log d)^2) + O(d)$ to compute the $\lfloor \log d \rfloor$th Graeffe iterate $g^{<\lceil \log d \rceil>}(z)$ and to evaluate $M^{<\lceil \log d \rceil>}(0,t)$. In conclusion the number of arithmetic operations is in $O\left(24d^2(\log d)^2 \log\left(\dfrac{3\sqrt{2}s_0}{r}\right)\right)$. We obtain a gain of a factor $d^2$ or $d^3$ compared to the complexity given in the introduction. Compared to the bound of arithmetic complexity given in [28], our bound is multiplied by a factor of $\log d$. The modified Weyl's algorithm of Pan [28] use many ingredients. In particular this algorithm computes Newton sums of roots of a Graeffe polynomial in order to perform Turan's test and combines the iterations with the generalized Newton method in the case of the multiple roots. All the now classical but tedious techniques of fast computation are used in this modified Weyl's algorithm.

In the case of real root computation, the arithmetic complexity of the bisection-exclusion algorithm using Graeffe iterates is in $O\left(pd(\log d)^2 \log\left(\frac{3s_0}{r}\right)\right)$ which is gain of a factor $d/(\log d)^2$ compared to the result of Mourrain et al. [25].

## 10. Practical comments, examples and numerical experiments

### 10.1. Bounds for roots

If we want to locate all the roots of a polynomial $g(z) = \sum_{k=0}^{d} g_k z^k$ we need a bound for the roots in order that to determine the initial square $S_0$. In the polynomial case there exits many bounds for the modulus of roots: Cauchy bound, Knuth bound etc... can be found in [23] or [17]. Let us remark each one are greater than the positive root of

$$|g_d|t^d - \sum_{k=0}^{d-1} |g_k|t^k,$$

which can be easily approximate by Newton's method.

### 10.2. Sums of polynomials and exponentials

In the case where the analytic function is $f(z) = \sum_{i=1}^{n} g_i(z)e^{c_i z}$ we show what kind of exclusion test we use in practice. In fact, we need to truncate the polynomial $M(x,t)$. It is why we will use a new exclusion polynomial $\bar{M}(x,t)$ whose the exclusion function $\bar{m}(x,t)$ defined below has a similar local behavior that of the exclusion function $m(x)$.

**Proposition 10.1.** *Let us consider the analytic function be defined by* $f(z) = \sum_{i=1}^{n} g_i(z)e^{c_i z}$, *where the* $g_i(z)$*'s are complex polynomials and the* $c_i$*'s are complex numbers. We note by* $\eta_i = |c_i|$, $d_i$ *the degree of* $g_i(z)$ *and d an integer such that* $d \geqslant \max_i d_i$. *We note by* $\theta_i(x) = \frac{1}{(d+1-d_i)!}\left|\sum_{j=0}^{d_i} \frac{g_i^{(j)}(x)}{j!} c_i^{d_i-j}\right|$. *Let us introduce*

$$\bar{M}(x,t) = |f(x)| - \sum_{k=1}^{d} \frac{|f^{(k)}(x)|}{k!} t^k - \sum_{i=1}^{n} \theta_i(x)\eta_i^{d+1-d_i}|e^{c_i x}|e^{\eta_i t}t^{d+1}.$$

*Let $\bar{m}(x,t)$ the exclusion function associated to $f$ with respect $\bar{M}(x,t)$: $\bar{M}(x,\bar{m}(x)) = 0$.*

1. *Let $r = \sqrt{2}\,s$. If $\bar{M}(x,r) > 0$ the function $f$ has not zeros in the square $S(x,s)$. Moreover $m(x) > \bar{m}(x)$.*
2. *Let $\bar{D}_m(\zeta,\rho)$ be a cluster of zeros of $f$ such that the assumptions of theorem 6.2 be satisfied and $L_{m,\delta}(u) + \Lambda(x)d(x,Z)^{d+1} < 1$, where $\Lambda(x) = \dfrac{1}{f^{(m)}(\zeta)} \displaystyle\sum_{i=1}^{n} \theta_i(x)\eta_i^{d+1-d_i}$ $|e^{c_i x}|e^{\eta_i d(x,Z)}$. Then*

$$\forall x \in D(\zeta,r)\backslash\bar{D}_m(\zeta,\rho), \quad (2 - L_{m,\delta}(u) - \Lambda(x)d(x,Z)^{d+1})^{\frac{1}{m}} - 1 \leqslant \frac{\bar{m}(x)}{|x-\zeta|} \leqslant 1.$$

**Proof.** The proof is done in the Appendix A.8. □

The figures of the introduction have been obtained with $d = 11$ and $f(x) = g_1(x)e^{ix} + g_2(x)e^{-1+2ix}$ where

$$\begin{aligned}
g_1(x) = {} & -.868997847226346338418217\!8 - .8850265833480658945418317\,i \\
& + x + (-.2553311571377752315850749 + .2613419028288941541561861\,i)x^2 \\
& - (.0020793641674305159076864\!34 + .0770509920782732333490016\!1\,i)x^3 \\
& + (.0071375878150572502378635\!42 + .0064812500077394703965957\!80\,i)x^4 \\
& + (-.0005927422781839878774265663 + .0000462206838567031892357401\!7\,i)x^5,
\end{aligned}$$

$$\begin{aligned}
g_2(x) = {} & .8689978989542825532098086 + .8850268859278433176318700\,i \\
& + (.7540247256824407372715811 + .0160291106569927273965755\!2\,i)x \\
& + (.1403575042219709729549892 - .1303398517844059099237165\,i)x^2 \\
& + (.0014104281917106164859789\!73 - .0242967322096362491922411\!4\,i)x^3 \\
& - (.0008493897415001390619349884 + .0010111326931040241412553\!55\,i)x^4.
\end{aligned}$$

## 10.3. *Polynomial $x^m$*

We perform the bisection-exclusion algorithm with the polynomial $f(x) = x^m$ in the square $S(0,s)$, (respectively, interval $I(0,s)$ ), $s > 0$. A numerical experiment shows that the bound for the number of retained squares is closed to the bound given in Theorem 4.2 computed by Matlab. The Table 1 before gives the number of retained squares (resp., intervals).

