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# COMPUTING THE DISTANCE FROM A POINT TO AN ALGEBRAIC HYPERSURFACE

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ABSTRACT. We generalize the Dandelin-Graeffés method to the multivariate case and apply it to compute the distance from a point to an algebraic surface. For that we introduce the positive root of a certain concave which is a good lower bound of this distance. We also illustrate this theoretical fact by a numerical example.

## 1. INTRODUCTION

This paper is devoted to the problem of computing the distance in $\mathbb{C}^n$ from a point $u$ to an algebraic hypersurface $\mathcal{Z} = \{z \in \mathbb{C}^n \ : P(z) = 0\}$ where $P(z)$ is a polynomial in $\mathbb{C}[z_1, \dots, z_n]$, the distance in $\mathbb{C}^n$ corresponding to the norm

$$\|z\| = \max_{1 \leq k \leq n} |z_i|.$$

By shifting the variable $z$, we can restrict in the case $u = 0$. In fact we compute a sequence of lower bounds converging to $d(0, \mathcal{Z})$. Such lower bounds for the the distance from $u$ to $\mathcal{Z}$ are particulary useful to give an approximation of $\mathcal{Z}$ via an exclusion-bisection algorithm. In the univariate case such lower bounds, also called proximity tests, are given by Weyl [8], Henrici-Gargantini [4], Schönhage [6], Turan [7]. One may consult about this subject the recent survey writen by Pan [5]. In the multivariate case a proximity test based on Taylor's formula is studied by Dedieu-Yakoubsohn [2].

The test presented here is based on both Taylor's formula and a generalization of Dandelin-Graeffe's process to the multivariate case (see [1] or [6] for the univariate case). It consists essentially in computing the $N$-th Graeffe iterate of $P(z)$ (see Definition 1), which has the form

$$P^{[N]}(z) = \sum_{j \geq 0} B_j(z)$$

where the $B_j(z)$ are homogeneous polynomials of degree $2^N j$, and then computing the non negative root $\rho_N$ of the equation in $\rho$

$$\|B_0\| = \sum_{j \geq 1} \|B_j\| \, \rho^j,$$

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the norms $\|B_j\|$ being the sum of the absolute values of coefficients of $B_j$. Then $r_N = \rho_N^{2^{-N}}$ tends rapidly to $d(0, \mathcal{Z})$.

More precisely the following theorem is proved :

**Theorem 1.** *Let $\rho_N$ be the unique nonnegative solution of*

$$|P^{[N]}(0)| = \sum_{j=1}^{d} \|B_j^{[N]}\| \rho^j \tag{1}$$

*The distance from 0 to $\mathcal{Z}$ satisfies*

$$r_N \leq d(0, \mathcal{Z}) \leq \kappa_N r_N, \tag{2}$$

*where*

$$r_N = \rho_N^{2^{-N}} \quad \text{and} \quad \kappa_N = \left( \frac{1}{2^{1/d} - 1} \sqrt{\frac{2^N + n - 1}{n - 1}} \right)^{1/2^N}.$$

*Moreover* $\lim_{N\to\infty} \kappa_N = 1$*, which implies* $\lim_{N\to\infty} r_N = d(0, \mathcal{Z})$.

## 2. The Graeffe process.

The purpose of this part is to generalize the classical univariate Graeffe process to the multivariate case. In the univariate case, the Graeffe iterate of polynomial $P(z)$ is defined as the unique polynomial $Q(z)$ such that $Q(z^2) = P(z)P(-z)$. In the multivariate case, the polynomial $P(z)P(-z)$ can not be written as $Q(z^2)$ where $Q(z)$ is a polynomial, thus we need to slightly modify the definition.

