Computing the real roots of a polynomial by the exclusion algorithm

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We describe a new algorithm for localizing the real roots of a polynomial $P(x)$. This algorithm determines intervals on which $P(x)$ does not possess any root. The remainder set contains the real roots of $P(x)$ and can be arbitrarily small.

Keywords: Exclusion, polynomial, root.

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1. Introduction

The main goal of this account is to describe and to study a new algorithm for finding the real roots of a polynomial: the exclusion algorithm. The localization of a general algebraic variety in $\mathbb{R}^n$ has been studied in [3] by a similar method. We study here the case $n = 1$ where more precise results can be proved. Let us explain the main idea of this process. Let $P(x) = \sum_{k=0}^{d} a_k x^k$ be a polynomial in $\mathbb{R}[x]$ with degree$(P) = d$. We denote by $Z = \{r \in \mathbb{R} : P(r) = 0\}$ and for any $x \in \mathbb{R}$ we define the following polynomial of the variable t:

$$M(x,t) = |P(x)| - \sum_{k=1}^{d} \frac{|P^{(k)}(x)|}{k!} t^k.$$

It appears in [10] with the same expression and in [11] with $x \in \mathbb{R}^n$. The polynomial $M(x,t)$ possesses a unique positive root $m(x)$ which satisfies the following properties:

(1) $m(x) = 0$ if and only if $P(x) = 0$ (proposition 2.1.1);
(2) if $P(x) \neq 0$ the interval $]x - m(x), x + m(x)[$ does not contain any root of $P(x)$ (proposition 2.1.2);
(3) $m(x)$ is Lipschitz: $|m(x) - m(y)| \leqslant |x - y|$ for each $x, y \in \mathbb{R}$ (corollary 2.6);

(4) if $Z \neq \emptyset$ there is a constant $\alpha > 0$ such that $\alpha d(x, Z) \leqslant m(x) \leqslant d(x, Z)$ for each $x \in \mathbb{R}$ (proposition 2.8).

The function $m(x)$ is called the exclusion function associated with $P(x)$. Let $\epsilon > 0$ be given and let $\rho > 0$ be any bound for the modulus of the roots of $P(x)$. Our aim is to compute a set F_ϵ satisfying

$$Z \subset F_\epsilon \subset Z + [-K\epsilon, K\epsilon],$$

where K is a constant independent of ϵ. This is done via the following algorithm:

Initialization: $x_0 = -\rho$ and $F_\epsilon = \emptyset$.
At *step p* we compute an approximation $\mu(x_p)$ of $m(x_p)$ such that

$$m(x_p) - \frac{\epsilon}{2} \leqslant \mu(x_p) \leqslant m(x_p).$$

If $\mu(x_p) \geqslant \epsilon$, by (2), the interval $]x_p - \mu(x_p), x_p + \mu(x_p)[$ does not contain any root of $P(x)$: we define $x_{p+1} := x_p + \mu(x_p)$ and $F_\epsilon := F_\epsilon \setminus]x_p - \mu(x_p), x_p + \mu(x_p)[$.

If $\mu(x_p) < \epsilon$ the interval $[x_p, x_p + \epsilon]$ may contain a root of $P(x)$ and we define $x_{p+1} := x_p + \epsilon$ and $F_\epsilon := F_\epsilon \cup [x_p, x_p + \epsilon]$.

This algorithm stops when $x_p \geqslant \rho$.

The main properties of this algorithm are the following:

(5) if $Z \neq \emptyset$ then $Z \subset F_\epsilon \subset Z + [-2\epsilon/\alpha, 2\epsilon/\alpha]$, α defined in (4) (proposition 3.2.1.);
(6) this objective is reached in $O(|\log \epsilon|)$ steps of the algorithm (theorem 3.4.1);
(7) each step of this algorithm requires $O(\log|\log \epsilon|)$ multiplications (3.8).

Moreover this method is stable, easy to implement and computes all the roots of $P(x)$ even if their multiplicities are greater than 1. This algorithm has been implemented in float arithmetic and has given excellent results even for difficult examples: the roots of the polynomial $\prod_{i=1}^{10}(x-i) = x^{10} - 55x^9 + 1320x^8 - 18150x^7 + 157773x^6 - 902055x^5 + 3416930x^4 - 8409500x^3 + 12753576x^2 - 10628640x + 3628800$ are computed in less than CPU 0.1 seconds with an accuracy of 10^{-6}.

Another process works like the exclusion algorithm: Weyl's method which first appeared in [12]. An implementation of this process has been given by Henrici and Gargantini in [6]. More recently a parallel algorithm has been studied by Coleman in [1]. This method is based on

$$m_W(x) = \frac{|P(x)|}{d|a_d|(2\rho)^{d-1}}$$

instead of $m(x)$. We prove, in section 4, that the exclusion algorithm works better than Weyl's.

Another famous algorithm computes the real roots of a polynomial: Sturm's method [2,7,8]. This method reaches the accuracy ϵ in $O(|\log \epsilon|)$ steps, like for the exclusion algorithm. Studying numerical examples in float arithmetic shows that Sturm's method works faster than exclusion in the case of low degree polynomials and does not work for high degrees when the coefficient size is large like in the previous example (overflow problems). Moreover, Sturm's algorithm is not stable, contrary to the exclusion algorithm.

The exclusion algorithm can compute the complex roots of $P(x)$! The definition of $M(x,t)$ and $m(x)$ can be extended to $x \in \mathbb{C}$ and the main results of this paper are still valid. A complex algorithm can be given which computes the roots of $P(x)$ in $\mathbb{C}$. This will be done in another paper.

2. The exclusion function associated with a polynomial

Let $P(x) = \sum_{k=0}^{d} a_k x^k$ be a polynomial in $\mathbb{R}[x]$ with degree d. We consider the following polynomial in $\mathbb{R}[t]$:

$$M(x,t) = |P(x)| - \sum_{k=1}^{d} \frac{|P^{(k)}(x)|}{k!} t^k .$$

Note that the degree of $M(x,t)$ is d. This polynomial, as a function of t, is concave and decreasing over $[0,\infty[$. Since $M(x,0) = |P(x)| \geqslant 0$, this polynomial has a unique positive root which is denoted by $m(x)$. This root, as a function of x, is called the exclusion function associated with $P(x)$. The main properties of $m(x)$ are the following:

PROPOSITION 2.1

For each $x \in \mathbb{R}$ we have:

(1) $m(x) = 0$ if and only if $P(x) = 0$.

(2) If $P(x) \neq 0$ then $P(y) \neq 0$ for each y satisfying $|x-y| < m(x)$.

Proof

The first property is easy. Let us prove the second. From Taylor's formula and the triangle inequality we get:

$$|P(y)| \geqslant |P(x)| - \sum_{k=1}^{d} \frac{|P^{(k)}(x)|}{k!} |y-x|^k ,$$

that is,

$$|P(y)| \geqslant M(x, |y-x|) .$$

If $P(x) \neq 0$ we have $m(x) > 0$, so that $M(x, |y-x|) > 0$ for each y satisfying $|y-x| < m(x)$ since $M(x,t)$ decreases over $[0,\infty[$; thus the inequality $|P(y)| > 0$ holds and this proves our proposition. □

PROPOSITION 2.2

The exclusion function $m(x)$ associated with $P(x)$ is continuous.

Proof

Let $\epsilon > 0$ be given. Since $M(x, m(x)) = 0$ and $M(x, t)$ is a strictly decreasing function of t we have $M(x, m(x) + \epsilon) < 0 < M(x, m(x) - \epsilon)$. Since $M(x, t)$ is continuous we have $M(y, m(x) + \epsilon) < 0 < M(y, m(x) - \epsilon)$ for any y in a neighbourhood of x. Hence $m(x) - \epsilon < m(y) < m(x) + \epsilon$ since $M(x, t)$ is a strictly decreasing function of t. □

Let Z be the set of real roots of $P(x)$. We denote by $d(x, Z)$ the distance of x from Z.