## 10.4. *Bisection-exclusion linked with Graeffe iteration*

To illustrate how works the improvement given in Section 9 we consider a polynomial of degree 10 which has the same clusters that of the analytic function given in the introduction. If we perform the bisection-exclusion algorithm with the exclusion test of the introduction, we obtain same results as in Fig. 1. Now using the exclusion test associated with the $\lceil \log d \rceil$th

Table 1

| $m$ | Retained squares | $q(b_m(0))$ |
|---|---|---|
| 2 | 12 | 12 |
| 3 | 24 | 27 |
| 4 | 44 | 46 |
| 5 | 76 | 76 |
| 6 | 112 | 112 |
| 7 | 148 | 151 |
| 8 | 192 | 198 |
| 9 | 248 | 256 |
| 10 | 308 | 313 |
| 11 | 376 | 382 |
| 12 | 448 | 454 |
| 13 | 532 | 540 |
| 14 | 608 | 621 |
| 15 | 708 | 716 |
| 16 | 812 | 813 |
| 17 | 912 | 920 |
| 18 | 1020 | 1037 |
| 19 | 1124 | 1152 |
| 20 | 1272 | 1280 |
| $m$ | Retained intervals | $\lfloor b_m(0) \rfloor$ |
| 2 | 2 | 3 |
| 3 | 4 | 4 |
| 4 | 6 | 6 |
| 5 | 6 | 7 |
| 6 | 8 | 9 |
| 7 | 10 | 10 |
| 8 | 12 | 12 |
| 9 | 12 | 13 |
| 10 | 14 | 14 |
| 11 | 16 | 16 |
| 12 | 16 | 17 |
| 13 | 18 | 19 |
| 14 | 20 | 20 |
| 15 | 22 | 22 |
| 16 | 22 | 23 |
| 17 | 24 | 25 |
| 18 | 26 | 26 |
| 19 | 26 | 27 |
| 20 | 28 | 29 |

Graeffe iterate associated to $g(z) = f(x+z)$ at each step of the algorithm, we obtain the following figure skipping the steps 1 and 2 where all the squares are retained. (Fig. 3).

## 11. Conclusion and further research

In this paper, we have precisely studied how works the bisection-exclusion algorithm with a test based on the Taylor formula. To do that we have performed the $\alpha$-theory of Smale. Nevertheless some questions have not been here treated.

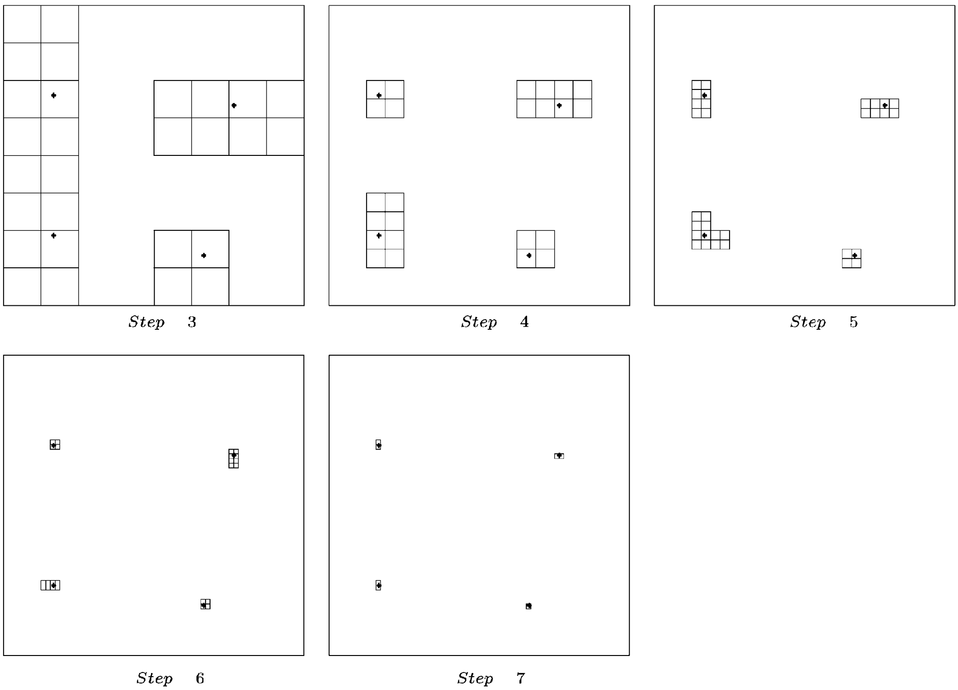


Fig. 3.

The first question is to quantify the change of local behavior of the exclusion function near a cluster. This question is related to the behavior of the generalized Newton operator or Schroeder operator near a cluster of root. A precise study of this fact can be found in [16].

The second question is the study of the bit complexity of this algorithm. An answer is to generalize the work of Pardo and its collaborators [5] in the case of clusters of roots. In fact the authors have linked the notion of approximate zeros [31] with the bit complexity in the case of simple roots.

The third question is to construct a fast Graeffe process in the analytic case.

## Index of symbols

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## Appendix A. Proofs

### A.1. Proof of Theorem 4.2

Let us consider a grid of squares contained in the closed disk $\bar{D}(0, \sqrt{2}\,br)$ as in Fig. 4, i.e., the center of the disk is not, in general, a vertex of the grid. Each square has a size $r$. Our objective is to count the number of squares of the grid contained in the disk. Let us consider the grid's point of coordinate $(r_1, r_2)$ nearest to the center of the disk defined by: $0 \leqslant r_1 < 2r$, $0 \leqslant r_2 < 2r$. We then introduce $s = r_1/r$ and $t = r_2/r$. Let us consider the points $A_k(0, 2kr + r_2)$ for $0 \leqslant k \leqslant k_1(b, s, t)$ and $B_k(0, -2kr - 2r + r_2)$ for $0 \leqslant k \leqslant k_1(b, s, 2-t)$ as in Fig. 4. It is easy to see the quantity $k_1(b, s, t)$ (respectively, $k_1(b, s, 2-t)$ ) is the largest integer $k$ such that there exists a square $S$ with size $r$ included in the disk and $A_k \in S$ (respectively, $B_k \in S$). In fact, from an easy geometric argument based on the Pythagoras formula, we find the $A_k$'s satisfy the two inequalities

$$2kr + r_2 \leqslant \sqrt{2b^2r^2 - r_1^2} \quad \text{and} \quad 2kr + r_2 \leqslant \sqrt{2b^2r^2 - (2r - r_1)^2}.$$

