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Contraction, Robustness, and Numerical Path-Following Using Secant Maps

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Secant type methods are useful for finding zeros of analytic equations that include polynomial systems. This paper proves new results concerning contraction and robustness theorems for secant maps. It is also shown that numerical path-following using secant maps has the same order of complexity that numerical path-following using Newton's map to approximate a zero. Such an algorithm was implemented and some numerical experiments are displayed. © 2000 Academic Press

Key Words: Regula Falsi; secant method; Newton method; approximate zero; contraction map; robustness; α-theory; homotopy method; complexity.

1. INTRODUCTION

Let f be an analytic function

$$f\colon E \to F$$

with E and F two real or complex Banach spaces. In the recent book "Complexity and Real Computation" [1, Chap. 8] the authors investigate the Newton method in a modern exposition: they give conditions of convergence to a root of f on a fixed input. This analysis is done with the quantities

$$\beta(f, x) = \|Df(x)^{-1} f(x)\|,$$

$$\gamma(f, x) = \sup_{k \geqslant 2} \left(\frac{\|Df(x)^{-1} D^k f(x)\|}{k!} \right)^{1/(k-1)},$$

$$\alpha(f, x) = \beta(f, x)\, \gamma(f, x),$$

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with $x \in E$ such that $Df(x)^{-1}$ exists. The Newton map is denoted by

$$N_f(x) = x - Df(x)^{-1} f(x).$$

the way proposed in [1] consists of three points.

(1) The computation of a ball centered in a root ζ of f which only contains approximate zeros. More precisely

N-GAMMA THEOREM [1, p. 156]. *Suppose that $f(\zeta) = 0$ and that $Df(\zeta)^{-1}$ exists. If*

$$\|x - \zeta\| \leqslant \frac{3 - \sqrt{7}}{2\gamma(f, \zeta)},$$

then x is an approximate zero of f with associated zero ζ, i.e., the sequence

$$x_0 = x, \qquad x_{k+1} = N_f(x_k), \qquad k \geqslant 0,$$

is well defined and satisfies

$$\|x_k - \zeta\| \leqslant (\tfrac{1}{2})^{2^k - 1} \|x - \zeta\|, \qquad k \geqslant 0.$$

(2) The computation of a ball in which N_f is a contraction map.

N-CONTRACTION THEOREM [1, Corollary 2, p. 164]. *Let $x \in E$ and $u > 0$ such that the two conditions hold*:

(1) $c = 2(\alpha(f, x) + u)/\psi(u)^2 < 1$, *with* $\psi(u) = 1 - 4u + 2u^2$.

(2) $\alpha(f, x) + cu \leqslant u$.

Then N_f is a contraction map of the ball $B(x, \frac{u}{\gamma(f, x)})$ into itself with contraction constant c. Hence there is a unique root ζ of f in $B(x, \frac{u}{\gamma(f, x)})$ and for all $y \in B(x, \frac{u}{\gamma(f, x)})$ tend to ζ under iteration of N_f.

(3) The computation of a neighborhood of a fixed input x which is contained in the ball of the N-Gamma Theorem. So

N-ROBUST α THEOREM [1, Theorem 4, p. 164]. *Let u_0 and α_0 be two real positive numbers such that*

(1) $c_0 := \dfrac{2(\alpha_0 + u_0)}{\psi(u_0)^2} < \dfrac{1}{2}$.

(2) $\alpha_0 + c_0 u_0 \leqslant u_0$.

$$(3)\quad \left(\frac{\alpha_0}{1-c_0}+u_0\right)\left(\frac{1}{\psi(\alpha_0/(1-c_0))(1-\alpha_0/(1-c_0))}\right)\leqslant\frac{3-\sqrt{7}}{2}.$$

$$(4)\quad \frac{1}{\psi(\alpha_0/(1-c_0))(1-\alpha_0/(1-c_0))}\leqslant\frac{1}{2c_0}.$$

If $\alpha(f, x)\leqslant\alpha_0$ *then there is a root* ζ *of* f *such that*

$$B\left(x, \frac{u_0}{\gamma(f, x)}\right)\subset B\left(\zeta, \frac{3-\sqrt{7}}{2\gamma(f, \zeta)}\right)$$

and N_f *maps* $B(x, \frac{u_0}{\gamma(f, x)})$ *into* $B(\zeta, \frac{u_0}{\gamma(f, \zeta)})$ *with contraction constant less than or equal to* $1/2$.

The first goal of this paper is to give a contraction theorem and a robust α theorem for secant type maps. The secant type methods to solve numerically analytic equations have been studied in [3]. In this paper the author defines the divided difference operator for analytic functions

$$[y, x]f=\sum_{k\geqslant 1}\frac{D^kf(x)}{k!}(y-x)^{k-1}$$

which satisfies the functional equation

$$f(y)-f(x)=([y, x]f)(y-x).$$

Moreover $[x, x]f=Df(x)$.

If $([y, x]f)^{-1}$ makes sense we introduce the secant map

$$S_f(y, x):=y-([y, x]f)^{-1}f(y)=x-([y, x]f)^{-1}f(x).$$

We also will denote for x fixed

$$R_{f, x}(y)=S_f(y, x).$$

Then we can define two sequences: the Regula Falsi sequence

$$x_0, x_1 \text{ given in } E,\qquad x_{k+1}=R_{f, x_0}(x_k),\qquad k\geqslant 1, \tag{RF}$$

and the secant sequence

$$x_0, x_1 \text{ given in } E,\qquad x_{k+1}=S_f(x_k, x_{k-1}),\qquad k\geqslant 1. \tag{S}$$

Using α-theory we will given new conditions so that these two maps are contraction maps.

THEOREM 1.1 (RF-Contraction). *Let $x \in E$ and $u > 0$ such that*

$$\frac{1-u}{1-2u}\alpha(f, x) \leqslant u, \qquad \text{and} \qquad \frac{1-u}{(1-2u)^2(1-3u)}\alpha(f, x) < 1.$$

Then

(1) *$R_{f,x}$ maps $B(x, \frac{u}{\gamma(f,x)})$ into itself.*

(2) *$R_{f,x}$ is a contraction map in that ball with contraction constant $((1-u)/(1-2u)^2\,(1-3u))\,\alpha(f, x)$.*

(3) *There is a unique root ζ of f such that*

$$\|x - \zeta\| \leqslant \frac{u}{\gamma(f, x)}.$$

With restrictions on $\alpha(f, x)$ and u, the ball $B(x, \frac{u}{\gamma(f,x)})$ is composed of approximate zeros. More precisely

THEOREM 1.2 (RF-Robust α Theorem). *Let u_0, α_0 be two real positive numbers such that*:

(1) $c_0 = \dfrac{1-u_0}{(1-2u_0)^2\,(1-3u_0)}\alpha_0 < \dfrac{1}{2}.$

(2) $\dfrac{1-u_0}{1-2u_0}\alpha_0 \leqslant u_0.$

(3) $\left(u_0 + \dfrac{\alpha_0}{1-c_0}\right)\left(\dfrac{1}{\psi(\alpha_0/(1-c_0))(1-\alpha_0/(1-c_0))}\right) \leqslant \dfrac{3-\sqrt{7}}{2}.$

(4) $\dfrac{1}{\psi(\alpha_0/(1-c_0))(1-(\alpha_0/(1-c_0))} \leqslant \dfrac{1}{2c_0}.$

If $\alpha(f, x) \leqslant \alpha_0$ then there is a root ζ of f such that

$$B\left(x, \frac{u_0}{\gamma(f, x)}\right) \subset B\left(\zeta, \frac{3-\sqrt{7}}{2\gamma(f, \zeta)}\right).$$

Moreover $R_{f,x}$ maps $B(x, \frac{u_0}{\gamma(f,x)})$ into $B(\zeta, \frac{u_0}{\gamma(f,\zeta)})$ with contraction constant less than or equal to $1/2$.

