Approximating the zeros of analytic functions by the exclusion algorithm

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We give a practical version of the exclusion algorithm for localizing the zeros of an analytic function and in particular of a polynomial in a compact of $\mathbb{C}$. We extend the real exclusion algorithm to a Jordan curve and give a method which excludes discs without any zero. The result of this algorithm is a set of discs arbitrarily small which contains the zeros of the analytic function.

Keywords: Exclusion, analytic function, polynomial, path, zeros.

AMS classification: 26C10, 65H05, 12D10.

0. Introduction

Let f be a function analytic at a point $a \in \mathbb{C}$, the set of complex numbers, such that $f(a) \neq 0$. The purpose of this study is the localization of the zeros of f in a compact set E: this paper is the announced continuation of [5] where we only approximate the real roots of a polynomial. The boundary of E is supposed to be a Jordan curve. We know that the set Z of the zeros of f is a finite set inside E. In order to localize Z, we define in section 2 an exclusion function $m(z)$ which verifies the following properties:

(1) $z \in Z$ iff $m(z) = 0$.

(2) If $f(z_0) \neq 0$ then $f(z) \neq 0$ for each $z \in B(z_0, m(z_0))$.

Furthermore, this function verifies Lipschitz and Łojasiewicz conditions: this study, similar to [5] with specific differences for the analytic case, is the purpose of section 2. The existence of such an exclusion function permits us to give in section 1 the general algorithm: the fundamental difference with [5] is the strategy used to exclude the open disc $B(z_0, m(z_0))$ of the compact set E since we cannot use a natural order as in the real exclusion algorithm. Because of this, we introduce the exclusion along a path in section 1.1. In section 3 we deal with the multiplicity of the zeros and separation of two zeros: the idea is to find another exclusion function associated with f when $m(z)$ is small enough. Sections 4 and 5 are devoted to the practical computation of $m(z)$ and to the estimation of the remainder set Z_ϵ which approximates the zeros with a given accuracy ϵ. We show in section 6 that this set Z_ϵ is obtained with a number $O(\mathrm{Log}(1/\epsilon))$ of steps (i.e. number of computations of the exclusion function). We illustrate this study with numerical examples in section 7. The method presented here is different from those of [2] or [6] in two points:

(1) The definition of the Weyl exclusion function: unfortunately, as is shown in [5], the numerical behavior of the Weyl function is bad near the zeros.

(2) The Weyl algorithm is based on dichotomy.

1. Description of the exclusion algorithm

Our aim is to localize the roots of an analytic each one f in a compact set E. We suppose that the set E is a given union of sets A_i the boundary of which is a Jordan curve. We first give the general form of this algorithm, each point will be discussed in more detail afterwards.

Inputs: $\epsilon > 0$, $Z_\epsilon = \emptyset$, $E = \cup_{i=1}^{n} A_i$, n is the number of connected components of E.
While $n > 0$ **do**
  begin
  Choose $z_0 \in$ boundary of E
  Compute $m(z_0)$, the exclusion function of f at z_0
  If $m(z_0) > \epsilon$ **then** $r_0 := m(z_0)$ **else** $r_0 := \epsilon$, $Z_\epsilon := Z_\epsilon(Z_0, r_0)$
  Compute the connected components of the set $E - B(z_0, r_0)$, and hence a new value for n.
end

PROPOSITION 1.1.
This algorithm stops in a finite number of steps.

Proof
In other words, this algorithm consists in constructing a strictly decreasing sequence of compact sets $E_0 = E$, $E_{i+1} = E_i - B(z_i, r_i)$ where z_i lies on the boundary of E_i. Since the radii $r_i \geqslant \epsilon$ and the z_i lie on the boundaries of the sets E_i, we have $\cap_i E_i = \emptyset$. Hence for some index p, the set E_p has only one connected component which is included in the disc $B(z, \epsilon)$ for every z belonging to the boundary of E_p. Consequently $E_{p+1} = \emptyset$ and the conclusion of this proposition follows. □

1.1. EXCLUSION ALONG A PATH

We now describe how we compute the connected components of the set $E - B(z_0, r_0)$. In other words, how to obtain fig. 2 from fig. 1.

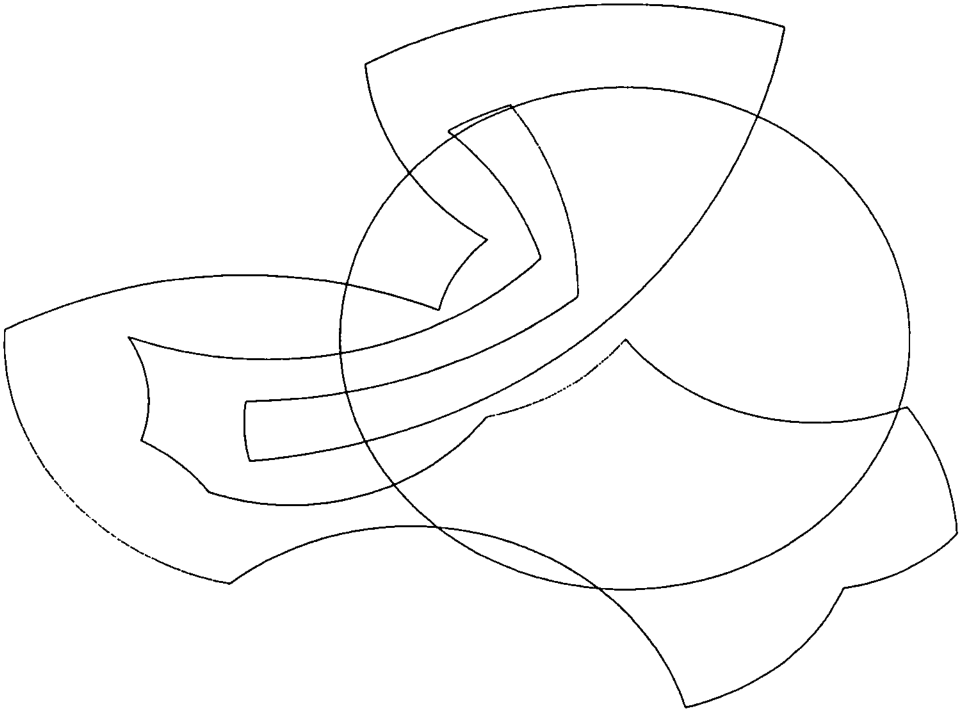


Fig. 1.

We first make precise the notation and hypotheses. Without loss of generality we can suppose that E is a connected compact subset of $\mathbb{C}$ the boundary of which is a given path γ. By path, we mean a finite sequence of curves $\gamma = \{\gamma_1, \ldots, \gamma_l\}$. Each curve γ_i is defined on an interval $[a_i, b_i] \subset \mathbb{R} \to \mathbb{C}$ and is assumed to be class C^1. Furthermore, we have $\gamma_i(b_i) = \gamma_{i+1}(a_{i+1})$, $1 \leqslant i \leqslant l-1$. All the paths considered here will be closed, i.e. $\gamma_l(b_l) = \gamma_1(a_1)$ and simple, i.e. $\gamma_i(r) = \gamma_j(s) \Leftrightarrow i = j$ *and* $r = s$. By the Jordan theorem [1] a simple closed path γ separates $\mathbb{C}$ into two connected components and we denote by $\mathfrak{Int}\gamma$ the compact one. We define an ordering on the path γ. Let y and z belonging to γ so that $y = \gamma_i(r)$, $z = \gamma_j(s)$. We say

$$y \preceq_\gamma z \Leftrightarrow \begin{cases} i<j & \text{if } i \neq j, \\ \left.\begin{array}{ll} r \leqslant s & \text{if } a_i < b_i \\ r \geqslant s & \text{if } a_i > b_i \end{array}\right\} & \text{if } i = j. \end{cases}$$

Also, if $y \preceq_\gamma z$, $\gamma_{[y,z]}$ is the path included in γ, where y is the initial point and z the end point. The meaning of notations such as $\gamma_{]y,z[}\ldots$, is clear. We use the notation (see [1]) $W(z, \gamma)$ for the winding number of the path γ with respect to z. The union of two paths $\gamma_{[x,y]}$ and $\omega_{[y,z]}$ is denoted by $\gamma_{[x,y]} \cup \omega_{[y,z]}$. The complement of the sub-path $\gamma_{[x,y]}$ in γ with the initial point x and the end point y is denoted by $\gamma - \gamma_{[x,y]}$.