Hence $k \leqslant \min\left(\dfrac{\sqrt{2b^2 - s^2} - t}{2}, \dfrac{\sqrt{2b^2 - (2-s)^2} - t}{2}\right)$. It follows the value for $k_1(b, s, t)$. A similar way gives the value $k_1(b, s, 2-t)$. Always from the Pythagoras formula, we deduce the number of squares of size $r$ in the band defined by the points $A_{k-1}$ and $A_k$ included in the disk is: $1 + q_k(b, s, t) + q_k(b, 2-s, t) = 1 + q_k(b, s, t) + q_k(b, -s, t) - 1 = q_k(b, s, t) + q_k(b, -s, t)$. In the same way, the number of squares of size $r$ in the band defined by the points $B_{k-1}$ and $B_k$ included in the disk is: $1 + q_k(b, s, 2-t) + q_k(b, 2-s, 2-t) = q_k(b, s, 2-t) + q_k(b, -s, 2-t)$. In the band defined by the points $A_0$ and $B_0$, the number

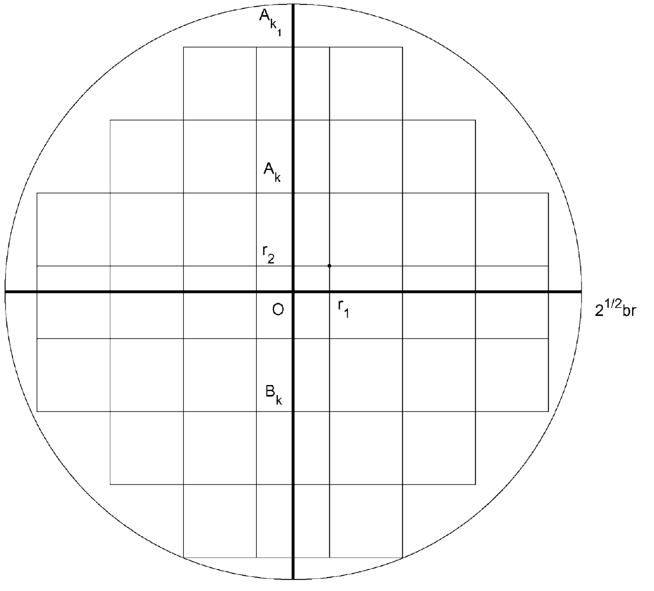


Fig. 4.

of squares is: $1+\min(q_0(b,s,t), q_0(b,s,2-t))+\min(q_0(b,2-s,t), q(b,2-s,2-t)) = 1+k_1(b,t,s)+k_1(b,t,2-s)$. Finally, the maximum number of squares included in the disk is

$$q(b) = 1 + \max_{0\leqslant s,t\leqslant 2} \left(k_1(b,t,s) + k_1(b,t,2-s) + \bar{q}_1(b,s,t) + \bar{q}_1(b,s,2-t)\right).$$

Writing the area of squares is less than the area of the disk we find $4q(b)r^2 \leqslant 2\pi b^2 r^2$. Hence $q(b) \leqslant \pi b^2/2$. We are done. □

*A.2. Proof of Theorem 5.3*

Let us prove the 1. Let $g(x) = \sum_{k\geqslant m} \frac{f^{(k)}(\zeta)}{k!}(x-\zeta)^k$. Using Rouché's theorem, we show that $f$ and $g$ have the same number of zeros (counting multiplicity) in $D(\zeta,\rho)$. Let $w$ be such that $|w-\zeta| = \rho$. Since $\alpha_m = \gamma_m\rho\,\beta_m/\rho \leqslant 1/9$ and $\beta_m/\rho = \frac{1}{3}$, it follows $\gamma_m\rho \leqslant \frac{\beta_m}{\rho} = \frac{1}{3}$. Using both Taylor series expansion at $\zeta$ for $f(w)-g(w)$ and $g(w)$ and triangle inequality, we obtain

$$\begin{aligned} |f(w)-g(w)| &\leqslant \frac{|f^{(m)}(\zeta)|}{m!}\rho^m \left(\sum_{k<m} (\beta_m/\rho)^{k-m}\right) \\ &< \frac{|f^{(m)}(\zeta)|}{m!}\rho^m \left(1 - \frac{1/3}{1-1/3}\right) \end{aligned}$$

$$\leqslant \frac{|f^{(m)}(\zeta)|}{m!}\rho^m\left(1-\frac{\gamma_m\rho}{1-\gamma_m\rho}\right)$$
$$\leqslant \frac{|f^{(m)}(\zeta)|}{m!}\rho^m\left(1-\sum_{k>m}(\gamma_m\rho)^{k-m}\right)$$
$$\leqslant |g(w)|.$$

Hence $|f(w)-g(w)|<|g(w)|$ for all $w$, such that $|w-\zeta|=\rho$ and the Rouché theorem applies. We now prove $\zeta$ is the only one zero with multiplicity $m$ of the analytic function $g(x)$ in $D(\zeta,\rho)$. It is clear we have $|g(w)|>\dfrac{|f^{(m)}(\zeta)|}{2m!}|w-\zeta|^m$ for all $w$ such that $|w-\zeta|<\rho$. Hence $g(w)\neq 0$ for $w\neq\zeta$.

Let us prove the part 2. Let $f(w)=0$ and $w\notin D_m(\zeta,\rho)$. Let $s=|w-\zeta|$. If $\gamma_m s\geqslant 1$ it follows $s\geqslant\dfrac{1}{\gamma_m}$. Since $\alpha_m\leqslant\frac{1}{9}$ we have $\dfrac{1}{\gamma_m}\geqslant 9\beta_m\geqslant 3\beta_m$ which implies $s\geqslant\dfrac{1}{\gamma_m}>\dfrac{1}{2\gamma_m}-\dfrac{3}{2}\beta_m\geqslant 0$.

