AAECC 2, 239–256 (1992)

AAECC
Applicable Algebra in Engineering, Communication and Computing

Localization of an Algebraic Hypersurface by the Exclusion Algorithm

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Received November 13, 1990; revised version September 12, 1991

Abstract. We describe a new algorithm for the localization of an algebraic hypersurface V in R^n or C^n. This algorithm computes a decreasing sequence of closed sets whose intersection is V. In the particular case of an hypersurface without any point at infinity, the notion of the asymptotic cone is used to determine a compact set containing this hypersurface. We give also a numerical version of this algorithm.

Keywords: Algebraic hypersurface, Localization, Approximation, Exclusion algorithm

1. Introduction

What is the exclusion algorithm? Let us explain the main idea of this process in a simple case. Suppose that you want to compute the different real solutions of a polynomial of a single variable: you start by giving a bounded interval $[-M, M]$ containing the different real roots. For a polynomial

$$P(x) = a_0 + a_1 x + \cdots + a_d x^d$$

and for any root r you have (Mignotte [6])

$$r \leqq 1 + \max_{0 \leqq i \leqq d-1} \left| \frac{a_i}{a_d} \right| = M.$$

In a second step you "eat" in the "cake" $[-M, M]$ "slices" which do not contain any root of P. What remains gives an approximation for the roots of the equation. What do "eating" and "slices" mean? We define them in the following way. For any x we consider the polynomial

$$M(x, t) = |P(x)| - \sum_{k=1}^{d} \frac{|P^{(k)}(x)|}{k!} t^k.$$

This polynomial possesses only one positive root, $m(x)$. We will show that $P(x) \neq 0$

and $|x-y|<m(x)$ imply $P(y)\neq 0$. Therefore, if x is not a root, the set $]x-m(x), x+m(x)[$ does not contain any root. "Eating a slice" means remove $]x-m(x), x+m(x)[$ from $[-M, M]$, you start again with another point and so on. This is exactly the exclusion algorithm!

In this paper we deal with an algebraic hypersurface V of K^n ($K=R$ or C) defined by its equation $P(x)=0$ for which we generalize the previous problem. For any $t\in K$, $|t|$ is either the absolute value or the modulus of t. For any $x\in K^n$, we will denote by $B(x,r)$, $\|x\|$, and $d(x,y)$, the open ball centered at x with radius r, the norm and the distance corresponding to

$$\|x\| = \max_{1\leqq i\leqq n} |x_i|.$$

In Sect. 2 we define the polynomial $M(x,t)$ and its positive root $m(x)$. This polynomial seems to appear for the first time in Cauchy's work. More recently F. Ronga [8] has introduced it in the case of several variables. This paper revolves around the following theorem:

For any $x, y\in K^n$ *we have* "$P(x)\neq 0$ *and* $\|x-y\|<m(x)\Rightarrow P(y)\neq 0$". The same outcome would be obtained if you substitute $d(x, V)$, the distance of x to V, for $m(x)$. But the consideration of $m(x)$ instead of $d(x, V)$ or $|P(x)|$ permits easier computations. The quantities $m(x)$, $d(x, V)$ and $|P(x)|$ are in some sense equivalent as is shown by Lojasiewicz's inequality. In Sect. 3 we describe the exclusion algorithm in the affine case. The problem is to approximate the intersection of the hypersurface V with a general closed set F. We prove that the exclusion algorithm stops in a finite number of steps if and only if F is compact and $F\cap V$ is the empty set. Next, in Sect. 4 we give a practical version of the exclusion algorithm in the case where F is a semi-algebraic compact set. Using Lojasiewicz's inequality we study the accuracy and the complexity of this process. In the last section we define the exclusion algorithm in the projective case which permits the localization of an algebraic hypersurface without any point at infinity. This new algorithm is obtained from the exclusion algorithm in the affine case by an homogeneization process on both the polynomial $P(x)$ and the set F.

2. Preliminaries

Let P be a polynomial in $K[x]$, $x=(x_1,\ldots,x_n)$, with degree $(P)=d$. We consider the following polynomial in $R[t]$:

$$M(x,t) = |P(x)| - \sum_{k=1}^{d} b_k t^k$$

where the coefficients b_k are given by

$$b_k = \frac{1}{k!}\sum_{1\leqq i_1,\ldots,i_k\leqq n}\left|\frac{\partial^k P(x)}{\partial x_{i_1}\cdots\partial x_{i_k}}\right|.$$

Note that the degree of $M(x,t)$ in t is d. This polynomial is concave and decreasing for $t\geqq 0$. As $M(x,0)\geqq 0$, this polynomial has a unique positive root which is denoted by $m(x)$. A first estimation for this root is given by

Proposition 2.1. *For each $j = 1, \dots, n$ such that $b_j \neq 0$ we have*

$$I_1 = \left(\frac{|P(x)|}{b_j}\right)^{1/j} \frac{|P(x)|}{\sum_{k=1}^{d} b_k S_1^k} \leqq m(x) \leqq \left(\frac{|P(x)|}{b_j}\right)^{1/j} = S_1.$$

Proof. The right bound is obtained by

$$|P(x)| = \sum_{k=1}^{d} b_k m(x)^k \geqq b_j m(x)^j.$$

The left bound is given by the intersection of the t axis with the line joining the points $(0, M(x,0))$ and $(S_1, M(x, S_1))$. □

The number $m(x)$ possesses many useful properties:

Proposition 2.2. *We have $m(x) = 0$ if and only if $P(x) = 0$. Moreover $\dot{x}$ is a singular point of V if and only if $m(x) = 0$ is a root of multiplicity $\geqq 2$ of $M(x, t)$.*

The proof is easy and left to the reader.

