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The Sturm method in the complex case

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Abstract

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Let $Z_K^0(F)$, $Z_K^\infty(F)$ be the number of zeros and the number of poles with their multiplicities of a complex rational fraction lying inside a compact K of $\mathbb{C}$ the boundary of which is a Jordan curve parametrized piecewise by rational curves. We compute the difference $Z_K^0(F) - Z_K^\infty(F)$ extending the Sturm method in the complex case.

1. Introduction and notations

Let F be an irreducible complex rational fraction and K be a compact set in $\mathbb{C}$. The boundary ∂K is assumed to be a connected simple closed Jordan curve parameterized piecewise by rational curves. Furthermore, ∂K is oriented counterclockwise. The purpose of this note is computing the right-hand side of the formula of the Principle of Argument written in the following suggestive fashion [1]

$$\int_{\partial K} \frac{F'}{F} = 2i\pi(\text{number of zeros} - \text{number of poles}),$$

using the method of Sturm sequences. In the case where F is a complex polynomial with no root on the real axis, Marden in Chapter 9 of [3] uses the Sturm sequences to find the number of zeros of F in the upper and lower half-planes. But only an upper bound of the number of zeros of a complex

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polynomial is given in a sector of plane. In this note we generalize the results obtained in [3] and we compute exactly the number of zeros.

First we precise the notations and hypotheses.

1.1. The boundary ∂K is the union of rational curves denoted by $\gamma_j(t)$ defined on the interval $[a_j, b_j]$, $0 \leq j \leq n-1$, such that

$$\gamma_j(b_j) = \gamma_{j+1}(a_{j+1})$$

with the convention $\gamma_n = \gamma_0$ and $a_n = a_0$. We define the real rational fractions $F_{0j}(t)$ and $F_{1j}(t)$ such that

$$F(\gamma_j(t)) = F_{0j}(t) + \mathrm{i}F_{1j}(t) \quad \text{for } t \in [a_j, b_j]\,,\ 0 \leq j \leq n-1\,,$$

and we consider the polynomials P_{0j} and P_{1j} such that

$$\frac{F_{1j}(t)}{F_{0j}(t)} = \frac{P_{1j}(t)}{P_{0j}(t)} \quad \text{for } t \in [a_j, b_j]\,,\ 0 \leq j \leq n-1\,.$$

1.2. Let $Z_K^0(F)$ and $Z_K^\infty(F)$ be the number of zeros of F and the number of poles of F respectively with their multiplicities lying inside K.

1.3. Let f, g be real rational fractions and a, b be real numbers. We define the quantity

$$\theta(f, g, a, b) = \tfrac{1}{2}(\operatorname{sign} f(a^+) - \operatorname{sign} g(b^-))\,.$$

When $f = g$ and $a = b = t$, this quantity is the Cauchy index of f at the point t: in this case we shall write $\theta(f, t)$. Also we adopt the convention that sign $0 = 0$. This function θ appears naturally at the end of the proof of Theorem 1.6.

1.4. We recall an algorithm to construct a Sturm sequence and the principal result concerning them. Given two real polynomials $P_0(t)$ and $P_1(t)$, the associated Sturm sequence $\mathrm{sturm}_i(P_0, P_1)$ is defined in the following way:

– If $\mathrm{degree}(P_0) \geq \mathrm{degree}(P_1)$ then

$$\mathrm{sturm}_0(P_0, P_1) = P_0\,, \qquad \mathrm{sturm}_1(P_0, P_1) = P_1$$

else

$$\mathrm{sturm}_0(P_0, P_1) = P_0\,, \qquad \mathrm{sturm}_1(P_0, P_1) = \mathrm{rem}(P_1, P_0)$$

where $\mathrm{rem}(P_1, P_0)$ is the remainder of the euclidean division of P_1 by P_0.