PROPOSITION 2.3

For any $x \in \mathbb{R}$ we have $m(x) \leqslant d(x, Z)$.

Proof

When $x \in Z$, that is $P(x) = 0$, we have $m(x) = d(x, Z) = 0$. When $x \notin Z$, that is $P(x) \neq 0$, we apply proposition 2.1. □

PROPOSITION 2.4

For any $x \in \mathbb{R}$ such that $P^{(k)}(x) \neq 0$ for $k = 0, \ldots, d - 1$, the exclusion function $m(x)$ possesses a derivative which is given by

$$m'(x) = \frac{\sum_{k=1}^{d} P^{(k)}(x)\epsilon_{k-1}(x) m(x)^{k-1}/(k-1)!}{\sum_{k=1}^{d} |P^{(k)}(x)| m(x)^{k-1}/(k-1)!},$$

with $\epsilon_0(x) = sign(P(x))$ and $\epsilon_k(x) = -sign(P^{(k)}(x))$ for $k = 1, \ldots, d$. For any x such that $P(x) \neq 0$ and $P^{(k)}(x) = 0$ for some $k = 1, \ldots, d - 1$, $m(x)$ possesses right and left derivatives which are the right and left limits of the previous formula. For any $x \in \mathbb{R}$ with $P(x) \neq 0$ we have $|m'_+(x)|$ and $|m'_-(x)| \leqslant 1$.

Proof

The function $m(x)$ is defined by $m(x) \geqslant 0$ and

$$M(x, m(x)) = |P(x)| - \sum_{k=1}^{d} \frac{|P^{(k)}(x)|}{k!} m(x)^k = 0.$$

Under the hypotheses we can differentiate this equality:

$$\frac{\partial M}{\partial x}(x, m(x)) + m'(x) \frac{\partial M}{\partial t}(x, m(x)) = 0.$$

Since $m(x) \neq 0$ the quantity $\partial M(x, m(x))/\partial t$ is non-zero and we obtain the previous formula. The second part of this proposition is easy: for any x such that $P(x) \neq 0$ the quantity $\sum_{k=1}^{d} (|P^{(k)}(x)|/(k-1)!) m(x)^{k-1}$ is non-zero and the previous formula possesses right and left limits at such a point. In both cases these derivatives are clearly bounded by 1. □

PROPOSITION 2.5

When r is a root of $P(x)$ with multiplicity p ($P^{(k)}(r) = 0$ for $k = 0, \ldots, p-1$ and $P^{(p)}(r) \neq 0$) we have

$$\lim_{x \to r} \frac{m(x)}{|x - r|} = 2^{1/p} - 1 ,$$

so that $m(x)$ possesses right and left derivatives at r which are given by

$$m'_{\pm}(r) = \pm(2^{1/p} - 1) .$$

Proof

We first study the case $p = 1$. Using proposition 2.4 we obtain:

$$\lim_{x \to r_{\pm}} m'(x) = \lim_{x \to r_{\pm}} \frac{P'(x)\epsilon_0(x)}{|P'(x)|} ,$$

and this quantity is equal to $+1$ for $x > r$ and -1 for $x < r$. Suppose now $p > 1$. For any $k = 0, \ldots, p$ we have

$$P^{(k)}(x) = \frac{(x - r)^{p-k}}{(p - k)!} (P^{(p)}(r) + \eta_k(x)) ,$$

with $\lim_{x \to r} \eta_k(x)/(|x - r|) = 0$. The equality $M(x, m(x)) = 0$ becomes:

$$\left| \frac{(x - r)^p}{p!} (P^{(p)}(r) + \eta_0(x)) \right| - \sum_{k=1}^{p} \frac{(x - r)^{p-k}}{(p - k)!} (P^{(p)}(r) + \eta_k(x)) \frac{m(x)^k}{k!} - \sum_{k=p+1}^{d} |P^{(k)}(x)| \frac{m(x)^k}{k!} = 0 .$$

We divide this equation by

$$\left| \frac{(x - r)^p}{p!} P^{(p)}(r) \right|$$

and obtain

$$\left| 1 + \frac{\eta_0(x)}{P^{(p)}(r)} \right| - \sum_{k=1}^{p} \binom{p}{k} \left| 1 + \frac{\eta_k(x)}{P^{(p)}(r)} \right| \left(\frac{m(x)}{|x - r|} \right)^k - \sum_{k=p+1}^{d} \frac{p!}{k!} \left| \frac{P^{(k)}(x)}{P^{(p)}(r)} \right| \frac{m(x)^k}{|x - r|^p} = 0 .$$

We now take the limit for $x \to r$. Using proposition 2.3, since $\lim_{x \to r} m(x) = 0$, we see that the second sum tends to zero. Suppose now that α is a cluster value of $m(x)/(|x - r|)$ when $x \to r$: such values exist since, by proposition 2.3, $0 \leqslant m(x)/(|x - r|) \leqslant 1$. Since $\lim_{x \to r} \eta_k(x)/P^{(p)}(r) = 0$ we obtain

$$1 - \sum_{k=1}^{p} \binom{p}{k} \alpha^k = 0 ,$$

so that $\alpha = 2^{1/p} - 1$. Consequently $\lim_{x \to r} m(x)/(|x - r|)$ always exists and is equal to $\alpha = 2^{1/p} - 1$. Notice that $\alpha = 2^{1/p} - 1 = 1$ when $p = 1$. □

COROLLARY 2.6

The exclusion function $m(x)$ is Lipschitz: $|m(x) - m(y)| \leqslant |x - y|$ for any $x, y \in \mathbb{R}$.

Proof

By propositions 2.2, 2.4 and 2.5 $m(x)$ is continuous, possesses right and left derivatives for any $x \in \mathbb{R}$ and these derivatives satisfy $|m'(x)| \leqslant 1$. By the Mean Value Theorem, for all x and $y \in \mathbb{R}$, $x < y$, there are real numbers $c \in]x, y[$ and $\theta \in [0, 1]$ such that $m(x) - m(y) = (\theta m'_{\pm}(c) + (1 - \theta) m'_{-}(c))(x - y)$. Thus $m(x)$ is Lipschitz with a lipschitz constant equal to 1. □

The asymptotic values of $m(x)$ are given by the following:

PROPOSITION 2.7

We have

$$\lim_{x \to \pm\infty} \frac{m(x)}{|x|} = 2^{1/d} - 1 .$$

Proof

We divide the inequality $M(x, m(x)) = 0$ by $|x|^d$ and obtain:

$$\left|\frac{P(x)}{x^d}\right| - \sum_{k=1}^{d} \left|\frac{P^{(k)}(x)}{k! x^{d-k}}\right| \left(\frac{m(x)}{|x|}\right)^k = 0 .$$

Let β be a cluster value of $m(x)/|x|$ as $x \to \pm\infty$. By proposition 2.3 such a value always exists. Since for $k = 0, \ldots, d$,

$$\lim_{x \to \pm\infty} \left|\frac{P^{(k)}(x)}{k! x^{d-k}}\right| = d\,|a_d| ,$$

we get the following limit equation:

$$|a_d| - \sum_{k=1}^{d} \binom{d}{k} |a_d| \beta^k = 0 .$$

Since $a_d \neq 0$, this yields $\beta = 2^{1/d} - 1$. □

PROPOSITION 2.8

Suppose that $Z = \{r \in \mathbb{R} : P(r) = 0\}$ is non-void. There is a constant $\alpha > 0$ such that for each $x \in \mathbb{R}$

$$\alpha d(x, Z) \leqslant m(x) \leqslant d(x, Z) .$$

Proof

It suffices to prove the first inequality (proposition 2.3). Consider the function defined by

$$f(x) = \begin{cases} \dfrac{m(x)}{d(x,Z)} & \text{if } x \notin Z, \\ 2^{1/p} - 1 & \text{if } x \in Z \text{ with multiplicity } p. \end{cases}$$

This function is continuous over $\mathbb{R}$ (proposition 2.5) and never vanishes. Since $\lim_{x \to \pm\infty} f(x) = 2^{1/d} - 1 > 0$ (proposition 2.7), we have $0 < \alpha = \inf_{x \in \mathbb{R}} f(x)$, and this proves our assertion. □

EXAMPLES OF EXCLUSION FUNCTIONS

We give two examples of exclusion functions.