This previous result is a better criterion than one given in the N-robust α theorem since we can have simultaneously $((1-u)/(1-2u)^2\ (1-3u))\ \alpha(f,x)<1$ and $2(\alpha(f,x)+u)/\psi(u)^2>1$.

We now state a contraction theorem for the S_f map.

THEOREM 1.3 (S-Contraction). *Let $x_0 \in E$ and $u_0>0$ such that*

$$\psi(u_0)(1-u_0)-4u_0>0.$$

Suppose

$$(1)\quad c := \frac{2(\alpha(f,x_0)+u_0)(1-u_0)^2}{(\psi(u_0)(1-u_0)-4u_0)^2}+\frac{2u_0}{\psi(u_0)(1-u_0)-4u_0}<1.$$

$$(2)\quad \frac{(1-u_0)^2\,\alpha(f,x_0)+u_0^2(3-2u_0)}{\psi(u_0)}\leqslant u_0.$$

Then

(1) *S_f maps $B(x_0, \frac{u_0}{\gamma(f,x_0)})\times B(x_0,\frac{u_0}{\gamma(f,x_0)})$ into $B(x_0,\frac{u_0}{\gamma(f,x_0)})$.*

(2) *S_f is a contraction map with contraction constant c.*

(3) *There is a unique root ζ of f such that*

$$\|x_0-\zeta\|\leqslant\frac{u_0}{\gamma(f,x_0)}.$$

As corollary, we have a robust result for the S_f map.

THEOREM 1.4 (S-Robust α Theorem). *Let u_0, α_0 be two real positive numbers such that*:

$$(1)\quad c_0 := \frac{2(\alpha_0+u_0)(1-u_0)^2}{(\psi(u_0)(1-u_0)-4u_0)^2}+\frac{2u_0}{\psi(u_0)(1-u_0)-4u_0}<\frac{1}{2}.$$

$$(2)\quad \frac{(1-u_0)^2\,\alpha_0+u_0^2(3-2u_0)}{\psi(u_0)}\leqslant u_0.$$

$$(3)\quad \left(u_0+\frac{\alpha_0}{1-c_0}\right)\left(\frac{1}{\psi(\alpha_0/(1-c_0))(1-\alpha_0/(1-c_0))}\right)\leqslant\frac{3-\sqrt{7}}{2}.$$

$$(4)\quad \frac{1}{\psi(\alpha_0/(1-c_0))(1-\alpha_0/(1-c_0))}\leqslant\frac{1}{2c_0}.$$

If $\alpha(f, x_0) \leqslant \alpha_0$ *then there is a root* ζ *of* f *such that*

$$B\left(x_0, \frac{u_0}{\gamma(f, x_0)}\right) \subset B\left(\zeta, \frac{3-\sqrt{7}}{2\gamma(f, \zeta)}\right).$$

Moreover S_f *maps* $B(x_0, \frac{u_0}{\gamma(f, x_0)}) \times B(x_0, \frac{u_0}{\gamma(f, x_0)})$ *into* $B(\zeta, \frac{u_0}{\gamma(f, \zeta)})$ *with contraction constant less than or equal to* $1/2$.

The second part of this paper is devoted to studying the complexity of finding one zero of f with numerical path-following using the secant type method. For that consider the homotopy

$$f_t(x) = f(x) - tf(x_0), \qquad t \in [0, 1].$$

Suppose also there is a regular curve $\zeta_t \in E$, i.e.,

$$\forall t \in [0, 1], \qquad f_t(\zeta_t) = 0, \qquad \text{and} \qquad Df_t(\zeta_t)^{-1} \qquad \text{exists.}$$

Consider the sequences $t_0 = 1 > t_1 > \cdots > t_k$ and

$$y_0 \in B(x_0, r), \qquad y_i \in B(x_i, r), \qquad x_{i+1} = S_{f_{i+1}}(y_i, x_i), \qquad 0 \leqslant i \leqslant k-1,$$

with $r \geqslant 0$ and $f_i = f_{t_i}$. Denote $\zeta_i = \zeta_t$ and define the following quantities:

(1) $\gamma = \max_{0 \leqslant t \leqslant 1} \gamma(f, \zeta_t)$,

(2) $\beta = \max(\max_{0 \leqslant t \leqslant 1} \|Df(\zeta_t)^{-1} f(z_0)\|, 1), \qquad \alpha = \beta\gamma$,

(3) let $u_0 > 0$ be such that $M := 1 - u_0(1-2u_0)/\alpha(1-u_0) > 0$,

(4) for $u \geqslant 0$ and $r \geqslant 0$, let

$$T(u, r) = \frac{(1-3u/\psi(u)(1-u))^2}{\psi(3u/\psi(u)(1-u))}\left(\frac{\gamma r}{\psi(u)(1-u)-\gamma r} + \frac{3u}{\psi(u)(1-u)-3u}\right) \times \frac{\psi(u)(1-u)-\gamma r}{\psi(u)(1-u)-2\gamma r}.$$

The interest to deal with this homotopy is that $D^k f_t(x) = D^k f(x)$ and $\gamma(f_t, x) = \gamma(f, x)$. This property does not hold for linear homotopy $h_t(x) = (1-t)\, h_0 + th_1(x)$.

The complexity of this numerical path following is given by

THEOREM 1.5. *Let* $\lambda \geqslant 0$, $r_\lambda = 2\lambda(\lambda + \sqrt{7})/(2\lambda + \sqrt{7} - 1)$ *and* $u_0 \geqslant 0$ *be such that*:

(1) $u_0 \leqslant \frac{3-\sqrt{7}}{4} - \frac{\lambda}{2}$.

(2) $T(u_0, r_\lambda) \leqslant \frac{M}{2}$.

For $t_i = M^i$ *the sequence* $x_{i+i} = S_{f_{i+1}}(y_i, x_i)$, *with* $y_i \in B(x_i, r_\lambda/\gamma)$, $i \leqslant k$, *is well defined and satisfies*

(1) *For all* $i \leqslant k$, x_i *is an approximate zero of* f *with associated zero* ζ_i.

(2) $\beta(f, x_i) \leqslant 2\beta M^i$, $0 \leqslant i \leqslant k$.

(3) *Moreover for*

$$k > \left(1 + \frac{\alpha(1-u_0)}{u_0(1-2u_0)}\right)\left(\ln \beta + \ln \frac{1}{\varepsilon} + 1\right)$$

where $\varepsilon > 0$, *one has*

$$\beta(f, x_k) \leqslant \varepsilon.$$

Hence one can find a subdivision of the interval $[0, 1]$ such that $t_i = M^i$ and each point x_i, $0 \leqslant i \leqslant k$, is closed of the curve ζ_t. On the other hand, for $\lambda = 0$ and $y_i = x_i$ we get as a limit case, the complexity of numerical path following using the Newton map.

Theorem 1.5 is more precise than the one stated in the unidimensional case [1, Theorem 2, p. 174] which does not state that the x_i's are approximate zeros of f with associated zero as ζ_i's.

As application of this result, we suggest a practical algorithm to approximate one zero of a polynomial or analytic system in $\mathbb{C}^n$:

SNPF ALGORITHM (Secant Numerical Path Following).

Inputs: $u > 0$, $\varepsilon > 0$, $r \geqslant 0$, f be an analytic system in $\mathbb{C}^n$.
$x_0 \in \mathbb{C}^n$.
$t_0 = 1$; $t_1 = 1 - u$.
while $t_1 \geqslant 0$ **do**
  $k = 1$; $x_1 = x_0$; $y_1 \in \mathbb{C}^n$ be such that $\|y_1 - x_1\| \leqslant r$
  while $k \leqslant \lceil n \ln(10) \rceil$ and $\|([y_1, x_1] f_{t_1})^{-1} f_{t_1}(x_1)\| > \varepsilon$ **do**
    $x_1 = s_{f_{t_1}}(y_1, x_1)$
    $y_1 \in \mathbb{C}^n$ be such that $\|y_1 - x_1\| \leqslant r$
    $k = k + 1$
  end
  If $\|([y_1, x_1] f_{t_1})^{-1} f_{t_1}(x_1)\| \leqslant \varepsilon$ **then**
    If $t_1 = 0$ **then** **return**
    else $t = t_1$; $t_1 = \max(t_1 - 2(t_0 - t_1), 0)$; $t_0 = t$; $x_0 = x_1$; **end**
  else $t_1 = \frac{t_0 + t_1}{2}$; **end**
end
Output: x_1.