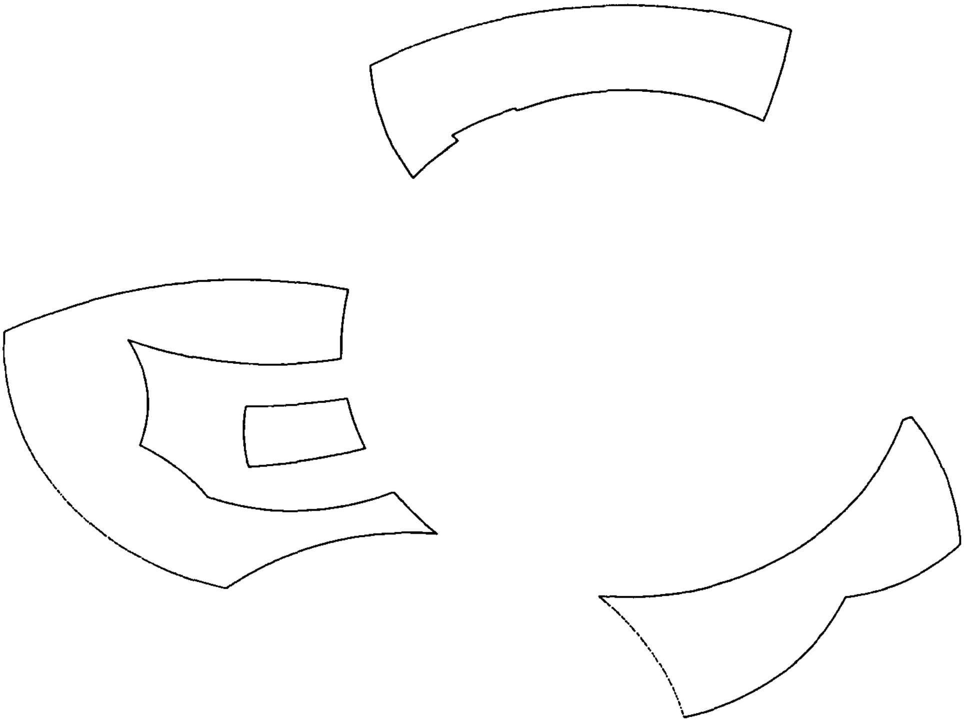


Fig. 2.

Let B_0 be the open disc of radius $r_0>0$ centered at $z_0=\gamma_1(a_1)$, C_0 the circle associated with B_0. Assume E is not included in B_0 and the number of intersection points of C_0 with the curves γ_i is finite. This is the case when the curves γ_i are rational. We consider the points $z_i\in\gamma\cap C_0$, $1\leqslant i\leqslant 2p$, verifying:

(1) if $x\in\gamma$ is such that $x\prec_\gamma z_i$ and $\gamma_{]x,z_i[}\subset B_0$ (resp. $\gamma_{]x,z_i[}\cap B_0=\emptyset$), then there exists $y\in\gamma$ such that $z_i\prec_\gamma y$ and $\gamma_{]z_i,y[}\cap B_0=\emptyset$ (resp. $\gamma_{]z_i,y[}\subset B_0$);

(2) $z_1\prec_\gamma z_2\prec_\gamma\dots\prec_\gamma z_{2p}$. We shall use the convention $z_i=z_{i\equiv k(2p+1)}$.

Observe that the number of such points in the intersection of $\gamma\cap C_0$ is even since the path γ is simple and closed. We only take into consideration the intersection points of this class: see remark 1.6. Furthermore, by the definition of the points z_i we have $\gamma_{]z_{2i-1},z_{2i}[}\cap B_0=\emptyset$ and $\gamma_{[z_{2i},z_{2i+1}]}\subset B_0$, $1\leqslant i\leqslant p$. Finally we give the following

DEFINITION 1.2

The arc of circle sub-tended by the sub-path $\gamma_{[z_i,z_{i+1}]}$ is the arc of the circle denoted by $\omega_{[z_i,z_{i+1}]}$ included in C_0 which verifies:

(1) if $\gamma_{[z_i,z_{i+1}]}\cap B_0=\emptyset$ then:

$$W(x,\gamma_{[z_i,z_{i+1}]}\cup\omega_{[z_{i+1},z_i]})=0,\text{for every } x\in\mathring{B}_0\,.$$

(2) if $\gamma_{]z_i,z_{i+1}[} \subset B_0$ there are two cases:
(2.1) if $z_{i+1} \notin \omega_{[z_{i-1},z_i]}$ then

$$W(x, \gamma_{[z_i,z_{i+1}]} \cup \omega_{[z_{i+1},z_i]}) = -W(y, \gamma_{[z_{i-1},z_i]} \cup \omega_{[z_i,z_{i-1}]}),$$

for every $x \in \overset{\circ}{\mathfrak{Int}}(\gamma_{[z_i,z_{i+1}]} \cup \omega_{[z_{i+1},z_i]})$ and $y \in \overset{\circ}{\mathfrak{Int}}(\gamma_{[z_{i-1},z_i]} \cup \omega_{[z_i,z_{i-1}]})$;
(2.2) if $z_{i+1} \in \omega_{[z_{i-1},z_i]})$ then

$$W(x, \gamma_{[z_i,z_{i+1}]} \cup \omega_{[z_{i+1},z_i]}) = W(y, \gamma_{[z_{i-1},z_i]} \cup \omega_{[z_i,z_{i-1}]}),$$

for every $x \in \overset{\circ}{\mathfrak{Int}}(\gamma_{[z_i,z_{i+1}]} \cup \omega_{[z_{i+1},z_i]})$ and $y \in \overset{\circ}{\mathfrak{Int}}(\gamma_{[z_{i-1},z_i]} \cup \omega_{[z_i,z_{i-1}]})$.

This formal definition corresponds to figs. 3, 4 and 5 where $\gamma_{[z_i,z_{i+1}]}$ is denoted by γ_i.
In the next theorem we characterize the sets $E - B_0$ which are connected.

THEOREM 1.3
Let E be a compact connected set whose boundary is a simple closed path γ. Let B_0, C_0 and the points z_i, $1 \leqslant i \leqslant 2p$, be defined as previously. The set $E - B_0$ is connected if and only if for all i, $1 \leqslant i \leqslant 2p$ we have

$$\gamma_{[z_i,z_{i+1}]} \cap B_0 = \emptyset \quad \text{or} \quad \omega_{[z_i,z_{i+1}]} \subset E. \tag{1}$$

To prove this theorem, we first give two lemmas.

LEMMA 1.4
Let $p > 1$ and E be a connected set. If $\omega_{]z_i,z_{i+1}[} \cap \omega_{]z_j,z_{j+1}[} = \emptyset$ for every $i \neq j$ then the set $E - B_0$ is not connected.

Proof
Suppose $\omega_{]z_{2p},z_1[} \cap E = \emptyset$ holds. We deduce from this that $\omega_{]z_{2i-1},z_{2i}[} \subset E$ and $\omega_{]z_{2i},z_{2i+1}[} \cap E = \emptyset$. In particular, the arcs of circle $\omega_{]z_1,z_2[}$ and $\omega_{]z_{2p-1},z_{2p}[}$ are separated by the arcs of circle $\omega_{]z_{2p},z_1[}$ and $\cup_{i=2}^{2p-2} \omega_{[z_i,z_{i+1}[}$. Consequently the two sets $\overset{\circ}{\mathfrak{Int}}(\gamma_{[z_1,z_2]} \cup \omega_{[z_2,z_1]})$ and $\overset{\circ}{\mathfrak{Int}}(\gamma_{[z_{2p-1},z_{2p}]} \cup \omega_{[z_{2p-1},z_{2p}]})$ containing a part of the set $E - B_0$

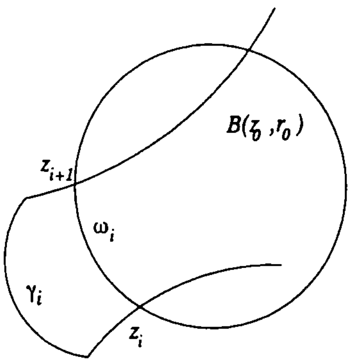


Fig. 3.