EXAMPLE 1

Consider the polynomial $P(x) = x^3 - x$. The derivative $m'(x)$ possesses five points of discontinuity, the roots of $P: -1, 0, 1$ and the roots of the derivative $P': \pm 1/\sqrt{3}$. Since the roots of P are simple, $|m'_{\pm}(x)| = 1$ for $x = \pm 1$ and $x = 0$. On each interval $[-1, -1/\sqrt{3}]$ and $[1/\sqrt{3}, 1]$, $m(x)$ is a segment. The equations of the asymptotes at $+\infty$ and $-\infty$ are respectively $y = (2^{1/3} - 1)x$ and $y = -(2^{1/3} - 1)x$. See fig. 1.

EXAMPLE 2

Consider the polynomial $P(x) = x^5 - 50x^3 + 625x$. The derivative $m'(x)$ possesses seven points of discontinuity, the roots of $P: -5, 0, 5$ and the roots of the derivatives of $P: \pm\sqrt{5}, \pm\sqrt{15}$. Since 0 is a simple root, $|m'_{\pm}(0)| = 1$. The roots ± 5 are of multiplicity two, we have $|m'_{\pm}(\pm 5)| = \sqrt{2} - 1$. The equations of the asymptotes at $+\infty$ and $-\infty$ are respectively $y = (2^{1/5} - 1)x$ and $y = -(2^{1/5} - 1)x$. See fig. 2.

3. The exclusion algorithm

3.1. DESCRIPTION

Our aim is to localize the set Z of real roots of $P(x)$, i.e. for a given precision $\epsilon > 0$ to compute a set F_ϵ such that $Z \subset F_\epsilon \subset Z + [-K\epsilon, K\epsilon]$ with K independent of ϵ. Let

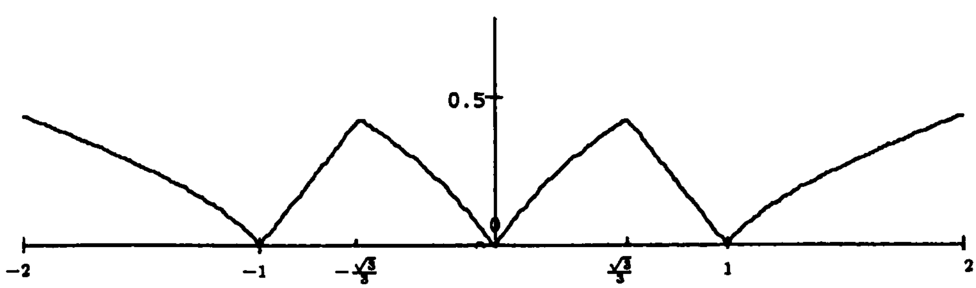

Fig. 1. The exclusion function of example 1.

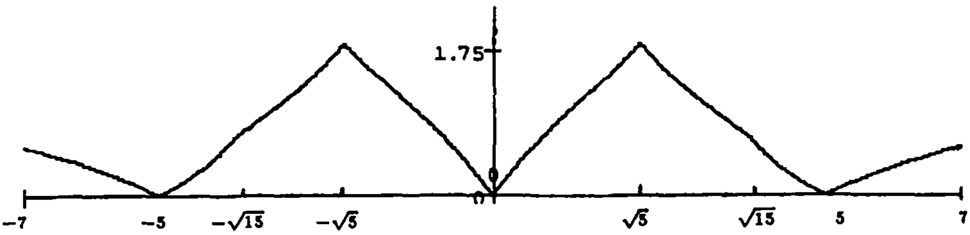

Fig. 2. The exclusion function of example 2.

$\rho > 0$ be such that $Z \subset [-\rho, \rho]$. Let $\epsilon > 0$ be given and for each $x \in \mathbb{R}$ let $\mu(x)$ be a positive real number such that

$$m(x) - \frac{\epsilon}{2} \leqslant \mu(x) \leqslant m(x).$$

The exclusion algorithm described in the introduction is given here in a pseudo program code. Each step will be studied in more detail in the sequel.

Begin
Compute a positive real number $\rho > 0$ such that $Z \subset [-\rho, \rho]$;
Initialization: $x_0 := -\rho, F_\epsilon = \emptyset, p := 0$;
While $x_p < \rho$ **do**
  Begin
  Compute the coefficients of $M(x_p, t)$;
  Compute $\mu(x_p)$;
  If $\mu(x_p) \geqslant \epsilon$ **then** $x_{p+1} := x_p + \mu(x_p)$ **else**
    Begin
    $x := x_p + \epsilon; i := 1;$
    While $\mu(x) < \epsilon$ **do** $i := i + 1$ *and* $x := x_p + i\epsilon$;
    $k_p := i; x_{p+1} := x;$
    $F_\epsilon := F_\epsilon \cup [x_{p-1} + \mu(x_{p-1}), x_{p+1} - \mu(x_{p+1}];$
    End;
  End;
End.

The following notation will be used in the sequel: in the case $\mu(x_{p-1}) \geqslant \epsilon$ and $\mu(x_p) < \epsilon$ the points $x_p + i\epsilon$, $0 \leqslant i \leqslant k_p$, are denoted by y_i^p. We have $y_0^p = x_p$, $y_{k_p}^p = x_{p+1}$ and $y_{i+1}^p = y_i^p + \epsilon$. Since $\mu(x_{p+1}) \geqslant \epsilon$ notice that $[x_{p-1} + \mu(x_{p-1}), x_{p+1} - \mu(x_{p+1})] \subset [y_0^p, y_{k_p-1}^p]$. See fig. 3.

3.2. PROPERTIES OF THE EXCLUSION ALGORITHM

PROPOSITION 3.2.1

(1) For each $p \geqslant 0$ we have $x_{p+1} - x_p \geqslant \epsilon$. The algorithm stops after a finite number of steps.

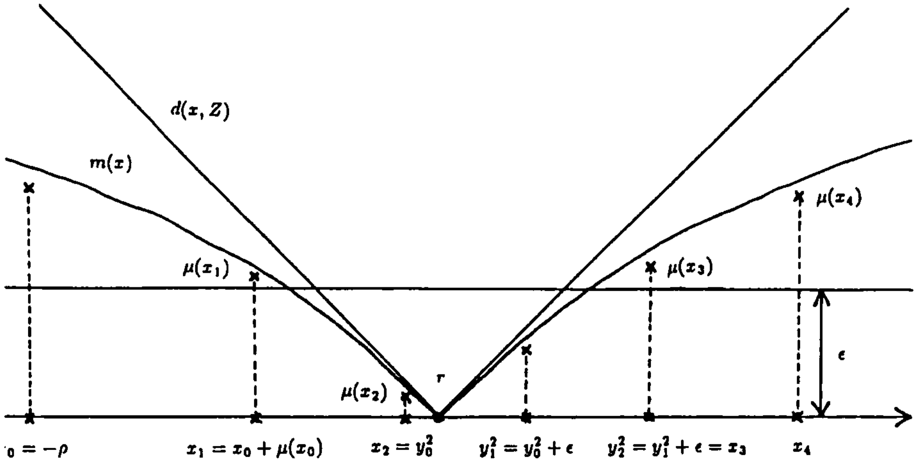

Fig. 3. Illustration of notation used.