– For $i \geq 1$ we compute

$$\mathrm{sturm}_{i+1}(P_0, P_1) = -\mathrm{rem}(\mathrm{sturm}_{i-1}(P_0, P_1), \mathrm{sturm}_i(P_0, P_1)) .$$

– We stop when there is an index p so that $\mathrm{sturm}_{p+1}(P_0, P_1) = 0$.

This construction appears in [3]. The previous sequence appears in [2] as being the signed remainder's sequence reserving to Sturm sequence of polynomials P and Q the signed remainder's sequence of polynomials P and $\mathrm{rem}(P'Q, P)$. Let us consider $\mathrm{Var}(P_0, \dots, P_n, t^{\pm})$, the number of consecutive variations of sign in a polynomial's sequence $P_0, \dots, P_n$ at $t^{\pm}$. If the previous sequence is the Sturm sequence $\mathrm{sturm}_0(P_0, P_1), \dots, \mathrm{sturm}_p(P_0, P_1)$, the number Var shall be denoted by $\mathrm{Var}(\mathrm{sturm}(P_0, P_1), t^{\pm})$. We denote by

$$\begin{aligned}&\mathrm{Var}(\mathrm{sturm}(P_0, P_1), a^+, b^-)\\ &\quad = \mathrm{Var}(\mathrm{sturm}(P_0, P_1), a^+) - \mathrm{Var}(\mathrm{sturm}(P_0, P_1), b^-) .\end{aligned}$$

If the polynomial P_1 is identically zero, we say that $\mathrm{Var}(P_0, P_1, t^{\pm}) = 0$. We have the following result:

Theorem 1.5. *Let a and b be real numbers with $a < b$ and P_0, P_1 be real polynomials. Then,*

$$\mathrm{Var}(\mathrm{sturm}(P_0, P_1), a^+, b^-) = \sum_{\{t \in]a,b[\,:\, P_0(t)=0\}} \theta\left(\frac{P_1}{P_0}, t\right) . \qquad \square$$

The proof of this theorem is based on the same ideas given in [2] or [3].
With these notations and properties, we shall prove the following theorem:

Theorem 1.6. *Let F be an irreducible complex rational fraction. Assume that F has neither zero nor pole on the boundary of a compact set K as introduced in* 1.1. *Let $F_{1j}(t)$ and $F_{0j}(t)$, P_{1j} and P_{0j}, $1 \leq j \leq n-1$, be defined as in* 1.1. *We have*

$$\begin{aligned}Z_K^0(F) - Z_K^{\infty}(F) = &-\tfrac{1}{2} \sum_{j=0}^{n-1} \mathrm{Var}(\mathrm{sturm}(P_{0j}, P_{1j}), a_j^+, b_j^-)\\ &-\tfrac{1}{2} \sum_{\substack{\{j:\, P_{0j}(b_j)=0,\\ 0 \leq j \leq n-1\}}} \theta\left(\frac{P_{1j+1}}{P_{0j+1}}, \frac{P_{1j}}{P_{0j}}, a_{j+1}, b_j\right) ,\end{aligned}$$

with the convention $P_{ln} = P_{l0}$, $l = 0,1$.

In [4], the previous formula is given without the second sum and only in the case where F is a polynomial. Example 3.1 illustrates that the second sum is actually necessary.

2. Proof of Theorem 1.6

First, we state a lemma.

Lemma 2.1. *Let γ be a rational curve defined on the real interval $[a, b]$ in C and F be a complex rational fraction which has neither zero nor pole in $\gamma([a, b])$. Define F_0 and F_1 to be real rational fractions verifying*

$$F(\gamma(t)) = F_0(t) + \mathrm{i}F_1(t)\,,$$

and consider the polynomials P_0 and P_1 so that

$$\frac{F_1}{F_0} = \frac{P_1}{P_0}\,.$$

We have

$$\int_\gamma \frac{F'(z)}{F(z)}\,\mathrm{d}z = \tfrac{1}{2}[\log(F_0^2(t) + F_1^2(t))]_a^b$$
$$+\,\mathrm{i}\left(\arctan\frac{P_1}{P_0}(b^-) - \arctan\frac{P_1}{P_0}(a^+)\right.$$
$$\left. -\,\pi\mathrm{Var}(\mathrm{sturm}(P_0, P_1), a^+, b^-)\right).$$