**Definition 1.** We call the $N$-th Graeffe iterate of $P(z) \in \mathbb{C}[z_1, \dots, z_n]$ the polynomial $P^{[N]}(z)$ defined by

$$P^{[N]}(z) = \prod_{j=0}^{2^N - 1} P(\omega^j z), \qquad \omega = \exp\left(\frac{2i\pi}{2^N}\right), \quad i^2 = -1. \tag{3}$$

When $P(z)$ is a univariate polynomial, we have $P^{[1]}(z) = P(z)P(-z) = P^{\langle 1 \rangle}(z^2)$ where $P^{\langle 1 \rangle}(z)$ is the classical Graeffe iterate of $P(z)$. More generally, the $N$-th classical univariate Graeffe iterate satisfy $P^{\langle N \rangle}(z) = P^{[N]}(z^{2^N})$ for all $N$. Remember that $P^{\langle N \rangle}(z)$ has the same degree as $P(z)$ and its roots are the $2^N$ powers of roots of $P(z)$ (see [Ba]).

Graeffe iterates satisfy several properties which make them easy to compute.

**Proposition 1.** *For all non negative integer $N$, the $N$-th Graeffe iterate of $P(z)$ writes as*

$$P^{[N]}(z) = \sum_{j \geq 0} B_j^{[N]}(z),$$

*where the $B_j^{[N]}$'s are homogeneous polynomials of degree $2^N j$. The $(N+1)$-st Graeffe iterate can be computed from the $N$-th thanks to the formula*

$$P^{[N+1]}(z) = P_0^{[N]}(z)^2 - P_1^{[N]}(z)^2, \qquad P_k^{[N]}(z) = \sum_{j \equiv k (\mathrm{mod} 2)} B_j^{[N]}(z). \tag{4}$$

*Proof.* Since the degrees of all the monomials in $P_0^{[N]}(z)^2$ and $P_1^{[N]}(z)^2$ are multiples of $2^{N+1}$, we need only to prove formula (2). For this, we notice that

$$P^{[N+1]}(z) = P^{[N]}(z)P^{[N]}(\omega z), \qquad \omega = \exp\left(\frac{2i\pi}{2^{N+1}}\right),$$

and since $P_0^{[N]}(\omega z) = P_0^{[N]}(z)$ and $P_1^{[N]}(\omega z) = -P_1^{[N]}(z)$, this implies

$$P^{[N+1]}(z) = \left(P_0^{[N]}(z) + P_1^{[N]}(z)\right)\left(P_0^{[N]}(z) - P_1^{[N]}(z)\right) = P_0^{[N]}(z)^2 - P_1^{[N]}(z)^2,$$

proving the result. □

## 3. The univariate case

In the univariate case, the distance $d(0, \mathcal{Z})$ from 0 to the set $\mathcal{Z}$ of zeros of $P(z)$ is also the smallest modulus of the roots of $P(z)$. Computing this distance is a classical task. It usually consists in using the Graeffe process together with a result giving an upper and a lower bound for $d(0, \mathcal{Z})$. Classical bounds are given in the following theorem, which can be found in [3] Theorems 6.4.d and 6.4.i.

**Theorem 2.** *Let $P(z) = \sum_{k=0}^{d} b_k z^k$ be a univariate complex polynomial, and $\rho(P)$ the nonnegative root of the equation*

$$|b_0| = \sum_{j=1}^{d} |b_j|\rho^j.$$

*Then*

$$\rho(P) \le d(0, \mathcal{Z}) \le \frac{1}{2^{1/d} - 1}\rho(P).$$

The value $\rho(P)$ is easily computable. When we apply this result to the $N$-th classical Graeffe iterate of $P(z)$, we obtain

$$\rho(P^{\langle N \rangle}) \le d(0, \mathcal{Z}_N) \le \frac{1}{2^{1/d} - 1}\rho(P^{\langle N \rangle}),$$

where $\mathcal{Z}_N$ is the set of roots of $P^{\langle N \rangle}(z)$. Since the roots of $P^{\langle N \rangle}(z)$ are the $2^N$-th powers of the roots of $P(z)$, we have $d(0, \mathcal{Z}_N) = d(0, \mathcal{Z})^{2^N}$, thus

$$(5) \qquad r_N \le d(0, \mathcal{Z}) \le \left(\frac{1}{2^{1/d} - 1}\right)^{2^{-N}} r_N, \qquad r_N = \rho(P^{\langle N \rangle})^{2^{-N}}.$$

The upper bound tends rapidly to the lower bound as $N$ increases, thus we have obtained an effective process to compute $d(0, \mathcal{Z})$.