(2) One has $Z \subset F_\epsilon$; if $F_\epsilon = \emptyset$ then $P(x)$ has no real root.

(3) If $Z \neq \emptyset$, then $F_\epsilon \subset Z + [-2\epsilon/\alpha, 2\epsilon/\alpha]$, where α is defined in proposition 2.8.

Proof

(1) If $\mu(x_p) \geqslant \epsilon$ then $x_{p+1} - x_p = \mu(x_p) \geqslant \epsilon$ else $x_{p+1} - x_p = k_p\epsilon \geqslant \epsilon$.

(2) Starting from $[-\rho, \rho]$ the algorithm removes intervals $[x_p, x_p + \mu(x_p)[$ with $\mu(x_p) \geqslant \epsilon$. Since $m(x_p) \geqslant \mu(x_p) \geqslant \epsilon$ such an interval does not contain any root of $P(x)$ (proposition 2.1).

(3) Consider $x \in F_\epsilon$. Thus x is in some interval $[y_i^p, y_{i+1}^p]$ with $\mu(y_i^p)$ and $\mu(y_{i+1}^p) < \epsilon$. In other words, one has $|x - y| \leqslant \epsilon/2$ for some y satisfying $\mu(y) < \epsilon$. From corollary 2.6 we get $m(x) \leqslant m(y) + |x - y| \leqslant \mu(y) + \epsilon/2 + |x - y| \leqslant \epsilon + \epsilon/2 + \epsilon/2 = 2\epsilon$. Proposition 2.8 gives $\alpha d(x, Z) \leqslant m(x)$, thus $d(x, Z) \leqslant 2\epsilon/\alpha$. □

3.3. STABILITY PROPERTIES OF THE EXCLUSION ALGORITHM

The exclusion algorithm possesses various stability properties: it is stable under modifications of the initial value x_0 and under rounding errors. These properties are established below.

Modifications of the initial value. Let x_0 and x_0' be two different initial values satisfying $x_0, x_0' \leqslant -\rho$ where ρ is any bound for the modulus of the roots. Our algorithm, starting from these initial values, gives two sets F_ϵ and F_ϵ' which both satisfy

$$Z \subset F_\epsilon, F_\epsilon' \subset Z + [-2\epsilon/\alpha, 2\epsilon/\alpha].$$

This proves the stability of our algorithm under modifications of the initial value.

Rounding errors. In one step, the algorithm needs the computation of $m(x)$ and consequently the computation of the considered polynomial $M(x,t)$ and all its derivatives. Since rounding errors always occur in float arithmetic the exclusion algorithm has been described via an approximation $\mu(x)$ of $m(x)$. This approximation has to satisfy the following inequalities: $m(x) - \epsilon/2 \leqslant \mu(x) \leqslant m(x)$, otherwise a real root of $P(x)$ can be missing in the remainder set F_ϵ. The computation of $\mu(x)$ is done via Newton's algorithm as it will be shown below. Such a computation is stable under rounding errors.

3.4. BOUNDS FOR THE NUMBER OF STEPS IN THE EXCLUSION ALGORITHM

By the exclusion algorithm, we have constructed sequences (x_p) and (y_i^p). Our aim is to give an upper bound for the number of points in these sequences.

THEOREM 3.4.1

Let $n = card(Z)$ be the number of real roots of P and $\rho > 0$ any number such that $Z \subset [-\rho, \rho]$. Let ϵ be given. Suppose that $Z \neq \emptyset$. The number of points constructed by the exclusion algorithm relative to ρ and ϵ is bounded by

$$n(3 + 4\alpha^{-1}) + \left(\frac{1}{\log 1 + \alpha/2} - \frac{1}{\log 1 - \alpha/2}\right) \log \frac{2\rho^n}{\epsilon^n n^n},$$

where the constant α is defined in 2.8.

Remark 3.4.2

This inequality proves that the accuracy $2\epsilon/\alpha$ for the approximation of Z is reached in at most $O(|\log \epsilon|)$ steps for the algorithm.

NOTATIONS

We denote by $z_1 < \ldots < z_n$ the different real roots of $P(x)$, by I_i the interval $[z_i - 2\epsilon/\alpha, z_i + 2\epsilon/\alpha]$ and by I the union of I_i, $1 \leqslant i \leqslant n$. Let us define by S the set of points x_p, y_i^p constructed by the exclusion algorithm. The proof of theorem 3.4.1 is divided into the following lemmas.

LEMMA 3.4.3

$card(S \cap I) \leqslant n(1 + 4\alpha^{-1})$.

Proof

Since the distance between two different points in S is always $\geqslant \epsilon$ there are at most $1 + 4\alpha^{-1}$ points of S in each interval $[z_i - 2\epsilon/\alpha, z_i + 2\epsilon/\alpha]$ and at most $n(1 + 4\alpha^{-1})$ such points in I. □

We now consider the points appearing in $S\backslash I$. According to the description of the algorithm given in section 3.1 such points are of type x_p with $\mu(x_p) \geqslant \epsilon$. Let us denote by

$$S_0 = \{x_p \in S\backslash I | x_p < z_1\},$$

$$S_i = \{x_p \in S\backslash I | z_i < x_p < z_{i+1}\}, \quad 1 \leqslant i \leqslant n-1,$$

$$S_n = \{x_p \in S\backslash I | x_p > z_n\}.$$

We will give an estimation for $card(S_i)$.

LEMMA 3.4.4

For each point x_p in $S\backslash I$ we have

(1) $d(x_p, Z) \geqslant m(x_p) \geqslant \mu(x_p) \geqslant \epsilon$,

(2) $\mu(x_p) \geqslant \frac{1}{2}\alpha d(x_p, Z)$.

Proof

For each x_p in $S\backslash I$, we have already noticed that the inequality $\mu(x_p) \geqslant \epsilon$ holds: this gives the first assertion. By theorem 2.8 we have $\alpha d(x_p, Z) \leqslant m(x_p)$; since $m(x_p) \geqslant \epsilon$ we get

$$\frac{\alpha}{2} d(x_p, Z) \leqslant \frac{1}{2} m(x_p) \leqslant m(x_p) - \frac{\epsilon}{2} \leqslant \mu(x_p),$$

and this proves our assertion. □

NOTATION

We now consider the points appearing in S_i, $1 \leqslant i \leqslant n-1$. The cases S_0 and S_{n+1} are similar. Let us denote them by $z_i < x_p < \ldots < x_{p+r} < z_{i+1}$. These points are given by the following iteration

$$x_{p+k+1} = x_{p+k} + \mu(x_{p+k}), \quad k = 0, \ldots, r-1.$$

Let us define the sequence (t_l), $0 \leqslant l \leqslant s$ by

$$t_0 = x_p,$$

$$t_{l+1} = t_l + \frac{\alpha}{2} d(t_l, Z),$$

where the index s is defined by

$$t_{s-1} \leqslant z_{i+1} - \epsilon < t_s < z_{i+1}.$$

Points t_l are in $]z_i, z_{i+1}[$ since $0 < \alpha/2 < 1$. See fig. 4.

LEMMA 3.4.5

For each $k = 0, \ldots, r$ we have $t_k \leqslant x_{p+k}$ so that $r < s$ with r defined above.