Proof. We obtain by a direct computation:

$$\int_\gamma \frac{F'(z)}{F(z)}\,\mathrm{d}z = \int_a^b \frac{F_0'(t) + \mathrm{i}F_1'(t)}{F_0(t) + \mathrm{i}F_1(t)}\,\mathrm{d}t$$
$$= \int_a^b \frac{F_0'(t)F_0(t) + F_1'(t)F_1(t)}{F_0^2(t) + F_1^2(t)}\,\mathrm{d}t$$
$$+\,\mathrm{i}\int_a^b \frac{F_1'(t)F_0(t) - F_0'(t)F_1(t)}{F_0^2(t) + F_1^2(t)}\,\mathrm{d}t\,.$$

Since F has neither zero nor pole in $\gamma([a, b])$, the first integral is equal to

$$\tfrac{1}{2}[\log F_0^2(t) + F_1^2(t)]_a^b\,.$$

Computing the second, we write

$$\frac{F_1}{F_0} = \frac{P_1}{P_0}\,,$$

where P_0 and P_1 are real polynomials. A short computation gives

$$A = \int_a^b \frac{F_1'(t)F_0(t) - F_0'(t)F_1(t)}{F_0^2(t) + F_1^2(t)}\,dt = \int_a^b \frac{P_1'(t)P_0(t) - P_0'(t)P_1(t)}{P_0^2(t) + P_1^2(t)}\,dt\,.$$

Let us consider the roots t_k of $P_0(t)$ in $]a, b[$ with $a < t_1 < t_2 < \cdots < t_l < b$. Then,

$$\begin{aligned} A = &\int_{a^+}^{t_1^-} \frac{P_1'(t)P_0(t) - P_0'(t)P_1(t)}{P_0^2(t) + P_1^2(t)}\,dt \\ &+ \sum_{k=1}^{l-1} \int_{t_k^+}^{t_{k+1}^-} \frac{P_1'(t)P_0(t) - P_0'(t)P_1(t)}{P_0^2(t) + P_1^2(t)}\,dt \\ &+ \int_{t_l^+}^{b^-} \frac{P_1'(t)P_0(t) - P_0'(t)P_1(t)}{P_0^2(t) + P_1^2(t)}\,dt\,. \end{aligned}$$

The integral A now becomes

$$\begin{aligned} A = &\arctan \frac{P_1}{P_0}(b^-) - \arctan \frac{P_1}{P_0}(a^+) \\ &+ \sum_{k=1}^{l} \arctan \frac{P_1}{P_0}(t_k^-) - \arctan \frac{P_1}{P_0}(t_k^+)\,. \end{aligned}$$

Then using the definition of the Cauchy index, we find that

$$\begin{aligned} &\arctan \frac{P_1}{P_0}(t_k^-) - \arctan \frac{P_1}{P_0}(t_k^+) \\ &\quad = \frac{\pi}{2}\left(\operatorname{sign} \frac{P_1}{P_0}(t_k^-) - \operatorname{sign} \frac{P_1}{P_0}(t_k^+)\right) = -\pi\theta\left(\frac{P_1}{P_0}, t_k\right). \end{aligned}$$

Hence,

$$A = \arctan \frac{P_1}{P_0}(b^-) - \arctan \frac{P_1}{P_0}(a^+) - \pi \sum_{k=1}^{l} \theta\left(\frac{P_1}{P_0}, t_k\right).$$

Applying Theorem 1.5 to the previous sum, we obtain finally

$$A = \arctan \frac{P_1}{P_0}(b^-) - \arctan \frac{P_1}{P_0}(a^+) - \pi \mathrm{Var}(\mathrm{sturm}(P_0, P_1), a^+, b^-)\,.$$

This achieves to prove the lemma. □

We shall use the following lemma, the proof of which is easy and left to the reader.