This algorithm constructs the following subdivision of the intervall $[0, 1]$

$$t_0 = 1, \qquad t_1 = 1 - u, \qquad t_{i+1} = \begin{cases} \max(t_i - 2(t_{i-1} - t_i), 0) \\ \qquad \text{if} \quad (\|[y_{i-1}, x_{i-1}] f_{t_i})^{-1} f_{t_i}(x_{i-1})\| \leqslant \varepsilon \\ \dfrac{t_{i-1} + t_i}{2} \qquad \text{else.} \end{cases}$$

We nest compute for any value of t_1 at most $n \ln(10) + 1$ iterates $x_1 := S_f(y_1, x_1)$ and we choose a new y_1 in the open ball $B(x_1, r)$ if the condition $\|([y_1, x_1] f_{t_1})^{-1} f_{t_1}(x_1)\| \leqslant \varepsilon$ is not satisfied.

The real numbers $M := 1 - u$ and r are not calculated as it is defined in Theorem 1.5. Using this result we will prove that the previous algorithm produces an approximate zero x_1 of f with associated zero ζ.

We will illustrate this algorithm with some examples of classical systems.

2. POINT ESTIMATES

To read easily this paper we first remember some technical lemmas.

Summation Lemma [1, Lemma 3, p. 161]. *For all* $0 \leqslant t < 1$, *and* $k \geqslant 0$, *we have* $\sum_{i \geqslant 0} \binom{k+i}{i} t^i = 1/(1-t)^{k+1}$.

From [2, p. 196, Theorem 1.16] we derive the following

Von Neuman Perturbation Lemma. *Let A be a bounded linear map from E into F. If* $\|I - A\| < 1$ *then A is invertible and*

$$\|A^{-1}\| \leqslant \frac{1}{1 - \|I - A\|}.$$

From [1, Proposition 5, p. 163] we can state the following

Fixed Point Lemma 1. *Let F a contraction map defined from a open ball* $B(x, r)$ *into itself with contraction constant* $c < 1$. *Then there exists* $\zeta \in B(x, r)$ *such that* $F(\zeta) = \zeta$ *and*

$$\|\zeta - x\| \leqslant \frac{1}{1 - c} \|F(x) - x\|.$$

We finally remember

LEMMA 2.1 [1, Proposition 3, p. 160]. *Let $x, y \in E$ and $u < 1 - \sqrt{2}/2$. For all y such that $u = \|y - x\| \gamma(f, x)$, we have*

$$(1) \quad \beta(f, y) \leqslant \frac{1-u}{\psi(u)} \left((1-u)\, \beta(f, x) + \|y - x\|\right).$$

$$(2) \quad \gamma(f, y) \leqslant \frac{\gamma(f, x)}{\psi(u)(1-u)}.$$

$$(3) \quad \alpha(f, y) \leqslant \frac{(1-u)\, \alpha(f, x) + u}{\psi(u)^2}.$$

LEMMA 2.2. *Let $x, y, x_1, y_1 \in E$ and $u = \gamma(f, x)\, \|x - y\|$, $u_1 = \gamma(f, x)\, \|x - x_1\|$, $v_1 = \gamma(f, x)\, \|y - y_1\|$, $v = \gamma(f, x)\, \|x - y_1\|$. Let us suppose that the previous quantities are strictly less than 1. We have*

$$(1) \quad \|Df(x)^{-1}([y_1, x_1] f - [y, x] f)\| \leqslant \frac{u_1 + v_1 - u_1(v_1 + u)}{(1-u_1)(1-v_1-u)(1-u)}.$$

$$(2) \quad \|Df(x)^{-1}([y_1, x_1] f - Df(x))\| \leqslant \frac{u_1 + v - u_1 v}{(1-u_1)(1-v)}.$$

Moreover if $2(1-u_1)(1-v) - 1 > 0$ then

$$(3) \quad \|[y_1, x_1] f^{-1}\, Df(x)\| \leqslant \frac{(1-u_1)(1-v)}{2(1-u_1)(1-v) - 1}.$$

Proof. (1) We have successively

$$\begin{aligned}
&[y_1, x_1] f - [y, x] f \\
&\quad = \int_0^1 (Df(ty_1 + (1-t)\, x_1) - Df(ty + (1-t)\, x))\, dt \\
&\quad = \int_0^1 \sum_{k \geqslant 1} (k+1) \frac{D^{k+1} f(ty + (1-t)\, x)}{(k+1)!} (t(y_1 - y) + (1-t)(x_1 - x))^k\, dt \\
&\quad = \int_0^1 \sum_{k \geqslant 1} (k+1) \left(\sum_{i \geqslant 0} \binom{k+i+1}{i} \frac{D^{k+i+1} f(x)}{(k+i+1)!} \right. \\
&\qquad \left. \times\, t^i (y - x)^i\, (t(y_1 - y) + (1-t)(x_1 - x))^k \right) dt.
\end{aligned}$$

Consequently using the summation lemma, we obtain

$$\begin{aligned}
&\|Df(x)^{-1}([y_1, x_1]f-[y, x]f)\| \\
&\quad\leqslant \int_0^1 \sum_{k\geqslant 1} (k+1)\left(\sum_{i\geqslant 0}\binom{k+i+1}{i}(tu)^i\right)(tv_1+(1-t)\,u_1)^k\,dt \\
&\quad\leqslant \int_0^1 \sum_{k\geqslant 1} (k+1)\,\frac{1}{(1-tu)^{k+2}}\,(tv_1+(1-t)\,u_1)^k\,dt \\
&\quad\leqslant \int_0^1 \left(\frac{1}{(1-tu-tv_1-(1-t)\,u_1)^2}-\frac{1}{(1-tu)^2}\right)dt \\
&\quad\leqslant \frac{u_1+v_1-u_1(v_1+u)}{(1-u_1)(1-v_1-u)(1-u)}.
\end{aligned}$$

(2) This part follows (1) with $x=y$.

Part (3) follows from the Von Neuman perturbation lemma. ∎

3. CONTRACTION AND ROBUSTNESS THEOREMS WITH R_f

In this section we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. From Lemma 2.2, part (3) with $u_1=v=u$, we have

$$\begin{aligned}
\|R_{f,x}(y)-x\| &= \|([y, x]f)^{-1}\,f(x)\| \\
&\leqslant \|([y, x]f)^{-1}\,Df(x)\| \quad \|Df(x)^{-1}\,f(x)\| \\
&\leqslant \frac{1-u}{1-2u}\,\beta(f, x).
\end{aligned}$$

Since $\frac{1-u}{1-2u}\,\beta(f, x)\leqslant\frac{u}{\gamma(f, x)}$, we have proved $R_{f,x}(B(x, r))\subset B(x, r)$ with $r=\frac{u}{\gamma(f, x)}$.

We now prove that $R_{f,x}$ is a contraction map with contraction constant $(1-u)\,\alpha(f, x)/(1-2u)^2\,(1-3u)$. We have

$$\begin{aligned}
&\|R_{f,x}(y)-R_{f,x}(z)\|\leqslant\|[z, x]f^{-1}\,Df(x)\| \\
&\|Df(x)^{-1}([y, x]f-[z, x]f)\| \quad \|[y, x]f^{-1}\,Df(x)\| \quad \|Df(x)^{-1}\,f(x)\|.
\end{aligned}$$

From Lemma 2.2 with $z:=y_1$, $u_1=0$, $\gamma(f, x)\,\|y-x\|\leqslant u$ and $\gamma(f, x)\,\|z-x\|\leqslant u$, each term of the previous inequality is bounded by

- $\|[z, x] f^{-1} Df(x)\| \leqslant \frac{1-u}{1-2u}$.
- $\|Df(x)^{-1}([y, x] f - [z, x] f)\| \leqslant v_1/(1-v_1-u)(1-u)$.
- $\|[y, x] f^{-1} Df(x)\| \leqslant \frac{1-u}{1-2u}$.

