JOURNAL OF COMPLEXITY **16**, 603–638 (2000)
doi:10.1006/jcom.2000.0555, available online at http://www.idealibrary.com on IDEAL®

Finding a Cluster of Zeros of Univariate Polynomials

Jean-Claude Yakoubsohn

Laboratoire MIP, Université Paul Sabatier, Toulouse, France
E-mail: yak@mip.ups-tlse.fr

Received September 1, 1999

A method to compute an accurate approximation for a zero cluster of a complex univariate polynomial is presented. The theoretical background on which this method is based deals with homotopy, Newton's method, and Rouché's theorem. First the homotopy method provides a point close to the zero cluster. Next the analysis of the behaviour of the Newton method in the neighbourhood of a zero cluster gives the number of zeros in this cluster. In this case, it is sufficient to know three points of the Newton sequence in order to generate an open disk susceptible to contain all the zeros of the cluster. Finally, an inclusion test based on a punctual version of the Rouché theorem validates the previous step. A specific implementation of this algorithm is given. Numerical experiments illustrate how this method works and some figures are displayed. © 2000 Academic Press

Contents.

1. INTRODUCTION

Let f be a univariate complex polynomial of degree d. The purpose of this paper is to detect numerically the existence of a zero cluster of f and to compute an open disk containing all the zeros of this cluster. When the computations are performed numerically, the coefficients of the polynomial are perturbed. Then, it is well known [26] that the roots are very sensitive to perturbations. Especially, the roots of multiplicity, say m, are decomposed into m roots. Therefore, in numerical analysis, it is more convenient

0885-064X/00 $35.00

to speak of clusters of zeros rather than multiple zeros. Obviously, the problem of computation of clusters is a particular case of more general root finding algorithms and many authors have investigated it. V. Pan's survey [19] summarizes the history of the algorithmic approach. We will describe a new algorithm which uses multiplicity to provide accurate outputs. This algorithm combines Newton's method and Rouché's theorem. The idea to combine Newton's method with another algorithm is not new and the present paper can be directly connected to Ostrowski [17], Renegar [22], Smale [24], Kim and Sutherland [14], Katz and Ying [13], Morgan *et al.* [16], and Dedieu and Shub [6]. In a more general context, other authors such as Reddien [21], Decker and Kelley [2, 3], Griewank [8], and Griewank and Osborne [9] have analyzed Newton's method in the neighborhood of singularities under certain conditions of regularity. For univariate polynomials, Theorem 3 below explains why it is difficult to obtain an accurate approximation of a cluster of m zeros. In fact, if the Newton method is convergent, the rate of convergence is geometric with limit ratio $\frac{m-1}{m}$. Therefore this ratio is known after many steps of the Newton method. We will say how fast we can compute this quantity. The knowledge of this value of m also permits modification of the Newton method. Schröder in [23] gives an analog of the Newton–Raphson method for multiple roots: Ostrowski in [17, Chap. 8] (see also Rall [20]) studies this process and the asymptotic behaviour close to a multiple root. But the Schröder iteration assumes knowledge of the multiplicity of the zero. For this reason in [17], the author also determines a suitable value of the multiplicity from numerical computations and only mentions the application in the case of cluster of zeros. Ostrowski next performs the numerical computations near the multiple root with an acceleration convergence rule based on the nonmodified Newton method. But this way does not prove the existence of a multiple root. This problem of finding numerically the number of zeros, counting multiplicities, has been studied by Renegar [22], who builds a hybrid algorithm around Newton's method and the Schur–Cohn algorithm. The idea of Renegar is to note there is exactly one zero of the derivative $f^{(m-1)}$ near a cluster of m zeros. Consequently the knowledge of an approximation of this zero will give a good approximation of a cluster. This is done via Newton's method and an approximate zero like Smale [24]. At this step, since the Schur–Cohn algorithm does not solve the problem of the number of zeros, counting multiplicities, Renegar approximates the winding number around the perimeter of a disk computed previously. The use of the argument principle is a natural way to determine the number of zeros of an analytic function, counting multiplicities, in a bounded domain. The difficulty is to control the discretization of closed contour. Authors such as Katz and Ying [13] propose a reliable numerical algorithm to do this. Morgan *et al.* [16] also deal with computations of the winding number in the context of path following methods to solve numerically nonlinear systems of equations. The method of these authors consists in numerically tracking a homotopy path near the singularity. Next, a contour is generated which permits computation of the winding number. These authors also mention the homotopy which will be used in this paper. More recently, Dedieu and Shub [6] deal with simple double zeros of an analytic function f of n variables. The situation of simple double zeros can appear when the rank of the derivative is equal to $n-1$. The idea of these authors is to say that the zeros of another analytic function g sufficiently close to f are badly conditioned. When there is a simple double zero x, this implies the existence of a certain inversible linear operator $A(f, x)$ associated with f and x. Then Dedieu and Shub estimate in terms of $A(f, x)$ the radius of a ball centered in x in which a small perturbation g of f has two zeros, counting multiplicities.

At the moment, the general problem of describing the clusters of m zeros of systems seems difficult and this paper will only investigate the univariate case. This paper will give a numerical answer to the existence of a cluster. For that, we will combine a property of the Newton sequence and a punctual version of Rouché's criterion. This will avoid the computation of the winding number by discretization of a closed contour.

On the other hand, experimental results show that the generic case for zero clusters allows us to consider mainly full zero m-clusters which are open disks containing m zeros of f and $m-1$ zeros of the derivative f'. We will state the convergence results for this class of zero clusters.

The main items which are exploited in this paper are the following:

- The use of a certain homotopy map to compute a point close to the cluster. This is explained in the next section.

- The behaviour of the Newton sequence at the neighbourhood of a zero cluster. As a matter of fact, Theorem 4 will show that the Newton sequence is close to a certain straight line. This provides an open disk susceptible to containing the cluster.

- A new punctual version of Rouché's theorem, see Theorem 2. This result will confirm if the previous disk does or does not contain the number of zeros of f previously predicted.

We have chosen to state the theoretical results using the technical background of the α-theory of Smale, see the book of Blum *et al.* [1]. Particularly, we will extend this theory to clusters of roots of univariate polynomials. We end this introduction by a remark of Renegar [22, p. 98]:

> we remark that terms like "far," "close," and "cluster" are relative to the unit of measurement.... Then the algorithm needs

> to be able to determine the unit of measurement at which a "cluster" it has approximated begins "breaking up" into small "clusters," and then approximate those smaller "clusters."

Our algorithm realizes this task. We also can be interested in the simultaneous computation of all clusters. This problem is studied in another paper by the author in [28].

2. THEORETICAL BACKGROUND AND MAIN RESULTS

In this section we only state the main theorems. The proofs will be given in Section 4. Denote by

$$N_f(x) = x - \frac{f(x)}{f'(x)}$$

the Newton map. Throughout this paper the sequence $(x_k)_{k \geqslant 0}$ is the Newton sequence starting at a point $x_0 \in \mathbb{C}$.

Let $m \geqslant 1$ be an integer and $z \in \mathbb{C}$. Define the two following quantities

$$\beta_m(f, z) = \max_{0 \leqslant k \leqslant m-1} \left| \frac{m!\, f^{(k)}(z)}{k!\, f^{(m)}(z)} \right|^{1/(m-k)},$$

$$\gamma_m(f, z) = \max_{m+1 \leqslant k \leqslant d} \left| \frac{m!\, f^{(k)}(z)}{k!\, f^{(m)}(z)} \right|^{1/(k-m)},$$

which extend the quantities given in [1, p. 156, 159]. Instead of $\beta_1(f, z)$ and $\gamma_1(f, z)$ we use the notations $\beta(f, z)$ and $\gamma(f, z)$. We will also use the polynomial

$$\psi(u) = 1 - 4u + 2u^2,$$

which appears both in the notion of approximate zero and in the point estimates of the α-theory. We give the following definition of an approximate zero of f; see [1, p. 155].

Definition 1. Let w be a sample zero of f. A point x_0 is an approximate zero of f associated with w, if the Newton sequence (x_k) starting at x_0 converges towards w and satisfies

$$|x_k - w| \leqslant \left(\tfrac{1}{2}\right)^{2^k - 1} |x_0 - w|, \qquad k \geqslant 0.$$

Actually, the background of the previous definition is the following γ-theorem; see [1, p. 156]:

THEOREM 1. *Let* $x_0 \in \mathbb{C}$ *and* $u = \gamma(f, w)\, |x_0 - w|$. *Suppose that* $\frac{u}{\psi(u)} < 1$. *Then the Newton sequence* (x_k) *starting at* x_0 *converges towards* w *and satisfies*

$$|x_k - w| \leqslant \left(\frac{u}{\psi(u)}\right)^{2^k - 1} |x_0 - w|, \qquad k \geqslant 0.$$

Moreover, if $u \leqslant (3 - \sqrt{7})/2$ *then* x_0 *is an approximate zero of* f *associated with* w.

We now define the notion of m-cluster.

DEFINITION 2. 1. An m-cluster of f is an open disk $D(z, r)$ which contains in zeros of f, counting multiplicities.

2. A full m-cluster of f is an m-cluster which contains $m-1$ zeros of f', counting multiplicities.

In the next lines, we generalize to the case of zero cluster the notion of approximate zero given above for a simple zero.

DEFINITION 3. Let $D(z, r)$ be an m-cluster of f. For $x_0 \in \mathbb{C}$, define $u_m = \gamma_m(f, z)\, |x_0 - z|$. A point x_0 is an approximate m-cluster of f associated with $D(z, r)$ if

1. there exists a real positive function $\varphi(u)$ such that $\lim_{u \to 0} \varphi(u) = \frac{m-1}{m}$.
2. $\varphi(u_m) < 1$.
3. the Newton sequence $(x_k)_{k \geqslant 0}$ satisfies

$$\begin{cases} |x_k - z| \leqslant \varphi(u_m)^k\, |x_0 - z| & \text{if} \quad m > 1, \\ |x_k - z| \leqslant \varphi(u_m)^{2^k - 1}\, |x_0 - z| & \text{if} \quad m = 1, \end{cases}$$

for all $k \geqslant 0$ such that $x_k \notin D(z, r)$.

The first result that we will state is a punctual version of Rouché's theorem. Many authors have proposed inclusion tests for zero clusters; see

Petković [18, p. 14]. These tests exploit the results given by Marden in [15] or compute the winding number; see Katz–Ying [13], for example. The test given here only uses the evaluation of the polynomial $R_m(f, z, r)$ defined by

$$R_m(f, z, r) = \frac{|f^{(m)}(z)|}{m!} r^m - \sum_{k=0}^{m-1} \frac{|f^{(k)}(z)|}{k!} r^k - \sum_{k=m+1}^{d} \frac{|f^{(k)}(z)|}{k!} r^k,$$

with $z \in \mathbb{C}$ and $r > 0$. We have:

THEOREM 2. *Let $z \in \mathbb{C}$ and $0 < r < \frac{1}{2\gamma_m(f, z)}$ for some $m \geqslant 1$. Assume that the inequality*

$$R_m(f, z, r) > 0$$

holds. Then $D(z, r)$ is an m-cluster of f.

In the case $m = 0$, this test, is an exclusion test, i.e., the disk $D(z, r)$ does not contain any zero of f. It has been used in the localization of an algebraic hypersurface; see Dedieu *et al.* [5] and Dedieu and Yakoubsohn [7].