Lemma 2.2. *Let a,b be real numbers and P_0, P_1, Q_0, Q_1 be real polynomials so that*

$$P_0(b)=0 \Leftrightarrow Q_0(a)=0\,,$$
$$\text{if } P_0(b)\neq 0 \quad \text{then } \frac{P_1}{P_0}(b)=\frac{Q_1}{Q_0}(a)\,.$$

Then we have

$$\arctan\frac{P_1}{P_0}(b^-)-\arctan\frac{Q_1}{Q_0}(a^+)$$
$$=\begin{cases}0\,, & \text{if } P_0(b)\neq 0\,,\\ -\pi\vartheta\left(\frac{Q_1}{Q_0},\frac{P_1}{P_0},a,b\right), & \text{otherwise}\,.\end{cases} \qquad \square$$

We prove now Theorem 1.6. By the Argument Principle applied to a complex rational fraction which has neither zero nor pole on the boundary ∂K we have:

$$Z_K^0(F)-Z_K^\infty(F)=\frac{1}{2i\pi}\int_K\frac{F'(z)}{F(z)}\,dz=\frac{1}{2i\pi}\sum_{j=0}^{n-1}\int_{\gamma_j}\frac{F'(z)}{F(z)}\,dz\,.$$

We apply Lemma 2.1. First we remark that

$$\sum_{j=0}^{n-1}[\log(F_{0j}^2(t)+F_{1j}^2(t))]_{a_j}^{b_j}=0\,,$$

since by construction the real rational fractions F_{0j} and F_{1j} verify $F_{0j}(b_j)=F_{0j+1}(a_{j+1})$ and $F_{1j}(b_j)=F_{1j+1}(a_{j+1})$.

Next we estimate the following sum:

$$\sum_{j=0}^{n-1}\arctan\frac{P_{1j}}{P_{0j}}(b_j^-)-\arctan\frac{P_{1j}}{P_{0j}}(a_{j+1}^+)$$
$$=\sum_{j=0}^{n-1}\arctan\frac{P_{1j}}{P_{0j}}(b_j^-)-\arctan\frac{P_{1j+1}}{P_{0j+1}}(a_{j+1}^+)\,.$$

Since the a_j and b_j are neither pole nor zero of F, it is easy to see that the polynomials P_{0j}, P_{1j}, P_{0j+1}, P_{1j+1} verify the hypotheses of Lemma 2.2 for all j, $0\leq j\leq n-1$. Consequently,

$$\arctan \frac{P_{1j}}{P_{0j}}(b_j^-) - \arctan \frac{P_{1j+1}}{P_{0j+1}}(a_{j+1}^+)$$
$$= \begin{cases} 0\,, & \text{if } P_{0j}(b_j) \neq 0\,, \\ -\pi\theta\left(\frac{P_{1j+1}}{P_{0j+1}}, \frac{P_{1j}}{P_{0j}}, a_{j+1}, b_j\right), & \text{otherwise}\,. \end{cases}$$

Finally we obtain

$$Z_K^0(F) - Z_K^\infty(F) = -\tfrac{1}{2}\sum_{j=0}^{n-1} \mathrm{Var}(\mathrm{sturm}(P_{0j}, P_{1j}), a_j^+, b_j^-)$$
$$-\tfrac{1}{2} \sum_{\substack{\{j:\, P_{0j}(b_j)=0,\\ 0\le j\le n-1\}}} \theta\left(\frac{P_{1j+1}}{P_{0j+1}}, \frac{P_{1j}}{P_{0j}}, a_{j+1}, b_j\right),$$