Finally,

$$\|R_{f,x}(y) - R_{f,x}(z)\| \leqslant \frac{1-u}{1-2u} \frac{\gamma(f, x) \|z-y\|}{(1-v_1-u)} \frac{1-u}{1-2u} \beta(f, x)$$

$$\leqslant \frac{1-u}{(1-2u)^2 (1-3u)} \alpha(f, x) \|z-y\|,$$

since $v_1 = \gamma(f, x)\|y-z\| \leqslant 2u$. We are done. ∎

Proof of Theorem 1.2. From Theorem 1.1 and assumptions (1) and (2) of Theorem 1.2, there is a root ζ of f in $B(x, \frac{u_0}{\gamma(f,x)})$. From the point fixed lemma, the inequality

$$\|x-\zeta\| \leqslant \frac{1}{1-c_0} \|x - R_{f,x}(x)\| = \frac{\beta(f, x)}{1-c_0}$$

holds. By the triangle inequality, for any y in the ball $B(x, \frac{u_0}{\gamma(f,x)})$, it follows

$$\|y-\zeta\| \leqslant \|y-x\| + \|x-\zeta\| \leqslant \frac{u_0}{\gamma(f, x)} + \frac{\beta(f, x)}{1-c_0}.$$

Hence

$$\gamma(f, x) \|y-\zeta\| \leqslant u_0 + \frac{\alpha(f, x)}{1-c_0}.$$

On the other hand by Lemma 2.1, part (2) we have

$$\frac{\gamma(f, \zeta)}{\gamma(f, x)} \leqslant \frac{1}{\psi(\gamma(f, x) \|x-\zeta\|)(1-\gamma(f, x) \|x-\zeta\|)}$$

$$\leqslant \frac{1}{\psi(\alpha_0/(1-c_0))(1-\alpha_0/(1-c_0))}.$$

Hence by assumption (3)

$$\gamma(f, \zeta) \|y-\zeta\| \leqslant \left(u_0 + \frac{\alpha_0}{1-c_0}\right) \left(\frac{1}{\psi((\alpha_0/(1-c_0))(1-\alpha_0/(1-c_0))}\right) \leqslant \frac{3-\sqrt{7}}{2}.$$

Then $y \in B(\zeta, (3-\sqrt{7})/2\gamma(f, \zeta))$. The point y is an approximate zero of f and $B(x, u_0/\gamma(f, x)) \subset B(\zeta, (3-\sqrt{7})/2\gamma(f, \zeta))$. Moreover if $y \in B(x, u_0/\gamma(f, x))$ then $\|y-\zeta\| \leqslant 2u_0/\gamma(f, x)$. From assumption (4) we get

$$\begin{aligned}\gamma(f, \zeta)\, \|R_{f,x}(y)-\zeta\| &\leqslant \gamma(f, \zeta)\, c_0\, \|y-\zeta\| \\ &\leqslant \frac{2c_0 u_0 \gamma(f, \zeta)}{\gamma(f, x)} \\ &\leqslant \frac{2c_0 u_0}{\psi(\alpha_0/(1-c_0))}\left(1-\frac{\alpha_0}{1-c_0}\right) \leqslant u_0.\end{aligned}$$

Hence $R_{f,x}$ maps $B(x, \frac{u_0}{\gamma(f,x)})$ into $B(\zeta, \frac{u_0}{\gamma(f,\zeta)})$ with contraction constant less than or equal to 1/2. ∎

4. CONTRACTION AND ROBUSTNESS THEOREMS FOR THE MAP S_f

We first state a point fixed lemma concerning the contraction map defined from $B(x_0, r) \times B(x_0, r)$ into $B(x_0, r)$.

FIXED POINT LEMMA 2. *Let a contraction map G be defined from $B(x_0, r) \times B(x_0, r)$ into $B(x_0, r)$ with contraction constant $c<1$, i.e.,*

$$\|G(y_1, x_1) - G(x, y)\| \leqslant c \max(\|y_1 - y\|, \|x_1 - x\|).$$

Then there exist ζ such that $G(\zeta, \zeta) = \zeta$ and

$$\|\zeta - x_0\| \leqslant \frac{1}{1-c}\, \|G(x_0, x_0) - x_0\|.$$

Proof. Let the sequence be defined by

$$x_1 = G(x_0, x_0), \qquad x_{n+1} = G(x_n, x_{n-1}).$$

Since G is a contraction map, it is easy to prove by induction

$$\begin{aligned} &\|x_{2p+1} - x_{2p}\| \leqslant c^p\, \|x_1 - x_0\| \\ \text{and} \qquad &\|x_{2p} - x_{2p-1}\| \leqslant c^p\, \|x_1 - x_0\|, \qquad p \geqslant 1. \end{aligned}$$

Consequently

$$\|x_{2p+2q+1} - x_{2q}\| \leqslant \sum_{i=2q}^{2p+2q} \|x_{i+1} - x_i\| \leqslant \frac{c^q}{1-c}\, \|x_1 - x_0\|.$$

A similar inequality holds for $\|x_{2p+2q}-x_{2q}\|$. Hence the sequence (x_n) is convergent and

$$\|\zeta-x_0\| \leqslant \frac{1}{1-c}\,\|x_1-x_0\|. \quad \blacksquare$$

To prove the map S_f is a contraction, we estimate the norm of its derivative $DS_f(y,x)$. Remember the norm of a linear operator A onto E is $\|A\|=\sup_{\|x=1\|}\|Ax\|$. The proofs of Theorems 1.3 and 1.4 need the following

LEMMA 4.1. *Let x_0 and u_0 as be defined in Theorem* 1.3. *In particular we have*

$$\psi(u_0)(1-u_0)-4u_0>0.$$

Let us consider x, y, x_1, $y_1 \in B(x_0, \frac{u_0}{\gamma(f,x_0)})$ and $u=\gamma(f,x)\,\|x-y\|$, $u_1=\gamma(f,x)\,\|x-x_1\|$, $v_1=\gamma(f,x)\,\|y-y_1\|$ and $v=\gamma(f,x)\,\|x-y_1\|$.

(1) $\|S_f(y_1,x_1)-S_f(y,x)\|$

$$\leqslant \frac{(1-u_1)(1-v)}{2(1-u_1)(1-v)-1}\left(\frac{2\alpha(f,x)}{(1-u_1)(1-v_1-u)(1-2u)} + \frac{u_1+v-u_1v}{(1-u_1)(1-v)}+\frac{u_1}{1-u_1}\right)\max(\|x_1-x\|,\|y_1-y\|).$$

(2) $$\|DS_f(y,x)\|\leqslant \frac{2\alpha(f,x)}{(1-2u)^2}+\frac{u}{1-2u}.$$

(3) $$\|DS_f(y,x)\|\leqslant \frac{2(\alpha(f,x_0)+u_0)(1-u_0)^2}{(\psi(u_0)(1-u_0)-4u_0)^2}+\frac{2u_0}{\psi(u_0)(1-u_0)-4u_0}.$$

Proof. (1) Expanding $f(y_1)$ we have

$$\begin{aligned} &S_f(y_1,x_1)-S_f(y,x)\\ &\quad = x_1-([y_1,x_1]f)^{-1}f(x_1)-x+([y,x]f)^{-1}f(x)\\ &\quad =([y_1,x_1]f)^{-1}\Big(([y_1,x_1]f-[y,x]f)([y,x]f)^{-1}f(x)\\ &\qquad +(y_1,x_1]f-Df(x))(x_1-x)-\sum_{k\geqslant 2}\frac{D^kf(x)}{k!}(x_1-x)^k\Big). \end{aligned}$$