Proposition 2.3. *The function $m(x)$ is continuous and semi-algebraic.*

Proof. In the case $K = C$, we consider $m(x)$ as function of $x \in R^{2n}$. Suppose now $K = R$. Considering the definition of $m(x)$, we have

$$Graph(m) = \{(x, \lambda) \in R^n \times R : \lambda \geqq 0 \quad \text{and} \quad M(x, \lambda) = 0\}.$$

This proves the semi-algebraicity of $m(x)$. For the continuity, recall that the roots of a monic polynomial are continuous functions of coefficients (see [5], [7]). We have to show that the coefficients of $M(x, t)$ are continuous functions of x and that the coefficient of t^d in $M(x, t)$ never vanishes. These verifications are easy. □

Combining the propositions 2.2 and 2.3 we obtain:

Corollary 2.4. *Let (x^p) be a sequence in K^n converging to x. Suppose that $\lim_{p \to \infty} m(x^p) = 0$ then we have $P(x) = 0$, that is $x \in V$.*

The number $m(x)$ can be interpreted as a measure for the distance of x to the hypersurface V. This is shown by the following results.

Proposition 2.5. *Let F be a compact subset of K^n. We have:*

$$c_1 |P(x)| \leqq m(x) \leqq c_2 |P(x)|^{1/d},$$

for each $x \in F$, with

$$c_1^{-1} = \max_{x \in F} \left(\sum_{k=1}^{d} b_k b_d^{(1-k)/d} |P(x)|^{(k-1)/d} \right) \quad \text{and} \quad c_2 = b_d^{-1/d}.$$

Proof. From proposition 2.1 with $j = d$ we have:

$$\frac{|P(x)|}{\sum_{k=1}^{d} b_k b_d^{(1-k)/d} |P(x)|^{(k-1)/d}} \leqq m(x) \leqq \left(\frac{|P(x)|}{b_d}\right)^{1/d}.$$

Notice that $b_d > 0$ and independent of x. This proves our proposition. □

The exclusion algorithm is based on the following result due to Ronga ([8], Lemma 1):

Proposition 2.6. *If $P(x) \neq 0$ then $P(y) \neq 0$ for each $y \in B(x, m(x))$.*

Proof. From Taylor's formula we have:

$$P(y) = P(x) + \sum_{k=1}^{d} \frac{1}{k!} \sum_{1 \leqq i_1, \ldots, i_k \leqq n} h_{i_1} \cdots h_{i_k} \frac{\partial^k P(x)}{\partial x_{i_1} \cdots \partial x_{i_k}},$$

with $h = (h_1, \ldots, h_n) = y - x$. From the triangle inequality we get

$$|P(y)| \geqq |P(x)| - \sum_{k=1}^{d} \frac{1}{k!} \|h\|^k \sum_{1 \leqq i_1, \ldots, i_k \leqq n} \left| \frac{\partial^k P(x)}{\partial x_{i_1} \cdots \partial x_{i_k}} \right|,$$

that is

$$|P(y)| \geqq M(x, \|y - x\|).$$

If $P(x) \neq 0$ we have $m(x) > 0$ from Proposition 2.2, so that $M(x, \|y - x\|) > 0$ for each y satisfying $\|y - x\| < m(x)$ as $M(x, t)$ decreases over $[0, +\infty[$. The inequality $|P(x)| > 0$ therefore holds and this proves our proposition. □

Proposition 2.7. *Let F be a semi-algebraic compact subset of K^n (in the case $K = C$, F is semi-algebraic as subset of R^{2n}). There is a constant a_1 strictly positive and an integer n_1 non zero such that*

$$a_1 d(x, V)^{n_1} \leqq m(x) \leqq d(x, V)$$

for each $x \in F$. Moreover, when each point of $V \cap F$ is non singular we can take $n_1 = 1$.

Proof. We only have to consider the case $K = R$. The inequality $m(x) \leqq d(x, V)$ is an easy consequence of Propositions 2.2 and 2.6. The other inequality

$$a_1 d(x, V)^{n_1} \leqq m(x)$$

is obtained via Lojasiewicz's inequality ([1], Corollaire 2.6.7) since $m(x)$ and $d(x, V)$ are continuous, semi-algebraic and have the same set of zeros. Let us now consider the non-singular case. In virtue of Proposition 2.5 we have:

$$c_1 |P(x)| \leqq m(x)$$

for each $x \in F$. Consequently it is sufficient to prove the following inequality

$$b_1 d(x, V) \leqq |P(x)|$$

for each $x \in F$. Let us denote by $\| \ \|_e$ (resp. $\langle \ , \ \rangle$ and d_e) the usual Euclidean norm (resp. the scalar product and the distance) in R^n. We will establish the previous inequality with $d_e(x, V)$ instead of $d(x, V)$. Consider the function defined by

$$f(x) = \begin{cases} \dfrac{|P(x)|}{d_e(x, V)} & \text{if } x \notin V \\ \|\nabla P(x)\|_e & \text{otherwise.} \end{cases}$$

Suppose f is continuous on F. From the hypothesis we get $f(x) > 0$ for each $x \in F$

so that

$$0 < b_1 = \inf_{x \in F} f(x).$$

We obtain easily

$$b_1 d_e(x, V) \leqq |P(x)|$$

for each $x \in F$. Let us now prove that f is continuous. For each $x \in F \backslash V$ let $r_x \in V$ be such that

$$d_e(x, r_x) = d_e(x, V).$$

From the Taylor formula we get

$$P(x) = P(r_x) + \langle x - r_x, \nabla P(r_x) \rangle + (x - r_x)^T H(y)(x - r_x)$$

where H is the Hessian and $y = \alpha x + (1 - \alpha) r_x$ for some α, $(0 < \alpha < 1)$. Since $r_x \in V$ we have $P(r_x) = 0$ so that

$$\left| \frac{|P(x)|}{\| x - r_x \|_e} - \frac{|\langle x - r_x, \nabla P(r_x) \rangle|}{\| x - r_x \|_e} \right| \leqq A \| x - r_x \|_e$$

where $A > 0$ majorizes $\| H(y) \|_e$ in a neighbourhood of F. Since each point of $F \cap V$ is non singular, for x close enough to V, r_x is a non singular point of V and by a classical optimization argument $x - r_x$ is orthogonal to the tangent space $T(r_x)$ we have

$$\frac{|\langle x - r_x, \nabla P(r_x) \rangle|}{\| x - r_x \|_e} = \| \nabla P(r_x) \|_e.$$

We obtain

$$\left| \frac{|P(x)|}{d_e(x, V)} - \| \nabla P(r_x) \|_e \right| \leqq A \| x - r_x \|_e,$$

and consequently, for each $r \in V$,

$$\lim_{x \to r} \frac{|P(x)|}{d_e(x, V)} = \| \nabla P(r) \|_e.$$

This proves the continuity of f and completes the proof. □

Remark 2.8. An estimation for the integer n_1 appearing in Lojasiewicz inequality is given by P. Solerno in [9].