## 4. The multivariate case

Thanks to the multivariate Graeffe process, we easily generalize the univariate algorithm to compute $d(0, \mathcal{Z})$ to the multivariate case.

**Theorem 3.** *Let $P(z)$ be a polynomial in $\mathbb{C}[z_1, \dots, z_n]$ of total degree $d$. Let $P^{[N]}(z) = \sum_{j\geq 0} B_j^{[N]}(z)$ be its $N$-th Graeffe iterate and $R_N$ the non-negative solution $R$ of the equation*

$$|P^{[N]}(0)| = \sum_{j\geq 1} \|B_j^{[N]}\|_\infty R^j, \tag{6}$$

*where $\|B_j^{[N]}\|_\infty = \sup_{\|z\|=1} \|B_j(z)\|$. Then we have*

$$r_N \leq d(0, \mathcal{Z}) \leq \left(\frac{1}{2^{1/d}-1}\right)^{2^{-N}} r_N, \qquad r_N = R_N^{2^{-N}} \tag{7}$$

*Proof.* If $P(0) = 0$, there is nothing to prove since $r_N = 0$. Otherwise, we have $P^{[N]}(0) \neq 0$. We prove first the left part of the inequality. Let $z \in \mathbb{C}^n$, $\|z\| < r_N = R_N^{2^{-N}}$. We have

$$|P^{[N]}(z)| \geq |P^{[N]}(0)| - \sum_{j\geq 1} \|B_j^{[N]}\|_\infty \|z\|^{2^N j} > |P^{[N]}(0)| - \sum_{j\geq 1} \|B_j^{[N]}\|_\infty R_N^j = 0,$$

thus $P^{[N]}(z)$ does not vanish in the open ball centered in zero with radius $r_N$, and since $P(z)$ is a factor of $P^{[N]}(z)$, this is also the case for $P(z)$. Thus $r_N \leq d(0, \mathcal{Z})$.

Now we prove the right inequality of 7. For all $y \in \mathbb{C}^n$ such that $\|y\| = 1$, we define the univariate polynomial $P_y(t) = P(ty)$. We have

$$P_y^{\langle N\rangle}(t) = \sum_{j=0}^{d} B_j^{[N]}(y)\, t^j. \tag{8}$$

Let $\mathcal{Z}_N(y)$ denotes the set of zeros of $P_y^{\langle N\rangle}(t)$. Since

$$d(0, \mathcal{Z}_N(y)) = d(0, \mathcal{Z}_0(y))^{2^N} \geq d(0, \mathcal{Z})^{2^N},$$

formula 8 together with Lemma 1 below yield for all $j$

$$|B_j^{[N]}(y)| \leq |P^{[N]}(0)| \binom{d}{j} \frac{1}{d(0, \mathcal{Z})^{2^N j}}. \tag{9}$$

The right side of the inequality is independant of $y$ such that $\|y\| = 1$, thus inequality 9 remains valid with $|B_j^{[N]}(y)|$ replaced by $\|B_j^{[N]}\|_\infty$. Plugging this information into equation 6 defining $R_N$, we obtain

$$|P^{[N]}(0)| \leq \sum_{j\geq 1} |P^{[N]}(0)| \binom{d}{j} \frac{R_N^j}{d(0, \mathcal{Z})^{2^N j}},$$

that is

$$1 \leq \left(1 + \frac{R_N}{d(0, \mathcal{Z})^{2^N}}\right)^d - 1,$$

leading to

$$\frac{d(0,\mathcal{Z})^{2^N}}{R_N} \leq \frac{1}{2^{1/d}-1},$$

proving the right inequality of 7. □

The following lemma was needed in the proof of Theorem 3. Its proof can be found in [3] chap. 6, 6.4-8.