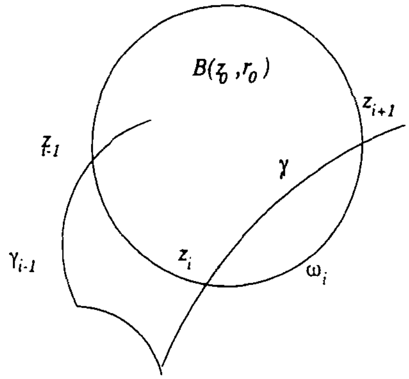


Fig. 4.

are separated by the disc B_0 containing itself the set $E \cap B_0$. This fact proves that $E - B_0$ is not connected.

To prove that $\omega_{]z_{2p},z_1[} \cap E = \emptyset$, let us assume the converse. By the hypothesis there is no point z_i lying in $\omega_{]z_{2p},z_1[}$. Hence $\omega_{]z_{2p},z_1[} \subset E$. Let $y \in \omega_{]z_{2p},z_1[}$ and consider the half line $D = \{y + r(y - z_0) : r \geqslant 0\}$: since E is a compact set, we have $D \cap \gamma \neq \emptyset$. Hence there exists $z \in D \cap \gamma$ such that $\|z - y\| < \|x - y\|$ for every $x \in D$ and $x \notin E$. We can now consider the index i such that $z_i \preceq_\gamma z \preceq_\gamma z_{i+1}$. We have $\gamma_{[z_i,z_{i+1}]} \cap B_0 = \emptyset$ since by definition of the points z_i, the only points of intersection of C_0 and $\gamma_{[z_i,z_{i+1}]}$ are z_i and z_{i+1} and since $x \notin B_0$.

Hence the segment $[y, z]$ is contained in the set $\mathring{\mathcal{I}nt}(\gamma_{[z_i,z_{i+1}]} \cup \omega_{[z_{i+1},z_i]})$. By definition 1.2 we conclude that $\omega_{[z_{2p},z_1]} \subset \omega_{[z_{i+1},z_i]}$, which gives a contradiction to the hypothesis. This proves the lemma. □

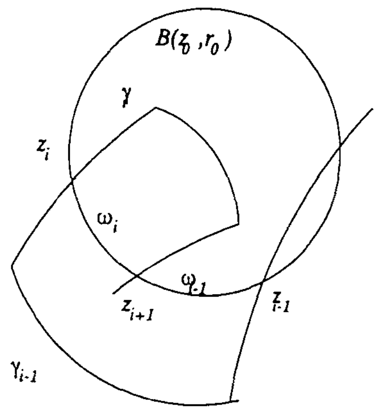


Fig. 5.

LEMMA 1.5

Let $p>1$ and assume the assertion (1) of the theorem. If E is a connected set there exists some index i such that $E - B_0 \subset \mathring{\mathfrak{Int}}(\gamma_{[z_i,z_{i+1}]} \cup \omega_{[z_{i+1},z_i]})$.

Proof

Define $i = \min\{k : \omega_{[z_k,z_{k+1}]} \cap \omega_{[z_{k-1},z_k]} \neq \emptyset\}$. We are going to show that $\omega_{[z_k,z_{k+1}]} \subset \omega_{[z_i,z_{i+1}]}$ for every $k \neq i$. With this in view, let us suppose that $C_0 - \omega_{[z_i,z_{i+1}]}$ contains some point z_j. In particular, we consider the two points z_k and z_l lying in $C_0 - \omega_{[z_i,z_{i+1}]}$ which are respectively the "nearest" z_i and z_{i+1}. Suppose that $z_k \preceq_\gamma z_l$. The arc of circle $\delta_{[z_i,z_k]}$ (respectively $\delta_{[z_l,z_{i+1}]}$) contained in $C_0 - \omega_{[z_i,z_{i+1}]}$ with initial point z_i (resp. z_l) and end point z_k (resp. z_{i+1}) has an empty intersection with E. Since the points z_{k-1} or z_{l+1} belong to $\omega_{[z_i,z_{i+1}]}$, the arcs of circle $\omega_{[z_{k-1},z_k]}$ or $\omega_{[z_l,z_{l+1}]}$ sub-tended by the respective sub-paths $\gamma_{[z_{k-1},z_k]}$ or $\gamma_{[z_l,z_{l+1}]}$ contained in the disc B_0 do not verify the assertion (1). Consequently $C_0 - \omega[z_i, z_{i+1}] \cap E = \emptyset$ and the lemma is proved. □

Proof of the theorem

In the case $p = 1$ it is easy to see that $E - B_0$ is connected and formula (1) holds. Let $p>1$. We first show that if there exists some i which verifies $\gamma_{[z_i,z_{i+1}]} \cap B_0 \subset E$ and $\omega_{[z_i,z_{i+1}]}) \not\subset E$, then the set $E - B_0$ is not connected. There can be two cases. In the case where $\omega_{[z_i,z_{i+1}]} \cap E = \emptyset$, the two sets $\mathring{\mathfrak{Int}}(\gamma_{[z_i-1,z_i]} \cup \omega_{[z_i,z_{i-1}]})$ and $\mathring{\mathfrak{Int}}(\gamma_{[z_{i+1},z_{i+2}]} \cup \omega_{[z_{i+2},z_{i+1}]})$ containing part of the set $E - B_0$ are separated by the disc B_0 containing itself the set $E \cap B_0$. Hence $E - B_0$ is not connected. In the case where $\omega_{[z_i,z_{i+1}]} \cap E \neq \emptyset$ there are two consecutive points, say z_k and z_{k+1}, on γ such that $\omega_{[z_k,z_{k+1}]} \cap E = \emptyset$ and $\omega_{[z_k,z_{k+1}]} \subset \omega_{[z_i,z_{i+1}]}$. Substituting i by k, we then come back to the previous case.

We proceed by induction to prove the converse. Denote by E_p a connected set which has p points of intersection with B_0 and suppose that if formula (1) holds for $1 \leqslant i \leqslant p-1$ then $E_{p-1} - B_0$ is connected. Applying lemma 1.5 to a set E_p, we consider the index i such that $\omega_{[z_k,z_{k+1}]} \subset \omega_{[z_i,z_{i+1}]}$ for all $k \neq i$. We have $\omega_{[z_{i-1},z_i]} \subset B_0$. We can now construct a simple path δ which separates the set E_p into two sets E_2 and E_{p-1} so that:

(1) both the points z_{i-1} and z_i belong to E_2 and not to E_{p-1},

(2) the sets $\mathring{E}_2$ and $\mathring{E}_{p-1}$ form a partition of the set $\mathring{E}_p - \delta$.

By the induction hypothesis, the set $E_{p-1} - B_0$ is connected as well as $E_2 - B_0$ since $E_2 \cap B_0 = \{z_i, z_{i+1}\}$. Hence $E_p - B_0$ is connected and the theorem is proved. □

We now describe the boundary of the set $E - B_0$ between two consecutive points z_i and z_{i+1} in γ. For this purpose we introduce the permutation σ and its inverse σ^{-1} which is defined by

$$z_{\sigma(1)} \preceq_{C_0} z_{\sigma(2)} \preceq_{C_0} \cdots \preceq_{C_0} z_{\sigma(2p)}\,.$$

We adopt the following convention:

$$\sigma(i) = \begin{cases} \sigma(i \equiv 2p) & \text{if } 2p < i, \\ \sigma(2p + i) & \text{if } i < 1, \\ \sigma(i) & \text{otherwise.} \end{cases}$$

Under these conditions we have

$$z_{\sigma(\sigma^{-1}(i)-1)} \preceq_{C_0} z_i \preceq_{C_0} z_{\sigma(\sigma^{-1}(i)+1)}, \quad \text{for all } 1 \leqslant i \leqslant 2p.$$

The boundary of $E - B_0$ is composed successively of:

(1) a sub-path $\gamma_{[z_{2i-1}, z_{2i}]}$, $1 \leqslant i \leqslant p$.