Multiplying judiciously by $Df(x)^{-1}\, Df(x)$, we get from the triangle inequality

$$\begin{aligned}
&\|S_f(y_1, x_1) - S_f(y, x)\| \\
&\quad \leqslant \|([y_1, x_1] f)^{-1}\, Df(x)\| \\
&\qquad \times \Big(\|Df(x)^{-1}([y, x] f - [y_1, x_1] f)\|\ \|([y, x] f)^{-1}\, f(x)\| \\
&\qquad + \|Df(x)^{-1}(Df(x) - [y_1, x_1] f)(x_1 - x)\| \\
&\qquad + \sum_{k \geqslant 2} \frac{\|Df(x)^{-1}\, D^k f(x)\|}{k!}\ \|x_1 - x\|^k \Big).
\end{aligned}$$

Using Lemma 2.2, we bound each term in the previous inequality

- $\|([y_1, x_1] f)^{-1}\, Df(x)\| \leqslant \dfrac{(1-u_1)(1-v)}{2(1-u_1)(1-v)-1}$,
- $\|Df(x)^{-1}([y, x] f - [y_1, x_1] f)\|$
  $\leqslant \dfrac{u_1 + v_1 - u_1(v_1 + u)}{(1-u_1)(1-v_1-u)(1-u)}$,
- $\|([y, x] f)^{-1}\, f(x)\| \leqslant \|([y, x] f)^{-1}\, Df(x)\|\ \|Df(x)^{-1}\, f(x)\|$
  $\leqslant \dfrac{1-u}{1-2u}\beta(f, x)$, from Lemma 2.2, part (3) with $u_1 = 0$ and $v = u$,
- $\|Df(x)^{-1}\, (Df(x) - [y_1, x_1] f)\| \leqslant \dfrac{u_1 + v - u_1 v}{(1-u_1)(1-v)}\ \|x_1 - x\|$,
- $\displaystyle\sum_{k \geqslant 2} \frac{\|Df(x)^{-1}\, D^k f(x)\|}{k!}\ \|x_1 - x\|^k \leqslant \frac{u_1}{1-u_1}\ \|x_1 - x\|$.

Since $u_1 + v_1 \leqslant 2\gamma(f, x) \max(\|x - x_1\|, \|y - y_1\|)$ we obtain

$$\begin{aligned}
&\|S_f(y_1, x_1) - S_f(y, x)\| \\
&\quad \leqslant \frac{(1-u)(1-v)}{2(1-u_1)(1-v)-1} \\
&\qquad \times \Big(\frac{u_1 + v_1 - u_1(v_1 + u)}{(1-u_1)(1-v_1-u)(1-u)}\ \frac{1-u}{1-2u}\beta(f, x)
\end{aligned}$$

$$+\frac{u_1+v-u_1v}{(1-u_1)(1-v)}\,\|x_1-x\|+\frac{u_1}{1-u_1}\,\|x_1-x\|\Bigg)$$

$$\leqslant\frac{(1-u_1)(1-v)}{2(1-u_1)(1-v)-1}\left(\frac{2\alpha(f,x)}{(1-u_1)(1-v_1-u)(1-2u)}\right.$$

$$\left.+\frac{u_1+v-u_1v}{(1-u_1)(1-v)}+\frac{u_1}{1-u_1}\right)\max(\|x-x_1\|,\|y-y_1\|).$$

(2) When $(x_1, y_1)\to(x, y)$ we have $u_1\to 0$, $v_1\to 0$ and $v\to u$. From (1) we have

$$\|DS_f(y,x)\|=\left\|\lim_{(x_1,y_1)\to(x,y)}\frac{S_f(y_1,x_1)-S_f(y,x)}{\max(\|x-x_1\|,\|y-y_1\|)}\right\|\leqslant\frac{2\alpha(f,x)}{(1-2u)^2}+\frac{u}{1-2u}.$$

(3) From (2) and Lemma 2.1, part (2) we have

$$u=\gamma(f,x)\,\|y-x\|\leqslant\gamma(f,x)(\|y-x_0\|+\|x-x_0\|)\leqslant\frac{2u_0}{\psi(u_0)(1-u_0)}.$$

using Lemma 2.1, part (3) we bound $\alpha(f,x)$ by $(\alpha(f,x_0)+u_0)/\psi(u_0)^2$. Then we get part (3) of this lemma. ▮

Proof of Theorem 1.3. (1) We first prove $S_f(y,x)\in B(x_0,r)$ for x, $y\in B(x_0,r)$,

$$S_f(y,x)-x_0=x-x_0-([y,x]f)^{-1}f(x)$$

$$=[y,x]f^{-1}\Bigg(([y,x]f-Df(x_0))(x-x_0)-f(x_0)$$

$$-\sum_{k\geqslant 2}\frac{D^kf(x_0)}{k!}(x_0-x)^k\Bigg).$$

Since $\gamma(f,x_0)\,\|x-x_0\|\leqslant u_0$ and $\gamma(f,x_0)\,\|y-x_0\|\leqslant u_0$, it follows

- $\|[y,x]f^{-1}\,Df(x_0)\|\leqslant(1-u_0)^2/\psi(u_0$ (Lemma 2.2, part (3) with $u_1=v\leqslant u_0$ and the function $u\to(1-u)^2/\psi(u)$ is an increasing function for $0\leqslant u<1-\sqrt{2}/2)$.
- $\|Df(x_0)^{-1}([y,x]f-Df(x_0))\|\leqslant(2-u_0)\,u_0/(1-u_0)^2$, Lemma 2.2, part (2) with $u_1=v\leqslant u_0$.

Hence

$$\begin{aligned}
&\|S_f(y, x) - x_0\| \\
&\quad \leqslant \|[y, x] f^{-1} Df(x_0)\| \Bigg(\|Df(x_0)^{-1}([y, x] f \\
&\qquad - Df(x_0))\| \, \|x - x_0\| + \|Df(x_0)^{-1} f(x_0)\| \\
&\qquad + \sum_{k \geqslant 2} \frac{\|Df(x_0)^{-1} D^k f(x_0)\|}{k!} \|x_0 - x\|^k \Bigg) \\
&\quad \leqslant \frac{(1-u_0)^2}{\psi(u_0)} \left(\frac{(2-u_0)\, u_0}{(1-u_0)^2} \|x - x_0\| + \beta(f, x_0) + \frac{u_0}{1-u_0} \|x - x_0\| \right) \\
&\quad \leqslant \frac{(1-u_0)^2 \beta(f, x_0) + u_0(3 - 2u_0) \|x - x_0\|}{\psi(u_0)}.
\end{aligned}$$

By assumption (2) this previous quantity is less than $u_0/\gamma(f, x_0)$. In conclusion $S_f(y, x) \in B(x_0, u_0/\gamma(f, x_0))$.

(2) Prove now the contraction constant is c. We have

$$\begin{aligned}
&\|S_f(y_1, x_1) - S_f(y, x)\| \\
&\qquad \leqslant \max_{x_2,\, y_2 \in B(x_0, r)} \|DS_f(y_2, x_2\| \max(\|x - x_1\|, \|y - y_1\|).
\end{aligned}$$

Applying Lemma 4.1, part (3) we are done.

(3) From Fixed Point Lemma 2 there exists $\zeta \in B(x_0, u_0/\gamma(f, x_0))$ such that $S_f(\zeta, \zeta) = \zeta$. ∎

The proof of Theorem 1.4 uses Theorem 1.3 and Fixed Point Lemma 2. It is similar to the one of Theorem 1.2 and left to the reader.

5. NUMERICAL PATH-FOLLOWING USING THE SECANT MAP

Consider the notations given in the Introduction. We need some lemmas. The first is a consequence of [3, Theorem 7].