**Lemma 1.** *Let $P(z) = \sum_{j=0}^{d} b_j z^j$ be a univariate complex polynomial, $\mathcal{Z}(P)$ the set of its roots, $b_0 \neq 0$. Then*

$$|b_j| \leq |b_0| \binom{d}{j} \frac{1}{d(0,\mathcal{Z}(P))^j}.$$

Theorem 3 can not be applied directly to approach $d(0,\mathcal{Z})$ in the practice, since the norms $\|\cdot\|_\infty$ are difficult to compute. Instead, we make use of the norm

$$\left\| \sum_\alpha a_\alpha z^\alpha \right\| = \sum |a_\alpha|,$$

easy to compute. Our main result is stated using this norm. We now shall prove the theorem 1.

## 5. Proof of the theorem 1

For the left part of inequality 2, we proceed as in the proof of Theorem 3 replacing the norm $\|\cdot\|_\infty$ with $\|\cdot\|$. We can do that because $\|\cdot\|_\infty \leq \|\cdot\|$ (see Lemma 2 below).

Now we prove the right part of 2. Lemma 2 and Lemma 3 below give for all $j$,

$$\|B_j^{[N]}\| \leq \sqrt{\binom{j2^N+n-1}{n-1}}\|B_j^{[N]}\|_\infty \leq \alpha^j \|B_j^{[N]}\|_\infty, \qquad \alpha = \sqrt{\binom{2^N+n-1}{n-1}}$$

thus

$$|P^{[N]}(0)| = \sum_{j=1}^{d} \|B_j^{[N]}\|\, \rho_N^j \leq \sum_{j=1}^{d} \|B_j^{[N]}\|_\infty\, (\alpha \rho_N)^j.$$

This implies

$$R_N \leq \alpha\, \rho_N,$$

where $R_N$ satisfies equation 6. The inequality 7 satified by $R_N$ now entails

$$d(0,\mathcal{Z}) \leq \left(\frac{1}{2^{1/d}-1}\right)^{2^{-N}} R_N^{2^{-N}} \leq \left(\frac{\alpha}{2^{1/d}-1}\right)^{2^{-N}} \rho_N^{2^{-N}},$$

proving the result.

In order to prove Theorem 1, we needed a lemma comparing the norm $\|\cdot\|$ on polynomials with the intrinsec norm $\|\cdot\|_\infty$.

**Lemma 2.** *Let $A \in \mathbb{C}[z_1, \ldots, z_n]$ be an homogeneous polynomial of degree $k$, $z \in \mathbb{C}^n$. Then*

$$\max_{\|z\|=1} |A(z)| = \|A\|_\infty \leq \|A\| \leq \sqrt{\binom{k+n-1}{n-1}}\, \|A\|_\infty.$$

*Proof.* The left inequality is trivial. Let us show the right inequality. Let $A(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha$ and $\|A\|_2 = \left(\sum_{|\alpha|=j} |a_\alpha|^2\right)^{1/2}$. The Parseval identity

$$\|A\|_2^2 = \left(\frac{1}{2\pi}\right)^n \int_{[0,2\pi]^n} |A(e^{i\theta_1}, \dots, e^{i\theta_n})|^2 \, d\theta_1 \cdots d\theta_n$$

implies $\|A\|_2 \le \|A\|$. To conclude, we use Cauchy-Schwarz inequality

$$\|A\| \le \sqrt{K} \, \|A\|_2,$$

where $K$ is the total number of $\alpha \in \mathbb{N}^n$ such that $|\alpha| = k$, that is $K = \binom{k+n-1}{n-1}$. □

The following combinatorial lemma was also needed in the proof of Theorem 1.

**Lemma 3.** *For all positive integer $j$, $K$ and $p$, we have*

$$\binom{jK+p}{p} \le \binom{K+p}{p}^j.$$

*Proof.* The inequality

$$\binom{jK+p}{p}\binom{K+p}{p}^{-j} = \prod_{\ell=1}^{p} \frac{(jK+\ell)\ell^{j-1}}{(K+\ell)^j} \le 1,$$

holds because each term in the product is $\le 1$, this being true because $(jK + \ell)\ell^{j-1} = \ell^j + jK\ell^{j-1}$ represents the first two terms in the binomial expansion of $(K+\ell)^j$ □