In terms of the quantities $\beta_m(f, z)$ and $\gamma_m(f, z)$, Theorem 2 becomes:

COROLLARY 1. *Let $D(z, r)$ be an open disk such that $f^{(m)}(z) \neq 0$. Assume that the radius r satisfies $0 < r < 1/2\gamma_m(f, z)$ and that the following inequality*

$$\beta_m(f, z) < \frac{1 - 2\gamma_m(f, z)\, r}{2 - 3\gamma_m(f, z)\, r}$$

holds. Then $D(z, r)$ is an m-cluster of f.

In the following, we are mainly interested in the full m-clusters. This is reached by the following:

COROLLARY 2. *Let $D(z, r)$ be an m-cluster of f such that $R_m(f, z, r) > 0$ and $0 < r < 1/2\gamma_m(f, z)$. Denote $u_m = \gamma_m(f, z)\, r$ and consider x such that $|x - z| = r$. Suppose*

$$\beta_m(f, z) \leqslant \frac{\psi(u_m)}{(1 - u_m)^2 + \psi(u_m)} r, \tag{1}$$

then $D(z, r)$ is a full zero m-cluster.

We now state a γ-theorem for a full zero cluster. For that purpose, we introduce the function

$$\begin{cases} \varphi_m(u) = \dfrac{m-1+\delta u}{m\psi(u)(1-(m-1)/mu)} & \text{if} \quad m \geqslant 2 \\ \varphi_1(u) = \dfrac{(\delta+1)\,u}{\psi(u)}, & \text{if} \quad m=1 \end{cases}$$

with $\delta = 0$ if $f(z)=0$ and $\delta = 1$ if $f(z) \neq 0$.

THEOREM 3. *Let $D(z, r)$ be an m-cluster of f with $f^{(m)}(z) \neq 0$. Let u_m and R be two positive real numbers such that:*

1. $u_m = \gamma_m(f, z)\, R.$
2. $r \leqslant u_m R.$
3. $\varphi_m(u_m) < 1.$
4. $\beta_m(f, z) \leqslant \psi(u_m)/((1-u_m)^2 + \psi(u_m))\, r.$

Suppose also that the set $D_0 = \{x \in D(z, R) : r \leqslant u_m\, |x-z|\}$ is nonempty. Let $x_0 \in D_0$. Then x_0 is an approximate m-cluster of f associated with $D(z, r)$. More precisely for all $k \geqslant 0$ such that $x_k \in D_0$, we have

$$\begin{cases} |x_k - z| \leqslant \varphi_m(u_m)^k\, |x_0 - z| & \text{if} \quad m > 1, \\ |x_k - z| \leqslant \varphi(u_1)^{2^k-1}\, |x_0 - z| & \text{if} \quad m = 1. \end{cases}$$

In Theorem 1, the quadratic convergence holds in the disk of center the root and of radius $R = \frac{u}{\gamma(f, w)}$. In Theorem 3, the radius R of the convergence disk is given by the condition $\varphi_m(u_m) < 1$. If $m = 1$, Theorem 3 generalizes Theorem 1 since z is not a zero of f. Notice that the condition $|x_k - z| \leqslant (\frac{1}{2})^{2^k}\, |x_0 - z|$ holds with $u_1 \leqslant 1/2\delta + 3/2 - 1/2\sqrt{\delta^2 + 6\delta + 7}$.

Under the assumptions of Theorem 3, we now describe the behaviour of the Newton method. We state that the Newton sequence starting at x_0 in the neighborhood of an m-cluster $D(z, r)$ is close to the straight line $[x_0, z]$.

THEOREM 4. *Suppose that the assumptions of Theorem 3 hold with $m > 1$. Let (x_k) be the Newton sequence starting at $x_0 \in D_0$. Introduce the sequence (y_k) defined by:*

$$y_0 = x_0, \qquad y_{k+1} - z = \frac{m-1}{m}(y_k - z), \qquad k \geqslant 0.$$

Then, for all $k \geqslant 0$ *such that* $x_k \in D_0$, *we have*

$$|x_{k+1} - y_{k+1}| \leqslant \frac{(m+\delta)\, u_m}{m^2 \psi(u_m) \left(1 - \frac{m-1}{m} u_m\right) (1 - \varphi_m(u_m))} \varphi_m(u_m)^k R.$$

In Section 3, we describe a practical algorithm for approximating an m-cluster of f. This algorithm will use both Theorem 2 and Theorem 4.

Now the problem is to find an approximate m-cluster. This is reached with the homotopy map defined in Blum *et al.* [1]. This homotopy is first introduced in Smale [25] as *global Newton*, see also Hirsch and Smale [11]. Authors such as Morgan *et al.* [16] use it for practical computations of the singular solutions of nonlinear systems. Theoretical aspects of this global Newton homotopy can be found in Guillemin and Pollack [10]. In this paper, we will state a result of complexity to find an approximate m-cluster. This result generalizes the theorem given in [1, p. 156]. The global Newton homotopy is defined by

$$f_t(x) = f(x) - tf(x_0),$$

where $x_0 \in \mathbb{C}$ is given and $t \in [0, 1]$.

We will suppose that there exists a smooth solution curve w_t on the interval $]0, 1]$, i.e.,

$$\forall t \in \,]0, 1], \qquad f_t(w_t) = 0, \qquad \text{and} \qquad f_t'(w_t) \neq 0.$$

The curve w_t continues at $t = 0$. We will assume w_0 is contained in an m-cluster $D(z, r)$ of f which satisfies the assumptions of Theorem 3. If $m = 1$ then w_t is a smooth curve on $[0, 1]$ and we will denote $z = w_0$. Remember that

$$u_m = \gamma_m(f, z)\, R.$$

Given $x_0 \in \mathbb{C}$, let us now introduce the sequences $t_0 = 1 > t_1 > \cdots > t_k > \cdots > 0$ and

$$z_0 = x_0 \in \mathbb{C}, \qquad z_{k+1} = N_{f_{k+1}}(z_k), \qquad k \geqslant 0,$$

with $f_k = f_{t_k}$. Denote also $w_k = w_{t_k}$. We are estimating an index k_0 which provides z_{k_0} as an approximate m-cluster of f associated with $D(z, r)$. To state the main result, define the following quantities for some $u > 0$:

1. $t^+ \in]0, 1]$ such that $|z - w_{t^+}| = R - \frac{2u}{g}$.
2. $g = \max_{t^+ \leqslant t \leqslant 1} \gamma(f, w_t)$.
3. $b = \max(\max_{t^+ \leqslant t \leqslant 1} |\frac{f(x_0)}{f'(w_t)}|, 1)$, $a = bg$.
4. $M = 1 - \frac{u(1-2u)}{a(1-u)} > 0$.
5. $T(u) = 3u(\psi(u)(1-u) - 3u)/\psi(u)^2\,(1-u)^2\,\psi(3u/\psi(u)(1-u))$.

Then we have:

THEOREM 5. *Under the assumptions of Theorem* 3 *and the notations above, suppose that u satisfies*:

1. $0 < u \leqslant (3 - \sqrt{7})/4$.
2. $T(u) \leqslant \frac{M}{2}$.
3. $r \leqslant (r - \frac{4u}{g})\, u_m$.

Let us consider the sequence (t_k) *defined by* $t_k = M^k$, $k \geqslant 0$. *Let* k_0 *be such that* $t_{k_0 - 1} \geqslant t^+ > t_{k_0}$.

Then the following assertions hold:

1. *The sequence* $z_k = N_{f_k}(z_{k-1})$, $1 \leqslant k \leqslant k_0$, *is well defined and each* z_k *is an approximate zero of* f_k *associated with* w_k.
2. $\beta(f, z_k) \leqslant \frac{2(1-u)^2\, b}{\psi(u)} M^k$, $0 \leqslant k \leqslant k_0 - 1$.
3. *The point* z_{k_0} *is an approximate m-cluster of f associated with* $D(z, r)$. *Obviously*

$$k_0 \leqslant \frac{\log t^+}{\log M} + 1.$$

4. *The value* t^+ *is bounded by*:

$$\frac{\psi(u_m)}{(1-u_m)^2} \frac{|f^{(m)}(z)|}{m!\,|f(x_0)|} \left(R - \frac{2u}{g}\right)^m \leqslant t^+ \leqslant \frac{1}{(1-u_m)^2} \frac{|f^{(m)}(z)|}{m!\,|f(x_0)|} \left(R - \frac{2u}{g}\right)^m.$$

In the case $m = 1$ the curve w_t is a smooth curve on the interval $[0, 1]$, Theorem 5 becomes

THEOREM 6. *Let us now define* $g = \max_{0 \leqslant t \leqslant 1} \gamma(f, w_t)$, $b = \max(\max_{0 \leqslant t \leqslant 1} |f(x_0)/f'(w_t)|, 1)$, *and* $a = bg$. *Suppose that*:

1. $0 \leqslant u \leqslant (3 - \sqrt{7})/6$.
2. $T(u) \leqslant \frac{M}{2}$.

Let us consider the sequence $t_k = M^k$, $k \geqslant 0$ *and define* t^+ *such that* $|z - w_{t^+}| \leqslant (3 - \sqrt{7})/6$. *Let* k_0 *be such that* $t_{k_0 - 1} \geqslant t^+ > t_{k_0}$.

Then the following assertions hold:

1. *The sequence* $z_k = N_{f_k}(z_{k-1})$, $1 \leqslant k$, *is well defined and each* z_k *is an approximate zero of* f_k *associated with* w_k.

2. $\beta(f, z_k) \leqslant (2(1-u)^2\, b/\psi(u))\, M^k$, $0 \leqslant k$.

3. *The point* z_{k_0} *is an approximate zero of* f *associated with* z. *Obviously*

$$k_0 \leqslant \frac{\log t^+}{\log M} + 1.$$

4. *The value* t^+ *is bounded by*:

$$\frac{2\sqrt{7}-21}{3}\,\frac{|f'(z)|}{g\,|f(x_0)|} \leqslant t^+ \leqslant (8-3\sqrt{7})\,\frac{|f'(z)|}{g\,|f(x_0)|}.$$

3. ALGORITHMS AND NUMERICAL EXPERIMENTS

3.1. *Practical Fast Computation of m-Clusters*

We are combining three results.

1. From Theorem 3, we know that the behaviour of the Newton sequence in the neighbourhood of a zero cluster gives the number of zeros of this cluster, counting multiplicities. More precisely from x_0, we compute the Newton iterates x_1 and x_2. Next we determine the integer m which minimizes the quantity

$$\left| \frac{|x_2 - x_1|}{|x_1 - x_0|} - \frac{m-1}{m} \right|.$$

2. From Theorem 4, we know that the direction of the zero cluster can be found from some point by the Newton method. We are then able to predict the existence of an m-cluster. More precisely, we compute the point z be such that $z - x_2 = \frac{m-1}{m}(z - x_1)$, i.e., $z = mx_2 - (m-1)\,x_1$, and the value $r = 1/2\gamma_m(f, z)$. The disk $D(z, r)$ provides a probable m-cluster $D(z, r)$.