LEMMA 5.1 [3]. *Let* $\lambda \geqslant 0$ *and* $u = (3 - \sqrt{7})/4 - \lambda/2 > 0$. *Let* ζ *be a zero of* f *and* $r_\lambda = 2\lambda(\lambda + \sqrt{7})/(2\lambda + \sqrt{7} - 1)$. *For all* $x \in B(\zeta, u/\gamma(f, \zeta))$ *and* $y \in B(x, r_\lambda/\gamma(f, \zeta))$, *the point* $z = S_f(y, x)$ *is well defined and verifies*

$$\|z - \zeta\| \leqslant \tfrac{1}{2} \|x - \zeta\|.$$

Remember $\gamma = \max_{0 \leqslant t \leqslant 1} \gamma(f, \zeta_t)$.

LEMMA 5.2. *Suppose the assumptions of Theorem* 1.5 *hold. Let i be given and $y_i \in B(x_i, r_\lambda/\gamma)$, $x_{i+1} = S_{f_{i+1}}(y_i, x_i)$ be well defined such that $\gamma \|x_i - x_{i+1}\| \leqslant 3u_0$ and $\gamma \|x_i - \zeta_i\| \leqslant u_0$. Then*

$$\beta(f_{i+1}, x_{i+1}) \leqslant T(u_0, r_\lambda)\, \beta(f_{i+1}, x_i) \leqslant \frac{M}{2}\, \beta(f_{i+1}, x_i).$$

Proof. We have

$$\begin{aligned}\beta(f_{i+1}, x_{i+1}) &= \|Df_{i+1}(x_{i+1})^{-1} f_{i+1}(x_{i+1})\| \\ &\leqslant \|Df_{i+1}(x_{i+1})^{-1} Df_{i+1}(x_i)\|\ \|Df_{i+1}(x_i)^{-1} f_{i+1}(x_{i+1})\|.\end{aligned}$$

We first bound $\|Df_{i+1}(x_{i+1})^{-1} Df_{i+1}(x_i)\|$. We know $\gamma(f_{i+1}, x) = \gamma(f, x)$. Remember $\zeta_{t_i} = \zeta_i$. Since $\gamma(f, \zeta_i) \leqslant \gamma$ and $\gamma \|x_i - \zeta_i\| \leqslant u_0$, we have from Lemma 2.1, part (2), $\gamma(f, x_i) \leqslant \gamma/\psi(u_0)(1 - u_0)$. Using Lemma 2.2, part (3) with $y_1 = x_1 := x_{i+1}$ and $x := x_i$ we get

$$\|Df_{i+1}(x_{i+1})^{-1} Df_{i+1}(x_i)\| \leqslant \frac{(1 - \gamma(f, x_i)\ \|x_{i+1} - x_i\|)^2}{\psi(\gamma(f, x_i)\ \|x_{i+1} - x_i\|)}.$$

The function $u \to (1-u)^2/\psi(u)$ is an increasing function when $0 \leqslant u < 1 - \sqrt{2}/2$. Since $\gamma \|x_i - x_{i+1}\| \leqslant 3u_0$ we have

$$\begin{aligned}\gamma(f, x_i)\ \|x_{i+1} - x_i\| &\leqslant \frac{\gamma(f, \zeta_i)\ \|x_{i+1} - x_i\|}{\psi(\gamma(f, \zeta_i)\ \|x_i - \zeta_i\|)(1 - \gamma(f, \zeta_i)\ \|x_i - \zeta_i\|)} \\ &\leqslant \frac{3u_0}{\psi(u_0)(1-u)}.\end{aligned}$$

Hence

$$\|Df_{i+1}(x_{i+1})^{-1} Df_{i+1}(x_i)\| \leqslant \frac{(1 - 3u_0/\psi(u_0)(1-u_0))^2}{\psi(3u_0/\psi(u_0)(1-u_0))}.$$

We now bound $\|Df_{i+1}(x_i)^{-1} f_{i+1}(x_{i+1})\|$. From Taylor's formula and the definition of x_{i+1} we have

$$\begin{aligned}\|Df_{i+1}(x_i)^{-1} f_{i+1}(x_{i+1})\| \leqslant \Big(&\|Df_{i+1}(x_i)^{-1}[y_i, x_i] f_{i+1} - I\| \\ &+ \sum_{k \geqslant 2} (\gamma(f, x_i)\ \|x_{i+1} - x_i\|)^{k-1} \Big) \|([y_i, x_i] f_{i+1})^{-1} f_{i+1}(x_i)\|.\end{aligned}$$

We bound each term of the previous inequality with $\|y_i - x_i\| \leqslant r_\lambda$.

- $\|Df_{i+1}(x_i)^{-1} f_{i+1}(x_{i+1})\| \leqslant \dfrac{\gamma(f, x_i)\, r_\lambda}{1-\gamma(f, x_i)\, r_\lambda}$, Lemma 2.2, part (2).
- $\sum_{k\geqslant 2} (\gamma(f, x_i)\, \|x_{i+1}-x_i\|)^{k-1} \leqslant \dfrac{\gamma(f, x_i)\, \|x_{i+1}-x_i\|}{1-\gamma(f, x_i)\, \|x_{i+1}-x_i\|}$.
- $\|([y_i, x_i] f_{i+1})^{-1} f_{i+1}(x_i)\|$

$$\leqslant \|([y_i, x_i] f_{i+1})^{-1} Df(x_i)\| \, \|Df(x_i)^{-1} f_{i+1}(x_i)\|$$

$$\leqslant \frac{1-\gamma(f, x_i)\, r_\lambda}{1-2\gamma(f, x_i)\, r_\lambda}\, \beta(f_{i+1}, x_i), \qquad \text{Lemma 2.2, part (3).}$$

We then get

$$\|Df_{i+1}(x_i)^{-1} f_{i+1}(x_{i+1})\| \leqslant \left(\frac{\gamma(f, x_i)\, r_\lambda}{1-\gamma(f, x_i)\, r_\lambda} + \frac{\gamma(f, x_i)\, \|x_{i+1}-x_i\|}{1-\gamma(f, x_i)\, \|x_{i+1}-x_i\|}\right) \frac{1-\gamma(f, x_i)\, r_\lambda}{1-2\gamma(f, x_i)\, r_\lambda}\, \beta(f_{i+1}, x_i)$$

$$\leqslant \left(\frac{\gamma r_\lambda}{\psi(u_0)(1-u_0)-\gamma r_\lambda} + \frac{3u_0}{\psi(u_0)(1-u_0)-3u_0}\right) \times \frac{\psi(u_0)(1-u_0)-\gamma r_\lambda}{\psi(u_0)(1-u_0)-2\gamma r_\lambda}\, \beta(f_{i+1}, x_i).$$

Consequently $\beta(f_{i+1}, x_{i+1}) \leqslant T(u_0, r_\lambda)\, \beta(f_{i+1}, x_i)$. ∎

LEMMA 5.3. *If $u_0 \leqslant 1-\sqrt{2}/2$ then $\gamma\, \|\zeta_i - \zeta_{i+1}\| \leqslant u_0$ for all i.*

Proof. We have $f_i(\zeta_i) = f_{i+1}(\zeta_{i+1}) = 0$. By definition of $f_t(x)$, it follows $f(\zeta_i) - t_i f(x_0) = f(\zeta_{i+1}) - t_{i+1} f(x_0)$. Hence $f(\zeta_i) - f(\zeta_{i+1}) = (t_i - t_{i+1}) f(x_0)$.