(2) an arc of circle $\omega_{[z_{2i}, z]}$ where $z \in \{z_{\sigma(\sigma^{-1}(i)-1)}, z_{\sigma(\sigma^{-1}(i)+1)}\}$ such that $\omega_{[z_{2i}, z]} \subset E$, for $1 \leqslant i \leqslant p$.

We now describe in pseudo language code the algorithm of exclusion along a path which computes the boundary of $E - B_0$:

Inputs: $\gamma_1 = \{\gamma_{11}, \dots \gamma_{1l}\}$ simply closed path such that $\mathcal{I}nt\gamma := E$
$z_0 := \gamma_1(a_1)$, C_0 arc of circle of radius $r_0 > 0$ centered at z_0.

```
Begin
Compute {z_1, ..., z_2p} = γ ∩ C_0
if p = 0 then n := 0                      E ⊂ B_0
else
  begin
  Compute σ and σ^{-1}, n := 0
  While p > 0 do
    begin
    i := 1, j := p, n := n + 1            the number of connected components
                                          increases by one.
    While j ≠ 1 do
      begin
      γ_ni = γ_[z_i, z_{i+1}]
      Compute j ∈ {σ(σ^{-1}(i) − 1), σ(σ^{-1}(i) + 1)} so that ω_[z_2i, z] ⊂ E
      γ_{ni+1} = ω_[z_{i+1}, z_j]
      z = {z_1, ..., z_2p} − {z_i, z_{i+1}},  we exclude the points z_i and z_{i+1} from the
                                          list.
      p := p − 2
      i := j                              we continue with the point z_j while j ≠ 1.
      end
    end
  end
end
```

Remark 1.6

If there are double points in the intersection of $\gamma \cap C_0$ it is sufficient to divide the boundaries of the connected components of $E - B_0$ correctly in order to obtain Jordan curves.

Remark 1.7

In practice, if we take circle arcs for the γ_{1i}'s, $1 \leqslant i \leqslant l$ which initialize the previous algorithm, we proceed in the following way. Such a circle arc, say $\gamma_{[a_i,b_i]}$, is represented by six parameters: center a, radius r, argument $(a_i - a)$, argument $(b_i - a)$, $W(a, C(a,r))$ and position of E with regard to the disc $B(a,r)$. To compute the intersection $\gamma \cap B_0$, we first test whether $B_0 \cap C(a,r) \neq \emptyset$. In the affirmative, we compute this intersection, say $\{x, y\}$. We next determine whether or not these points belong to $\gamma_{[a_i,b_i]}$ and we order the points x and y in γ. In view of that, we have the following:

LEMMA 1.8

Introduce the notation: $n(u,v) = (u-v)/\|u-v\|$. Also define $u \times v = u_1v_2 - u_2v_1$ and $u \cdot v = u_1v_1 + u_2v_2$. We have

(1) $x \in \gamma[a_i, b_i]$ iff $(n(a_i,b_i) \times n(a_i,x))W(a, C(a,r)) < 0$.

(2) Let x and y by in $\gamma_{[a_i,b_i]}$: $x \preceq_\gamma y$ iff $n(a_i,b_i) \cdot n(a_i,x) < n(a_i,b_i) \cdot n(a_i,y)$.

The proof of this elementary lemma is left to the reader.

2. The exclusion function

Denote $\|z\|$ the Euclidian norm in $\mathbb{C}$ and $B(z,r)$ the associated open disc of radius r centered at z. Let f be an analytic function and R_a the radius of convergence of the power series $\sum_{k=0}^{\infty}(f^{(k)}(a)/k!)(z-a)^k$. Let E be a compact set contained in $B(a, R_a)$ and z_0 an element of E and consider the power series in t defined by

$$M(z_0, t) = \|f(z_0)\| - \sum_{k=1}^{\infty} \frac{\|f^{(k)}(z_0)\|}{k!} t^k .$$

We deduce from [1, p. 410] that a lower bound of the radius of convergence of the power series $M(z_0,t)$ is $R_a - \|z_0 - a\|$. Further, $M(z_0,t)$ is concave, strictly decreasing with respect to $t \in [0, R_a - \|z_0 - a\|]$. This fact leads to the following definition for the exclusion function:

DEFINITION 2.1

Let f be an analytic function and a, R_a, E defined as previously. The exclusion function associated with an analytic function f is the function from E to $\mathbb{R}^+$ defined by

$$m(z) = \begin{cases} R_a - \|z-a\|, & \text{if } M(z, R_a - \|z-a\|) \geqslant 0, \\ \text{the positive root of } M(z,t) & \text{otherwise.} \end{cases}$$

We then have

PROPOSITION 2.2

The exclusion function verifies on the compact set E the following properties:

(1) $m(z_0) = 0$ iff $f(z_0) = 0$;

(2) if $f(z_0) \neq 0$ then $f(z) \neq 0$ for all $z \in B(z_0, m(z_0))$;

(3) $m(z)$ is continuous.

Proof

Using successively Taylor's formula at the point z and the triangle inequality we obtain $f(z) \geqslant M(z_0, \|z-z_0\|)$, and assertions 1 and 2 follow easily. To prove the continuity, we first suppose that $M(z_0, R_a - \|z_0 - a\|) > 0$. Since $M(z,t)$ is continuous at z, for every $\epsilon > 0$ there exists η such that $z \in B(z_0, \eta)$ implies $M(z, R_a - \|z_0 - a\|) > 0$. So, we can choose η such that $M(z, R_a - \|z_0 - a\| + \eta) > 0$ by continuity of $M(z,t)$ in t. Further, we have

$$R_a - \|z_0 - a\| - \eta < R_a - \|z-a\| < R_a - \|z_0 - a\| + \eta .$$

The strictly decreasing nature of $M(z,t)$ implies that

$$0 < M(z, R_a - \|z_0 - a\| + \eta) < M(z, R_a - \|z-a\|) < M(z, R_a - \|z_0 - a\| - \eta) .$$

Consequently for every $\epsilon > 0$ there exists η so that $m(z) = R_a - \|z-a\|$ for all $z \in B(z_0, \eta)$. The continuity of $R_a - \|z-a\|$ follows.

We now study the case $M(z_0, R_a - \|z_0 - a\|) \leqslant 0$. The arguments used for the continuity and strictly decreasing nature of $M(z,t)$ allow us to say: for every $\epsilon > 0$, there exists η such that $z \in B(z_0, \eta)$ implies

$$M(z, m(z_0) + \epsilon) < M(z_0, m(z_0)) = 0 < M(z, m(z_0) - \epsilon) .$$

If $z \in B(z_0, \eta)$ there are two cases. First, if $M(z, R_a - \|z-a\|) > 0$ then we have

$$M(z, m(z_0) + \epsilon) < 0 < M(z, R_a - \|z-a\|) < M(z, m(z_0) - \epsilon) ,$$

by definition of the radius of convergence of $M(z,t)$. Next, if $M(z, R_a - \|z-a\|) \leqslant 0$ then we have

$$M(z, m(z_0) + \epsilon) < 0 = M(z, m(z)) < M(z, m(z_0) - \epsilon) .$$

Applying the strictly decreasing nature of $M(z,t)$, we conclude that $m(z)$ in both cases is continuous. This proves the proposition. □

In fact the exclusion function $m(z)$ satisfies a Lipschitz condition:

PROPOSITION 2.3

Let z_1 and z_2 in E be such that $M(z_1, m(z_1))M(z_2, m(z_2)) \geqslant 0$. We have

$$|m(z_1) - m(z_2)| \leqslant \|z_1 - z_2\| .$$

Proof

If $m(z_i) = R_a - \|z_i - a\|$ for $i = 1, 2$, it is obvious. In the case where $M(z_i, R_a - \|z_i - a\|) \leqslant 0$ for $i = 1, 2$, the proof is based on the following lemma:

LEMMA 2.4

Let z in E be such that $M(z, R_a - \|z - a\|) \leqslant 0$.