3. The Exclusion Algorithm in the Affine Case

3.1. Description of the Algorithm

Let F be a closed subset of K^n and let

$$V = \{x \in K^n \colon P(x) = 0\}.$$

Our aim is to localize the set $V \cap F$ in F, that is, roughly speaking, to find in F a subset a bit bigger than $V \cap F$. What is our algorithm? Pick up any point $x \in F$. If

$x \notin V$ we know from Proposition 2.6 that $B(x, m(x)) \cap V = \phi$, or in other words

$$V \cap F \subset F \backslash B(x, m(x)).$$

The set $F \backslash B(x, m(x))$, obtained by excluding $B(x, m(x))$ from F, is smaller than F and always countains $V \cap F$. Now you consider $F \backslash B(x, m(x))$ instead of F and start again with another point...etc...

We will now give a more formal description of this algorithm. We denote by (r_p) and (ε_p) two sequences of strictly positive real numbers such that $\lim_{p \to \infty} r_p = \lim_{p \to \infty} \varepsilon_p = 0$. We will construct a decreasing sequence (F_p) of closed subsets of K^n whose intersection will be $F \cap V$.

We define $F_0 = F$.

Starting from F_{p-1} we choose n_p points $x_i^p \in K^n$, $1 \leqq i \leqq n_p$, such that

$$F_{p-1} \subset \bigcup_{1 \leqq i \leqq n_p} B(x_i^p, r_p).$$

The integer n_p is finite if F_{p-1} is compact, infinite in the other case. For each i, $1 \leqq i \leqq n_p$, when $P(x_i^p) \neq 0$, we compute an approximation s_i^p of $m_i^p = m(x_i^p)$ satisfying

$$m_i^p - \varepsilon_p \leqq s_i^p \leqq m_i^p. \tag{1}$$

Let us denote by

$$B_i^p = \begin{cases} B(x_i^p, s_i^p) & \text{if} \quad P(x_i^p) \neq 0, \\ \varnothing & \text{otherwise,} \end{cases}$$

and we define

$$F_p = F_{p-1} \backslash \bigcup_{1 \leqq i \leqq n_p} B_i^p.$$

Stopping criterion. This algorithm stops when $F_p = \varnothing$, otherwise we construct an infinite decreasing sequence (F_p) of closed sets.

Theorem 3.2. *The sequence (F_p) is decreasing and*

$$\bigcap_{p \geqq 0} F_p = F \cap V.$$

Proof. We have clearly $F \cap V \subset \bigcap F_p$ as, from Proposition 2.6, for each p and i, we have $V \cap B_i^p = \varnothing$. Let us show the other inclusion, that is

$$\bigcap_{p \geqq 0} F_p \subset V.$$

If $\bigcap F_p = \varnothing$ this inclusion is obvious. Let $x \in \bigcap F_p$. For each p, we have $x \in B(x_i^p, r_p)$ for some i. As $\lim_{p \to \infty} r_p = 0$ we obtain

$$x = \lim_{p \to \infty} x_i^p. \tag{2}$$

Now, as $x \in F_p$, we have $x \notin B_i^p$, that is

$$s_i^p \leqq d(x, x_i^p). \tag{3}$$

The relations (1), (2), (3) and $\lim_{p\to\infty} \varepsilon_p = 0$ give

$$\lim_{p\to\infty} m_i^p = 0. \tag{4}$$

By (4), (2) and Corollary 2.5 we obtain $P(x)=0$, that is $x \in V$. □

Corollary 3.3. *The exclusion algorithm stops if and only if F is compact and $F \cap V = \emptyset$. In this case the set $\{B_i^p : p \geqq 0, 1 \leqq i \leqq n_p\}$ is a finite open covering of F.*

Proof. Let us suppose that $F_p = \emptyset$. That means $F_{p-1} \subset \bigcup_{1 \leqq i \leqq n_p} B_i^p$ so that

$$F \subset \bigcup_{\substack{1 \leqq k \leqq p \\ 1 \leqq i \leqq n_k}} B_i^k.$$

This inclusion proves that F is compact and $F \cap V = \emptyset$ (for each non-void B_i^k we have $B_i^k \cap V = \emptyset$). Suppose now that F is compact and $F \cap V = \emptyset$. From Theorem 3.2 we have $\bigcap F_p = \emptyset$; as F is compact $F_p = \emptyset$ for some index p and the algorithm stops. □

Proposition 3.4. *Let F be a semi-algebraic set such that $F \cap V = \emptyset$. Take $r_p = \varepsilon_p = 1/p$ in the exclusion algorithm. Then $F_p = \emptyset$ at step*

$$p \geqq \frac{2}{a_1 d(F, V)^{n_1}},$$

where a_1 and n_1 are defined in Proposition 2.7.