In the contrary case $\gamma_m s<1$, we write

$$0=f(w)=f(\zeta)+\sum_{k\geqslant 1}\frac{f^{(k)}(\zeta)}{k!}(w-\zeta)^k.$$

Since $s>\rho=3\beta_m$ it follows $\beta_m/s<\beta_m/\rho=1/3$. Since $\gamma_m s<1$ and $\beta_m/s<1$, we get from the previous Taylor's formula

$$\begin{aligned}0&\geqslant 1-\sum_{k=0}^{m-1}\frac{m!|f^{(k)}(\zeta)|}{k!|f^{(m)}(\zeta)|}|w-\zeta|^{k-m}-\sum_{k>m}\frac{m!|f^{(k)}(\zeta)|}{k!|f^{(m)}(\zeta)|}|w-\zeta|^{k-m}\\&\geqslant 1-\sum_{k=0}^{m-1}\left(\frac{\beta_m}{|w-\zeta|}\right)^{m-k}-\sum_{k>m}\left(\gamma_m|w-\zeta|\right)^{k-m}\\&\geqslant 1-\frac{\beta_m/s}{1-\beta_m/s}-\frac{\gamma_m s}{1-\gamma_m s}\\&\geqslant\frac{(1+3\alpha_m)s-2\gamma_m s^2-2\beta_m}{s(1-\gamma_m s)(1-\beta_m/s)}=-\frac{1}{\gamma_m}e(\gamma_m s),\end{aligned}$$

where $e(u)=2u^2-(1+3\alpha_m)u+2\alpha_m$. Hence $e(\gamma_m s)\geqslant 0$. Since $9\alpha_m\leqslant 1$, the polynomial $e(u)$ has two zeros. An easy computation shows that $e(3\alpha_m)=e(1/2-3\alpha_m/2)=(9\alpha_m-1)\alpha_m\leqslant 0$. But we know that $e(\gamma_m s)\geqslant 0$, $\gamma_m s>3\alpha_m$ and $3\alpha_m\leqslant 1/2-3\alpha_m/2$ (from $9\alpha_m\leqslant 1$ ). Hence, $\gamma_m s$ is greater than the largest root of the polynomial $e(u)$. It follows that the inequality $\gamma_m s\geqslant 1/2-3\alpha_m/2$ holds. We are done. □

### A.3. Proof of Theorem 6.2

Since $L_{m,\delta}(u) < 1$ it follows $u < 1$ and $f^{(m)}(\zeta) \neq 0$. The function $m(x)$ is defined by $M(x, m(x)) = 0$. From Taylor's formula in $\zeta$, we get successively

$$\begin{aligned}
0 &= |f(x)| - \sum_{k \geqslant 1} \frac{\left|f^{(k)}(x)\right|}{k!} m(x)^k \\
&\geqslant \frac{\left|f^{(m)}(\zeta)(x-\zeta)^m\right|}{m!} - \sum_{k \neq m} \frac{\left|f^{(k)}(\zeta)(x-\zeta)^k\right|}{k!} \\
&\quad - \sum_{k=1}^{m} \frac{\left|f^{(m)}(\zeta)(x-\zeta)^{m-k}\right|}{k!(m-k)!} m(x)^k \\
&\quad - \sum_{k+j<m,\, k \geqslant 1} \frac{\left|f^{(k+j)}(\zeta)(x-\zeta)^j\right|}{k!j!} m(x)^k \\
&\quad - \sum_{k+j>m,\, k \geqslant 1} \frac{\left|f^{(k+j)}(\zeta)(x-\zeta)^j\right|}{k!j!} m(x)^k.
\end{aligned}$$

It is equivalent to

$$\begin{aligned}
0 &\geqslant 1 - \sum_{k \neq m} \left|\frac{m! f^{(k)}(\zeta)}{k! f^{(m)}(\zeta)}\right| |x-\zeta|^{k-m} - \sum_{k=1}^{m} \binom{m}{k} \left(\frac{m(x)}{|x-\zeta|}\right)^k \\
&\quad - \sum_{k+j<m,\, k \geqslant 1} \binom{k+j}{j} \left|\frac{m! f^{(k+j)}(\zeta)}{(k+j)! f^{(m)}(\zeta)}\right| |x-\zeta|^{j-m} m(x)^k \\
&\quad - \sum_{k+j>m,\, k \geqslant 1} \binom{k+j}{j} \left|\frac{m! f^{(k+j)}(\zeta)}{(k+j)! f^{(m)}(\zeta)}\right| |x-\zeta|^{j-m} m(x)^k.
\end{aligned}$$

We write this previous inequality under the form

$$0 \geqslant 1 - A - B - C - D$$

and we bound in the sequel the four sums $A$, $B$, $C$, $D$ with respect to $u$. For that we will use the inequality $\beta_m(f;\zeta) \leqslant \delta u |x-\zeta|$ which goes from $x \notin \bar{D}_m(\zeta, \rho)$.

The quantity $A = \sum_{k \neq m} \left|\frac{m! f^{(k)}(\zeta)}{k! f^{(m)}(\zeta)}\right| |x-\zeta|^{k-m}$ is bounded by

$$\begin{aligned}
A &\leqslant \sum_{k=0}^{m-1} \left(\frac{\beta_m(f;\zeta)}{|x-\zeta|}\right)^{m-k} + \sum_{k \geqslant m+1} \left(\gamma_m(f;\zeta)|x-\zeta|\right)^{k-m} \\
&\leqslant \frac{\beta_m(f;\zeta)/|x-\zeta|}{1-\beta_m(f;\zeta)/|x-\zeta|} + \frac{\gamma_m(f;\zeta)|x-\zeta|}{1-\gamma_m(f;\zeta)|x-\zeta|} \leqslant \frac{\delta u}{1-u} + \frac{u}{1-u}.
\end{aligned}$$

The quantity $B=\sum_{k=1}^{m}\binom{m}{k}\left(\frac{m(x)}{|x-\zeta|}\right)^k$ is equal to $\left(1+\frac{m(x)}{|x-\zeta|}\right)^m-1$.