Then

$$-(\zeta_{i+1}-\zeta_i) - \sum_{k\geqslant 2} Df(\zeta_i)^{-1} \frac{D^k f(\zeta_i)}{k!} (\zeta_{i+1}-\zeta_i)^k = (t_i - t_{i+1})\, Df(\zeta_i)^{-1} f(x_0).$$

From the triangle inequality we obtain

$$\|\zeta_{i+1}-\zeta_i\| - \sum_{k\geqslant 2} \gamma^{k-1} \|\zeta_{i+1}-\zeta_i\|^k \leqslant (t_i - t_{i+1}) \|Df(\zeta_i)^{-1} f(x_0)\|.$$

With $u=\gamma \|\zeta_i-\zeta_{i+1}\|$, $\beta=\max(\max_{0\leqslant t\leqslant 1} \|Df(\zeta_t)^{-1} f(x_0)\|, 1)$ and $\alpha=\gamma\beta$, it follows

$$u - \sum_{k\geqslant 2} u^k \leqslant (t_i - t_{i+1})\,\alpha.$$

But $t_i - t_{i+1} \leqslant t_0 - t_1 = 1 - M = u_0(1-2u_0)/\alpha(1-u_0)$. Hence

$$u - \frac{u^2}{1-u} = \frac{u(1-2u)}{1-u} \leqslant (1-M)\,\alpha = \frac{u_0(1-2u_0)}{1-u_0}.$$

The function $u \to (1-2u)/(1-u)$ is an increasing function for $u \in [0, 1-\sqrt{2}/2]$. Consequently $u \leqslant u_0$ and, we are done. ∎

Proof of Theorem 1.5. (1) Prove by induction the x_i's are approximate zeros of f_i with associated zero ζ_i. It is obvious for $i=0$. Suppose $\gamma \|x_i - \zeta_i\| \leqslant u_0$ and prove now that $\gamma \|x_{i+1} - \zeta_{i+1}\| \leqslant u_0$. By the triangle inequality, the inductive assumption, and Lemma 5.3 we have

$$\begin{aligned}\gamma \|x_i - \zeta_{i+1}\| &\leqslant \gamma \|x_i - \zeta_i\| + \gamma \|\zeta_i - \zeta_{i+1}\| \\ &\leqslant u_0 + u_0 = 2u_0.\end{aligned}$$

We know $2u_0 \leqslant (3-\sqrt{7})/2 - \lambda$ and $y_i \in B(x_i, r_\lambda/\gamma)$. From Lemma 5.1 the point x_{i+1} is well defined with

$$\gamma \|x_{i+1} - \zeta_{i+1}\| = \gamma \|S_{f_{i+1}}(y_i, x_i) - \zeta_{i+1}\| \leqslant \frac{\gamma}{2} \|x_i - \zeta_{i+1}\| \leqslant u_0.$$

(2) We first establish by induction the statement

$$\beta(f_i, x_i) \leqslant (t_i - t_{i+1})\,\beta.$$

Since $\beta(f_0, x_0) = 0$, the previous statement holds for $i=0$. For i given, we know that the point x_i is an approximate zero of f_i with associated zero ζ_i.

Then $\|x_i - x_{i+1}\| \leqslant \|x_i - \zeta_i\| + \|\zeta_i - \zeta_{i+1}\| + \|\zeta_{i+1} - x_{i+1}\| \leqslant 3u_0$. We apply Lemma 5.2 and we get

$$\beta(f_{i+1}, x_{i+1}) \leqslant \frac{M}{2} \beta(f_{i+1}, x_i).$$

On the other hand $f_{i+1}(x_i) = f_i(x_i) + (t_i - t_{i+1}) f(x_0)$. From the inductive assumption we get

$$\beta(f_{i+1}, x_i) \leqslant \beta(f_i, x_i) + (t_i - t_{i+1}) \|Df(x_i)^{-1} f(x_0)\| \leqslant 2(t_i - t_{i+1}) \beta.$$

Hence

$$\beta(f_{i+1}, x_{i+1}) \leqslant \frac{M}{2} \beta(f_{i+1}, x_i) \leqslant M(t_i - t_{i+1}) \beta = (t_{i+1} - t_{i+2}) \beta.$$

We prove now that

$$\beta(f_i, x_i) \leqslant (t_i - t_{i+1}) \beta \Rightarrow \beta(f, x_i) \leqslant 2M^i \beta.$$

One has $f_i(x_i) = f(x_i) - t_i f(x_0)$. Hence

$$\beta(f, x_i) - t_i \beta \leqslant \beta(f_i, x_i) \leqslant (t_i - t_{i+1}) \beta.$$

So, $\beta(f, x_i) \leqslant (2t_i - t_{i+1}) \beta < 2M^i \beta$. Part (2) is proved.

(3) The inequality $\beta(f, x_k) \leqslant \varepsilon$ holds if $2M^k \beta \leqslant \varepsilon$, i.e.,

$$k \ln \frac{1}{M} > \ln \beta + \ln \frac{1}{\varepsilon} + 1.$$

From the inequality $\frac{1}{\ln(1+s)} \leqslant 1 + \frac{1}{s}$ for $s > 0$, we get

$$k > \left(1 + \frac{\alpha(1 - u_0)}{u_0(1 - 2u_0)}\right) \left(\ln \beta + \ln \frac{1}{\varepsilon} + 1\right).$$

We are done. ∎

6. NUMERICAL EXPERIMENTS

THEOREM 6.1. *The SNPF algorithm stops. A lower bound for the number of steps is* $\ln(1 + \frac{1}{u})$. *The number of computation of iterates* x_1 *is bounded by the number of steps times* $\lceil n \ln(10) \rceil$.

Proof. If the sequence (t_i) is strictly decreasing then $t_i = 1-(2^i-1)\,u$. For $i \geqslant \ln(1+\frac{1}{u})$ we have $t_i = 0$ and the algorithm stops.

In the other case, there is an index j such that $\|([y_{j-1}, x_{j-1}] f_j)^{-1} f_j(x_{j-1})\| \leqslant \varepsilon$ and $\|([y_j, x_j] f_{j+1})^{-1} f_{j+1}(x_j)\| > \varepsilon$. By definition of t_i's and Theorem 1.5, there exists $i_j > j$ with $t_i < t_{i_j} < t_{i+1}$ and $\|([y_{i_j-1}, x_{i_j-1}] f_{i_j})^{-1} f_{i_j}(x_{i_j-1})\| \leqslant \varepsilon$. Consequently there is a strictly decreasing sequence $1 > t_{i_1} > \cdots > t_{i_j} > \cdots$ and the algorithm stops. ∎

The SNPF algorithm has been implemented with MATLAB. The computations work with the complex numbers. A lot of examples given on the Web site www.inria.fr/SAGA/POL have been tested. We only present numerical experiments with random quadratic systems and a symmetric system.

We first show an easy example how the algorithm works in practice.

EXAMPLE 6.1. *Two ellipses,*

$$3z_1^2 + 2z_2^2 - 5 = 0$$

$$2z_1^2 + 3z_2^2 - 5 = 0.$$

The inputs are $x_0 = (1+2i, 2+i)$, $u = 0.1$, $r = 0$, and $\varepsilon = 0.05$. Here we replace $2\log(10)$ by 2 in the SNPF algorithm. We give the values of t_1, t_0 and $\beta = \|([y_1, x_1] f)^{-1} f(x_1)\|$ before to test if $\beta \leqslant \varepsilon$.

t_1	t_0	β
0.9	1	4×10^{-3}
0.7	0.9	2×10^{-2}
0.3	0.7	1×10^{-2}
0	0.3	9×10^{-2}
0.15	0.3	3×10^{-1}

t_1	t_0	β
0.225	0.3	1×10^{-1}
0.2625	0.3	2×10^{-2}
0.1875	0.2625	2×10^{-2}
0.0375	0.1875	1×10^{-2}
0	0.0375	7×10^{-3}

We obtain $x_1 = (0.99998 + i7\times10^{-6}, 0.999997 - i4\times10^{-6})$ as approximate zero.

The following table shows that the iteration number increases with the radius of the ball $B(x_1, r)$.

r	0	0.1	0.2	0.3	0.4	0.5	0.6
nit	10	10	15	17	17	37	52

EXAMPLE 6.2. *Quadratic polynomials systems,*

$$f\colon x \in \mathbb{C}^n \to f(x) = x^T A x + Bx + C \in \mathbb{C}^n,$$

where $A=(A_1, ..., A_n)$ is a vector of n matrix $n \times n$, B is a matrix $n \times n$, and C is a vector in $\mathbb{C}^n$ with the convention $(x^T A x)_i = x^T A_i x$. The numerical experiments consist of choosing randomly A, B, and C and to count with respect to n the number of variables, the following quantities:

(1) the cpu-time.