3. Finally, Theorem 2 can be used to decide if the open disk calculated previously in an m-cluster. that is described in the algorithm below.

m-cluster Algorithm.
Inputs: a polynomial f, a point $x_0 \in \mathbb{C}$, an integer $n_{it} \geqslant 2$.
Compute $x_1 = N_f(x_0)$, $k = 1$.
while $k \leqslant n_{it}$ **do**
  Compute $x_2 = N_f(x_1)$
  Determine the integer m which minimizes the quantity $\left|\frac{|x_2 - x_1|}{|x_1 - x_0|} - \frac{m-1}{m}\right|$.
  Compute $z = mx_2 - (m-1)\, x_1$, $r = \frac{1}{2\gamma_m(f, z)}$,
  $x_0 = x_1$, $x_1 = x_2$,
  $k = k + 1$
end
if $R_m(z, r) > 0$ **then Output** "$D(z, r)$ is an m-cluster" **else Output** "$D(z, r)$ is not an m-cluster".

In the case where $D(z, r)$ is an m-cluster, we will say point x_0 provides an m-cluster.

3.2. *Numerical Experiments of m-cluster Algorithm*

The numerical experiments are performed with Matlab. We first show how the m-cluster algorithm works numerically. Let us consider the polynomial of degree 24:

$$\begin{aligned} f(x) = {} & (-0.0043 + 0.0095i)\, x^{24} + (-0.0771 + 00092i)\, x^{23} \\ & + (-0.1022 - 0.2038i)\, x^{22} + (0.1469 + 0.0528i)\, x^{21} \\ & + (-0.12760 + 0.850i)\, x^{20} \\ & + (-0.1038 - 2.3716i)\, x^{19} + (1.5977 + 0.1609i)\, x^{18} \\ & + (-0.6833 + 0.0160i)\, x^{17} + (-1.2528 - 1.2595i)\, x^{16} \\ & + (1.3196 + 0.3469i)\, x^{15} \\ & + (-1.4812 + 0.0969i)\, x^{14} + (-1.2981 - 0.6038i)\, x^{13} \\ & + (-0.5567 + 0.5488i)\, x^{12} + (-0.7638 - 0.6068i)\, x^{11} \\ & + (-0.5337 - 0.4766i)\, x^{10} \\ & + (-0.4883 + 0.4865i)\, x^{9} + (-1.2791 - 1.3822i)\, x^{8} \\ & + (0.7608 + 1.1392i)\, x^{7} + (-2.6292 - 1.1942i)\, x^{6} \\ & + (1.801 - 0.1687i)\, x^{5} \\ & + (-0.0016 - 0.0005i)\, x^{4} + (0.0000136 + 0.0000045i)\, x^{3} \\ & + (0.0000000056 - 0.00000000108i)\, x^{2} \\ & + (1.3 \times 10^{-11} - 2.9 \times 10^{-11} i)\, x \\ & - 4.0 \times 10^{-14} + 1.0 \times 10^{-14} i. \end{aligned}$$

This polynomial has a 5-cluster $D(0.3\times 10^{-3})$. The five roots given by Matlab are

$$(1.48-2.11i)\,\eta,\qquad (-1.67-0.88)\,\eta,\qquad (-1.22+1.74i)\,\eta,$$
$$(1.54+0.44i)\,\eta,\qquad (0.73+1.16i)\,\eta,$$

with $\eta=10^{-3}$.

The m-cluster algorithm is initialized at $x_0=-0.6+0.5i$. Figure 1 shows the Newton sequence (x_k). We see that the x_k's are close to a straight line. When $k=10$, we find that the value $m=5$ minimizes the quantity $||x_{k+1}-x_k|/|x_k-x_{k-1}|-m-1/m|$. Moreover, $R_5((-3+5i)\,\eta, 0.322)>0$. Consequently the point x_{10} provides an 5-cluster of f. Figure 2 shows the points $x_5, ..., x_{10}$ and the m-cluster $D(-3+5i)\,\eta, 0.322)$. Table 1 gives the corresponding numerical results.

Remark 1. When the m-cluster algorithm finds a cluster of zeros, we know the number of zeros in the cluster. The radius given in this algorithm is equal to $r=1/2\gamma_m(f, z)$. If we are interested in the knowledge of all roots of the cluster with a precision ε it is sufficient to replace $r_k=1/2\gamma_m(f, z_k)$ by $r_k=\varepsilon$ in the m-cluster algorithm.

For example, if the m-cluster algorithm works with $r=3\times 10^{-3}$ with the same inputs as in Table 1, we find $D(-3\times 10^{-7}+3\times 10^{-4}\,i, 0.003)$ to be a 5-cluster at the iteration 17.

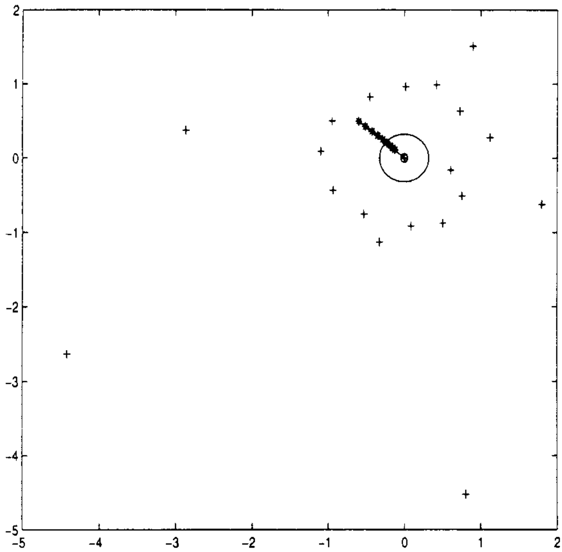

FIG. 1. Behaviour of the Newton method at the neighborhood of the 5-cluster. Roots, +; Newton's iterates, *.

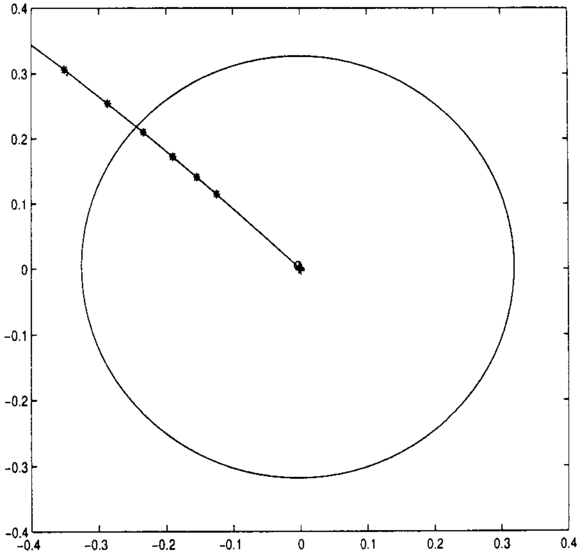

FIG. 2. Enlargement of the Fig. 1 with the 5-cluster. Roots, +; Newton's iterates, *.

On the other hand, if we restart the m-cluster algorithm with the previous point $z = -3 \times 10^{-7} + 3 \times 10^{-4}i$, the process converges towards a simple root of f. The value after five iterations is $0.00073309350863 + 0.00116487319846i$ to compare with $0.00073309350870 + 0.00116487319850i$, the value given by Matlab.

TABLE 1

Numerical Behaviour of the Newton Sequence near a 5-Cluster

k	m	x_k	z_k	r_k	$R_m(z_k, r_k)$
1		$-0.600 + 0.500i$			
1		$-0.512 + 0.430i$			
2	23	$-0.427 + 0.365i$	$1.462 - 1.058i$	21.672	$-3e30$
3	9	$-0.351 + 0.306i$	$0.255 - 0.163i$	0.133	-0.013
4	7	$-0.287 + 0.255i$	$0.097 - 0.055i$	0.151	-0.001
5	6	$-0.234 + 0.210i$	$0.032 - 0.012i$	0.667	-0.205
6	6	$-0.190 + 0.173i$	$0.029 - 0.016i$	0.663	-0.198
7	6	$-0.154 + 0.141i$	$0.026 - 0.017i$	0.673	-0.223
8	6	$-0.124 + 0.115i$	$0.023 - 0.016i$	0.675	-0.228
9	6	$-0.100 + 0.093i$	$0.020 - 0.015i$	0.663	-0.198
10	5	$-0.081 + 0.075i$	$-0.003 + 0.005i$	0.322	0.002
11	5	$-0.065 + 0.061i$	$-0.002 + 0.003i$	0.319	0.002
12	5	$-0.052 + 0.049i$	$-0.001 + 0.002i$	0.317	0.002

3.3. *Computing m-cluster using Global Newton Path Following Algorithm*

Obviously, any x_0 does not permit the computation of an approximate zero if f associated with a cluster. Here, we adapt the SNPF algorithm studied in Yakoubsohn [27]. This algorithm is a predictor–corrector method. For that, we need three ingredients:

1. The global Newton homotopy $f_t(x) = f(x) - tf(x_0)$ where $x_0 \in \mathbb{C}$ is given.

2. The Newton method which is used to correct numerically the point computed at the previous step. More precisely, if z_{k-1} is the point at the step $k-1$ corresponding to the value t_{k-1}, the point z_k is obtained after n_{it} iterations of the Newton method starting at z_{k-1} applied to f_k:

$$z_0 = x_0, \qquad y_0 = z_{k-1}, \qquad y_i = N_{f_k}(y_{i-1}),$$

$$1 \leqslant i \leqslant n_{it}, \qquad z_k = y_{n_{it}}, \qquad k \geqslant 0.$$

Let us denote $\beta_k = |f_k(z_k)/f'(z_k)|$.

3. The subdivision of the interval $[0, 1]$ in connection with the value of β and the m-cluster algorithm:

3.1. If $\beta_k > \varepsilon$ then perform the m-cluster algorithm described above. If z_k is an approximate m-cluster then the algorithm stops and we have computed an m-cluster $D(z, r)$. Otherwise, the value of t_{k+1} is given by $t_{k+1} = (t_k + t_{k-1})/2$.

3.2. If $\beta_k \leqslant \varepsilon$, and $t_k > 0$ then $t_{k+1} = \max(t_k - 2(t_{k-1} - t_k), 0)$.

3.3. If $\beta_k \leqslant \varepsilon$, $t_k = 0$ and $R_1(f, z_k, 1/2\gamma_1(f, z_k) > 0$ then the disk $D(z_k, 1/2\gamma_1(f, z_k))$ contains only one root and the algorithm stops. Otherwise, the value of t_{k+1} is given by $t_{k+1} = (t_k + t_{k-1})/2$.

The result of the algorithm is an m-cluster $D(z, r)$. A formal description of this algorithm is given below.

Global newton Path Following Algorithm.