We bound $C=\sum_{k+j<m,\,k\geqslant 1}\binom{k+j}{j}\left|\frac{m!f^{(k+j)}(\zeta)}{(k+j)!f^{(m)}(\zeta)}\right||x-\zeta|^{j-m}m(x)^k$ using $m(x)\leqslant|x-\zeta|$ and the definition of $\beta_m(f;\zeta)$.

$$\begin{aligned}
C&=\sum_{k=1}^{m-1}\sum_{j=0}^{m-k-1}\binom{k+j}{j}\left|\frac{m!f^{(k+j)}(\zeta)}{(k+j)!f^{(m)}(\zeta)}\right||x-\zeta|^{j-m}m(x)^k\\
&\leqslant\sum_{k=1}^{m-1}\sum_{j=0}^{m-k-1}\binom{k+j}{j}\left|\frac{m!f^{(k+j)}(\zeta)}{(k+j)!f^{(m)}(\zeta)}\right||x-\zeta|^{k+j-m}\\
&\leqslant\sum_{k=1}^{m-1}\sum_{j=0}^{m-1-k}\binom{k+j}{j}\delta u^{m-k-j}.
\end{aligned}$$

Since the function $j\to\binom{k+j}{j}$ increases we deduce

$$\begin{aligned}
C&\leqslant\delta\sum_{k=1}^{m-1}\binom{m-1}{m-k-1}\sum_{j=0}^{m-k-1}u^{m-k-j}\\
&\leqslant\delta\sum_{k=1}^{m-1}\binom{m-1}{m-k-1}\frac{u}{1-u}\\
&\leqslant(2^{m-1}-1)\frac{\delta u}{1-u}.
\end{aligned}$$

Finally, we bound $D=\sum_{k+j>m,\,k\geqslant 1}\binom{k+j}{j}\left|\frac{m!f^{(k+j)}(\zeta)}{(k+j)!f^{(m)}(\zeta)}\right||x-\zeta|^{j-m}m(x)^k$. We get

$$D\leqslant\sum_{k=1}^{m}\sum_{j\geqslant m-k+1}\binom{k+j}{j}u^{k+j-m}+\sum_{k\geqslant m+1}\sum_{j\geqslant 0}\binom{k+j}{j}u^{k+j-m}.$$

First we have

$$\sum_{k=1}^{m}\sum_{j\geqslant m-k+1}\binom{k+j}{j}u^{k+j-m}\leqslant\sum_{k=1}^{m}\sum_{j\geqslant 1}\binom{j+m}{j+m-k}u^j.$$

We remark that $\binom{j+m}{j+m-k}\leqslant\binom{m}{k}\binom{k+j}{j}$. This follows from the fact that the function $j\to l_j=\binom{j+m}{k}\binom{k+j}{j}^{-1}$ decreases. Hence $l_j\leqslant l_0=\binom{m}{k}$. Hence we can write

$$\sum_{k=1}^{m}\sum_{j\geqslant m-k+1}\binom{k+j}{j}u^{k+j-m}\leqslant\sum_{k=1}^{m}\binom{m}{k}\sum_{j\geqslant 1}\binom{k+j}{j}u^j$$

$$
\begin{aligned}
&\leqslant \sum_{k=1}^{m} \binom{m}{k} \left( \frac{1}{(1-u)^{k+1}} - 1 \right) \\
&\leqslant \frac{(2-u)^m}{(1-u)^{m+1}} - \frac{u}{1-u} - 2^m.
\end{aligned}
$$

On the other hand using $\sum_{j \geqslant 0} \binom{k+j}{j} u^j = \frac{1}{(1-u)^{k+1}}$, we get

$$
\begin{aligned}
\sum_{k \geqslant m+1} \sum_{j \geqslant 0} \binom{k+j}{j} u^{k+j-m} &\leqslant \sum_{k \geqslant m+1} \frac{u^{k-m}}{(1-u)^{k-m}} \frac{1}{(1-u)^{m+1}} \\
&\leqslant \frac{u}{(1-u)^{m+1}(1-2u)}.
\end{aligned}
$$

Finally

$$
D \leqslant \frac{(2-u)^m}{(1-u)^{m+1}} - \frac{u}{1-u} - 2^m + \frac{u}{(1-u)^{m+1}(1-2u)}.
$$

We now can to collect the previous point estimates on $A$, $B$, $C$ and $D$. The inequality $0 \geqslant 1 - A - B - C - D$ becomes

$$
\begin{aligned}
0 &\geqslant 2 - \frac{(\delta+1)u}{1-u} - \left(1 + \frac{m(x)}{|x-\zeta|}\right)^m - \frac{(2^{m-1}-1)\delta u}{1-u} - \frac{(2-u)^m}{(1-u)^{m+1}} \\
&\quad + \frac{u}{1-u} + 2^m - \frac{u}{(1-u)^{m+1}(1-2u)} \\
&\geqslant 2 - \left(1 + \frac{m(x)}{|x-\zeta|}\right)^m - L_{m,\delta}(u).
\end{aligned}
$$

Finally

$$
\frac{m(x)}{|x-\zeta|} \geqslant (2 - L_{m,\delta}(u))^{\frac{1}{m}} - 1.
$$

We are done. $\square$

### A.4. Proof of Theorem 6.4

Let $x$ be such that $f(x) \neq 0$. Hence $d(x, Z) \neq 0$ and $m(x) \neq 0$. Remember the quantities $\frac{|g^{(k)}(x)|}{k!|g(x)|}$ are related to the distance function to the set of the zeros $Z$ by, see [17, p. 454],

$$
\frac{|g^{(k)}(x)|}{k!|g(x)|} \leqslant \binom{d}{k} \frac{1}{d(x, Z)^k}. \tag{13}
$$