and the conclusion of Theorem 1.6 holds. □

3. Examples

Example 3.1. Let us consider the rectangle $K = [-\mathrm{i}, 1-\mathrm{i}, 1+\mathrm{i}, \mathrm{i}]$ and $P(z) = z^2 - z + 1$ with roots $(1 - \mathrm{i}\sqrt{3}/2)$ and $(1 + \mathrm{i}\sqrt{3}/2)$. The curves are defined on $[0, 1]$ by (see Fig. 1):

$$\gamma_0(t) = -\mathrm{i}(1-t) + (1-\mathrm{i})t\,, \qquad \gamma_1(t) = (1-\mathrm{i})(1-t) + (1+\mathrm{i})t\,,$$
$$\gamma_2(t) = (1+\mathrm{i})(1-t) + \mathrm{i}t\,, \qquad \gamma_3(t) = \mathrm{i}(1-t) - \mathrm{i}t\,.$$

On the segment $[-\mathrm{i}, 1-\mathrm{i}]$ we have $P_{00}(t) = t^2 - t$, $P_{10}(t) = -2t + 1$. The associate Sturm sequence is: P_{00}, P_{10}, 1; and $\mathrm{Var}(\mathrm{sturm}(P_{00}, P_{10}), 0^+, 1^-) = 0$.

On the segment $[1-\mathrm{i}, 1+\mathrm{i}]$ we have $P_{01}(t) = -t^2 + t$, $P_{11}(t) = 2t - 1$. The associate Sturm sequence is: P_{01}, P_{11}, -1; and $\mathrm{Var}(\mathrm{sturm}(P_{0.1}, P_{11}), 0^+, 1^-) = 0$.

The result on the segment $[1+\mathrm{i}, \mathrm{i}]$ (resp. $[\mathrm{i}, -\mathrm{i}]$) is the same as that on the segment $[-\mathrm{i}, 1-\mathrm{i}]$ (resp. $[1-\mathrm{i}, 1+\mathrm{i}]$). Since $P_{0j}(1) = 0$, $1 \le j \le 4$, we compute the function θ at the summits of the rectangle. We obtain $\theta(\frac{P_{1j-1}}{P_{0j-1}}, \frac{P_{1j}}{P_{0j}}, 0, 1) = -1$. Applying Theorem 1.6 we find $Z_K^0(P) = 2$. The first sum of the formula of Theorem 1.6 is zero and does not compute the number $Z_K^0(P)$ as it is asserted in [4].

Example 3.2. Let us consider ∂K composed of

$$\gamma_0(t) = \frac{-2t}{t^2+1} + \mathrm{i}\,\frac{1-t^2}{1+t^2}\,, \quad -1 \le t \le 1\,,$$

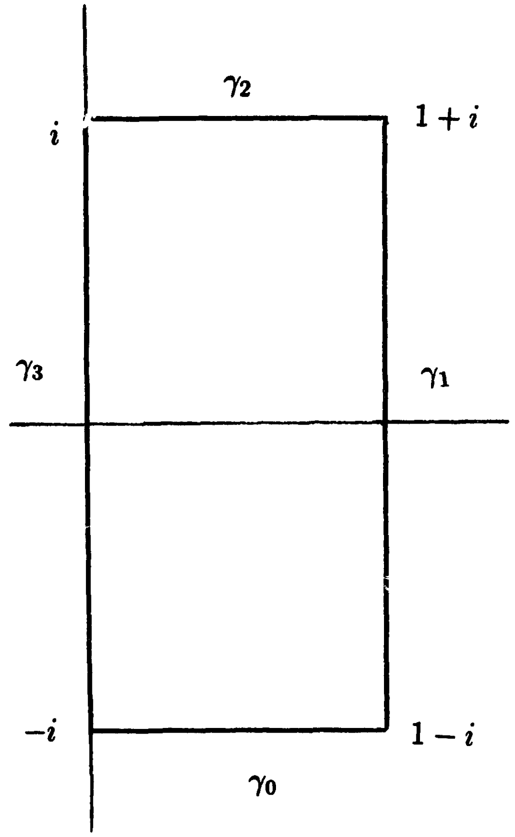

Fig. 1.

$$\gamma_1(t) = -(1-t) - \mathrm{i}t\,, \qquad 0 \le t \le 1\,,$$

$$\gamma_2(t) = -\mathrm{i}(1-t) + t\,, \qquad 0 \le t \le 1\,,$$

(see Fig. 2) and $F(z) = (2z^2 + 1)/z$ with for roots $\pm \mathrm{i}\sqrt{2}/2$ and 0 for pole.