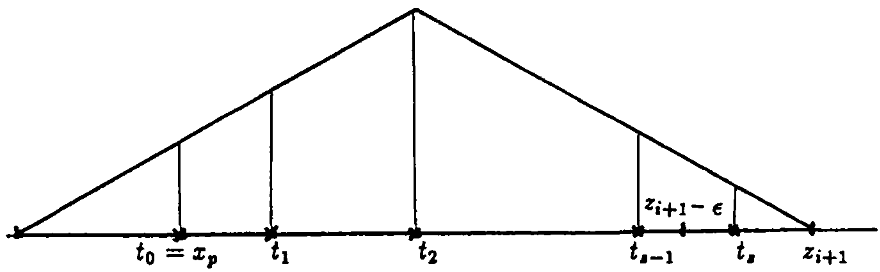

Fig. 4. Illustration of notation used.

Proof

Suppose that $t_k \leqslant x_{p+k}, k = 0, \ldots, r$. Since $d(x_{p+k}, Z) \geqslant \epsilon$ (lemma 3.4.4) we have

$$\epsilon \leqslant d(x_{p+r}, Z) \leqslant z_{i+1} - x_{p+r} \leqslant z_{i+1} - t_r ,$$

so that $t_r \leqslant z_{i+1} - \epsilon$. Since $z_{i+1} - \epsilon < t_s$ we obtain $r < s$. We now prove the inequality $t_k \leqslant x_{p+k}$. For $k = 0$ this inequality holds. Let us suppose that $t_k \leqslant x_{p+k}$. We have:

$$x_{p+k+1} - t_{k+1} = x_{p+k} + \mu(x_{p+k}) - t_k - \frac{\alpha}{2} d(t_k, Z)$$
$$= (x_{p+k} - t_k) + \left(\mu(x_{p+k}) - \frac{\alpha}{2} d(x_{p+k}, Z)\right) + \frac{\alpha}{2}\left(d(x_{p+k}, Z) - d(t_k, Z)\right).$$

We now consider three different cases.

(1) If $z_i < t_k \leqslant x_{p+k} \leqslant (z_{i+1} + z_i)/2$, we have

$$x_{p+k+1} - t_{k+1} = \left(1 + \frac{\alpha}{2}\right)(x_{p+k} - t_k) + \left(\mu(x_{p+k}) - \frac{\alpha}{2} d(x_{p+k}, Z)\right) \geqslant 0$$

by the recurrency hypothesis and lemma 3.4.4.

(2) If $(z_{i+1} + z_i)/2 \leqslant t_k \leqslant x_{p+k} < z_{i+1}$ we have

$$x_{p+k+1} - t_{k+1} = \left(1 - \frac{\alpha}{2}\right)(x_{p+k} - t_k) + \left(\mu(x_{p+k}) - \frac{\alpha}{2} d(x_{p+k}, Z)\right) \geqslant 0$$

by the same argument as before.

(3) If $z_i < t_k \leqslant (z_{i+1} + z_i)/2 \leqslant x_{p+k} < z_{i+1}$ we have

$$x_{p+k+1} - t_{k+1} = \left(1 - \frac{\alpha}{2}\right)(x_{p+k} - t_k) + \alpha\left(\frac{z_{i+1} + z_i}{2} - t_k\right)$$
$$+ \mu(x_{p+k}) - \frac{\alpha}{2} d(x_{p+k}, Z) \geqslant 0 ,$$

as before. This achieves the proof of the lemma. □

LEMMA 3.4.6

With the same notations as in lemma 3.4.5:

$$card(S) \leqslant 2 + \left(\frac{1}{\log 1 + \alpha/2} - \frac{1}{\log 1 - \alpha/2}\right) \log \frac{z_{i+1} - z_i}{2\epsilon} .$$

Proof

According to the definition of t_l we have

$$t_l = z_i + \left(1 + \frac{\alpha}{2}\right)^l (t_0 - z_i) ,$$

while $t_l \leqslant (z_{i+1} + z_i)/2$. Let q be the first index such that $(z_{i+1} + z_i)/2 < t_q$; we have

$$t_{q+l} = z_{i+1} - \left(1 - \frac{\alpha}{2}\right)^l (z_{i+1} - t_1) .$$

The index q satisfies the following inequality:

$$z_i + \left(1 + \frac{\alpha}{2}\right)^{q-l} (t_0 - z_i) \leqslant \frac{z_{i+1} + z_i}{2} ,$$

so that

$$q \leqslant 1 + \frac{\log \dfrac{z_{i+1} - z_i}{2(t_0 - z_i)}}{\log 1 + \alpha/2} .$$

Since $t_0 = x_p$ and $d(x_p, Z) \geqslant \epsilon$ (lemma 3.4.4) we obtain

$$q \leqslant 1 + \frac{\log \dfrac{z_{i+1} - z_i}{2\epsilon}}{\log 1 + \alpha/2} .$$

The index s has been defined by $t_{s-1} \leqslant z_{i+1} - \epsilon \leqslant t_s < z_{i+1}$; this gives

$$z_{i+1} - \left(1 - \frac{\alpha}{2}\right)^{s-q-1} (z_{i+1} - t_q) \leqslant z_{i+1} - \epsilon ,$$

so that

$$s - q - 1 \leqslant \frac{\log \dfrac{\epsilon}{z_{i+1} - t_1}}{\log 1 - \alpha/2} .$$

Since $z_{i+1} - t_q \leqslant (z_{i+1} - z_i)/2$ we obtain

$$s - q - 1 \leqslant \frac{\log \dfrac{2\epsilon}{z_{i+1} - z_i}}{\log 1 - \alpha/2} .$$

and consequently:

$$s \leqslant 2 + \left(\frac{1}{\log 1 + \alpha/2} - \frac{1}{\log 1 - \alpha/2}\right) \log \frac{z_{i+1} - z_i}{2\epsilon} .$$

□

LEMMA 3.4.7

The following inequalities hold:

$$card(S_i) \leqslant 2 + \left(\frac{1}{\log 1 + \alpha/2} - \frac{1}{\log 1 - \alpha/2}\right) \log \frac{z_{i+1} - z_i}{2\epsilon}, \quad 1 \leqslant i \leqslant n-1,$$

$$card(S_0) \leqslant 1 + \frac{\log \dfrac{\epsilon}{z_1 + \rho}}{\log 1 - \alpha/2},$$

$$card(S_n) \leqslant 1 + \frac{\log \dfrac{\rho - z_n}{\epsilon}}{\log 1 + \alpha/2}.$$

Proof

We have $S_i = \{x_p, \ldots, x_{p+r}\}$ so that $card(S_i) = r + 1$. Since $r < s$ the first inequality comes from lemma 3.4.6. The second and the third inequalities have been obtained by a similar argument. □

Proof of theorem 3.4.1

Let $N = card\left(\cup_{i=0}^{n} S_i\right)$; by the previous lemma

$$N \leqslant 2n + \frac{1}{\log 1 + \alpha/2}\left(\log 2 + \log \frac{\rho - z_n}{2\epsilon} + \sum_{i=1}^{n-1} \log \frac{z_{i+1} - z_i}{2\epsilon}\right)$$

$$- \frac{1}{\log 1 - \alpha/2}\left(\log 2 + \log \frac{\rho + z_1}{2\epsilon} + \sum_{i=1}^{n-1} \log \frac{z_{i+1} - z_i}{2\epsilon}\right).$$

Since log is concave, we get

$$N \leqslant 2n + \frac{1}{\log 1 + \alpha/2}\left(\log 2 - n \log \epsilon + n \log \frac{\rho - z_1}{2n}\right)$$

$$- \frac{1}{\log 1 - \alpha/2}\left(\log 2 - n \log \epsilon + n \log \frac{\rho + z_n}{2n}\right).$$

We also use the inequalities $\rho - z_1$ and $\rho + z_n \leqslant 2\rho$ so that

$$N \leqslant 2n + \left(\frac{1}{\log 1 + \alpha/2} - \frac{1}{\log 1 - \alpha/2}\right) \log \frac{2\rho^n}{\epsilon^n n^n}.$$