(1) If there exists k such that $f^{(k)}(z) = 0$, then for each direction $w \in \mathbb{C}$, the exclusion function admits a right and left directional derivative in the direction w satisfying

$$m'_{\pm}(z; w) = \frac{w_0 f'(z) - \displaystyle\sum_{k=2}^{\infty} w_{k-1} \frac{f^{(k)}(z)}{(m-1)!} m^{k-1}(z)}{\displaystyle\sum_{k=1}^{\infty} \frac{\|f^{(k)}(z)\|}{(k-1)!} m^{k-1}(z)},$$

where

$$w_k = \begin{cases} \dfrac{f^{(k)}(z)}{\|f^{(k)}(z)\|} & \text{if } f^{(k)}(z) \neq 0 \\ \pm \dfrac{w}{\|w\|} & \text{else}, \end{cases} \qquad \text{for each } k \geqslant 0 .$$

(2) If $f^{(k)}(z) \neq 0$ for each k, then the exclusion function is differentiable. The expression for $m'(z)$ is deduced from the previous expression.

Suppose that this lemma holds. The mean value theorem applied to the directional derivative of $m(z)$ in the direction $z_1 - z_2$ implies that there exists $z = \theta z_1 + (1 - \theta) z_2$ with $0 \leqslant \theta \leqslant 1$ such that

$$m(z_1) - m(z_2) = \langle m'_{\pm}(z; z_1 - z_2), z_1 - z_2 \rangle .$$

Since by the previous lemma $\|m'_{\pm}(z; z_1 - z_2)\| < 1$ we conclude that $|m(z_1) - m(z_2)| \leqslant \|z_1 - z_2\|$ and proposition 2.3 is proved. □

Proof of lemma 2.4

We deal with the identity $M(z, m(z)) \equiv 0$ and proceed as in [5] but with directional derivatives and analytic functions. □

We now prove a Łojasiewicz inequality.

PROPOSITION 2.5

Let E be a compact set contained in $B(a, R_a)$ of the analytic function f. Then there exists $\alpha > 0$ such that

$$\alpha d(z, Z) \leqslant m(z) \leqslant d(z, Z),$$

for each $z \in E$.

The background of this proposition is

LEMMA 2.6

Let z_0 be a zero of f with multiplicity p. We have

$$\lim_{z \to z_0} \frac{m(z)}{\|z - z_0\|} = 2^{1/p} - 1.$$

On the other hand, $m(z)$ possesses right and left directional derivatives at z_0 in the direction w, which are equal to

$$m'_{\pm}(z_0; w) = \pm(2^{1/p} - 1)\frac{w}{\|w\|}.$$

Proof

This lemma is established in the same way as in [5]. □

Proof of proposition 2.5

Let us consider the function $\phi(z)$ defined by

$$\phi(z) = \begin{cases} m(z)/d(z, Z) & \text{if } x \in E - Z, \\ 2^{1/p} - 1 & \text{if } x \in Z. \end{cases}$$

As this function is continuous and never vanishes on the compact set E, we deduce the assertion of the proposition easily. □

Remark 2.7

If f is a polynomial of degree d we can replace the compact set E by $\mathbb{C}$. In this case, $R_a = \infty$ and

$$\lim_{\|z\| \to \infty} \left\| \frac{f(z)}{z} \right\| = 2^{1/d} - 1,$$

as is shown in [5]. Since $\lim_{d \to \infty} 2^{1/d} - 1 = 0$, we justify a posteriori the fact that the exclusion function vanishes on the boundary of the disk of convergence.

An example of the exclusion function is the following: $P(z) = z^2 + z + 1$ with roots $(-1 + \sqrt{3}I)/2, (-1 - \sqrt{3}I)/2$. Then

$$M(x,y,t) = ((x^2-y^2+x+1)^2+(2xy+y)^2)^{1/2} - ((2x+1)^2+y^2)^{1/2}t - t^2,$$

where we have substituted z by $x+yI$. We obtain the surface as shown in fig. 6.

3. Separation and multiplicity of zeros

We can define the exclusion function $m_p(z)$ associated to $f^{(p)}(z)$ by introducing the power series

$$M_p(z,t) = \|f^{(p)}(z)\| - \sum_{k=1}^{\infty} \frac{\|f^{(k+p)}(z)\|}{k!} t^k.$$

Further, if we are sufficiently near to a zero, we can determine the multiplicity p numerically by computing the successive exclusion function $m_k(z)$ while $m_k(z)$ is small enough. More precisely we have

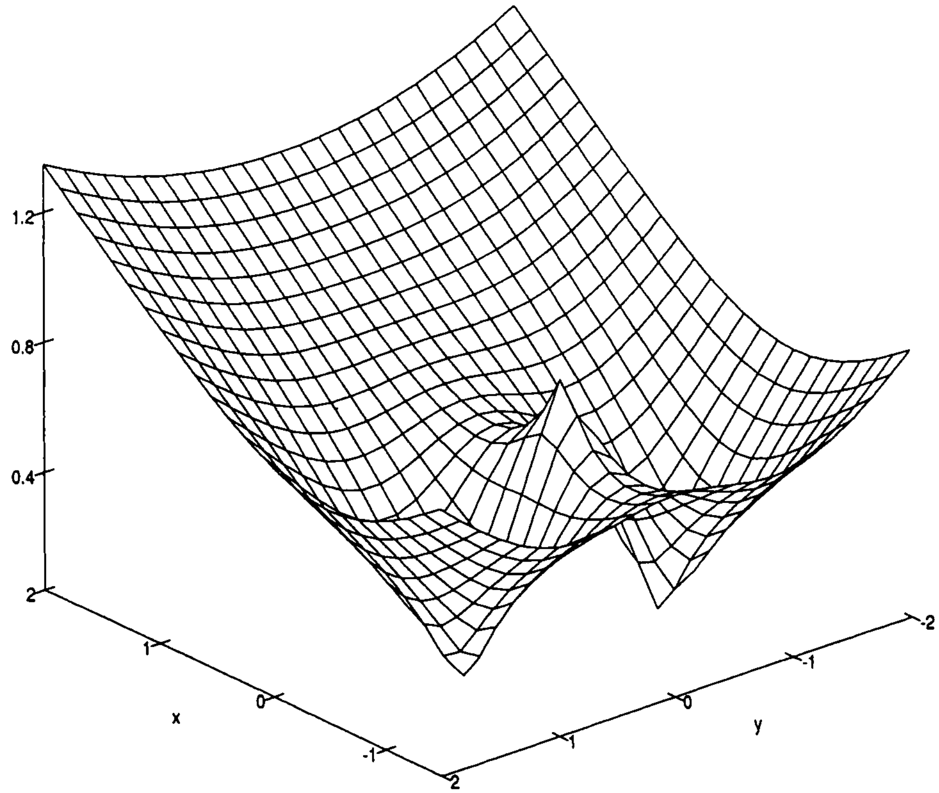


Fig. 6.

PROPOSITION 3.1

Let $\epsilon > 0$ and z_0 be a zero with multiplicity p. There exists $\eta > 0$ such that for all $z \in B(z_0, \eta)$ we have

$$\frac{m_i(z)}{m(z)} \simeq \frac{2^{1/(p-i)} - 1}{2^{1/p} - 1}, \quad 1 \leqslant i \leqslant p - 1 .$$

Proof

This is a direct consequence of lemma 2.6. □

Consequently, we can determine numerically the multiplicity of a zero. Since the application $i \to [2^{1/(p-i)} - 1]/[2^{1/p} - 1]$ is strictly decreasing we compute

$$p = \max\left\{ i : \frac{m_1(z)}{m(z)} \leqslant \frac{2^{1/(i-1)} - 1}{2^{1/i} - 1} \right\},$$

when $m(z)$ is "sufficiently small".