On γ_0 we have $P_{00}(t) = 6t$, $P_{10}(t) = t^2 - 1$. The associate Sturm sequence is: $6t$, -1; and $\mathrm{Var}(\mathrm{sturm}(P_{00}, P_{10}), -1^+, 1^-) = -1$.

On γ_1 we have $P_{01}(t) = (t-1)(4t^2 - 4t + 3)$, $P_{11}(t) = -t(4t^2 - 4t + 1)$. The associate Sturm sequence is:

$$P_{01}\,,\ P_{11}\,,\ 4t^2 - 6t + 3\,,\ 2t - 3\,,\ -3\,;$$

and $\mathrm{Var}(\mathrm{sturm}(P_{00}, P_{10}), 0^+, 1^-) = 0$.

On γ_2 we have $P_{02}(t) = t(4t^2 - 4t + 3)$, $P_{12}(t) = (t-1)(4t^2 - 4t + 1)$. The associate Sturm sequence is:

$$P_{02}\,,\ P_{12}\,,\ -4t^2 + 2t - 1\,,\ -2t - 1\,,\ 3\,;$$

and $\mathrm{Var}(\mathrm{sturm}(P_{02}, P_{12}), 0^+, 1^-) = 0$.

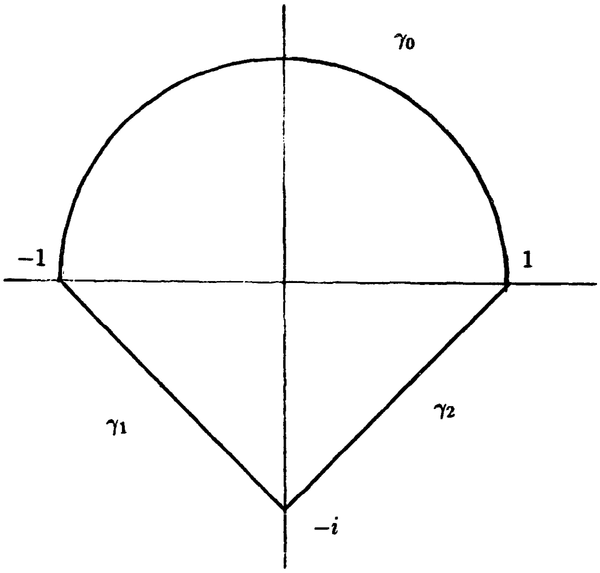

Fig. 2.

Furthermore, $P_{01}(1) = P_{02}(0) = 0$ and

$$\theta\left(\frac{P_{12}}{P_{02}}, \frac{P_{11}}{P_{01}}, 0, 1\right) = -1 .$$

Applying Theorem 1.6 we find $Z_K^0(F) - Z_K^\infty(F) = 1/2 + 1/2 = 1$.

Example 3.3. Let us consider the sector of boundary ∂K composed of

$$\gamma_0(t) = \frac{24+7i}{25}\, t , \qquad 0 \le t \le 1 ,$$

$$\gamma_1(t) = \frac{-2t}{t^2+1} + i\,\frac{1-t^2}{1+t^2} , \qquad -\tfrac{3}{4} \le t \le -\tfrac{1}{4} ,$$

$$\gamma_2(t) = \frac{8+15i}{17}\,(1-t) , \qquad 0 \le t \le 1 ,$$

(see Fig. 3) and $P(z) = 4z^3 - (6+4i)z^2 + (2+4i)z - i$ with $(1+i)/2$ as double root inside K.