Inputs: f a polynomial of degree d, $x_0 \in \mathbb{C}$, ε a positive real number, $n_{it} \geqslant 2$ an integer.
$\beta = 2\varepsilon, \qquad t_0 = 1, \qquad t_1 = 1 - \varepsilon, \qquad z_0 = x_0$
while $\beta > \varepsilon$ **or** $t_0 \geqslant 0$

Compute the points $y_0 = z_0$, $y_i = N_{f_{t_1}}(y_{i-1})$, $1 \leqslant i \leqslant n_{it}$, $z_1 = y_{n_{it}}$.
$\beta = |f_{t_1}(z_1)/f'(z_1)|$
if $\beta > \varepsilon$
Compute $D(z, r) =$ **m-cluster**(f, z_1, n_{it})
if $R_m(f, z, r) > 0$ return the m-cluster $D(z, r)$ and stop
else replace t_1 by $(t_1 + t_0)/2$.
If $\beta \leqslant \varepsilon$ **and** $t_1 = 0$
$r = 1/2\gamma(f, z_1)$.
if $R_1(f, z_1, r) > 0$ then $z = z_1$, return the 1-cluster $D(z, r)$ and stop
else replace t_1 by $(t_1 + t_0)/2$.
If $\beta \leqslant \varepsilon$ **and** $t_1 > 0$ replace t_0 by t_1, t_1 by $\max(3t_1 - 2t_0, 0)$ and z_0 by z_1.
end
Output: m-cluster $D(z, r)$

We then have the following

PROPOSITION 1. *Let f be a polynomial which has p clusters of zeros. For all m-cluster $D(z, r)$, $m \geqslant 1$, let us define R such that for all $y \in D(z, R)$, y provides an m-cluster containing the zeros of $D(z, r)$. For all $k \geqslant 0$, suppose the inequality $\beta_k \leqslant \varepsilon$ implies the existence of a unique root w_k of f_k such that $\gamma(f_k, w_k)\,|w_k - z_k| < 1$. If $\varepsilon \leqslant (1 - \sqrt{6}/3)\,R/2$ then the global Newton Path Following algorithm stops.*

The proof of this proposition is given in Section 4.6.

Consider one more the polynomial of the numerical example given in 3.2. We start the algorithm with $x_0 = -3.5 - 3.1i$ and $n_{it} = 3$. There are 106 iterations in the algorithm. We find five zeros counting multiplicities in the disk $D(0.0107 - 0.013i, 0.2828)$ provided by the point $-0.1701 - 0.2331i$. Figure 3 shows the numerical path-following by the iterates and the disk $D(z, r)$.

An application of this algorithm is to determine the basins of attraction of each root or zero-cluster relative to the global Newton homotopy. For each point

$$x_{jk} = -5 + hj + (-5.5 + hk)\, i, \qquad 0 \leqslant j, k \leqslant 160, \quad h = 1/20$$

which belongs to the square $[-5, 3] \times [-5.5, 2.5]$, we determine the zero provided by this initial point.

We next attribute different colors at each point following the result obtained. For example, we decide the red color when the initial point goes back to the 5-cluster. Each root of the polynomial is displayed in Fig. 4 with an "*".

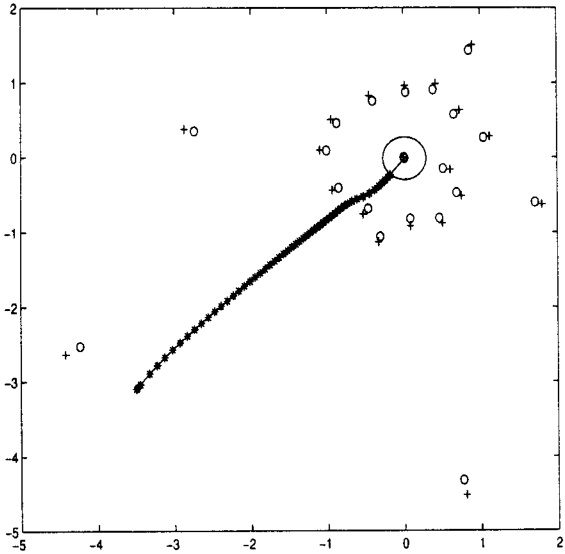

FIG. 3. Global Newton path following, $n_{it} = 3$. Roots of f, $+$; roots of f', $\circ$; Newton's iterates, *.

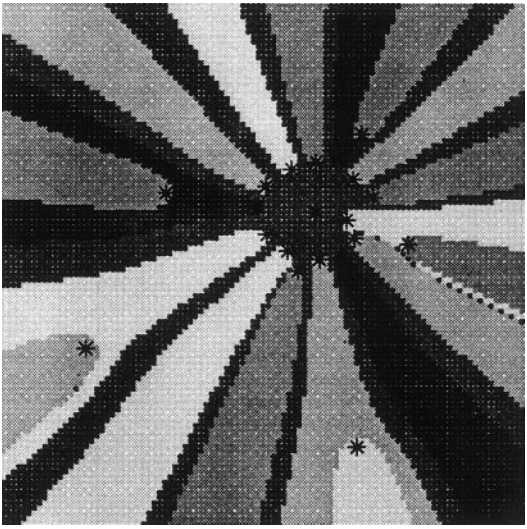

FIG. 4. Dynamic of the Newton homotopy.

4. PROOFS

4.1. *Proof of Theorem* 2

First, we state a result of separation of roots which extends those given by Dedieu in [4] and Dedieu and Shub in [6].

LEMMA 1. *Let z be a zero of f of multiplicities m exactly. For another zero $w \neq z$ we have*:

$$|w-z| \geqslant \frac{1}{2\gamma_m(f,z)}.$$

Proof. From Taylor's formula it follows:

$$0 = f(w) = \frac{f^{(m)}(z)}{m!}(w-z)^m + \sum_{k \geqslant m+1} \frac{f^{(k)}(z)}{k!}(w-z)^k.$$

By the triangle inequality, we get:

$$0 \geqslant |w-z|^m \left(\frac{|f^{(m)}(z)|}{m!} - \sum_{k \geqslant m+1} \frac{|f^{(k)}(z)|}{k!} |w-z|^{k-m} \right).$$

Hence, using the definition of $\gamma_m(f,z)$, it follows

$$0 \geqslant 1 - \sum_{k \geqslant m+1} (\gamma_m(f,z)\,|w-z|)^{k-m}$$

$$0 \geqslant \frac{1-2\gamma_m(f,z)\,|w-z|}{1-\gamma_m(f,z)\,|w-z|}.$$

The lemma follows. ∎

Proof of Theorem 2. Let us consider the polynomial $g(x) = (x-z)^m \times (\sum_{k \geqslant m} \frac{f^{(k)}(z)}{k!}(w-z)^{k-m})$. If $w \neq z$ is another zero if g, we have from Lemma 1

$$|w-z| \geqslant \frac{1}{2\gamma_m(g,z)} = \frac{1}{2\gamma_m(f,z)}.$$

On the other hand, if the inequality

$$|f(x) - g(x)| < |g(x)|$$

holds for all x such that $|x-z|=r$, Rouché's theorem asserts that the polynomials f and g have the same the number of zeros in the disk $D(z,r)$, counting multiplicities. Therefore, if $r<1/2\gamma_m(f,z)$ then the polynomial f has m roots in $D(z,r)$, counting multiplicities.

Now, it is easy to show for $|x-z|=r$:

$$|f(x)-g(x)| \leqslant \sum_{k=0}^{m-1} \frac{|f^{(k)}(z)|}{k!} r^k,$$

and

$$\left(\frac{|f^{(m)}(z)|}{m!} - \sum_{k=m+1}^{d} \frac{|f^{(k)}(z)|}{k!} r^{k-m}\right) r^m \leqslant |g(x)|.$$

Therefore, the inequality $R_m(f,z,r)>0$ implies Rouché's theorem and the disk $D(z,r)$ is an m-cluster. ∎

Proof of Corollary 1. We prove $\beta_m(f,s)<((1-2\gamma_m(f,z)\,r)/(2-3\gamma_m(f,z)\,r))\,r$ implies $R_m(f,z,r)>0$. Let $|x-z|=r$. Using the definition of $\beta_m(f,z)$, we bound the following sum:

$$\sum_{k=0}^{m-1} \frac{|f^{(k)}(z)|}{k!} r^k \leqslant \sum_{k=0}^{m-1} \left(\frac{\beta_m(f,z)}{r}\right)^{m-k} \frac{|f^{(m)}(z)|}{m!} r^m$$

$$\leqslant \frac{\dfrac{\beta_m(f,z)}{r}}{1-\dfrac{\beta_m(f,z)}{r}} \frac{|f^{(m)}(z)|}{m!} r^m.$$

The inequality $\beta_m(f,z)<(1-2\gamma_m(f,z)\,r/2-3\gamma_m(f,z)\,r)\,r$ implies

$$\frac{\dfrac{\beta_m(f,z)}{r}}{1-\dfrac{\beta_m(f,z)}{r}} < \frac{1-2\gamma_m(f,z)\,r}{1-\gamma_m(f,z)\,r}.$$

Hence

$$\sum_{k=0}^{m-1} \frac{|f^{(k)}(z)|}{k!} r^k < \frac{1-2\gamma_m(f,z)\,r}{1-\gamma_m(f,z)\,r} \frac{|f^{(m)}(z)|}{m!} r^m.$$

On the other hand we have:

$$\sum_{k=m+1}^{d} \frac{|f^{(k)}(z)|}{k!} r^k \leqslant \sum_{k=m+1}^{d} (\gamma_m(f, z)\, r)^{k-m} \frac{|f^{(m)}(z)|}{m!} r^m$$
$$\leqslant \frac{\gamma_m(f, z)\, r}{1-\gamma_m(f, z)\, r} \frac{|f^{(m)}(z)|}{m!} r^m.$$

Moreover, $f^{(m)}(z) \neq 0$. Then we have

$$R_m(f, z, r) > \left(1 - \frac{1-2\gamma_m(f, z)\, r}{1-\gamma_m(f, z)\, r} - \frac{\gamma_m(f, z)\, r}{1-\gamma_m(f, z)\, r}\right) \frac{|f^{(m)}(z)|}{m!} r^m = 0.$$

We are done. ∎

Proof of Corollary 2. Since $R_m(f, z, r) > 0$, we have $f^{(m)}(z) \neq 0$. We first observe that

$$\gamma_m(f, z) < \max_{m \leqslant k \leqslant d-1} \left(\frac{k+1}{m} \frac{m!\, f^{(k+1)}(z)}{(k+1)!\, f^{(m)}(z)}\right)^{1/(k-m+1)} = \gamma_{m-1}(f', z).$$

Hence $r < 1/2\gamma_m(f, z) < 1/2\gamma_{m-1}(f', z)$.