Proof. We have to show that for each $x \in F_{p-1}$ there is an i, $1 \leqq i \leqq n$ such that $x \in B(x_i^p, s_i^p)$. From the hypothesis we have

$$r_p = \frac{1}{p} \leqq a_1 d(x, V)^{n_1} - \varepsilon_p,$$

and from Proposition 2.7.

$$r_p \leqq m(x) - \varepsilon_p.$$

Therefore by (1), for each i

$$r_p \leqq m(x_i^p) - \varepsilon_p \leqq s_i^p.$$

As we have $F_{p-1} \subset \bigcup_{1 \leqq i \leqq n_p} B(x_i^p, r_p)$ the same inclusion remains with s_i^p instead of r_p and this proves our proposition. □

Example 3.5. Localizing a circle. In this easy example we take $K = R$ and:

$$P(x,y) = x^2 + y^2 - 1, \quad F = \{(x,y): 0 \leqq x, y \leqq 2\}.$$

We have:

$$M(x,y,t) = |x^2 + y^2 - 1| - 2(|x| + |y|)t - 2t^2.$$

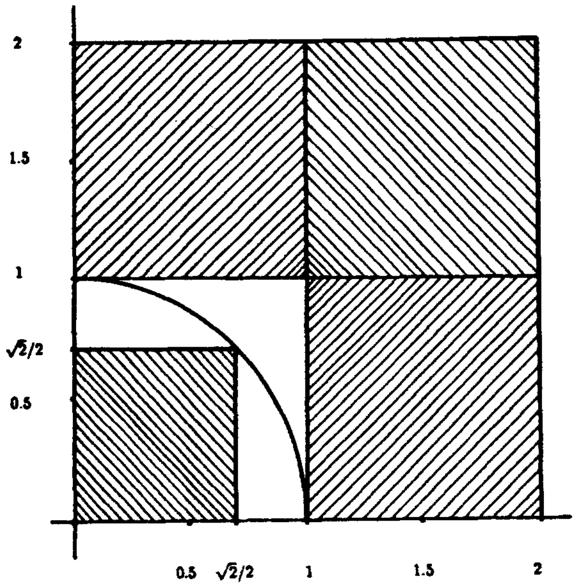


Fig. 1

so that

$$M(0,0,t) = 1 - 2t^2 \qquad m(0,0) = \sqrt{2}/2$$

$$M(1.5,0.5,t) = 2.5 - 4t - 2t^2 \quad m(1.5,0.5) = 0.5$$

$$M(1.5,1.5,t) = 3.5 - 6t - 2t^2 \quad m(1.5,1.5) = 0.5$$

$$M(0.5,1.5,t) = M(1.5,0.5,t).$$

We obtain the following Fig. 1:

4. The Exclusion Algorithm in Practice

In the previous section we have described a "theoretic" version of the exclusion algorithm. How can we implement it? Since some details need to be spelled out, we shall investigate a more practical situation. We only consider the case $K = R$. Let us define

$$F = \{x \in R^n \mid 0 \leqq x_k \leqq 1,\ 1 \leqq k \leqq n\}.$$

Our aim is to localize V in F.

Let p be a given integer. We consider the following open cover of F:

$$F \subset \bigcup_{1 \leqq i \leqq n_p} B\left(x_i^p, \frac{1}{p}\right),$$

with

$$x_i^p = \left(\frac{2i_1+1}{p+1}, \ldots, \frac{2i_n+1}{p+1}\right), \quad 0 \leqq i_1 \cdots i_n \leqq \frac{p}{2}.$$

We have

$$n_p = \left(\left[\frac{p}{2}\right]+1\right)^n.$$

For each i, we compute an approximation s_i^p of $m(x_i^p)$:

$$m(x_i^p) - \frac{1}{p} \leqq s_i^p \leqq m(x_i^p).$$

Such an approximation can be computed using numerical analysis as it will be shown later. Let us define

$$E_i^p = \begin{cases} B(x_i^p, s_i^p) & \text{if} \quad s_i^p \geqq \frac{1}{p}, \\ \varnothing & \text{otherwise,} \end{cases}$$

and

$$V_p = F \setminus \bigcup_{1 \leqq i \leqq n_p} E_i^p.$$

The set V_p is an approximation of $F \cap V$:

Theorem 4.1. *For each $x \in V_p$ we have*

$$d(x, V) < \frac{1 + (2a_1^{-1})^{1/n_1}}{p^{1/n_1}}$$

where a_1 and n_1 are the constants appearing in Proposition 2.7.

Proof. From Proposition 2.7 we have

$$a_1 d(x, V)^{n_1} \leqq m(x)$$

for each $x \in F$. Let i be such that

$$\left(\frac{2}{a_1 p}\right)^{1/n_1} \leqq d(x_i^p, V).$$

We have

$$\frac{2}{p} \leqq m(x_i^p),$$

and we obtain $1/p \leqq s_i^p$. Consequently for such an i, $E_i^p = B(x_i^p, s_i^p)$. For any $x \in V_p$ there is an i such that $x \in B\left(x_i^p, \frac{1}{p}\right)$ with $d(x_i^p, V) < (2/a_1 p)^{1/n_1}$. We obtain

$$d(x, V) \leqq d(x, x_i) + d(x_i, V) \leqq \frac{1}{p} + \left(\frac{2}{a_1 p}\right)^{1/n_1} \leqq \frac{1 + (2a_1^{-1})^{1/n_1}}{p^{1/n_1}}. \quad \square$$

Corollary 4.2. *Let $\varepsilon > 0$ be given. If*

$$p \geqq \left(\frac{1 + (2a_1^{-1})^{1/n_1}}{\varepsilon} \right)^{n_1},$$

then for each $x \in V_p$ we have $d(x, V) < \varepsilon$.

Corollary 4.3. *The sequence (V_p) converges to V in the following sense:*

$$\lim_{p \to \infty} \sup_{x \in V_p} d(x, V) = 0.$$

The proofs of these corollaries are easy and left to the reader.