We now deal with the separation of zeros. Let z_0 be a given zero, the problem is to find a disc centered at z_0 which does not contain another zero of f. To this end, we introduce the power series

$$L_p(z, t) = \frac{\|f^{(p)}(z)\|}{p!} - \sum_{k=1}^{\infty} \frac{\|f^{(p+k)}(z)\|}{(p+k)!} t^k ,$$

and we denote by $l_p(z)$ the associated exclusion function. This is justified by the following:

PROPOSITION 3.2

Let z_0 be a zero of f of multiplicity p. We have

$$\|z_1 - z_0\| > l_p(z_0) \quad \text{for every zero } z_1 \neq z_0 .$$

Proof

If $\|z_1 - z_0\| > R_a - \|z_0 - a\|$, it is obvious. Let $z_1 \neq z_0$ be some zero of f such that $z_1 \in B(z_0, R_a - \|z_0 - a\|)$. By Taylor's formula, we have

$$0 = f(z_1) = \frac{f^{(p)}(z_0)}{p!}(z_0 - z_1)^p + \sum_{k=1}^{\infty} \frac{f^{(p+k)}(z_0)}{(p+k)!}(z_1 - z_0)^{p+k} .$$

Hence,

$$\left\| \frac{f^{(p)}(z_0)}{p!} + \sum_{k=1}^{\infty} \frac{f^{(p+k)}(z_0)}{(p+k)!}(z_1 - z_0)^k \right\| \cdot \|z_1 - z_0\|^p = 0 .$$

Since $z_0 \neq z_1$, we can write

$$\frac{\|f^{(p)}(z_0)\|}{p!} = \left\| \sum_{k=1}^{\infty} \frac{f^{(p+k)}(z_0)}{(p+k)!} (z_1 - z_0)^k \right\| \leqslant \sum_{k=1}^{\infty} \frac{\|f^{(p+k)}(z_0)\|}{(p+k)!} \|z_1 - z_0\|^k .$$

This implies that $L_p(z_0, \|z_1 - z_0\|) \leqslant 0$. We conclude that $\|z_1 - z_0\| \geqslant l_p(z_0)$ and the proposition is proved. □

In the particular case of the localization of real roots of a polynomial of degree d, it is easy to see that $l_p(z_0) > m_p(z_0)$.

We give the following example: $P(z) = z^2 - 1$, $M(z, t) = |z^2 - 1| - 2|z| - t^2$, $M_1(z, t) = 2|z| - 2t$ and $L_1(z, t) = 2|z| - t$. We have $m_1(\pm 1) = 1$ and $l_1(\pm 1) = 2$. See also numerical example 1.

4. Practical computation of the exclusion function and of Z_ϵ

When we deal with a power series, the coefficients are often the result of another computation. We have the following property concerning the stability:

PROPOSITION 4.1

Let $M_f(x, t)$ and $M_g(y, t)$ be the exclusion polynomials associated respectively with f and g at points x and y and $m_f(x)$ and $m_g(y)$ their respective exclusion functions. Let $\epsilon > 0$ be given. Then we have

$$\|y - x\| \leqslant \frac{\epsilon}{e^{m_f(x)+1} \max_k \|g^{(k)}(x)\|} \Rightarrow |M_g(y, m_f(x))| \leqslant \epsilon .$$

Proof

We only study the case where $M_f(x, R_a - \|x - a\|) < 0$. We first write

$$\begin{aligned} M_g(y, t) &= \|g(y)\| - \sum_{k=1}^{\infty} \frac{\|g^{(k)}(y)\|}{k!} t^k \\ &= \left\| \sum_{i=0}^{\infty} \frac{g^{(i)}(x)}{i!} (y - x)^i \right\| - \sum_{k=1}^{\infty} \frac{1}{k!} \left\| \sum_{i=0}^{\infty} \frac{g^{(k+i)}(y)}{i!} (y - x)^i \right\| t^k . \end{aligned}$$

We collect the terms with index $i = 0$ and use the triangle inequality. We obtain:

$$M_g(x, t) - \|y - x\| R(x) \leqslant M_g(y, t) \leqslant M_g(x, t) + \|y - x\| R(x) ,$$

where

$$R(x) = \left\| \sum_{i=1}^{\infty} \frac{g_i(x)}{i!} (y - x)^{i-1} \right\| + \sum_{k=1}^{\infty} \frac{1}{k!} \left\| \sum_{i=1}^{\infty} \frac{g^{(k+i)}(y)}{i!} (y - x)^{i-1} \right\| t^k .$$

Let us consider $G(x) = \max_i g^{(i)}(x)$. We write:

$$R(x) \leqslant G(x)\mathrm{e}^{\|y-x\|} + G(x)\mathrm{e}^{\|y-x\|}(\mathrm{e}^t - 1) = G(x)\mathrm{e}^{\|y-x\|}\mathrm{e}^t .$$

For $0 \leqslant s \leqslant 1$, we have $0 \leqslant s\mathrm{e}^s \leqslant s\mathrm{e}$. We substitute t by $m_f(x)$ in the previous inequality and we obtain the inequality of the proposition for every y such that

$$\|y - x\| < \frac{\epsilon}{\mathrm{e}^{m_f(x)+1} G(x)} .$$

□

Let us suppose that the calculation of $f^{(k)}(z)/k!$ is a solved problem. We have to calculate $M(z,t)$. For $\epsilon > 0$ we deal with

$$\bar{M}_n(z,t) = \|f(z)\| - \sum_{k=0}^{n} \frac{\|f^{(k)}(z)\|}{k!} t^k$$

instead of $M(z,t)$ such that $|\overline{M}_{n+1}(z,t) - M_n(z,t)| < \epsilon$. To compute $m(z)$, we adopt the following method when $M(z, R_a - \|z-a\|) < 0$:

(1) A dichotomy method is first used until we find a value $\bar{t}$ which verifies $M(z,\bar{t}) < 0$. This method is initialized with $t = \min(1, (R_a - \|z-a\|)/2)$.

(2) Next we use the Newton method following the process described in [5].

The principal reason of this is that the intersection of the tangent to the curve of the equation $s(t) = M(z,t)$ at the point $(t, M(z,t))$ with the x axis can be greater than $R_a - \|z-a\|$ when the Newton method is initialized with a value of t such that $M(z,t) > 0$.

The set Z_ϵ is a union of discs $B(z, m(z))$ with $m(z) \leqslant \epsilon$. Each disc can contain a root of f. We first discuss the theoretical estimation of Z_ϵ. We have the following estimation:

PROPOSITION 4.2

$$Z \subset Z_\epsilon \subset Z + B(0, 2\epsilon/\alpha) .$$

Proof

Let $y \in Z_\epsilon$. Then there exists z_0 such that $m(z_0) < \epsilon$ and $y \in B(z_0, \epsilon)$. The Łojasiewicz inequality and the fact that $m(z)$ is Lipschitz imply

$$\alpha d(y, Z) \leqslant m(y) \leqslant m(z) + \|y - z_0\| < 2\epsilon .$$

The proposition follows easily. □

But this localization requires a great number of discs to describe only one root of f: it is better to have a small number of discs for each root. In view of this, we proceed in the following way. Suppose that $Z_\epsilon = \cup_{i=1}^{p} B(z_i, r_i)$ at one step of the algorithm. Let z be such that $m(z) \leqslant \epsilon$. We collect the disc $B(z, m(z))$ with any disc of the previous union if for some i we have $r_i - \epsilon < d(z, z_i) \leqslant r_i$: then we replace in Z_ϵ the disc $B(z_i, r_i)$ by the disc

$$B(\tau(z_i - z), r)$$

where

$$r = \frac{r_i + d(z_i, z) + \epsilon}{2} \quad \text{and} \quad \tau = \frac{r}{d(z, z_i)} .$$

Obviously we do not count the disc $B(z, m(z)$ if $B(z, m(z)) \subset B(z_i, r_i)$. If for every $1 \leqslant i \leqslant p$ we have $d(B(z_i, r_i), B(z, \epsilon)) > \epsilon$, then we define $B(z_{p+1}, r_{p+1}) = B(z, \epsilon)$.