The proof consists to show that $\beta_m(f, z) \leqslant \psi(u_m)/((1-u_m)^2 + \psi(u_m))\, r$ implies $R_{m-1}(f', z, r) > 0$. Remember $u_m = \gamma_m(f, z)\, |x-z|$ with $|x-z| = r$. We have

$$R_{m-1}(f', z, r) = \frac{m\, |f^{(m)}(z)|}{m!} |x-z|^{m-1} - \sum_{k=1}^{m-1} k \frac{|f^{(k)}(z)|}{k!} |x-z|^{k-1}$$
$$- \sum_{k=m+1}^{d} k \frac{|f^{(k)}(z)|}{k!} |x-z|^{k-1}.$$

First, we bound

$$\sum_{k=1}^{m-1} k \frac{|f^{(k)}(z)|}{k!} |x-z|^{k-1} \leqslant \sum_{k=1}^{m-1} k \left(\frac{\beta_m(f, z)}{|x-z|}\right)^{m-k} \frac{|f^{(m)}(z)|}{m!} |x-z|^{m-1}$$
$$\leqslant (m-1) \frac{\dfrac{\beta_m(f, z)}{|x-z|}}{1 - \dfrac{\beta_m(f, z)}{|x-z|}} \frac{|f^{(m)}(z)|}{m!} |x-z|^{m-1}.$$

Condition (1) implies

$$\frac{\dfrac{\beta_m(f, z)}{|x-z|}}{1-\dfrac{\beta_m(f, z)}{|x-z|}} \leqslant \frac{\psi(u_m)}{(1-u_m)^2}.$$

Therefore

$$\sum_{k=1}^{m-1} k \frac{|f^{(k)}(z)|}{k!} |x-z|^{k-1} \leqslant (m-1) \frac{\psi(u_m)}{(1-u_m)^2} \frac{|f^{(m)}(z)|}{m!} |x-z|^{m-1}.$$

We now find an upper bound for the quantity

$$\begin{aligned} \sum_{k=m+1}^{d} & k \frac{|f^{(k)}(z)|}{k!} |x-z|^{k-1} \\ & \leqslant \sum_{k=m+1}^{d} k(\gamma_m(f, z) \, |x-z|)^{k-m} \frac{|f^{(m)}(z)|}{m!} |x-z|^{m-1} \\ & \leqslant \left(\sum_{k\geqslant 1} (k+m) \, u_m^k \right) \frac{|f^{(m)}(z)|}{m!} |x-z|^{m-1} \\ & \leqslant \left(\frac{u_m}{(1-u_m)^2} + \frac{m u_m}{1-u_m} \right) \frac{|f^{(m)}(z)|}{m!} |x-z|^{m-1}. \end{aligned}$$

Therefore

$$\begin{aligned} & R_{m-1}(f', z, r) \\ & \quad \geqslant \left(m-(m-1) \frac{\psi(u_m)}{(1-u_m)^2} - \frac{u_m}{(1-u_m)^2} - \frac{m u_m}{1-u_m} \right) \frac{|f^{(m)}(z)|}{m!} |x-z|^{m-1} \\ & \quad \geqslant \frac{(m-1) \, u_m + \psi(u_m)}{(1-u_m)^2} \frac{|f^{(m)}(z)|}{m!} |x-z|^{m-1}. \end{aligned}$$

For $u_m < 1-\sqrt{2}/2$ we have $\psi(u_m) < 0$. Hence, $R_{m-1}(f', z, |x-z|) > 0$. We are done. ∎

4.2. *Proof of Theorem* 3

We first need to state the following point estimate. Denote by $D(z, r)$ an m-cluster of f.

LEMMA 2. *Denote* $u_m = \gamma_m(f, z)\,|x-z|$. *Suppose* $u_m < 1-\sqrt{2}/2$ *and* $\beta_m(f, z) \leqslant \psi(u_m)/((1-u_m)^2+\psi(u_m))\,r$. *Consider* x *such that* $r \leqslant u_m\,|x-z|$.

1. *Then the derivative* $f'(x)$ *is nonzero. More precisely*:

$$\frac{m!\,|f'(x)|}{|f^m(z)|\,|x-z|^{m-1}} \geqslant \frac{m\psi(u_m)}{(1-u_m)^2}\left(1-\frac{m-1}{m}\,u_m\right).$$

2. *Hence* $N_f(x)$ *is well defined and satisfies*:

$$|N_f(x)-z| \leqslant \begin{cases} \dfrac{m-1+\delta u_m}{m\psi(u_m)(1-(m-1/m)\,u_m)}\,|x-z| & \textit{if} \quad m>1 \\ \dfrac{(\delta+1)\,u_m}{\psi(u_m)}\,|x-z| & \textit{if} \quad m=1, \end{cases}$$

with $\delta=0$ *if* $f(z)=0$ *and* $\delta=1$ *if* $f(z)\neq 0$.

Proof. From Corollary 2 the polynomial f' has $m-1$ zeros in the disk $D(z, r)$. Moreover, for $x \in D_0 = \{y \in D(z, R) : r \leqslant u_m\,|y-z|\}$, we get from Taylor's formula and triangle inequality:

$$\begin{aligned} &\frac{m!\,|f'(x)|}{|f^m(z)|\,|x-z|^{m-1}} \\ &\quad \geqslant m - \sum_{k\geqslant m+1} k\left|\frac{m!\,f^{(k)}(z)}{k!\,f^{(m)}(z)}\right|\,|x-z|^{k-m} - \sum_{k=1}^{m-1} k\left|\frac{m!\,f^{(k)}(z)}{k!\,f^{(m)}(z)}\right|\,|x-z|^{k-m} \\ &\quad \geqslant m - \sum_{k\geqslant m+1} k(\gamma_m(f, z)\,|x-z|)^{k-m} - \sum_{k=1}^{m-1} k\left(\frac{\beta_m(f, z)}{|x-z|}\right)^{m-k}. \end{aligned}$$

According to

$$\beta_m(f, z) \leqslant \frac{\psi(u_m)}{(1-u_m)^2+\psi(u_m)}\,r$$

and $r \leqslant u_m\,|x-z|$, we bound the previous sums as in the proof of Corollary 2. We obtain:

$$\begin{aligned} \frac{m!\,|f'(x)|}{|f^m(z)|\,|x-z|^{m-1}} &\geqslant m - \frac{u_m}{(1-u_m)^2} - \frac{mu_m}{1-u_m} - \frac{(m-1)\,u_m\psi(u_m)}{(1-u_m)^2} \\ &\geqslant \frac{m-(3m+1)\,u_m+2mu_m^2}{(1-u_m)^2} - \frac{(m-1)\,u_m\psi(u_m)}{(1-u_m)^2}. \end{aligned}$$

Using the inequality $m-(3m+1)\,u_m+2mu_m^2>m\psi(u_m)$, we can bound:

$$\frac{m!\,|f'(x)|}{|f^m(z)|\,|x-z|^{m-1}}\geqslant\frac{m\psi(u_m)}{(1-u_m)^2}\left(1-\frac{m-1}{m}\,u_m\right).$$

Part 1 of the lemma holds. We now prove part 2. the assumptions of Corollary 2 are satisfied. We can bound

$$\frac{\dfrac{\beta_m(f,z)}{|x-z|}}{1-\dfrac{\beta_m(f,z)}{|x-z|}}$$

by $\psi(u_m)\,u/(1-u_m)^2$. Then, we have

$$\begin{aligned}
\left|\frac{m!(f'(x)(x-z)-f(x))}{f^{(m)}(z)(x-z)^m}\right| &\leqslant \left|\frac{m!\,f(z)}{f^{(m)}(z)(x-z)^m}\right| \\
&\quad+\sum_{k=2}^{m-1}(k-1)\left(\frac{\beta_m(f,z)}{|x-z|}\right)^{m-k}+m-1 \\
&\quad+\sum_{k\geqslant m+1}(k-1)(\gamma_m(f,z)\,|x-z|)^{k-m} \\
&\leqslant \left|\frac{m!\,f(z)}{f^{(m)}(z)(x-z)^m}\right|+(m-2)_+\frac{\dfrac{\beta_m(f,z)}{|x-z|}}{1-\dfrac{\beta_m(f,z)}{|x-z|}} \\
&\quad+m-1+\frac{u_m}{(1-u_m)^2}+\frac{(m-1)\,u_m}{1-u_m} \\
&\leqslant \left|\frac{m!\,f(z)}{f^{(m)}(z)(x-z)^m}\right|+\frac{(m-2)_+\psi(u_m)\,u_m}{(1-u_m)^2} \\
&\quad+m-1+\frac{u_m}{(1-u_m)^2}+\frac{(m-1)\,u_m}{1-u_m},
\end{aligned}$$

with $(m-2)_+=0$ if $m\leqslant 2$ and $(m-2)_+=m-2$ if $m\geqslant 2$. If $f(z)=0$ and $m>1$, a straightforward computation gives

$$\left|\frac{m!(f'(x)(x-z)-f(x))}{f^{(m)}(z)(x-z)^m}\right| \leqslant \frac{m-1-2u_m^2(m-2)(2-u_m)}{(1-u_m)^2}$$

$$\leqslant \frac{m-1}{(1-u_m)^2}.$$

If $f(z)=0$ and $m=1$, it is obvious that $|m!(f'(x)(x-z)-f(x))/f^{(m)}(z)(x-z)^m| \leqslant u_m/(1-u_m)^2$. Hence

$$\left|\frac{m!(f'(x)(x-z)-f(x))}{f^{(m)}(z)(x-z)^m}\right| \leqslant \begin{cases} \dfrac{m-1}{(1-u_m)^2}, & \text{if} \quad m>1, \\ \dfrac{u_m}{(1-u_m)^2}, & \text{if} \quad m=1. \end{cases}$$

On the other hand, if f(z) $\neq 0$ and $m>1$, we have in a similar way with $|m!\ f(z)/f^{(m)}(z)(x-z)^m| \leqslant \psi(u_m)\, u_m/(1-u_m)^2$:

$$\left|\frac{m!(f'(x)(x-z)-f(x))}{f^{(m)}(z)(x-z)^m}\right| \leqslant \frac{\psi(u_m)\, u_m}{(1-u_m)^2} + \frac{(m-2)\,\psi(u_m)\, u_m}{(1-u_m)^2} + m - 1$$

$$+ \frac{u_m}{(1-u_m)^2} + \frac{(m-1)\, u_m}{1-u_m}$$

$$\leqslant \frac{m-1+u_m(1-4(m-1)\, u_m + 2(m-1)\, u_m^2)}{(1-u_m)^2}$$

$$\leqslant \frac{m-1+u_m\psi(u_m)}{(1-u_m)^2} \leqslant \frac{m-1+u_m}{(1-u_m)^2}.$$

The previous inequality holds because $1-4(m-1)\, u_m + 2(m-1)\, u_m^2 \leqslant \psi(u_m) < 1$. The case $f(z) \neq 0$ and $m=1$ is easily bounded. Summarizing the case $f(z) \neq 0$, we get

$$\left|\frac{m!(f'(x)(x-z)-f(x))}{f^{(m)}(z)(x-z)^m}\right| \leqslant \begin{cases} \dfrac{m-1+u_m}{(1-u_m)^2}, & \text{if} \quad m>1, \\ \dfrac{2u_m}{(1-u_m)^2}, & \text{if} \quad m=1. \end{cases}$$

Now using part 1 of this lemma, we obtain:

$$
\begin{aligned}
|N_f(x)-z| &= \frac{|f'(x)(x-z)-f(x)|}{|f'(x)|} \\
&\leqslant \left|\frac{m!(f'(x)(x-z)-f(x))}{f^{(m)}(z)(x-z)^m}\right| \left|\frac{f^{(m)}(z)(x-z)^{m-1}}{f'(z)}\right| |x-z| \\
&\leqslant \begin{cases} \dfrac{m-1+\delta u_m}{m\psi(u_m)(1-(m-1)/mu_m)}\,|x-z|, & \text{if} \quad m<1, \\ \dfrac{(\delta+1)\,u_m}{\psi(u_m)}\,|x-z|, & \text{if} \quad m=1, \end{cases}
\end{aligned}
$$

with $\delta=0$ if $f(z)=0$ and $\delta=1$ if $f(z)\neq 0$. The lemma is proved. ∎

Proof of Theorem 3. Using Lemma 2, we prove it by induction for all $x_k \in D_0$. The case $k=0$ is obvious. Suppose $x_k \in D_0$ is well defined and satisfies

$$
\begin{cases} |x_k-z| \leqslant \varphi_m(u_m)^k\,|x_0-z| & \text{if} \quad m>1, \\ |x_k-z| \leqslant \varphi(u_1)^{2^k-1}\,|x_0-z| & \text{if} \quad m=1. \end{cases}
$$