(2) the number N_{it} of computations of iterates x_1.

In each case the initial point x_0 is chosen randomly. For $\varepsilon=0.01$ and $u=0.1$, we get Figs. 1 and 2.

EXAMPLE 6.3. *Symmetric systems.* Let us consider $\sigma_k = \sum_{0 \leqslant i_1 < \cdots < i_k \leqslant n} x_{i_1} ... x_{i_k}$, $1 \leqslant k \leqslant n$ for $x=(x_1, ..., x_n)$ and

$$f(x) = (\sigma_1(x), ..., \sigma_{n-1}(x), \sigma_n(x) - 1).$$

The roots of f are the n-uplets constituted of roots of the univariate polynomial $z^n - 1$.

As in a previous example when the number n increases we get Figs. 3 and 4 with $\varepsilon=0.01$ and $u=0.1$.

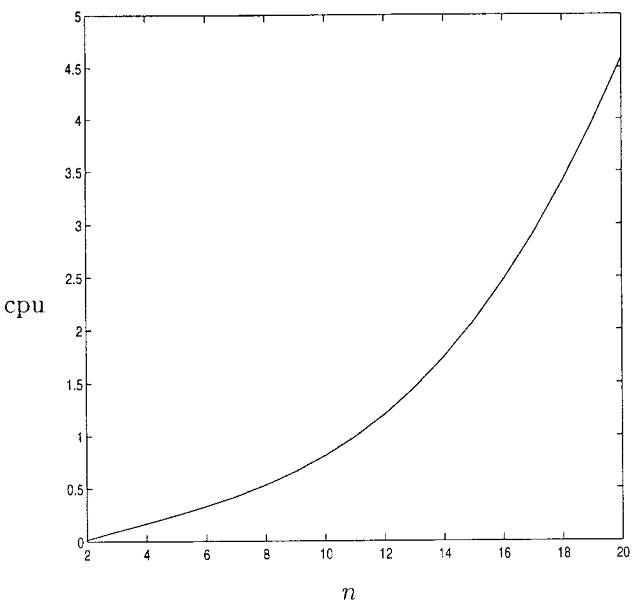

FIGURE 1

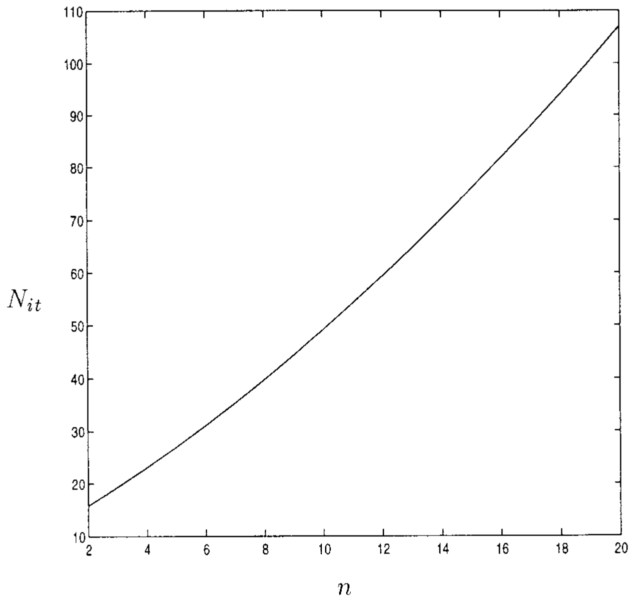

FIGURE 2

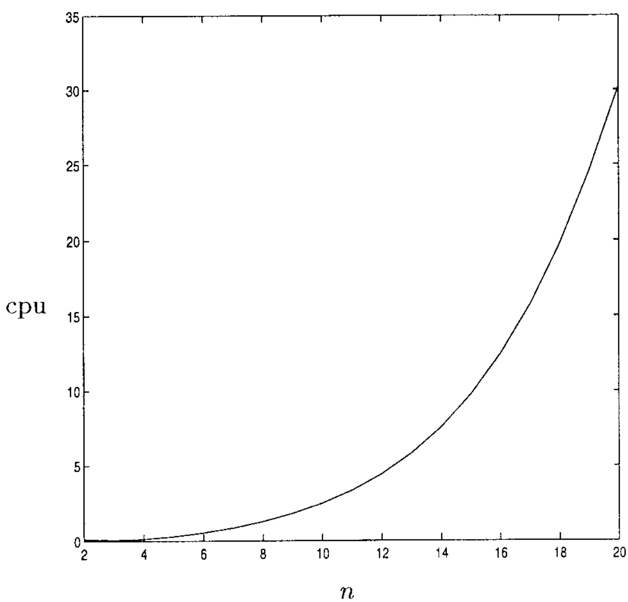

FIGURE 3

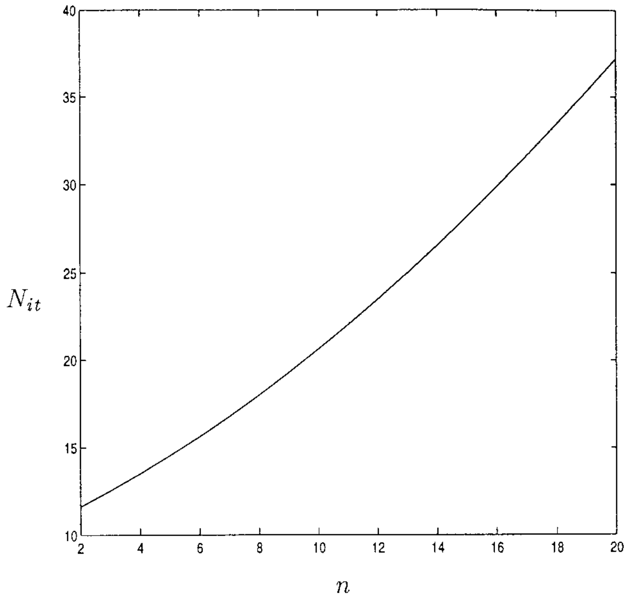

FIGURE 4

The numerical experiments show the growth of cpu time is due to the evaluation of the system at each step of the method. The number of steps seems a linear increase in the number of steps due to the hard-coded $n\log(10)$ bound (Theorem 6.1).

APPENDIX: MATLAB CODE

```
function x1=SNPF (systeme,n,u,r,eps)
x0=randn(1,n)+i * randn(1,n);
S=feval(systeme,x0,r); F0=S(1:1,1:n);
t0=1;t1=1-u;
while t1>=0
  k=0;x1=x0;beta=2 * eps;
  F=S(1:1,1:n)-t1 * F0;J=S(2:n+1,1:n);
  while k<=floor(n * log(10)) & beta>eps
    k=k+1;
    correction=(J\F.').';
    x1=x1-correction; beta=norm(correction);
    S=feval(systeme,x1,r); F=S(1:1,1:n)-t1 * F0;
     J=S(2:n+1,1:n);
  end
```

```
  [t1,t0,beta]; %print if you want to see how the
  algorithm works
    if beta< =eps
    if t1==0 t1=-1;
    else
      t=t1; t1=max(3*t1-2*t0,0);t0=t; x0=x1;
      S=feval(systeme,x0,r);
    end
  else t1=(t0+t1)/2;
  end
end
function S=two_ellipses(x,r)
S=[3*x(1)**2+2*x(2)**2-5  2*x(1)**2+3*x(2)**2-5];
y=x-r;
S=[S;3*(x(1)+y(1))  2*(x(2)+y(2))];
S=[S;2*(x(1)+y(1))  3*(x(2)+y(2))];
```

REFERENCES

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2. T. Kato, “Perturbation Theory for Linear Operators,” Springer-Verlag, New York/Berlin, 1976.
3. J. C. Yakoubsohn, Finding zeros of analytic functions: α-Theory for secant type methods, *J. Complexity* **15** (1999), 239–281.