5. Complexity

The complexity of this algorithm is the estimation of the number of steps necessary to exclude entirely the initial set E. In other words, each calculation of the exclusion function represents one elementary step of the algorithm. But the direct study of the complexity with the exclusion function seems to be difficult. Since we have the inequality

$$\alpha d(z, Z) \leqslant m(z) ,$$

we deal with the function $\alpha d(z, Z)$ instead of $m(z)$. Further, the function $m(z)$ is equivalent to $(2^{1/p} - 1)d(z, Z)$ in a neighborhood of a zero z_0 with multiplicity p. Consequently, the complexity of the exclusion algorithm using $m(z)$ and that using $\alpha d(z, Z)$ is the same size.

THEOREM 5.1

Let $\overline{B}(a, \rho)$ be a disc containing d roots of the analytic function f. An upper bound for the number of steps in the exclusion algorithm is equal to

$$d\left(\left[\frac{\pi}{\arctan\frac{\alpha}{\sqrt{4-\alpha^2}}}\right] + 1\right)\left(\left[\frac{\log\frac{\epsilon}{2\rho}}{\log\beta}\right] + 1\right) + O(1) ,$$

with

$$\beta = \sqrt{1 - \frac{\alpha^2}{4}} - \frac{\sqrt{3}}{2}\alpha .$$

To prove this theorem, we first establish

LEMMA 5.2

Let a be a root of f and β be defined as in the previous theorem.

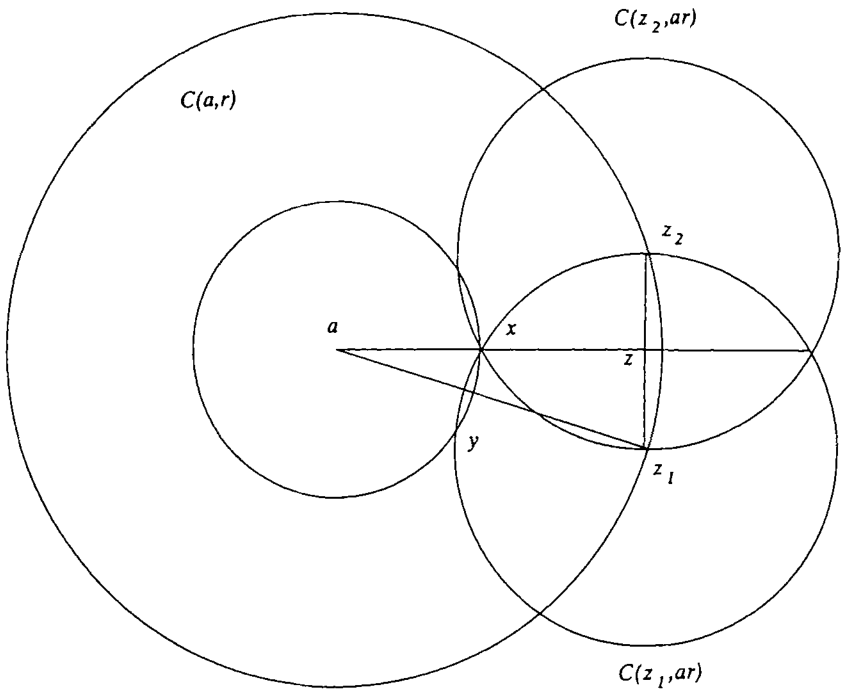


Fig. 7.

(1) An upper bound for the number of discs which cover the circle $C(a,r)$ in the exclusion algorithm using $\alpha d(z,Z)$ is equal to

$$s=\left[\frac{\pi}{\arctan\frac{\alpha}{\sqrt{4-\alpha^2}}}\right]+1\,.$$

(2) The circle $C(a,\beta r)$ is included in the set $\cup_{i=1}^{s}\overline{B}(z_i,\alpha r)$, where

$$z_1\in C(a,r),\quad z_i\in \bar{C}(a,r)\cap C(z_{i-1},\alpha r),\quad z_{i-1}\prec_{C(a,r)} z_i\,.$$

(3) Let $\omega_{[x,y]}$ be an arc of circle of the boundary of $\overline{B}(a,r)-\cup_{i=1}^{s}\overline{B}(z_i,\alpha r)$. Then we have

$$\|x-y\|=\alpha\beta r\,.$$

Proof

Let $C(z_1,\alpha r)$ and $C(z_2,\alpha r)$ be two circles such that $z_1\in C(a,\alpha r)$ and $z_2\in C(z_1,\alpha r)\cap C(a,r)$. By considerations of elementary geometry we have

(1) $\|a-z\|^2 = \|a-z_1\|^2 - \|z-z_1\|^2$, since the triangle azz_1 is right-angled. Thus $\|a-z\| = r\sqrt{1-\alpha^2/4}$, since $\|a-z\| = r$ and $\|z-z_1\| = \alpha r/2$.
(2) $\|z-x\| = \sqrt{3}\alpha r/2$, since the triangle $z_1 z_2 x$ is equilateral.
(3) $\|a-x\| = \|a-z\| - \|z-x\| = r(\sqrt{1-\alpha^2/4} - \sqrt{3}\alpha/2)$.
Then we have

$$\tan\frac{\theta}{2} = \frac{\|z_1 - z\|}{\|a-z\|} = \frac{\alpha}{\sqrt{4-\alpha^2}},$$

where $\theta/2$ is the angle zaz_1.

Consequently an upper bound for the number s of discs $B(z_i, \alpha r)$ which cover the circle $C(a,r)$ where the z_i are defined as previously, verifies $2\pi \leqslant s\theta$, i.e.

$$s = \left[\frac{\pi}{\arctan\dfrac{\alpha}{\sqrt{4-\alpha^2}}}\right] + 1.$$

Since $r_1 = \|ax\|$ is the maximum of the distance between the center a and the boundary of the set $\overline{B}(a,r) - \overline{B}(a,r) \cap \cup_{i=1}^{s} B(z_i, \alpha r)$, it follows that the circle $C(a,r) \subset \cup_{i=1}^{s} \overline{B}(z_i, \alpha r)$. To prove the last point of the lemma, we write $\|(x-y)/2\|^2 = \|z_1 - x\|^2 - \|z_1 - h\|^2$, where h is the middle of the segment xy. An easy computation shows that $\|z_1 - h\| = \frac{1}{2}(\alpha^2 + 1 - \beta^2)r$. Hence

$$\|\frac{x-y}{2}\|^2 = \left(\alpha^2 - \frac{(\alpha^2+1-\beta^2)^2}{4}\right)r^2 = \frac{-\beta^4 - \alpha^4 - 1 + 2\alpha^2 + 2\beta^2 + 2\alpha^2\beta^2}{4} r^2.$$

But by definition, β is the solution of the equation $\beta^4 + \alpha^4 + 1 - 2\alpha^2 - 2\beta^2 - \alpha^2\beta^2 = 0$. So, $\|x-y\| = \alpha\beta r$ and this lemma is proved. □

LEMMA 5.3

Let s be as before. An upper found for the number of discs which cover the set $\overline{B}(a,r) - \overline{B}(a,\epsilon)$ is given by

$$s\left[\frac{\log\frac{\epsilon}{r}}{\log\beta}\right] + 1.$$

Proof

The consequence of point 3 of lemma 5.2 is that we can replace the boundary of $B(a,r) - \cup_{i=1}^{s} \overline{B}(z_i, \alpha r)$ by the circle $C(a, \beta r)$ in the following step of thè exclusion algorithm. Then we define the sequence of circles $C(a, r_i)$

$$r_0 = r, \quad r_i = \beta r_{i-1}.$$

An index i such that $r_i \leqslant \epsilon$ verifies

$$i \geqslant \left[\frac{\log\frac{\epsilon}{r}}{\log\beta}\right] + 1 .$$

Applying part 1 of lemma 5.2, we find the desired upper bound. □

Proof of the theorem

Denote by a_i, $1 \leqslant i \leqslant d$, the zeros of the analytic function in the disc $\overline{B}(a, \rho)$. There exists a finite number N which does not depend on ϵ ("sufficiently small") such that the set $E - \cup_{i=1}^{N} B(z_i, \alpha d(z_i, Z))$ has d connected components A_i with the property that $d(z, Z) = \|z - a_i\|$ for every $z \in A_i$, $1 \leqslant i \leqslant d$. We claim that the number of steps needed to exclude all the A_i's is less than the number of steps needed to exclude separately the discs $B(a_i, 2p)$ with the exclusion function $\alpha d(a_i, Z)$, $1 \leqslant i \leqslant d$. We multiply the upper bound of the lemma by d and we substitute r by 2ρ to obtain the desired result. □

Remark 5.4

The complexity of the computation of the exclusion function $m(x)$ has been studied in [5]. We recall that this complexity is $O(\log\log 1/\epsilon)$.