The assumptions of Lemma 2 are satisfied. Consequently x_{k+1} is well defined and verifies the inequalities

$$
\begin{aligned}
|x_{k+1}-z| &\leqslant \begin{cases} \dfrac{m-1+\delta u_{mk}}{m\psi(u_{mk})\left(1-\dfrac{m-1}{m}u_{mk}\right)}\,|x_k-z|, & \text{if} \quad m>1, \\ \dfrac{(\delta+1)\,u_{mk}}{\psi(u_{mk})}\,|x_k-z|, & \text{if} \quad m=1, \end{cases} \\
&\leqslant \begin{cases} \dfrac{m-1+\delta u_m}{m\psi(u_m)\left(1-\dfrac{m-1}{m}u_m\right)}\,|x_k-z|, & \text{if} \quad m>1, \\ \dfrac{(\delta+1)\,\gamma_m(f,z)}{\psi(u_m)}\,|x_k-z|^2, & \text{if} \quad m=1, \end{cases}
\end{aligned}
$$

with $u_{mk} = \gamma_m(f, z)\, |x_k - z| < u_m$. Applying the inductive hypothesis we get

$$|x_{k+1} - z| \leqslant \begin{cases} \left(\dfrac{m-1+\delta u_m}{m\psi(u_m)\left(1 - \dfrac{m-1}{m}\, u_m\right)} \right)^{k+1} |x_0 - z|, & \text{if} \quad m > 1, \\ \dfrac{(\delta+1)\, \gamma_m(f, z)}{\psi(u_m)} \left(\dfrac{(\delta+1)\, u_m}{\psi(u_m)} \right)^{2^{k+1}-2} |x_0 - z|^2, & \text{if} \quad m = 1, \end{cases}$$

$$\leqslant \begin{cases} \left(\dfrac{m-1+\delta u_m}{m\psi(u_m)\left(1 - \dfrac{m-1}{m}\, u_m\right)} \right) |x_k - z|, & \text{if} \quad m > 1, \\ \left(\dfrac{(\delta+1)\, u_m}{\psi(u_m)} \right)^{2^{k+1}-1} |x_0 - z|, & \text{if} \quad m = 1. \end{cases}$$

This proves the theorem. ▌

4.3. *Proof of Theorem* 4

We need the following lemma

LEMMA 3. *let* $\bar{x} = z + \frac{m-1}{m}(x - z)$. *Under the assumptions of Lemma* 2 *with* $m > 1$, *we have*

$$|N_f(x) - \bar{x}| \leqslant \frac{(m+\delta)\, u_m}{m^2 \psi(u_m)\left(1 - \dfrac{m-1}{m}\, u_m\right)}\, |x - z|,$$

where $u_m = \gamma_m(f, z)\, |x - z|$.

Proof. A straightforward computation shows that

$$N_f(x) - \bar{x} = \frac{f'(x)(x-z) - mf(x)}{mf'(x)}.$$

From Taylor's formula we have:

$$|f'(x)(x-z) - mf(x)|$$
$$= \left| \sum_{k=0}^{m-1} (k-m)\, \frac{f^{(k)}(z)}{k!}(x-z)^k + \sum_{k \geqslant m+1} (k-m)\, \frac{f^{(k)}(z)}{k!}\, (x-z)^k \right|.$$

As in the proof of Lemma 2, we bound the quantity

$$\begin{aligned}
&\sum_{k=0}^{m-1} (m-k) \left| \frac{m!\, f^{(k)}(z)}{k!\, f^{(m)}(z)} \right| |x-z|^k \\
&\quad \leqslant m \left| \frac{m!\, f(z)}{f^{(m)}(z)} \right| + \sum_{k=1}^{m-1} (m-k) \left| \frac{m!\, f^{(k)}(z)}{k!\, f^{(m)}(z)} \right| |x-z|^k \\
&\quad \leqslant m\delta \left(\frac{\beta_m(f, z)}{|x-z|} \right)^m |x-z|^m + \sum_{k=1}^{m-1} (m-k) \left(\frac{\beta_m(f, z)}{|x-z|} \right)^{m-k} |x-z|^m \\
&\quad \leqslant (m-1+\delta) \frac{\dfrac{\beta_m(f, z)}{|x-z|}}{1-\dfrac{\beta_m(f, z)}{|x-z|}} |x-z|^m \\
&\quad \leqslant (m-1+\delta) \frac{u_m \psi(u_m)}{(1-u_m)^2} |x-z|^m,
\end{aligned}$$

with $\delta = 0$ if $f(z) = 0$ and $\delta = 1$ if $f(z) \neq 0$. The sum $\sum_{k \geqslant m+1} (k-m) |(m!\, f^{(k)}(z)/k!\, f^{(m)}(z))(x-z)^k|$ is bounded by $u_m/((1-u_m)^2)|x-z|^m$.

Using the estimates of part 1 of Lemma 2 for $f'(x)$, we get finally:

$$\begin{aligned}
|N_f(x) - \bar{x}| &\leqslant \frac{\dfrac{(m-1+\delta)\, u_m \psi(u_m)}{(1-u_m)^2} + \dfrac{u_m}{(1-u_m)^2}}{\dfrac{m^2 \psi(u_m) \left(1 - \dfrac{m-1}{m} u_m\right)}{(1-u_m)^2}} |x-z|, \\
&\leqslant \frac{(m+\delta)\, u_m}{m^2 \psi(u_m) \left(1 - \dfrac{m-1}{m} u_m\right)} |x-z|.
\end{aligned}$$

The lemma follows. ∎

We state the following easy lemma:

LEMMA 4. *Let a, b, t be three positive real numbers. Let the sequence (s_k) be defined by*

$$s_0 = 0, \qquad s_{k+1} = a t^k + b s_k, \qquad k \geqslant 0.$$

Then $s_{k+1} = a \sum_{i=0}^{k} b^i t^{k-i}$.

Proof of Theorem 4. We first introduce the intermediate sequence $\bar{x}_{k+1} = z + \frac{m-1}{m}(x_k - z)$, $k \geqslant 0$. We also denote $u_{km} = \gamma_m(f, z)\,|x_k - z| \leqslant \gamma_m(f, z)\,|x_0 - z| \leqslant u_m = \gamma_m(f, z)\,R$. To estimate the quantity $|x_{k+1} - y_{k+1}|$, we write

$$x_{k+1} - y_{k+1} = x_{k+1} - \bar{x}_{k+1} + \bar{x}_{k+1} - y_{k+1}.$$

Using Lemma 3, we bound $|x_{k+1} - \bar{x}_{k+1}|$ by

$$|x_{k+1} - \bar{x}_{k+1}| \leqslant \frac{(m+\delta)\,u_{km}}{m^2 \psi(u_{km})\left(1 - \dfrac{m-1}{m}\,u_{km}\right)}\,|x_k - z|.$$

On the other hand, we have $\bar{x}_{k+1} - y_{k+1} = \frac{m-1}{m}(x_k - y_k)$. From Theorem 3 we have $|x_k - z| \leqslant \varphi_m(u_m)^k\,R$. then we can estimate $|x_{k+1} - y_{k+1}|$ using $u_{km} \leqslant u_m$. We find:

$$\begin{aligned}
|x_{k+1} - y_{k+1}| &\leqslant \frac{(m+\delta)\,\gamma_m(f, z)}{m^2 \psi(u_{km})\left(1 - \dfrac{m-1}{m}\,u_{km}\right)}\,|x_k - z|^2 + \frac{m-1}{m}\,|x_k - y_k|,\\
&\leqslant \frac{(m+\delta)\,u_m}{m^2 \psi(u_m)\left(1 - \dfrac{m-1}{m}\,u_m\right)}\,\varphi_m(u_m)^{2k}\,R + \frac{m-1}{m}\,|x_k - z_k|.
\end{aligned}$$

From Lemma 4, we get using the inequality $\frac{m-1}{m} \leqslant \varphi_m(u_m)$:

$$\begin{aligned}
|x_{k+1} - y_{k+1}| &\leqslant \frac{(m+\delta)\,u_m}{m^2 \psi(u_m)\left(1 - \dfrac{m-1}{m}\,u_m\right)}\,R \sum_{i=0}^{k} \varphi_m(u_m)^{2(k-i)} \left(\frac{m-1}{m}\right)^i\\
&\leqslant \frac{(m+\delta)\,u_m}{m^2 \psi(u_m)\left(1 - \dfrac{m-1}{m}\,u_m\right)}\,R \sum_{i=0}^{k} \varphi_m(u_m)^{2k-i}\\
&\leqslant \frac{(m+\delta)\,u_m}{m^2 \psi(u_m)\left(1 - \dfrac{m-1}{m}\,u_m\right)}\,\frac{\varphi_m(u_m)^k}{1 - \varphi_m(u_m)}\,R.
\end{aligned}$$

The theorem follows. ∎

4.4. *Proof of Theorem* 5

We need some lemmas to prove this theorem. First, say that the derivatives of the global Newton homotopy f_t are those of the polynomial f. This is fundamental in the proof and in the practical experiments. We rewrite Lemma 2 in the particular case $m=1$:

LEMMA 5. *Let* $u \leqslant (3-\sqrt{7})/2$ *and* w *be a simple root of* f. *For all* $x \in D(w, \frac{u}{\gamma(f,w)})$ *the point* $N_f(x)$ *is well defined and satisfies*

$$|N_f(x) - w| \leqslant \tfrac{1}{2} |x - w|.$$

We have the classical point estimates [1, p. 160]:

LEMMA 6. *Let* $y = N_f(x) \in \mathbb{C}$ *be such that* $u = \gamma(f, x)\,|y - x| < 1 - \sqrt{2}/2$. *Then* $f'(y) \neq 0$ *and the point estimates hold*:

1. $\beta(f, y) \leqslant (u(1-u))/\psi(u)\, \beta(f, x)$,
2. $\gamma(f, y) \leqslant \gamma(f, x)/(\psi(u)(1-u))$.

Remember $g = \max_{t^+ \leqslant t \leqslant 1} \gamma(f, w_t)$.

LEMMA 7. *Let* $k \geqslant 0$. *Under the assumptions of Theorem* 5, *suppose* $z_{k+1} = N_{f_{k+1}}(z_k)$ *is well defined with* $g\,|z_k - z_{k+1}| \leqslant 3u$ *and* $g\,|z_k - w_k| \leqslant u$. *Then*

$$\beta(f_{k+1}, z_{k+1}) \leqslant T(u)\, \beta(f_{k+1}, z_k) \leqslant \frac{M}{2}\, \beta(f_{k+1}, z_k).$$

Proof. From Lemma 6 we have respectively

$$\beta(f_{k+1}, z_{k+1}) \leqslant \frac{(1 - \gamma(f_{k+1}, z_k)\,|z_{k+1} - z_k|)\, \gamma(f_{k+1}, z_k)\,|z_{k+1} - z_k|}{\psi(\gamma(f_{k+1}, z_k)\,|z_{k+1} - z_k|)}\, \beta(f_{k+1}, z_k)$$

and

$$\gamma(f_{k+1}, z_k) = \gamma(f, z_k) \leqslant \frac{\gamma(f, w_k)}{\psi(\gamma(f, w_k)\,|w_k - z_k|)(1 - \gamma(f, w_k)\,|w_k - z_k|)}.$$

We use both the inequalities $\gamma(f, z_k) \leqslant g$, $g\,|z_k - z_{k+1}| \leqslant 3u$ and $g\,|z_k - w_k| \leqslant u$ to obtain

$$\beta(f_{k+1}, z_{k+1}) \leqslant \frac{3u(\psi(u)(1-u)-3u)}{\psi(u)^2(1-u)^2\,\psi\left(\dfrac{3u}{\psi(u)(1-u)}\right)}\,\beta(f_{k+1}, z_k)$$

$$= T(u)\,\beta(f_{k+1}, z_k).$$

Since by Hypothesis 2 of Theorem 5, $T(u) \leqslant \frac{M}{2}$, we are done. ∎

Remember k_0 is the index verifying $M^{k_0-1} \geqslant t^+ > M^{k_0}$.