On γ_0 we have

$$P_{00}(t) = \tfrac{10296}{3125}t^3 - \tfrac{909}{250}t^2 + t , \qquad P_{10}(t) = \tfrac{23506}{3125}t^3 - \tfrac{2062}{125}t^2 + 11t - 1 .$$

The associate Sturm sequence is:

$$P_{00} ,\ P_{10} ,\ -16875t^2 + 17950t - 5148 ,\ \tfrac{4714}{9}t - \tfrac{3623}{25} ,\ 1 ;$$

and $\mathrm{Var}(\mathrm{sturm}(P_{00}, P_{10}), 0^+, 1^-) = 0$.

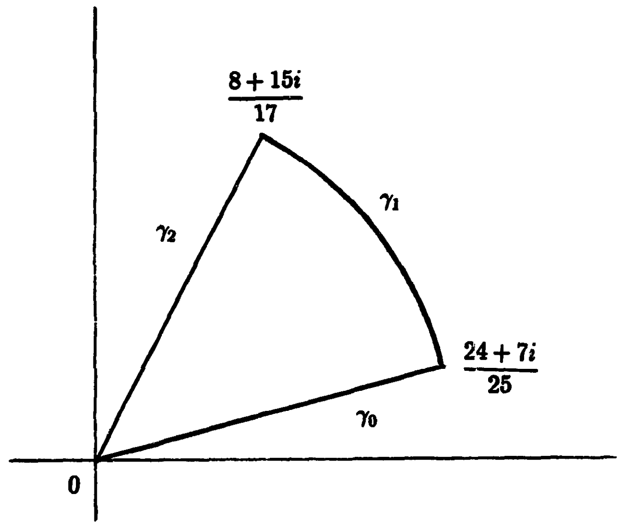

Fig. 3.

On γ_1 we have

$$P_{00}(t) = 5t^6 + 18t^5 - 13t^4 - 44t^3 - 17t^2 + 2t + 1\,,$$
$$P_{10}(t) = 5t^6 - 32t^5 - 85t^4 - 16t^3 + 39t^2 + 16t + 1\,.$$

The associate Sturm sequence is:

$$P_{00}\,,\ P_{10},\ -25t^5 - 12t^4 + 14t^3 + 28t^2 + 7t\,,\ 3219t^4 + 4044t^3 + 438t^2 - 628t - 125\,,$$
$$26678t^3 - \tfrac{82024}{3}t^2 - 591t + \tfrac{616}{3}\,,\ 481519t^2 + 356184t + 59255\,,\ 2831t + 10984\,,\ 1\,;$$

and $\mathrm{Var}(\mathrm{sturm}(P_{00}, P_{10}), 0^+, 1^-) = -2$.

On γ_2 we have

$$P_{00}(t) = \tfrac{4388}{289}(1-t)^3 + \tfrac{963}{34}(1-t)^2 + 11(t-1)\,,$$
$$P_{10}(t) = \tfrac{1980}{289}(1-t)^3 - \tfrac{796}{17}(1-t)^2 - 62t + 45\,.$$

The associate Sturm is:

$$P_{00}\,,\ P_{10}\,,\ -8381t^2 + 7208t - 1271\,,\ 34822t - \tfrac{223861}{17}\,,\ -1\,.$$

And $\mathrm{Var}(\mathrm{sturm}(P_{00}, P_{10}), 0^+, 1^-) = -1$.

Furthermore, we have $P_{00}(0) = P_{02} = 0$ and

$$\theta\left(\frac{P_{10}}{P_0}, \frac{P_{12}}{P_{02}}, 0, 1\right) = -1 .$$

Applying Theorem 1.6 we find $Z_K^0(P) = \frac{3}{2} + \frac{1}{2} = 2$.

Acknowledgment

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