4.4. Computing a Lower Bound of $m(x)$

The exclusion algorithm requires to compute a lower bound of $m(x)$ with a given accuracy ε. In this sub-section we describe an algorithm based on Newton's iteration which solves this problem and we compute the complexity of this algorithm. Let $f(t)$ be a real function defined over the interval $[0, +\infty[$ two times continuously differentiable, such that $f(0) > 0$ and the derivatives $f'(t), f''(t)$ strictly negative over $]0, +\infty[$. This function possesses a unique positive root denoted by m. Let α, β be such that: $0 < \alpha < m < \beta$. Let us consider the sequence (s_k) given by

$$s_1 = \beta, \quad s_{k+1} = s_k - \frac{f(s_k)}{f'(s_k)}.$$

Since f is a concave function, the sequence (s_k) is decreasing and converges to the root m. A lower bound of m is given by the following algorithm.

–Inputs: $f(t), \alpha, \beta$, and ε.

– Compute $s_{k+1} = s_k - \dfrac{f(s_k)}{f'(s_k)}$ while $f\left(s_k - \dfrac{f'(\beta)}{f''(\alpha)} \right) \leqq 0$. Let μ be the first index k such that: $f\left(s_k - \dfrac{f'(\beta)}{f''(\alpha)} \right) > 0$.

– Compute $s_{\mu+k}$ while $k \leqq \nu$ where ν is the first index such that

$$\nu > \frac{1}{\log 2} \log \left(\frac{\log \dfrac{2f'(\alpha)}{f''(\beta)} \dfrac{1}{\varepsilon}}{\log 2} \right).$$

We have the following result:

Proposition 4.4.1. *Let ε, ν, μ and $s_{\mu+\nu}$ be defined as before. Then*

$$m - \varepsilon \leqq s_{\mu+\nu} - \varepsilon < m.$$

The number of steps to obtain a lower bound of m is in

$$O\left(\log \log \frac{1}{\varepsilon} \right).$$

First we prove the following

Lemma 4.4.2. *Let a, b be two real numbers such that $\alpha \leqq a < m < b \leqq \beta$ and*

$$0 < C = \frac{f''(\beta)}{2f'(\alpha)}(b-a) < 1.$$

Let λ be the first index such that $s_\lambda \leqq b$. Then for each k greater than

$$\frac{1}{\log 2}\log\left(\frac{\log\frac{b-a}{C}\frac{1}{\varepsilon}}{\log\frac{1}{C}}\right),$$

we have $s_{\lambda+k} - m < \varepsilon$.

Proof. From the definition of $s_{k+\lambda}$ and the Taylor formula we deduce that $s_{k+\lambda} - m$ is equal to

$$\frac{f(m) - f(s_{k+\lambda-1}) - f'(s_{k+\lambda-1})(m - s_{k+\lambda-1})}{f'(s_{k+\lambda-1})} = \frac{f''(u)}{2f'(s_{k+\lambda-1})}(s_{k+\lambda-1} - m)^2$$

with $u \in]m, s_{k+\lambda-1}[$. Since the derivatives f' and f'' are descreasing and negative functions over $[\alpha, \beta]$ we have,

$$s_{k+\lambda} - m \leqq \frac{f''(\beta)}{2f'(\alpha)}(s_{k+\lambda-1} - m)^2.$$

We get successively,

$$s_{k+\lambda} - m \leqq \left(\frac{f''(\beta)}{2f'(\alpha)}\right)^{1+2+\dots+2^{k-1}}(s_\lambda - m)^{2^k} \leqq \frac{f''(\beta)^{2^k-1}}{2f'(\alpha)}(b-a)^{2^k} = \frac{2f'(\alpha)}{f''(\beta)}C^{2^k}$$

The conclusion of lemma follows immediately from the assumption $C < 1$. □

Proof of Proposition 4.4.1. Since the sequence (s_k) converges to m, there exists an index μ such that $0 < s_\mu - \dfrac{f'(\alpha)}{f''(\beta)} < m$. The index μ is determined by the signs of the quantities $f\left(s_k - \dfrac{f'(\beta)}{f''(\alpha)}\right)$. The hypotheses of the previous lemma are satisfied with $a = s_\mu - \dfrac{f'(\alpha)}{f''(\beta)}$ and $b = s_\mu$, since in this case $C = 1/2$. Thus the proposition is established. □

Remark 4.4.3. In the case $f(t) = M(x, t)$, we can choose $\beta = S_1$ and $\alpha = I_1$, see Proposition 2.1.

4.5. Complexity of this Algorithm

Let $\varepsilon > 0$ be a given accuracy, that is

$$\sup_{x \in V_P} d(x, V) \leqq \varepsilon.$$

According to Corollary 4.2 we have

$$p \cong \left(\frac{1 + (2a_1^{-1})^{1/n_1}}{\varepsilon} \right)^{n_1}.$$

Moreover, it has been shown that the open cover of F contains n_p balls with

$$n_p \cong \left(\frac{p}{2} \right)^n.$$

Consequently, our algorithm requires at most

$$n_p \cong \left(\frac{2^{-1/n_1} + a_1^{-1/n_1}}{\varepsilon} \right)^{nn_1}$$

steps, each of them consists in computing the approximation s_i^p of $m(x_i^p)$. In the non-singular case we can take $n_1 = 1$ (Proposition 2.7) so that

$$n_p \cong \left(\frac{2^{-1} + a_1^{-1}}{\varepsilon} \right)^n.$$

Since the computation of an approximation s_i^p of $m(x)$ runs in $O\left(\log\log \frac{1}{\varepsilon} \right)$ (Proposition 4.4.1), the exclusion algorithm requires $n_p O\left(\log\log \frac{1}{\varepsilon} \right)$ Newton's iterations of the function $M(x, .)$.

A sharper study of the complexity needs a lower bound of a_1 as a function of the degree and coefficient size of $P(x)$. Unfortunately we didn't reach this goal...

5. The Exclusion Algorithm in the Projective Case

The main disadvantage of the algorithm described in the previous sections is, in the case of a compact hypersurface V and a non-compact set F, that an infinite number of steps is needed to localize V in F. To avoid this difficulty we use an homogeneization process: the set F becomes compact in the projective space $P(K^{n+1})$ and the situation is once again favorable.

Homogeneization of P.