## 6. Convergence of the process

The sharpness of inequality 2 depends essentially of the rate of convergence of $\kappa_N$ to 1. In fact, this convergence appears to be fast. As an illustration, Table 1 shows, for different values of $n$ and $d$ , $1 \le n \le 10$ and $2 \le d \le 10$, the minimal value of $N$ such that

$$\rho_N \le d(0, \mathcal{Z}) \le 2\rho_N.$$

| $d \backslash n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 2 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 |
| 3 | 2 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 4 |
| 4 | 2 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 4 |
| 5 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 4 | 4 |
| 6 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 4 | 4 |
| 7 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 4 | 4 |
| 8 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 9 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 10 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |

Table 1.

The following result also gives an idea of how fast does $\kappa_N$ tend to 1.

**Proposition 2.** *The error coefficient $\kappa_N$ satisfies, for $N \geq 1$,*

$$1 \leq \kappa_N \leq (2d)^{1/2^N} 2^{N(n-1)/2^{N+1}}.$$

*Proof.* The inequality $1 \leq \kappa_N$ is trivial. For the other, we first write

$$\binom{2^N+n-1}{n-1} = \frac{2^N+1}{1} \cdot \frac{2^N+2}{2} \cdots \frac{2^N+n-1}{n-1},$$

thus

$$\binom{2^N+n-1}{n-1} = 2^{N(n-1)}(2^{-N}+1)\left(2^{-N}+\frac{1}{2}\right)\cdots\left(2^{-N}+\frac{1}{n-1}\right),$$

and since $N \geq 1$, this implies

$$\binom{2^N+n-1}{n-1} \leq 2^{N(n-1)}(2^{-N}+1) \leq 1.5 \cdot 2^{N(n-1)}.$$

Now, we have $2^{1/d} - 1 = e^{\log 2/d} - 1 \geq \log 2/d$, thus

$$\frac{1}{2^{1/d}-1} \leq \frac{d}{\log 2}.$$

This finally gives

$$\kappa_N \leq \left(\frac{d}{\log 2}\sqrt{1.5} \cdot 2^{N(n-1)/2}\right)^{1/2^N} \leq \left((2d)\, 2^{N(n-1)/2}\right)^{1/2^N},$$

yielding the result. □

In the practice, when $n$ is large, computation of $r_N$ becomes very expensive when $N$ gets large. The following result gives a bound on $\kappa_N$ for a reasonable value of $N$.

**Proposition 3.** *Let $N$ be such that $2^N \leq n < 2^{N+1}$. Then the error coefficient $\kappa_N$ satisfies*

$$1 \leq \kappa_N \leq 4(1.45d)^{2/n}.$$

*Proof.* We need to prove the second inequality. First, we notice that

$$\binom{2^N+n-1}{n-1} \leq \binom{2n-1}{n-1} \leq \binom{2n}{n} \leq 2^{2n},$$

and since $2^{1/d} - 1 \geq \log 2/d$ and $1/2^N \leq 2/n$, we obtain

$$\kappa_N \leq \left(\frac{d}{\log 2}\sqrt{2^{2n}}\right)^{1/2^N} \leq \left(\frac{d}{\log 2}\sqrt{2^{2n}}\right)^{2/n} \leq 4(1.45d)^{2/n},$$

proving the result. □

## 7. Examples

Theorem 1 provides an efficient way of computing the distance from a point to an algebraic hypersurface. We illustrate this result by computing the distance from 0 to a family of algebraic hypersuface $\mathcal{Z}_{n,d}$ for positive integer $n$ and $d$, defined by

$$\mathcal{Z}_{n,d} = \{z \in \mathbb{C}^n \ : \ P_{n,d}(z) = 0\}, \quad \text{where} \quad P_{n,d}(z) = \sum_{j=1}^{n}(1 - z_j)^d - 1.$$