Theorem 3.4.1 is obtained by adding this result and lemma 3.4.3. □

3.5. COMPUTING AN UPPER BOUND FOR THE MODULUS OF THE ROOTS

The initialisation of the exclusion algorithm requires the knowledge of an upper bound ρ for the modulus of the roots. Such a bound can be computed directly from the coefficients of $P(x)$ like Cauchy's bound

$$\rho_{Cauchy} = 1 + \max_{0 \leqslant k \leqslant d-1} \left| \frac{a_k}{a_d} \right| .$$

A better bound is given by the positive root ρ of the polynomial $P_0(x) = |a_d| t^d - \sum_{k=0}^{d-1} |a_k| t^k$ (see [9]). The derivative of $P_0(x)$ also possesses a unique positive root ρ' which satisfies $0 \leqslant \rho' \leqslant \rho$; the polynomial $P_0(x)$ is convex and increasing over $[\rho', \infty[$. We compute ρ by the classical Newton algorithm $t_{p+1} = t_p - P_0(x)(t_p)/(P_0(x)'(t_p))$ starting at $t_0 \geqslant \rho'$. More precisely, we propose the following process (we denote by ϵ_0 a small positive real number):

Begin
$t_0 := 1$;
While $P_0'(t_p) \leqslant 0$ **do** $t_0 := t_0 + 1$;
While $|t_p - t_{p+1}| \geqslant \epsilon_0$ **do** $t_{p+1} = t_p - \dfrac{P_0(t_p)}{P_0'(t_p)}$;
End.

3.6. COMPUTING THE COEFFICIENT OF $M(x, t)$

At each step of the exclusion algorithm we have to compute the coefficients $P^{(k)}(x_p)/k!$, $0 \leqslant k \leqslant d$ of the polynomial $M(x_p, t)$. These coefficients are computed via the complete Horner scheme [5]. This algorithm requires $d(d+1)/2$ multiplications.

3.7. COMPUTING A LOWER BOUND $\mu(x)$ OF $m(x)$

The exclusion algorithm requires to compute a lower bound $\mu(x)$ of $m(x)$ which satisfies

$$m(x) - \frac{\epsilon}{2} \leqslant \mu(x) \leqslant m(x) .$$

In this section we describe an algorithm based on Newton's iteration which solves this problem and we compute the complexity of this algorithm (proposition 3.7.1). Let $f(t)$ be a real function defined over the interval $[0, +\infty[$ two times continuously differentiable, such that $f(0) > 0$, $f'(t) < 0$ and $f''(t) < 0$ over $]0, +\infty[$. This function possesses a unique positive root denoted by m. Let α, β be such that $0 < \alpha < m < \beta$. Let us consider the sequence (s_k) given by

$$s_1 = \beta , \quad s_{k+1} = s_k - \frac{f(s_k)}{f'(s_k)} .$$

Since f is concave this sequence is decreasing and converges to the root m. A lower bound for m is given by the following algorithm

- **Inputs:** $f(t), \alpha, \beta$, and ϵ;
- **Compute** $s_{k+1} = s_k - f(s_k)/f'(s_k)$ **while** $f(s_k - f'(\beta)/f''(\alpha)) \leqslant 0$;

Let μ be the first integer k such that: $f(s_k - f'(\beta)/f''(\alpha)) > 0$;

- **Compute** $s_{\mu+k}$ **while** $k \leqslant \nu$ where ν is the first integer such that

$$\nu > \frac{1}{\log 2}\left(\log\log\frac{2f'(\alpha)}{\epsilon f''(\beta)} - \log\log 2\right).$$

We have the following result:

PROPOSITION 3.7.1

Let ϵ, ν, μ and $s_{\mu+\nu}$ be defined as before. Then

$$m - \epsilon \leqslant s_{\mu+\nu} - \epsilon < m.$$

The number of steps to obtain this lower bound for m is in

$$O(\log|\log\epsilon|).$$

We first prove the following

LEMMA 3.7.2

Let a, b be two real numbers such that $\alpha \leqslant a < m < b \leqslant \beta$ and

$$0 < C = \frac{f''(\beta)}{2f'(\alpha)}(b-a) < 1.$$

Let λ be the first index such that $s_\lambda \leqslant b$. Then for each k greater than

$$\frac{1}{\log 2}\log\left(\frac{\log\dfrac{b-a}{\epsilon C}}{\log\dfrac{1}{C}}\right),$$

we have $s_{\lambda+k} - m < \epsilon$.

Proof

From the definition of $s_{k+\lambda}$ and the Taylor formula we deduce that $s_{k+\lambda} - m$ is equal to

$$\frac{f(m)-f(s_{k+\lambda-1})-f'(s_{k+\lambda-1})(m-s_{k+\lambda-1})}{f'(s_{k+\lambda-1})}=\frac{f''(u)}{2f'(s_{k+\lambda-1})}(s_{k+\lambda-1}-m)^2,$$

with $u\in]m,s_{k+\lambda-1}[$. Since the derivatives f' and f'' are decreasing and negative functions over $[\alpha,\beta]$ we have

$$s_{k+\lambda}-m\leqslant\frac{f''(\beta)}{2f'(\alpha)}(s_{k+\lambda-1}-m)^2 .$$

We get successively,

$$s_{k+\lambda}-m\leqslant\left(\frac{f''(\beta)}{2f'(\alpha)}\right)^{1+2+\ldots+2^{k-1}}(s_\lambda-m)^{2^k}\leqslant\frac{f''(\beta)^{2^k-1}}{2f'(\alpha)}(b-a)^{2^k}=\frac{2f'(\alpha)}{f''(\beta)}C^{2^k} .$$

The conclusion follows immediately from the assumption $C<1$. □

Proof of proposition 3.7.1

Since the sequence (s_k) converges to m, there exists an integer μ such that $0<s_\mu-f'(\alpha)/f''(\beta)<m$. This integer is determined by the signs of the quantities $f(s_k-f'(\beta)/f''(\alpha))$. The hypotheses of the previous lemma are satisfied for $a=s_\mu-f'(\alpha)/f''(\beta)$ and $b=s_\mu$ since in this case $C=1/2$. Thus the proposition is established. □

Remark 3.7.3

In the case $f(t)=M(x,t)$, we can choose

$$\beta=\left(\frac{j!|P(x)|}{|P^{(j)}(x)|}\right)^{1/j},\quad \alpha=\frac{|P(x)|\beta}{|P(x)|-M(x,\beta)},$$

where j is the first index such that $P^{(j)}(x)\neq 0$.
The bound β comes from $M(x,m(x))=0$:

$$|P(x)|=\sum_{k=1}^{d}\frac{|P^{(k)}(x)|}{k!}m(x)^k\geqslant\frac{|P^{(j)}(x)|}{j!}m(x)^j .$$

The bound α is the value of the abscissa of the intersection point of the x' axis and the straight line passing through the points $(0,|P(x)|)$ and $(\beta,M(x,\beta))$.

3.8. COMPLEXITY OF THE EXCLUSION ALGORITHM

According to remark 3.4.2, the exclusion algorithm reaches the accuracy $2\epsilon/\alpha$ in $O(|\log\epsilon|)$ steps of the algorithm; each of them consists in the evaluation of the coefficients appearing in $M(x,t)$ and in computing the approximation $\mu(x)$ of $m(x)$. The coefficients in $M(x,t)$ can be evaluated in $d(d+1)/2$ multiplications (section 3.6). Computing $\mu(x)$ needs $O(\log|\log\epsilon|)$ Newton's iterations of $M(x,t)$ (proposition 3.7.1) and such an iteration needs $2d-1$ multiplications (the evaluations of $M(x, t_p)$ and $M'(x, t_p)$) and one division. Combining these results gives

$$O(d^2|\log\epsilon| + d|\log\epsilon| \times \log|\log\epsilon|)$$

multiplications.