Applying Leibniz's rule and bounding from (13), we find easily, for $1 \leqslant k \leqslant d$,

$$\begin{aligned}\frac{|f^{(k)}(x)|}{k!|f(x)|} &\leqslant \sum_{j=0}^{k-1} \frac{|g^{(j)}(x)|}{j!|g(x)|} \frac{|h^{(k-j)}(x)|}{(k-j)!|h(x)|} + \frac{|g^{(k)}(x)|}{k!|g(x)|} \\ &\leqslant \lambda \sum_{j=0}^{k-1} \binom{d}{j} \frac{\tau^{k-j-1}}{d(x,Z)^j} + \binom{d}{k} \frac{1}{d(x,Z)^k} \\ &\leqslant \frac{\lambda}{\tau d(x,Z)^k} \sum_{j=0}^{k-1} \binom{d}{j} (\tau d(x,Z))^{k-j} + \binom{d}{k} \frac{1}{d(x,Z)^k}.\end{aligned}$$

From (2) we have $\tau d(x,Z) \leqslant 2\sqrt{2} s_0 \tau \leqslant \frac{1}{2}$. It follows

$$\frac{|f^{(k)}(x)|}{k!|f(x)|} \leqslant \frac{2^d \lambda}{\tau(2d(x,Z))^k} \sum_{j=0}^{k-1} \binom{d}{j} \left(\frac{1}{2}\right)^{d-j} + \binom{d}{k} \frac{1}{d(x,Z)^k}.$$

Finally

$$\frac{|f^{(k)}(x)|}{k!|f(x)|} \leqslant \frac{(3^d - 2^d)\lambda}{\tau(2d(x,Z))^k} + \binom{d}{k} \frac{1}{d(x,Z)^k}. \tag{14}$$

In the same way, since $g^{(k)}(x) = 0$ for $k \geqslant d+1$, we find

$$\begin{aligned}\frac{|f^{(k)}(x)|}{k!|f(x)|} &\leqslant \sum_{j=0}^{d} \frac{|g^{(j)}(x)|}{j!|g(x)|} \frac{|h^{(k-j)}(x)|}{(k-j)!} \\ &\leqslant \lambda \sum_{j=0}^{d} \binom{d}{j} \frac{\tau^{k-j-1}}{d(x,Z)^j} \\ &\leqslant \frac{\lambda \tau^{k-d}}{\tau d(x,Z)^d} \sum_{j=0}^{d} \binom{d}{j} (\tau d(x,Z))^{d-j}.\end{aligned}$$

Hence we get

$$\frac{|f^{(k)}(x)|}{k!|f(x)|} \leqslant \frac{3^d \tau^{k-d} \lambda}{2^d \tau d(x,Z)^d}. \tag{15}$$

Dividing the identity $M(x, m(x)) = 0$ by $|f(x)| = |g(x)h(x)|$ and using the inequalities (14), (15) we obtain

$$\begin{aligned}0 \geqslant 1 &- \sum_{k=1}^{d} \binom{d}{k} \frac{m(x)^k}{d(x,Z)^k} - \frac{(3^d - 2^d)\lambda}{\tau} \sum_{k=1}^{d} \left(\frac{m(x)}{2d(x,Z)}\right)^k \\ &- \frac{\lambda 3^d}{2^d \tau} \sum_{k \geqslant d+1} \tau^{k-d} \frac{m(x)^k}{d(x,Z)^d}.\end{aligned}$$

Since we have both $m(x)\leqslant d(x,Z)$ and $\tau d(x,Z)\leqslant \frac{1}{2}$, the previous inequality becomes

$$\begin{aligned}
0 &\geqslant 2-\left(1+\frac{m(x)}{d(x,Z)}\right)^d-\frac{\lambda(3^d-2^d)m(x)}{2\tau d(x,Z)}\sum_{k=1}^{d}\left(\frac{1}{2}\right)^{k-1}\\
&\quad -\frac{\lambda 3^d m(x)}{2^d\tau d(x,Z)}\sum_{k\geqslant d+1}\left(\frac{1}{2}\right)^{k-d}\\
&\geqslant 2-\left(1+\frac{m(x)}{d(x,Z)}\right)^d-\frac{\lambda(3^d-2^d+(3/2)^d)}{\tau}\,\frac{m(x)}{d(x,Z)}\\
&\geqslant 2-\left(1+\frac{m(x)}{d(x,Z)}\right)^d-\frac{\lambda 3^d}{\tau}\,\frac{m(x)}{d(x,Z)}.
\end{aligned}$$

Finally thanks to Lemma .1 below the proposition follows. □

**Lemma .1.** *Let $c\geqslant 0$. The positive solution of the equation*

$$2-cu-(1+u)^d=0$$

*is greater than* $\dfrac{2^{1/d}-1}{c(2^{1/d}-1)+1}$.

**Proof.** The function $u\in[0,+\infty[\to 2-(1+u)^d\in\mathbb{R}$ is a concave function which is zero for $u=2^{1/d}-1$. The equation of the straight line joining the points $(0,1)$ and $(2^{1/d}-1,0)$ is $v=-\dfrac{1}{2^{1/d}-1}u+1$. Hence, the zero of the equation $cu=-\dfrac{1}{2^{1/d}-1}u+1$ is less than the zero of $cu=2-(1+u)^d$. We are done. □

*A.5. Proof of Theorem 8.1*

The proof is similar to that of the Theorem 4.1. The inequality $j_0<j_1$ follows from $4b\rho<r$. We first prove it needs $j_0$ steps in the algorithm to isolate the nearly real clusters of zeros in the interval $I_0$. Let $s_k=\dfrac{s_0}{2^k}$ and $I=I(x,s_k)$ be a non-excluded interval at a level $k\leqslant j_0$. We have from (H5) $ad(x,Z)\leqslant m(x)\leqslant s_k$.