Lemma 8. *Consider s, t belonging to the interval $[t_{k+1}, t_k]$, $0 \leqslant k \leqslant k_0 - 1$, with $t \geqslant t^+$. Then we have $g\,|w_s - w_t| \leqslant u$.*

Proof. It is fundamental here that $t \geqslant t^+$ in order to use the quantity g in the estimates below. From the definition of the global Newton homotopy, we have $f(w_s) - f(w_t) = (s-t)\,f(x_0)$. Hence

$$\sum_{k \geqslant 1} \frac{f^{(k)}(w_s)}{k!}\,(w_s - w_t)^k = (s-t)\,f(x_0).$$

Multiplying the previous equality by the inverse of $f'(w_t)$, using the triangle inequality and the definition of g, we get

$$g\,|w_s - w_t| - \sum_{k \geqslant 2} (g\,|w_s - w_t|)^k \leqslant |s-t|\,bg.$$

Since s and t belong to the interval $[t_{k+1}, t_k]$, we have $|s-t| \leqslant t_k - t_{k+1} = M^k(1-M) \leqslant 1-M$. From the definition of m, it follows that

$$g\,|w_s - w_t| - \frac{(g\,|w_s - w_t|)^2}{1 - g\,|w_s - w_t|} = \frac{g\,|w_s - w_t|(1 - 2g\,|w_s - w_t|)}{1 - g\,|w_s - w_t|}$$

$$\leqslant (1-M)\,a = \frac{u(1-2u)}{1-u}.$$

The function $u \to \frac{u(1-2u)}{1-u}$ is an increasing function in the interval for $u \in [0, 1 - \sqrt{2}/2]$. Hence $g\,|w_s - w_t| \leqslant u$. We are done. ∎

We now estimate a value of t in order that $\frac{r}{u_m} \leqslant |w_t - z| \leqslant R$. For this value of t, w_t is an approximate m-cluster of f associated with $D(z, r)$ as in Theorem 3.

LEMMA 9. *Under the assumptions of Theorem* 5, *let* t *be such that* $r/u_m \leqslant |w_t - z| \leqslant R$. *We have*

$$\frac{\psi(u_m)}{(1-u_m)^2} \frac{|f^{(m)}(z)|}{m!\,|f(x_0)|} |w_t - z|^m \leqslant t^+ \leqslant \frac{1}{(1-u_m)^2} \frac{|f^{(m)}(z)|}{m!\,|f(x_0)|} |w_t - z|^m.$$

Proof. We have $0 = f_t(w_t) = f(w_t) - tf(x_0)$. Remember $u_m = \gamma_m(f, z)\, R$. We have

$$\begin{aligned} tf(x_0) &= f(w_t) \\ &= \left(1 + \sum_{k=0}^{m-1} \frac{m!\, f^{(k)}(z)}{k!\, f^{(m)}(z)} (w_t - z)^{k-m} + \sum_{k \geqslant m+1} \frac{m!\, f^{(k)}(z)}{k!\, f^{(m)}(z)} (w_t - z)^{k-m}\right) \\ &\quad \times \frac{f^{(m)}(z)}{m!} (w_t - z)^m. \end{aligned}$$

Since $r/u_m \leqslant |w_t - z| \leqslant R$, we can bound the two sums of the previous expression as in the proof of Theorem 3. More precisely

$$\begin{aligned} \sum_{k=0}^{m-1} \left|\frac{m!\, f^{(k)}(z)}{k!\, f^{(m)}(z)}\right| |w_t - z|^{k-m} &\leqslant \sum_{k=0}^{m-1} \left(\frac{\beta_m(f, z)}{|w_t - z|}\right)^{m-k} \\ &\leqslant \frac{\dfrac{\beta_m(f, z)}{|w_t - z|}}{1 - \dfrac{\beta_m(f, z)}{|w_t - z|}} \\ &\leqslant \frac{\psi(u_m)\, u_m}{(1-u_m)^2}, \end{aligned}$$

and

$$\sum_{k \geqslant m+1} \left|\frac{m!\, f^{(k)}(z)}{k!\, f^{(m)}(z)}\right| |w_t - z|^{k-m} \leqslant \frac{u_m}{1 - u_m}.$$

Using Hypothesis 4 of Theorem 3 we get

$$\left(1-\frac{\psi(u_m)\,u_m}{(1-u_m)^2}-\frac{u_m}{1-u_m}\right)\frac{|f^{(m)}(z)|}{m!}\,|w_t-z|^m$$

$$\leqslant t\,|f(x_0)|\leqslant\left(1+\frac{\psi(u_m)\,u_m}{(1-u_m)^2}+\frac{u_m}{1-u_m}\right)\frac{|f^{(m)}(z)|}{m!}\,|w_t-z|^m$$

$$\frac{\psi(u_m)}{(1-u_m)^2}\,\frac{|f^{(m)}(z)|}{m!}|w_t-z|^m$$

$$\leqslant t\,|f(x_0)|\leqslant\frac{1}{(1-u_m)^2}\,\frac{|f^{(m)}(z)|}{m!}\,|w_t-z|^m.$$

We are done. ▮

Proof of Theorem 5. Let t^+ be such that $|z-w_{t^+}|=R-\frac{2u}{g}$. The fundamental property to begin the proof is that $f'(w_t)\neq 0$ for all $t\in[t^+,1]$. Therefore the quantity $g=\max_{t^+\leqslant t\leqslant 1}\gamma(f,w_t)$ is bounded. Let k_0 be such that $t_{k_0-1}\geqslant t^+>t_{k_0}$.

1. We first prove by induction that the z_k's are approximate zeros of f_k associated with w_k for $0\leqslant k\leqslant k_0$. It is obvious for $k=0$. Prove the inequality $g\,|z_k-w_k|\leqslant u$ implies $g\,|z_{k+1}-w_{k+1}|\leqslant u$. Successively, by triangle inequality, inductive hypothesis, and Lemma 8, we have for $0\leqslant k\leqslant k_0-1$:

$$\begin{aligned} g\,|z_k-w_{k+1}| &\leqslant g\,|z_k-w_k|+g\,|w_k-w_{k+1}|\\ &\leqslant u+u=2u. \end{aligned}$$

Since $2u\leqslant(3-\sqrt{7})/2$ we know from Lemma 5 that the point z_{k+1} is well defined with

$$g|z_{k+1}-w_{k+1}|\leqslant\frac{g}{2}\,|z_k-w_{k+1}|\leqslant u,\qquad 0\leqslant k\leqslant k_0-1.$$

2. We next prove by induction the inequality

$$\beta(f_k,z_k)\leqslant\frac{(1-u)^2\,b}{\psi(u)}\,(t_k-t_{k+1}),\qquad 0\leqslant k\leqslant k_0-1.$$

Since $\beta(f_{t_0}, x_0)=0$, the previous condition obviously holds for $k=0$. If for $k<k_0-1$, the point z_k is an approximate zero of f_k associated with w_k then z_{k+1} is well defined and satisfies

$$\begin{aligned} g\,|z_{k+1}-z_k| &\leqslant g\,|z_{k+1}-w_{k+1}|+g\,|w_{k+1}-w_k|+g\,|w_k-z_k| \\ &\leqslant u+u+u=3u. \end{aligned}$$

From Lemma 7, it follows for $k\leqslant k_0-1$:

$$\beta(f_{k+1}, z_{k+1})\leqslant\frac{M}{2}\,\beta(f_{k+1}, z_k).$$

On the other hand $f_{k+1}(z_k)=f_k(z_k)+(t_k-t_{k+1})\,f(x_0)$. From the induction hypothesis, we get:

$$\beta(f_{k+1}, z_k)\leqslant\beta(f_k, z_k)+(t_k-t_{k+1})\,\frac{|f(x_0)|}{|f'(z_k)|}.$$

Classical point estimate on $f'(z_k)$ gives $|f'(z_k)|\geqslant(\psi(u)/(1-u)^2)\,|f'(w_k)|$. Using inductive assumption, and the definition of b, it follows that

$$\beta(f_{k+1}, z_k)\leqslant\frac{2(1-u)^2\,b}{\psi(u)}\,(t_k-t_{k+1}),\qquad 0\leqslant k\leqslant k_0-1.$$

Hence,

$$\begin{aligned} \beta(f_{k+1}, z_{k+1}) &\leqslant\frac{M}{2}\,\beta(f_{k+1}, z_k)\leqslant M(t_k-t_{k+1})\,\frac{(1-u)^2\,b}{\psi(u)} \\ &=(t_{k+1}-t_{k+2})\,\frac{(1-u)^2\,b}{\psi(u)},\qquad 0\leqslant k\leqslant k_0-1. \end{aligned}$$

For $0\leqslant k\leqslant k_0-1$, let us now prove

$$\beta(f_k, z_k)\leqslant\frac{(1-u)^2\,b}{\psi(u)}\,(t_k-t_{k+1})\Rightarrow\beta(f, z_k)\leqslant\frac{2(1-u)^2\,b}{\psi(u)}\,M^k.$$

We have $f_k(z_k) = f(z_k) - t_k f(x_0)$. Remember $f'(z_k) = f'_k(z_k)$ and $|f'(z_k)| \geqslant (\psi(u)/(1-u)^2)\ |f'(w_k)|$. then, using the definition of b, we get

$$\beta(f, z_k) - \frac{(1-u)^2\, b}{\psi(u)}\, t_k \leqslant \beta(f_k, z_k).$$

Since $\beta(f_k, z_k) \leqslant ((1-u)^2\, b/\psi(u))(t_k - t_{k+1})$ we get with $2(t_k - t_{k+1}) \leqslant 2M^k(1-M) < 2M^k$,

$$\beta(f, z_k) \leqslant \frac{(1-u)^2\, b}{\psi(u)}\,(2t_k - t_{k+1}) < \frac{2(1-u)^2\, b}{\psi(u)}\, M^k.$$

Part 2 of the theorem follows.

3. Finally, we prove the point z_{k_0} is an approximate m-cluster of f associated with $D(z, r)$. for that it is sufficient from Theorem 3 to show that the inequalities $r/u_m \leqslant |z - z_{k_0}| \leqslant R$ hold.