Let us denote $(x_0, x) = (x_0, x_1, \ldots, x_n)$ a point of K^{n+1}. We define P^* by

$$P^*(x_0, x) = x_0^d P\left(\frac{x}{x_0} \right),$$

where $d = degree(P)$. We denote by $M^*(x_0, x, t)$ the polynomial in the variable t associated to P^* and by $m^*(x_0, x)$ its positive root. With these notations the Proposition 2.6 becomes:

Proposition 5.1. *For each x, if $P^*(0, x) \neq 0$, then we have $P(y) \neq 0$ for each $y = (x_1 + h_1, \ldots, x_n + h_n)/h_0$ with $\max_{0 \leq i \leq n} |h_i| < m^*(0, x)$ and $h_0 \neq 0$. Moreover $P^*(0, z) \neq 0$ for each $z = (x_1 + h_1, \ldots, x_n + h_n)$ with $\max_{1 \leq i \leq n} |h_i| < m^*(0, x)$ and $z \neq 0$.*

Proof. Apply Proposition 2.6 to the polynomial P^* at the point $(0,x)$. We get $P^*(h_0,z)\neq 0$ for each $(h_0,z)=(0+h_0,x_1+h_1,\dots,x_n+h_n)$ such that $\max_{0\leq i\leq n}|h_i|< m^*(0,x)$. For $y=z/h_0$, $h_0\neq 0$, we obtain $P(y)=h_0^{-d}P^*(h_0,z)\neq 0$, and this proves the first assertion. The second is obtained similarly with $h_0=0$. □

Remark 5.2. The Proposition 5.1 says that if $(0,x)$ is not a point at infinity for the hypersurface V then $P(y)\neq 0$ on the unbounded set

$$\left\{(x+h)/h_0\colon \max_{0\leq i\leq n}|h_i|<m^*(0,x)\right\}.$$

Homogeneization of F.

Let F be a closed subset of K^n. We will consider in the sequel the asymptotic cone F_∞ of F. This set has been defined by G. Choquet [2] in the convex case and by J. P. Dedieu [3], [4] for general sets. We now recall this construction. The set F_∞ consists of the cluster values of the sequences $(\varepsilon_p x_p)$ with $\varepsilon_p>0$, $\lim_{p\to\infty}\varepsilon_p=0$ and $x_p\in F$. It is a closed cone with its apex at the origin. Moreover, for

$$C(F)=\{\lambda(x,1)\in K^{n+1}\colon \lambda>0 \quad \text{and} \quad x\in F\}$$

we have

$$cl(C(F))=C(F)\cup(F_\infty\times\{0\}).$$

For $V=\{x\in K^n\colon P(x)=0\}$ we denote

$$V_{\inf}=\{x\in K^n\colon P^*(0,x)=0\}.$$

Notice that $V_\infty\subset V_{\inf}$. This inclusion can be strict: V_∞ (resp. $V_{\inf}$) corresponds to the closure of $C(V)$ for the Euclidean topology (resp. the Zariski topology).

Description of the exclusion algorithm in the projective case.

We use the same notations as before. The exclusion algorithm in the projective case is simply the exclusion algorithm in the affine case applied to the polynomial $P^*(0,x)$ and the set $F_\infty\cap S(0,1)$ where $S(0,1)$ is the unit sphere for the sup norm.

We define $G_0=F_\infty\cap S(0,1)$ and $L_0=F$.

We choose n_p points x_i^p, $1\leq i\leq n_p$ such that

$$G_{p-1}\subset\bigcup_{1\leq i\leq n_p}B(x_i^p,r_p).$$

If $P^*(0,x_i^p)\neq 0$, we compute s_i^p an approximation of $m^*(0,x_i^p)$ such that $m_i^{*p}-\varepsilon_p\leq s_i^p\leq m_i^{*p}$ with $m_i^{*p}=m^*(0,x_i^p)$. We now define the set

$$K_i^p=\begin{cases}\{x_i^p+h\in K^n\colon \max_{1\leq i\leq n}|h_i|<s_i^p\} & \text{if } P^*(0,x_i^p)\neq 0,\\ \varnothing & \text{otherwise,}\end{cases}$$

and the set

$$N_i^p=\begin{cases}\left\{\dfrac{x_i^p+h}{h_0}\in K^n\colon \max_{0\leq i\leq n}|h_i|<s_i^p, h_0\neq 0\right\} & \text{if } P^*(0,x_i^p)\neq 0,\\ \varnothing & \text{otherwise.}\end{cases}$$

The sets G_p and L_p are given by

$$G_p = G_{p-1} \setminus \bigcup_{1 \leqq i \leqq n_p} K_i^p,$$

$$L_p = L_{p-1} \setminus \bigcup_{1 \leqq i \leqq n_p} N_i^p.$$

Stopping Criterion. The algorithm stops when $G_p = \varnothing$. Otherwise we construct an infinite sequence of closed sets (G_p).

Proposition 5.3. *We have*:

$$\bigcap_{p \geqq 0} G_p = F_\infty \cap S(0,1) \cap V_{\text{inf}}, \quad \text{and} \quad \left(\bigcap_{p \geqq 0} L_p\right)_\infty \subset F_\infty \cap V_{\text{inf}}.$$