Since $\varepsilon<s_k$ we get for any $z\in I$, $d(z,Z)\leqslant d(z,x)+d(x,Z)\leqslant bs_k$. Let us consider $\bigcup_{i=1}^{p} I(\zeta_i,bs_k)$. The number of intervals of length $2s_k$ in $I(\zeta_i,bs_k)$ is bounded by $\lfloor b\rfloor$. Hence the total number $q_k$ of retained intervals at level $k$ is bounded by

$$q_k\leqslant p\lfloor b\rfloor. \tag{16}$$

The index $j_0$ was selected so that for all $z$ belonging to a retained interval $I$ the inequality $\dfrac{s_0b}{2^{j_0}}\leqslant r$ holds. Hence at level $j$, $j_0<j<j_1$, we have $d(z,Z)\leqslant r$. Since the $I(\zeta_i,r)'s$ are pairwise disjoint intervals, the set $Z_\varepsilon$ will be an union of $p$ pairwise disjoint sets: $Z_{\varepsilon,i}\subset I(\zeta_i,r)$, $1\leqslant i\leqslant p$. Moreover, for all $x\in\bar{I}_i$ we have $m(x)\leqslant d(x,Z)\leqslant\rho\leqslant\dfrac{s_0}{2^{j_1-1}}\leqslant\varepsilon$. Hence $\bar{I}_i\subset Z_{\varepsilon,i}$.

We now prove the part 1. To do that, we bound $d(z,\zeta_i)$ for some $z$ in a retained interval $I(x,s_j)$ at level $j$ included in $Z_{\varepsilon,i}$. There are two cases. First, if $x$ lies in $I(\zeta_i,r)\backslash\bar{I}_i$ it follows $d(z,\zeta_i)\leqslant d(z,x)+d(x,\zeta_i)$. Since $a_i d(x,\zeta_i)\leqslant m(x)\leqslant s_j$ we get $d(z,\zeta_i)\leqslant b_i s_j$. Next, if $x\in\bar{I}_i$ then since $\rho_i<s_j$ it implies $\zeta_i\in I(x,s_j)$. Hence, $d(z,\zeta_i)\leqslant d(z,x)+d(x,\zeta_i)\leqslant s_j+\rho_i$. Since $a_i\leqslant 1$ and $b_i=1+1/a_i\geqslant 2$, we have $\dfrac{s_j}{a_i}>\dfrac{\rho}{a_i}\geqslant\dfrac{\rho_i}{a_i}>\rho_i$. It follows

$$d(z,\zeta_i)\leqslant\max(b_i s_j,\ s_j+\rho_i)=b_i s_j.$$

The part 1 follows. We now prove the part 2. Hence, at level $j$ we have $Z_{\varepsilon,i}\subset I(\zeta_i,b_i s_j)\subset I(\zeta_i,r)$. Hence, the number $q_{\varepsilon,i}$ of retained intervals at level $j$ contained in $I(\zeta_i,b_i' s_j)$ verifies:

$$q_{\varepsilon,i}\leqslant\lfloor b_i\rfloor. \tag{17}$$

To prove the 3, we remark the numbers $p_k$ (respectively, $q_k$) of excluded (respectively, retained) intervals are also the numbers of True (respectively, False) at level $k$. They satisfy the relations

1. $p_0=0$, $q_0=1$.
2. $p_k+q_k=2q_{k-1},\quad k\geqslant 1$.

Using both the bounds (16) and (17), we deduce a bound for the total number $Q_\varepsilon$ of exclusion tests

$$\begin{aligned}Q_\varepsilon&=\sum_{k=0}^{j}p_k+q_k=1+\sum_{k=1}^{j}2q_{k-1}\leqslant 1+\sum_{k=1}^{j_0}2q_{k-1}+\sum_{k=j_0+1}^{j}2q_{k-1}\\&\leqslant 1+2j_0p\lfloor b\rfloor+2(j-j_0)\sum_{i=1}^{p}\lfloor b_i\rfloor.\end{aligned}$$

We are done. □

### A.6. Proof of Theorem 9.1

We begin the proof with the bound of Lemma 3.1 p. 69 and formula 5.3, p. 105 [18]

$$|\delta_k|\leqslant\left((1+u)^{2(d-k)}-1\right)\frac{\tilde{f}^{(k)}(|x|)}{k!}$$

Using Proposition 1 of Blum et al. [4, p. 267] we have

$$\frac{\tilde{f}^{(k)}(|x|)}{k!}\leqslant\binom{d}{k}\|f\|\,|x|_1^{d-k}.$$

Using $t\leqslant|x|_1$ and Lemma .2 below, we find

$$\sum_{k=0}^{d}|\delta_k|t^k\leqslant\|f\|\,|x|_1^d\sum_{k=0}^{d}\binom{d}{k}\left((1+u)^{2(d-k)}-1\right)$$

$$\leqslant \|f\|\,|x|_1^d\left((1+(1+u)^2)^d-2^d\right)$$
$$\leqslant \frac{3\|f\|\,|x|_1^d\,2^{d-1}u}{1-\frac{3(d-1)}{4}u}.$$

From Lemma .3, we have $\|f\|\leqslant\sqrt{3}h$. With the definition of $u$ and the assumption on $n$ the result follows easily. □

**Lemma .2.** *We have* $(1+(1+u)^2)^d-2^d\leqslant\dfrac{2^{d-1}3u}{1-\frac{3(d-1)u}{4}}$.

**Proof.** From [37, Lemma 2], we know that $\dfrac{d-1}{2}=\max\limits_{2\leqslant k\leqslant d}\left(\dfrac{1}{d}\binom{d}{k}\right)^{\frac{1}{k-1}}$. Hence for $v=3u$, we have $v<1$ and

$$(2+v)^d-2^d=2^{d-1}dv\left(1+\sum_{k=2}^{d}\frac{1}{d}\binom{d}{k}\left(\frac{v}{2}\right)^{k-1}\leqslant\frac{2^{d-1}dv}{1-\frac{d-1}{4}v}\right).$$

Since $(1+u)^2<1+3u$ the lemma follows. □

**Lemma .3.**

$$\|f\|\leqslant\sqrt{3}h.$$

**Proof.** Since $\|f\|^2\leqslant h^2\sum\limits_{k=0}^{d}\binom{d}{k}^{-1}$, we prove this previous sum is less than $\sqrt{3}$. In fact

$$\sum_{k=0}^{d}\binom{d}{k}^{-1}\leqslant 2+\sum_{k=1}^{d-1}\binom{d}{1}^{-1}\leqslant 2+\frac{d-1}{d}<3.$$

We are done. □

*A.7. Proof of Theorem 9.2*

From Theorem 1.2, the number of bits of precision to isolate the roots is given by $j_0=\left\lceil\log\left(46\sqrt{2}d\gamma(f)s_0\right)\right\rceil$. A bound for $\gamma(f)$ follows from Lemma .4. We are done. □