We know that $|z - w_{t^+}| = R - \frac{2u}{g}$. From Lemma 8, with $s = t_{k_0}$ and $t = t^+$, we have $g\ |w_{k_0} - w_{t^+}| \leqslant u$. On the other hand, from part 1 of this theorem, we also have $g\ |z_{k_0} - w_{k_0}| \leqslant u$. Therefore,

$$|z - z_{k_0}| \leqslant |z - w_{t^+}| + |w_t + w_{k_0}| + |w_{k_0} - z_{k_0}| = R - \frac{2u}{g} + \frac{u}{g} + \frac{u}{g} = R.$$

From assumption, we know $R = 4u/g \geqslant r/u_m$. Hence,

$$|z - z_{k_0}| \geqslant |z - w_{t^+}| - |w_{t^+} - w_{k_0}| - |w_{k_0} - z_{k_0}| \geqslant R - \frac{4u}{g} \geqslant \frac{r}{u_m}. \quad \blacksquare$$

4.5. *Proof of Theorem* 6

To prove Theorem 6, we replace Lemma 9 by the following.

LEMMA 10. *under the assumptions of Theorem* 6, *let* t^+ *be such that* $|z - w_{t^+}| = (3 - \sqrt{7})/6g$. *We have*

$$\frac{8\sqrt{7} - 21}{3g}\, \frac{|f'(z)|}{|f(x_0)|} \leqslant t^+ \leqslant (8 - 3\sqrt{7})\, \frac{|f'(z)|}{|f(x_0)|}.$$

Proof. We have $0=f(w_t)-f(z)=t(x_0)$. A point estimate, as in the proof of Lemma 9, shows that:

$$\frac{|w_t-z|\,(1-2g\,|w_t-z|)}{1-g\,|w_t-z|}\,|f'(z)|\leqslant t\leqslant\frac{|w_t-z|}{1-g\,|w_t-z|}\,|f'(z)|.$$

Hence, if $|w_t-z|=(3-\sqrt{7})/6g$, a straightforward computation gives the results. ∎

Using Lemma 10 instead of Lemma 9, the proof of Theorem 6 is made in the same way as in Theorem 5.

4.6. *Proof of Proposition* 1

Let us consider the notations of Section 3. We first prove the following estimate

LEMMA 11. *For some $k\geqslant 0$, let us suppose there exists a root w_k of f_k such that $\gamma(f_k, w_k)\,|w_k-z_k|<1$. If $\beta(f_k, z_k)\leqslant\varepsilon$ then $|z_k-w_k|\leqslant\varepsilon/(1-\sqrt{6}/3)$.*

Proof. Remember that for all $j\geqslant 1$, we have $f_k^{(j)}(z_k)=f^{(j)}(z_k)$. From Taylor's formula, a classical estimate gives under the assumption $\gamma(f_k, w_k)$ $|w_k-z_k|<1$:

$$\beta(f_k, z_k)=\left|\frac{f_k(z_k)}{f'(z_k)}\right|\geqslant\frac{1-\sum_{k\geqslant 2}(\gamma(f, w_k)\,|w_k-z_k|)^{k-1}}{1+\sum_{k\geqslant 1}k(\gamma(f, w_k)\,|w_k-z_k|)^{k-1}}\,|w_k-z_k|$$
$$\geqslant(1-2\gamma(f, w_k)\,|w_k-z_k|)(1-\gamma(f, w_k)\,|w_k-z_k|)|w_k-z_k|.$$

We verify easily the inequality $(1-2t)(1-t)\,t(1-\sqrt{6}/3)\,t$. Hence the inequality $(1-2\gamma(f, w_k)\,|w_k-z_k|)(1-\gamma(f, w_k)\,|w_k-z_k|)\,|w_k-z_k|\leqslant\beta(f_k, z_k)\leqslant\varepsilon$ implies $|z_k-w_k|\leqslant\varepsilon/(1-\sqrt{6}/3)$. ∎

Proof of Proposition 1. The global Newton Path Following algorithm computes a sequence (t_k) such that $t_{k+1}=\max(t_k-2(t_{k-1}-t_k), 0)$ if $\beta_k\leqslant\varepsilon$ and $t_{k+1}=(t_{k-1}+t_k)/2$ otherwise. Denote $r_k=1/(2\gamma_m(f, z_k))$ where (z_k) is the sequence introduced in Section 3.3. Let w_k be the root of f_k nearest z_k when $\beta_k\leqslant\varepsilon$. The algorithm stops in the two following cases: either there exists k such that $t_k=0$, $\beta_k\leqslant\varepsilon$, and $R_1(f, z_k, r_k)>0$ or there exists k such that $\beta_k>\varepsilon$ and such that z_k provides an m-cluster. For $t_0=1$, we have $\beta_0=0$. Let $t_{k_0}>0$ be such that $\beta_{k_0}\leqslant\varepsilon$ and $\beta_{k_0+1}>\varepsilon$. If z_{k_0+1} provides an m-cluster, the algorithm stops. Otherwise, for all $i>1$ such that $\beta_{k_0+i}>\varepsilon$, we have $t_{k_0+1}<t_{k_0+2}<\cdots<t_{k_0+i}<\cdots<t_{k_0}$. The same conclusion holds when we have $t_{k_0+1}=0$, $\beta_{k_0+1}\leqslant\varepsilon$, and $R_1(f, z_{k_0+1}, r_{k_0+1})\leqslant 0$.

By continuity of $\beta(f_t, z)$ at $(t, z) = (t_{k_0}, z_{k_0})$, there is an index j such that $\beta_{k_0+j} \leqslant \varepsilon$. Consequently, we have constructed a strictly decreasing subsequence (t_{k_j}) which verifies $t_{k_j} > 0$ and $\beta_{k_j} \leqslant \varepsilon$. At each point z_{k_j} is associated a root w_k, which verifies $\gamma(f_k, w_k)\,|w_{k_j} - z_{k_j}| < 1$. Since the subsequence (t_{k_j}) converges to 0, the subsequence (w_k) converges towards either an m-cluster or a simple root. Denote it by $D(z, r)$. This is provided respectively by the signs of the polynomials R_m and R_1 in the algorithm.

Finally, to prove there is a finite number of steps, let us introduce the value: $t^+ = \sup\{t : w_t$ provides an m-cluster and $|w_t - z| \leqslant R/2\}$. Since $t^+ > 0$, there is an index k such that $|w_k - z| \leqslant R/2$ and $\beta(f_k, z_k) \leqslant \varepsilon$. From assumption $\gamma(f_k, w_k)\,|w_k - z_k| < 1$. Since $\varepsilon \leqslant (1 - \sqrt{6/3})\,\frac{R}{2}$, Lemma 11 gives

$$|z_k - z| \leqslant |z_k - w_k| + |w_k - z| \leqslant R/2 + R/2 = R.$$

Hence $z_k \in D(z, R)$ and provides an m-cluster containing all the zeros of $D(z, r)$. We are done. ∎

ACKNOWLEDGMENTS

Part of this research was done while visiting Laboratoire GAGE at the École Polytechnique, Palaiseau, France. The author thanks Marc Guisti for this invitation and for helpful discussions. I am also thankful to Luis Miguel Pardo for several suggestions and valuable remarks.

REFERENCES

1. L. Blum, F. Cucker, M. Shub, and S. Smale, “Complexity and Real Computation,” Springer-Verlag, Berlin/New York, 1998.
2. D. W. Decker and C. T. Kelley, Newton’s method at singular points 1, *SIAM J. Numer. Anal.* **17** (1980), 66–70.
3. D. W. Decker and C. T. Kelley, Newton’s method at singular points 2, *SIAM J. Numer. Anal.* **17** (1980), 465–471.
4. J. P. Dedieu, Condition number analysis for sparse polynomial systems, *in* “Foundations of Computational Mathematics” (F. Cucker and M. Shub, Eds.), Springer-Verlag, Berlin/New York, 1997.
5. J. P. Dedieu, X. Gourdon, and J. C. Yakoubsohn, “Computing the Distance from a Point to an Algebraic Hypersurface,” Lectures in Applied Mathematics, Vol. 32, Amer. Math. Soc., Providence, RI, 1996.
6. J. P. Dedieu and M. Shub, On simple double zeros and badly conditioned zeros of analytic functions of n variables, *Math. Comput.*, in press.
7. J. P. Dedieu and J. C. Yakoubsohn, Localization of an algebraic hypersurface by the exclusion algorithm, *Appl. Algebra Eng. Commun. Comput.* **2** (1992), 239–256.

8. A. Griewank, On solving nonlinear equations with simple singularities or nearly singular solutions, *SIAM Rev.* **27** (1985), 537–563.
9. A. Griewank and M. R. Osborne, Analysis of Newton's method at irregular singularities, *SIAM J. Numer. Anal.* **20** (1983), 747–773.
10. V. Guillemin and A. Pollack, "Differential Topology," 1st ed., Prentice–Hall, Englewood Cliffs, NJ, 1981.
11. M. Hirsch and S. Smale, On algorithms for solving $f(x)=0$, *Comm. Pure Appl. Math.* **32** (1979), 281–312.
12. J. Hubbard, D. Schleicher, and S. Sutherland, How to really find roots of polynomials by Newton's method, preprint (1998).
13. I. N. Katz and X. Ying, A reliable principle algorithm to find the number of zeros of an analytic function in a bounded domain, *Numer. Math.* **53** (1988), 143–163.
14. M.-H. Kim and S. Sutherland, Polynomial root-finding algorithms and branched covers, *SIAM J. Comput.* **23** (1994), 415–436.
15. M. Marden, "Geometry of Polynomials," Amer. Math. Soc., Providence, RI, 1966.
16. A. P. Morgan, A. J. Sommese, and C. W. Wampler, Computing singular solutions to nonlinear systems, *Numer. Math.* **58** (1991), 669–684.
17. A. M. Ostrowski, "Solutions of Equations in Euclidean and Banach Spaces," 3rd ed. Academic Press, San Diego, 1973.
18. M. Petković, "Iterative Methods for Simultaneous Inclusion of Polynomial Zeros," Lecture Notes in Mathematics, Vol. 1387, Springer-Verlag, Berlin/New York, 1989.
19. V. Pan, solving a polynomial equation: Some history and recent progress, *SIAM Rev.* **39** (1997), 187–220.
20. L. B. Rall, Convergence of the Newton process to multiple solutions, *Numerische Mathematik* **9** (1966), 23–37.
21. C. W. Reddien, On Newton method for singular problem, *SIAM J. Numer. Anal.* **15** (1980), 993–986.
22. J. Renegar, On the worst case arithmetic complexity of approximating zeros of polynomials, *J. Complexity* **3** (1987), 90–113.
23. E. Schröder, Über unendlich viele Algorithmen zur Auflösung der Gleichungen, *Math. Ann.* **2** (1870), 317–365.
24. S. Smale, Newton's estimate from data at one point, *in* "the Merging of Disciplines: New Directions in Pure, Applied and Computational Mathematics" (R. Ewing, K. Goss, and G. Martin, Eds.), pp. 185–196, Springer-Verlag, New York, 1986.
25. S. Smale, A convergent process of price adjustment and global Newton methods, *J. Math. Econ.* **3** (1976), 107–120.
26. J. H. Wilkinson, The perfidious polynomial in Studies in numerical analysis, MAA Studies Mathematics, Vol. 24, pp. 1–28, Math. Assoc. America, Washington DC, 1984.
27. J. C. Yakoubsohn, Contraction, robustness and numerical path-following using secant maps, *J. Complexity* **16** (2000), 206–310.
28. J. C. Yakoubsohn, "Simultaneous Computation of All Zeros Clusters of A Univariate Polynomial," Tech. Report Laboratoire MIP, Université Paul Sabatier, Toulouse, France, 1999.

Printed in Belgium