4. Comparison with Weyl's method

The Weyl method is based on the following:

THEOREM 4.1

Let $\rho > 0$ be any number which bounds the modulus of the real or complex roots of $P(x)$. Let x and s be such that $s > 0$ and $|x| < \rho$. If

$$|P(x)| \geqslant sd|a_d|(2\rho)^{d-1}$$

then, for each y satisfying $|y - x| < s$ and $|y| < \rho$, we have $P(y) \neq 0$.

The proof of this theorem is based on the following:

LEMMA 4.2

For each $u \in [-\rho, \rho]$ we have

$$|P'(u)| \leqslant d|a_d|(2\rho)^{d-1}.$$

If r is a root of $P(x)$ then

$$|P'(r)| \leqslant |a_d|(2\rho)^{d-1}.$$

Proof

Let $z_1, \ldots, z_d$ be the roots of $P(x)$. We have:

$$P'(u) = a_d \sum_{i=1}^{d} \prod_{j \neq i} (u - z_j) \text{ and } P'(r) = a_d \prod_{z_j \neq r} (r - z_j).$$

Bounding these formulas gives our lemma since $|u|$, $|z_j|$ and $|r| < \rho$. □

Proof of theorem 4.1

We have $P(y) = P(x) + (y - x)P'(u)$ for some $u = \theta x + (1 - \theta)y$, $0 < \theta < 1$, so that $|P(y)| \geqslant |P(x)| - |y - x||P'(u)|$. By lemma 4.2 and since $|y - x| < s$ we have $|P(y)| > |P(x)| - sd|a_d|(2\rho)^{d-1}$ and the conclusion holds. □

COROLLARY 4.3

Let us define

$$m_W(x) = \frac{|P(x)|}{d|a_d|(2\rho)^{d-1}}.$$

If $|y| < \rho$ and $|y - x| < m_W(x)$ then $P(y) \neq 0$.

The proof is easy and left to the reader.

Remark 4.4

This corollary is similar to our proposition 2.1 and we can write easily a "Weyl exclusion algorithm" based on the radius $m_W(x)$ instead of $m(x)$. The asymptotic behaviour of $m(x)$ in the neighbourhood of a real root r is (proposition 2.5)

$$m(x) \simeq (2^{1/p} - 1)|x - r|,$$

where p is the multiplicity of r. In Weyl's case we have

$$m_W(x) \simeq \frac{|P^{(p)}(r)|}{p!d|a_d|(2\rho)^{d-1}}|x - r|^p.$$

For $p = 1$ this gives

$$m_W(x) \simeq \frac{|P'(r)|}{d|a_d|(2\rho)^{d-1}}|x - r|$$

and by lemma 6.2

$$\frac{|P'(r)|}{d|a_d|(2\rho)^{d-1}} \leqslant \frac{1}{d} < 1,$$

instead of

$$m(x) \simeq |x - r|.$$

Consequently our method is better than Weyl's one; this has been corroborated by several numerical examples.

5. Numerical examples

In the following examples we compute for a given precision ϵ, the number of iterations *Nit*, the lower and the upper bounds of the intervals which may contain the roots. We used float arithmetic on a CDC 4600. The CPU time for the exclusion algorithm is always less than 0.1 second. The bound ρ for the modulus of the roots is the bound given in section 3.5.

EXAMPLE 5.1

We consider $P(x) = x^3 - x$. The roots are -1, 0, 1. The upper bound for the modulus of the roots if $\rho = 1.52$. The CPU time in Weyl's method is equal to 1.1 second for an accuracy $\epsilon = 10^{-6}$. See table 1.

Table 1
Example 5.1.

ϵ	Exclusion			Weyl		
	Nit	Lower bound	Upper bound	*Nit*	Lower bound	Upper bound
10^{-3}	37	−1.0000481	0.9989999	305	−1.0071517	−0.9912501
		−0.0000214	0.0010000		0.0157156	0.0162603
		0.9999638	1.0010114		0.9927865	1.0086665
10^{-4}	49	−1.0000477	−0.9998999	451	−1.0007195	−0.9991301
		−0.0000000	0.0001000		−0.0015733	0.0016243
		0.9999999	1.0001001		0.9992807	1.0008698
10^{-5}	59	−1.0000000	−0.9999899	598	−1.0000731	−0.9999140
		−0.0000000	0.0000100		−0.0001572	0.0001625
		0.9999999	1.0000100		0.9999293	1.0000880
10^{-6}	69	−1.0000000	−0.9999989	744	−1.0000074	−0.9999915
		−0.0000000	0.0000010		−0.0000157	0.0000162
		0.9999999	1.0000010		0.9999920	1.0000080

EXAMPLE 5.2

We consider $P(x) = x^5 - 50x^3 + 625x$. The roots are $-5, -5, 0, 5, 5$. The upper bound for the modulus of the roots is $\rho = 7.77$. The CPU time in Weyl's method is 11.2 seconds for an accuracy $\epsilon = 10^{-3}$. See table 2.

Table 2
Example 5.2.

ϵ	Exclusion			Weyl		
	Nit	Lower bound	Upper bound	*Nit*	Lower bound	Upper bound
10^{-3}	93	−5.0024213	−4.9974836	8417	−5.6708516	−4.0658522
		−0.0000001	0.0010000		−0.4743324	0.4756657
		4.9983680	5.0030270		4.0671853	5.6721837
10^{-4}	118	−5.0001683	−4.9997050	23975	−5.2307093	−4.7457093
		−0.0000031	0.0001000		−0.0466286	0.0466712
		4.9998007	5.0002772		4.7457520	5.2307519
10^{-5}	144	−5.0000199	−4.9999721	67942	−5.0752038	−4.9224538
		−0.0000001	0.0000100		−0.0046567	0.0046632
		4.9999803	5.0000280		4.9224604	5.0752204
10^{-6}	169	−5.0000023	−4.9999974	202169	−5.0240257	−4.9757397
		−0.0000000	0.0000010		−0.0004661	0.0004668
		4.9999984	5.0000024		4.9757403	5.0240273

Table 3
Example 5.3.

ϵ	Exclusion			ϵ	Exclusion		
	Nit	Lower bound	Upper bound		*Nit*	Lower bound	Upper bound
10^{-3}	223	0.9990523	1.0010005	10^{-5}	295	0.9999950	1.0000100
		1.9997729	2.0010000			1.9999947	2.0000100
		2.9999217	3.001000			2.9999999	3.0000100
		3.9999949	4.010000			3.9999999	4.0000100
		4.9997263	5.0010000			4.9999999	5.0000100
		5.9996342	6.0010010			5.9999989	6.0000100
		6.9998937	7.0010044			6.9999954	7.0000100
		7.9999692	8.0010088			7.9999920	8.0000100
		8.9995256	9.0010079			8.9999999	9.0000100
		9.9999859	10.0010229			9.9999999	10.0000100
10^{-4}	260	0.9999950	1.0001000	10^{-6}	332	0.9999999	1.0000010
		1.9999612	2.0001000			1.9999999	2.0000010
		2.9999960	3.0001000			2.9999999	3.0000010
		3.9999505	4.0001000			3.9999999	4.0000010
		4.9999910	5.000100			4.9999999	5.0000009
		5.9999999	6.0001000			5.9999999	6.0000010
		6.9999999	7.0001000			6.9999999	7.0000010
		7.9999999	8.0001000			7.9999999	8.0000010
		8.9999999	9.0001001			8.9999999	9.0000010
		9.9999999	10.0001002			9.9999999	10.0000010

EXAMPLE 5.3

We consider $P(x) = x^{10} - 55x^9 + 1320x^8 - 18150x^7 + 157773x^6 - 902055x^5 + 3416930x^4 - 8409500x^3 + 12753576x^2 - 10628640x + 3628800$. The roots are the integers 1, 2, ..., 10. The upper bound for the modulus of the roots is $\rho = 75.92$. Weyl's method gives the interval $[-52.54, 63.50]$ in 127,538 iterations in CPU time 1

Table 4
Example 5.4.