Proof. The first assertion is a direct consequence of Theorem 3.2. Let us prove the second assertion. Let $a \in \left(\bigcap_{p \geqq 0} L_p\right)_\infty$, $a \neq 0$. As $\left(\bigcap_{p \geqq 0} L_p\right)_\infty$ and $F_\infty \cap V_{\text{inf}}$ are cones with their summits at the origin, we can suppose that $\|a\| = 1$. From the inclusion $\left(\bigcap_{p \geqq 0} L_p\right)_\infty \subset F_\infty$ we get $a \in F_\infty$ and consequently $a \in G_0$. Suppose now that $a \notin V_{\text{inf}}$. At some step of the algorithm we have found p, and i, $1 \leqq i \leqq n_p$ such that $a \in K_i^p$. As $a \in \left(\bigcap_{p \geqq 0} L_p\right)_\infty$, there are sequences (η_q) and (a_q) with $\eta_q > 0$, $\lim_q \eta_q = 0$, $a_q \in \bigcap_{p \geqq 0} L_p$, and $a = \lim_q \eta_q a_q$. The set K_i^p is open so that, as $a \in K_i^p$, $\eta_q a_q \in K_i^p$ for each q sufficiently large. Consequently $a_q \in N_i^p$ for each q such that $\eta_q < s_i^p$ and this proves that $a_q \notin L_p$ for each q large enough. This contradicts the fact $x_q \in \bigcap_{p \geqq 0} L_p$ for each q, and therefore the hypothesis $a \notin V_{\text{inf}}$ is false. □

Corollary 5.4. *The projective exclusion algorithm stops in a finite number of steps if and only if* $F_\infty \cap V_{\text{inf}} = \{0\}$. *In this case the set* $\bigcap_{p \geqq 0} L_p$ *is compact and contains* $F \cap V$.

Proof. The first assertion comes from Corollary 3.3. The inclusion $F \cap V \subset \bigcap_{p \geqq 0} L_p$ is given by Proposition 5.1, as $\bigcap_{p \geqq 0} L_p$ is obtained from F by excluding the sets N_i^p; for such a set $V \cap N_i^p = \varnothing$. We shall prove that $\bigcap_{p \geqq 0} L_p$ is compact. From Proposition 5.3 we have $\left(\bigcap_{p \geqq 0} L_p\right)_\infty = \{0\}$. Suppose that $\bigcap_{p \geqq 0} F_p$ is not compact. This set contains a sequence (a_q) such that $\lim_q \|a_q\| = +\infty$. Consider the sequence $b_q = \|a_q\|^{-1} a_q$. As $\|b_q\| = 1$ we can extract a converging subsequence (also denoted by (b_q)): $\lim_q b_q = b \neq 0$. We obtain $b = \lim_q \|a_q\|^{-1} a_q$ so that $b \in \left(\bigcap_{p \geqq 0} L_p\right)_\infty$ with $b \neq 0$, and this contradicts $\left(\bigcap_{p \geqq 0} L_p\right)_\infty = \{0\}$. □

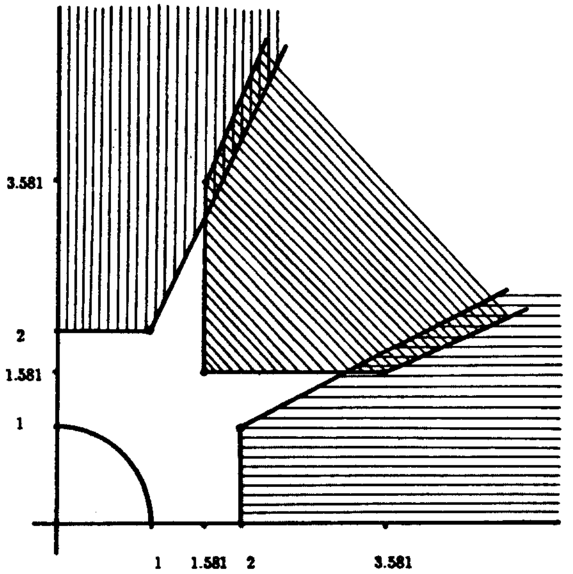


Fig. 2

Corollary 5.5. *Every hypersurface without any point at infinity can be localized in a finite number of steps.*

Proof. In this case $F = R^n$ and $V_{\text{inf}} = \{0\}$. We apply Corollary 5.4.

Example 5.6. Localizing a circle in R^2.

Let $P(x, y) = x^2 + y^2 - 1$ and $F = \{(x, y): x \geqq 0, y \geqq 0\}$. We have

$$P^*(w, x, y) = x^2 + y^2 - w^2,$$
$$M(w, x, y, t) = |x^2 + y^2 - w^2| - 2(|x| + |y| + |w|)t - 3t^2.$$

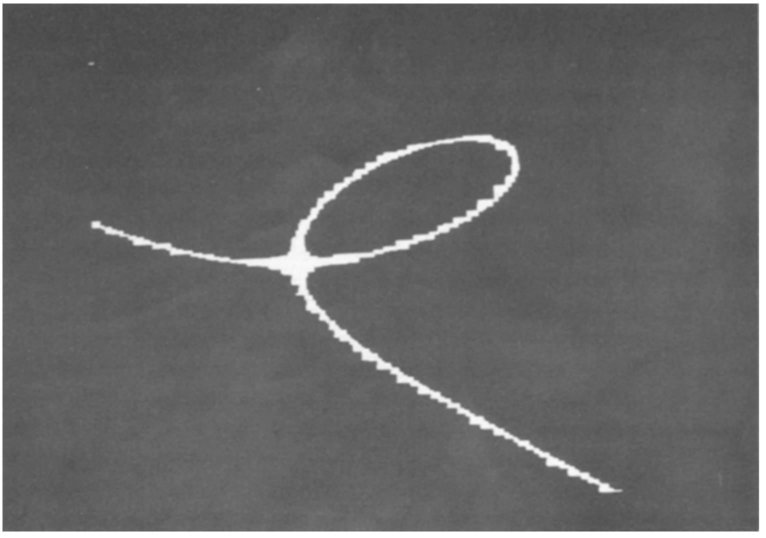
From Proposition 5.1, if $P^*(0, x, y) \neq 0$ then $P(x, y) \neq 0$ for each $(x + \lambda, y + \mu)/v$ with $\max\{|\lambda|, |\mu|, |v|\} < m^*(0, x, y)$. We have:

$$M(1,0,0,t) = M(0,1,0,t) = 1 - 2t - 3t^2 \quad m^*(1,0,0) = \tfrac{1}{3}$$
$$M(1,1,0,t) = 2 - 4t - 3t^2 \quad m(1,1,0) = \frac{-2 + \sqrt{10}}{3} \cong 0.387,$$

We obtain the following Fig. 2: the excluded regions are shaded.