**Lemma .4.** $\gamma(f)\leqslant\dfrac{\sqrt{3}\max(1,h)\sqrt{d}(d-1)}{2\min(1,d(f,\Sigma))}$.

**Proof.** Let $x$ be a root of the polynomial $f$. We first compute $d(f,\Sigma_x)$. For two complex polynomials $f(z)=\sum\limits_{k=0}^{d}a_kz^k$ and $g(z)=\sum\limits_{k=0}^{d}b_kz^k$, let us define the hermitian product

$< f, g >= \sum_{k=0}^{d} \binom{d}{k}^{-1} a_k \bar{b}_k$. We then have $\dfrac{f^{(k)}(x)}{k!} =< f(z), p_k(z) >$ with $p_k(z) = \binom{d}{k} z^k (1 + z\bar{x})^{d-k}$. These useful formulas to represent polynomials can be found in [8]. In this context the polynomials $p_0(z) = (1 + z\bar{x})^d$ and $p_1(z) = dz(1 + z\bar{x})^{d-1}$ are orthogonal to $\Sigma_x$ with respect the hermitian product $< .,. >$ above. Consequently the norm of the projection $\pi(f)$ of $f$ on the linear space generated by $p_0(z)$ and $p_1(z)$ is equal to $d(f, \Sigma_x)$. The projection $\pi(f)$ is defined by $< \pi(f), p_0 >=< f, p_0 >= f(x) = 0$ and $< \pi(f), p_1 >=< f, p_1 >= f'(x)$. A straightforward computation gives $\pi(f) = \dfrac{(- < p_1, p_0 > p_0 + < p_0, p_0 > p_1)\, f'(x)}{< p_0, p_0 >< p_1, p_1 > - |p_1(x)|^2}$. Since $< p_0, p_0 >= p_0(x)$, $< p_1, p_0 >= p_1(x)$ and $< p_1, p_1 >= p_1'(x) = d(1 + d|x|^2)(1 + |x|^2)^{d-2}$ it follows

$$d(f, \Sigma_x)^2 = \|\pi(f)\|^2 = \frac{p_0(x)|f'(x)|^2}{< p_0, p_0 >< p_1, p_1 > - |p_1(x)|^2} = \frac{|f'(x)|^2}{d(1 + |x|^2)^{d-2}}.$$

Let us bound $< p_k, p_k >$. We have

$$\langle p_k, p_k \rangle = \binom{d}{k}^2 \sum_{j=0}^{d-k} \binom{d-k}{j}^2 \binom{d}{k+j}^{-1} |x|^{2j} = \binom{d}{k} \sum_{j=0}^{d-k} \binom{d-k}{j} \binom{k+j}{k} |x|^{2j}.$$

Since $\binom{k+j}{j} \leqslant \binom{d}{k}$ it follows $< p_k, p_k > \leqslant \binom{d}{k}^2 (1 + |x|^2)^{d-k}$. With the notation $|x|_1^2 = (1 + |x|^2)$, we then deduce that

$$\frac{|f^{(k)}(x)|}{k!|f'(x)|} \leqslant \frac{\binom{d}{k} \|f\| \, ||p_k||}{\sqrt{d}\, d(f, \Sigma_x) |x|_1^{d-2}} \leqslant \frac{\binom{d}{k} \|f\|}{\sqrt{d}\, d(f, \Sigma_x) |x|_1^{k-2}}.$$

Using $\|f\| \leqslant \sqrt{3}h$ proved in Lemma .3 and $\left( \dfrac{1}{\sqrt{d}} \binom{d}{k} \right)^{1/(k-1)} \leqslant \dfrac{\sqrt{d}(d-1)}{2}$ [4, Chapter 14, Lemma 10], we find that

$$\gamma(f, x) = \sup_{k \geqslant 2} \left| \frac{f^{(k)}(x)}{k! f'(x)} \right|^{\frac{1}{k-1}} \leqslant \frac{\sqrt{3} \max(1, h) \sqrt{d}(d-1)}{2 \min(1, d(f, \Sigma_x))}.$$

Since $d(f, \Sigma_x) \geqslant d(f, \Sigma)$ the result follows. We are done. □

*A.8. Proof of Theorem 10.1*

From Leibniz's rule it follows:

$$\begin{aligned} f(z) &= \sum_{k=0}^{d} \frac{f^{(k)}(x)}{k!}(z-x)^k + \sum_{i=1}^{n} \sum_{k \geqslant d+1} \frac{1}{k!} \sum_{j=0}^{d_i} \binom{k}{j} g_i^{(j)}(x) c_i^{k-j} e^{c_i x}(z-x)^k \\ &= \sum_{k=0}^{d} \frac{f^{(k)}(x)}{k!}(z-x)^k + \sum_{i=1}^{n} \sum_{k \geqslant d+1} \sum_{j=0}^{d_i} \frac{g_i^{(j)}(x)}{j!} \frac{c_i^{k-j}}{(k-j)!} e^{c_i x}(z-x)^k. \end{aligned}$$

We bound the previous quantity using the definitions of $\theta_i$'s, $\eta_i$'s and the fact that $(k-j)! \geqslant (k-d-1)!(k-d)\ldots(k-d_i) \geqslant (k-d-1)!(d+1-d_i)!$ when $k \geqslant d+1$. A straightforward computation shows successively with $|z-x| \leqslant r$, that

$$\begin{aligned} |f(z)| &\geqslant |f(x)| - \sum_{k=1}^{d} \frac{|f^{(k)}(x)|}{k!} r^k - \sum_{i=1}^{n} \theta_i(x) \left( \sum_{k \geqslant d+1} \frac{(\eta_i r)^{k-d-1}}{(k-d-1)!} \right) \\ &\quad \times \eta_i^{d+1-d_i} |e^{c_i x}| r^{d+1} \geqslant \bar{M}(x,r). \end{aligned}$$

We have also proved $M(x,t) > \bar{M}(x,t)$. Hence $m(x) > \bar{m}(x)$.

The proof of the part 2 is the same one that of Theorem 6.2. We are done. □

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