ϵ	Exclusion			Weyl		
	Nit	Lower bound	Upper bound	*Nit*	Lower bound	Upper bound
10^{-1}	25	−1.5117868	−1.0170778	40	−1.9990224	2.0004797
10^{-2}	25	no root		400	−1.9990224	2.0002244
10^{-3}	25	no root		3999	−1.9990224	1.9991968
10^{-4}	25	no root		36879	−1.8221371	1.5790627

Table 5
Example 5.5.

ϵ	Exclusion			ϵ	Exclusion		
	Nit	Lower bound	Upper bound		*Nit*	Lower bound	Upper bound
10^{-1}	23	−2.4238712431	−1.6238712431	10^{-4}	66	−2.0003097695	−1.9996097695
		0.9432167819	1.2423167819			0.99995	1.00025
10^{-2}	37	−2.0363554473	−1.9563554473	10^{-5}	79	−2.0000343029	−1.9999543029
		0.9995	1.0025			0.9999995	1.0000025
10^{-3}	52	−2.0028281827	−1.9958281827	10^{-6}	94	−2.0000027403	−1.9999957403
		0.9995	1.0025			0.9999995	1.0000025

minute for an accuracy $\epsilon = 10^{-3}$. In this case $m_W(x) = 2.3 \times 10^{-21}|P(x)|$: this explains this bad result. See table 3.

EXAMPLE 5.4

We now consider $P(x) = \sum_{k=0}^{10} x_k$ which does not possess any real root. The upper bound for the modulus of the roots is $\rho = 2$. CPU time in Weyl's method is more than 1 minute for an accuracy $\epsilon = 10^{-4}$. See table 4.

Weyl's method fails in the following examples.

EXAMPLE 5.5

We consider $P(x) = x^5 + x^4 - 4x^3 + 2x^2 + 8x - 8$. The real roots are $-2, 1$ with respective multiplicity 2 and 1. The upper bound for the roots is $\rho = 3.98$. See table 5.

Table 6
Example 5.6.

ϵ	Exclusion			ϵ	Exclusion		
	Nit	Lower bound	Upper bound		*Nit*	Lower bound	Upper bound
10^{-1}	54	−2.5058676083	−1.7058676083	10^{-4}	89	−2.0002864328	−1.9995864328
		−1.7647613648	−1.6647613648			0.9999419351	1.0000173641
		0.8780919064	1.1780919064			2.9994457832	3.0006457832
		2.6596334933	3.8806170428				
		3.6675159389	3.8675159389				
10^{-2}	64	−2.035447372	−1.9553447372	10^{-5}	101	−2.0000315302	−1.9999615302
		0.9916508417	1.0216508417			0.9999873641	1.0000249911
		2.9501236250	3.0601236250			2.9999480774	3.0000580774
10^{-3}	78	−2.0026454369	−1.9956454369	10^{-6}	157	−2.0000034976	−1.9999954976
		0.9994872893	1.0024872893			0.9999994999	1.0000024929
		2.9951411552	3.0061411552			2.9999808483	3.0000068483

Table 7
Example 5.7.

	Exclusion				Exclusion		
ϵ	*Nit*	Lower bound	Upper bound	ϵ	*Nit*	Lower bound	Upper bound
10^5	182	−0.9969224559	−0.9968924559	10^{-6}	185	−0.9969179088	−0.996914179088
		−0.9723800487	−0.9723500487			−0.9723704209	−0.9723674209
		−0.9238864136	−0.9238564136			−0.9238800325	−0.9238770325
		−0.8526453975	−0.8526153975			−0.8526408898	−0.8526378898
		−0.7604110412	−0.7603810412			−0.7604065379	−0.7604035379
		−0.6494530780	−0.6494230780			−0.6494485764	−0.6494455764
		−0.5225035772	−0.5224735772			−0.5224990764	−0.5224960764
		−0.3826884374	−0.3826584374			−0.3826839370	−0.3826809370
		−0.2334503656	−0.2334203656			−0.2334458654	−0.2334428654
		−0.0784640960	−0.0784340960			−0.0784595960	−0.0784565960
		0.0784540957	0.0784840957			0.0784585957	0.0784615987
		0.2334403638	0.2334703638			0.2334448638	0.2334478638
		0.3826784324	0.3827084324			0.3826829324	0.3826859324
		0.6494430483	0.6494430483			0.6494475483	0.6494505483
		0.7604009656	0.7604309656			0.7604054656	0.7604084656
		0.8526351644	0.8526651644			0.8526396644	0.8526426644
		0.9238689366	0.9238989366			0.9238790325	0.9238820325
		0.9723641259	0.9723941259			0.9723686412	0.9723716412
		0.9969123337	0.9969423337			0.9969168337	0.9969198337

Table 8
Example 5.8.

ϵ	Exclusion			ϵ	Exclusion		
	Nit	Lower bound	Upper bound		*Nit*	Lower bound	Upper bound
10^{-1}	85	−28.3414702088	−28.0414702088	10^{-4}	125	−28.22334245460	−28.2231245460
		−1.1638584670	−0.6638584670	10^{-4}	125	−0.8656279041	−0.8653279041
		4.2680609779	4.6680609779			4.31780724878	4.3183724878
		8.2371332505	8.4921332505			8.2420832505	8.2423832505
10^{-2}	98	−28.2303569384	−28.2003569384	10^{-5}	138	−28.2233794548	−28.2233494548
		0.8754557864	−0.8454557864	10^{-5}	138	−0.8655885043	−0.8655585043
		4.313124373	4.3431245373			4.3181259427	4.3181559427
		8.2416332505	8.2671332505			8.2421282505	8.2421582505
10^{-3}	112	−28.2238756078	−28.2208756078	10^{-5}	138	−28.2233734762	−28.2233704762
		−0.8661281287	−0.8631281287			−0.8655783006	−0.8655753006
		4.3176318764	4.3206318764			4.3181319412	4.3181349412
		8.2416332505	8.246332505			8.2421327505	8.242135505

EXAMPLE 5.6

We consider $P(x) = x^8 - 8x^7 + 14x^6 + 38x^5 - 145x^4 + 82x^3 + 234x^2 - 432x + 216$. The real roots are $-2, -2, 1, 3, 3, 3$. The upper bound for the modulus of the roots is $\rho = 10.95$. See table 6.

EXAMPLE 5.7

We consider $P(x) = 524288x^{20} - 2621440x^{18} + 5570560x^{16} - 6553600x^{14} + 4659200x^{12} - 2050048x^{10} + 549120x^8 - 84480x^6 + 6600x^4 - 200x^2 + 1$, the first kind Tchebychev polynomial of degree 20. Its roots are real and localized in $[-1, 1]$. The upper bound for the roots is $\rho = 3.62$. See table 7.

EXAMPLE 5.8

We consider $P(x) = 0.1495836012x^{10} + 0.52152613x^9 - 67.0508637x^8 + 851.5688445x^7 - 5094.094050x^6 + 17111.78804x^5 - 32750.95865x^4 + 30269.80956x^3 + 3027.90601x^2 - 31283.35894x + 19455.89724$. This polynomial appears in [4]. Its real roots are approximately $-28.22, -0.86, 4.31, 8.24$. The upper bound for the modulus of the roots is $\rho = 29.23$. See table 8.

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