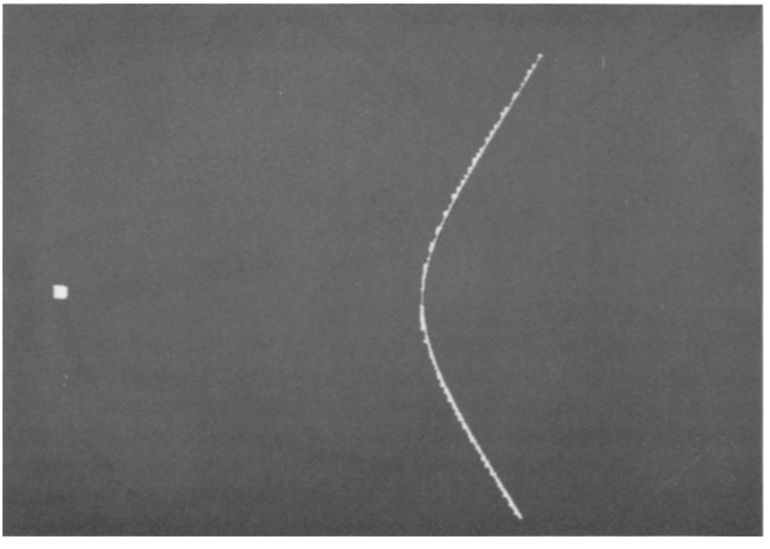
6. Examples

The following curves have been obtained from our practical exclusion algorithm in the affine case with $K = R$. For each of the following pictures, we give the equation of the curve, the coordinates of the rectangle in which the curve has been localized and the value of the accuracy $1/p$. We use float arithmetic on a Mackintosh 2.



The folium of Descartes:

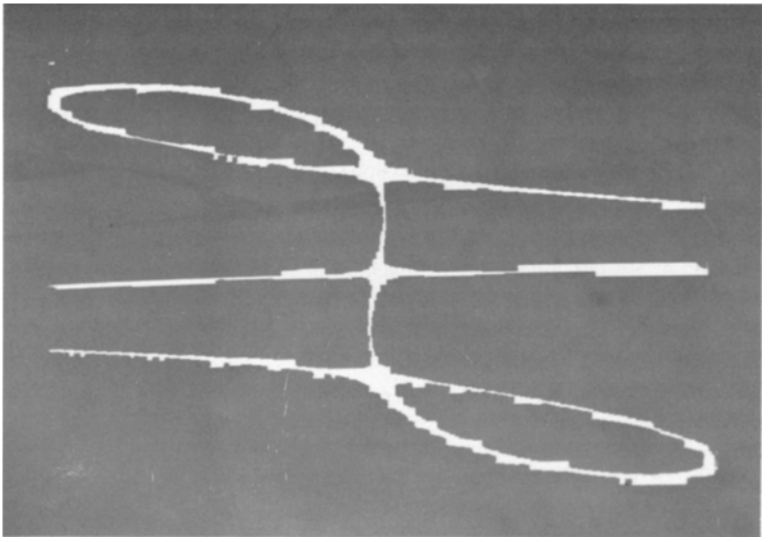
$P(x, y) = x^3 + y^3 - 2xy, \ -1 \leqq x \leqq 3, \ -2 \leqq y \leqq 2, p^{-1} = 0.02.$



The divergent parabola:

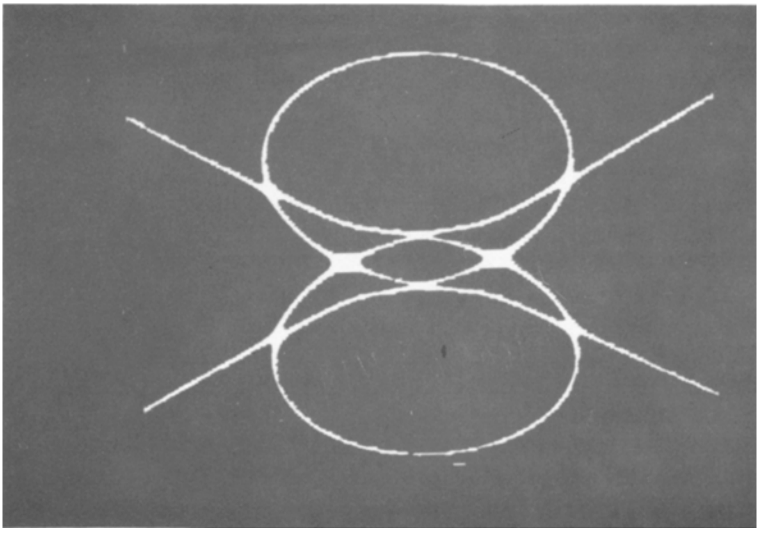
$P(x, y) = y^2 - x^3 + 2x^2, \ -0.5 \leqq x \leqq 3, \ -2 \leqq y \leqq 2, p^{-1} = 0.02.$

The isolated point $(0, 0)$ appears in a small rectangle.



A sextic:

$$P(x, y) = (4y^2 + xy - 1)^2 - (4y^2 - 1)^2(1 - y^2),$$
$$-3.7 \leqq x \leqq 3.7,\ -2 \leqq y \leqq 2,\ p^{-1} = 0.03.$$



The curve of Gergueb:

$$P(x, y) = -7x^8 - 12x^6y^2 + 28x^6 + 6x^4y^4 + 44x^4y^2 - 42x^4 + 20x^2y^6 + 68x^2y^4$$
$$-52x^2y^2 + 28x^2 + 9y^8 - 204y^6 + 70y^4 + 20y^2 - 7,$$
$$-2.5 \leqq x \leqq 2.5,\ -4 \leqq y \leqq 4,\ p^{-1} = 0.02.$$

This curve appears in the study of a geometrical problem via the Wu Wen-Tsün method.

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