The set $\mathcal{Z}_{n,d}$ is a sphere associated to the $d$-norm centered in $(1, \dots, 1)$, and its distance from 0 is found to be

$$d(0, \mathcal{Z}_{n,d}) = 1 - n^{-1/d}. \tag{10}$$

Let us prove this formula. If $\|z\| < 1 - n^{-1/d}$, then we have for all $j$ the inequality $|z_j| < 1 - n^{-1/d}$, thus

$$|P_{n,d}(z)| \geq \sum_{j=1}^{n} |1 - z_j|^d - 1 \geq \sum_{j=1}^{n}(1 - |z_j|)^d - 1 > \sum_{j=1}^{n} \frac{1}{n} - 1 = 0,$$

which proves that $P_{n,d}(z)$ does not vanish in the open ball centred in zero of radius $1 - n^{-1/d}$. Thus $d(0, \mathcal{Z}_{n,d}) \geq 1 - n^{-1/d}$. Since the point $z = (1 - n^{-1/d}, \dots, 1 - n^{-1/d})$ belongs to $\mathcal{Z}_{n,d}$, we have finally proved formula 10.

Below are tables giving for several values of $n$ the value of the ratio $r_N/d(0, \mathcal{Z}_{n,d})$ of Theorem t.principal for several values of $d$ and $N$. The computations were made in Maple.

| $d$ | $r_0/d$ | $r_1/d$ | $r_2/d$ | $r_3/d$ | $r_4/d$ |
|---|---|---|---|---|---|
| 2 | 0.7673 | 0.9725 | 0.9996 | 1.0000 | 1.0000 |
| 5 | 0.6525 | 0.9479 | 0.9973 | 1.0000 | 1.0000 |
| 7 | 0.6325 | 0.9400 | 0.9960 | 0.9999 | 1.0000 |
| 15 | 0.6067 | 0.9271 | 0.9938 | 0.9999 | 1.0000 |

*Table 2.* Some values of $r_N/d(0, \mathcal{Z}_{n,d})$ for $n = 2$.

| $d$ | $r_0/d$ | $r_1/d$ | $r_2/d$ | $r_3/d$ | $r_4/d$ |
|---|---|---|---|---|---|
| 2 | 0.6885 | 0.9386 | 0.9752 | 0.9785 | 0.9910 |
| 5 | 0.5453 | 0.6859 | 0.8546 | 0.9357 | 0.9610 |
| 7 | 0.5212 | 0.6475 | 0.8307 | 0.9270 | 0.9535 |

*Table 3.* Some values of $r_N/d(0, \mathcal{Z}_{n,d})$ for $n = 3$.

| $d$ | $r_0/d$ | $r_1/d$ | $r_2/d$ | $r_3/d$ |
|---|---|---|---|---|
| 2 | 0.6457 | 0.7847 | 0.8759 | 0.9326 |
| 5 | 0.4891 | 0.5612 | 0.7640 | 0.8311 |
| 7 | 0.4632 | 0.5268 | 0.7437 | 0.8117 |

*Table 4.* Some values of $r_N/d(0, \mathcal{Z}_{n,d})$ for $n = 4$.

| $d$ | $r_0/d$ | $r_1/d$ | $r_2/d$ | $r_3/d$ |
|---|---|---|---|---|
| 2 | 0.6180 | 0.7101 | 0.8384 | 0.8831 |
| 5 | 0.4533 | 0.4970 | 0.7031 | 0.7661 |

*Table 5.* Some values of $r_N/d(0,\mathcal{Z}_{n,d})$ for $n=5$.

| $d$ | $r_0/d$ | $r_1/d$ | $r_2/d$ | $r_3/d$ |
|---|---|---|---|---|
| 2 | 0.5832 | 0.6338 | 0.8108 | 0.8224 |
| 3 | 0.4802 | 0.5108 | 0.6478 | 0.7561 |

*Table 6.* Some values of $r_N/d(0,\mathcal{Z}_{n,d})$ for $n=7$.

| $d$ | $r_0/d$ | $r_1/d$ | $r_2/d$ | $r_3/d$ |
|---|---|---|---|---|
| 2 | 0.5534 | 0.5796 | 0.6779 | 0.7561 |

*Table 7.* Some values of $r_N/d(0,\mathcal{Z}_{n,d})$ for $n=10$ and $d=2$.

These examples show that the bound is quite good for a small value $N$ of Graeffe iterates.

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