We conclude this section by giving the evolution of the exclusion algorithm of the set $Z = \{1 + I, -1 + I, -I\}$ with the exclusion function $d(z, Z)/2$ and the initial set $E = B(0, 2)$. For each figure we show the set $E - B_0$ and the disc which will be excluded at the following step (see fig. 8).

6. Numerical examples

We give for each of the examples the number of steps *nit*, the ratio $m_1(z)/m(z)$ which permits to determine the multiplicity (proposition 3.1) and the positive root $l_p(z)$ as in proposition 3.2 which is called "separation" in the tables.

EXAMPLE 1

$f(z) = z^2 - 1, E = B(O, 1.5), \epsilon = 0.1$ We obtain $nit = 37$ (see table 1).

EXAMPLE 2

$f(z) = z^{10} - 1, E = B(0, 1.5), \epsilon = 0.1$. We obtain $nit = 665$ (see table 2).

Table 1
Results for example 1.

Root	Disc	m_1/m	Multiplicity	Separation
-1	$[-0.999, 0.0320, 0100]$	31.72	1	1.999
1	$[0.989, 0.008, 0.150]$	77.45	1	1.979

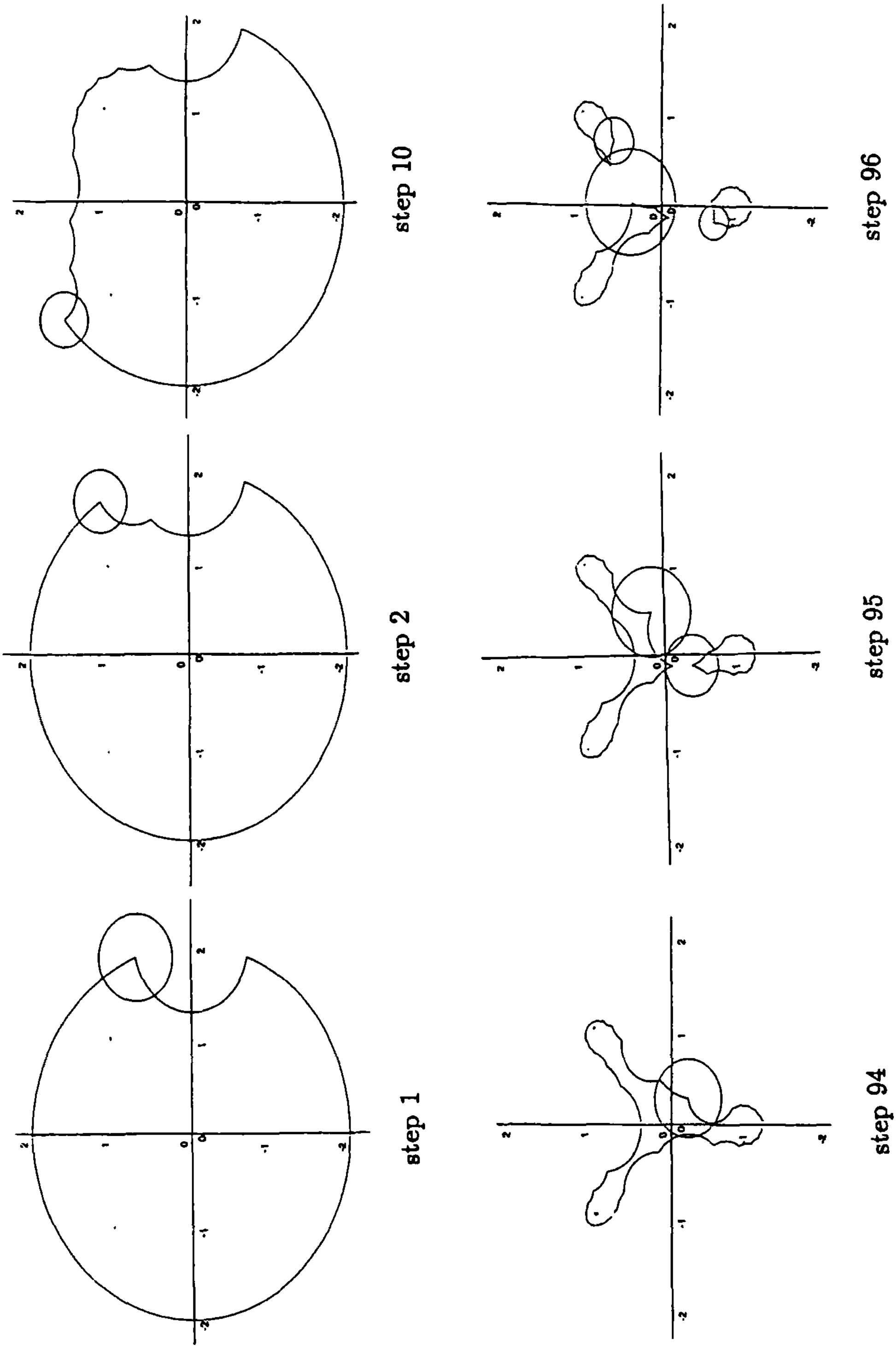
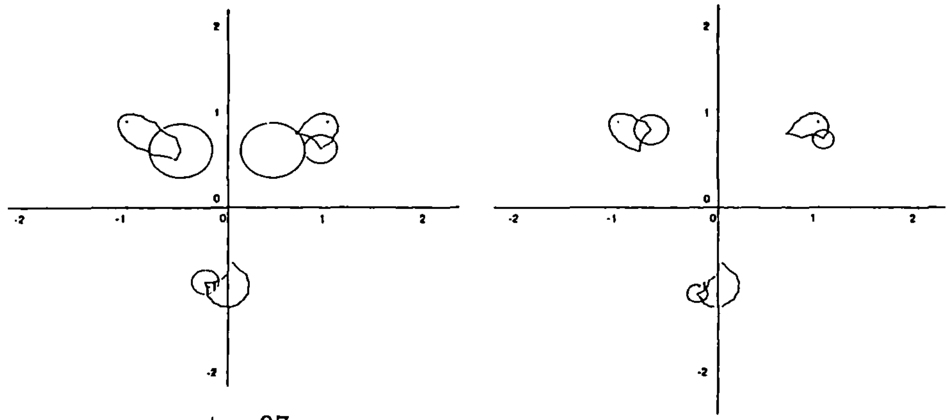


Fig. 8.



step 97 step 98

Fig. 8. Continued.

EXAMPLE 3

Let $P(z) = z^8 - 5/2(1+I)z^7 + (1+2I)z^6 + 3/4(-9+I)z^5 + (21/4I + 103/16)z^4 + (-1/32I - 25/32)z^3 + (21/4 - 13/16I)z^2 + (-29/16 - 43/16I)z + 1/2I - 3/8$, where the roots are $1+I, I/2, -1/2-I, 1/2+I/2$ with respective multiplicities $3, 2, 2, 1$. We take $\epsilon = 0.01$, $E = B(0, 1.5)$. We obtain $nit = 707$ (see table 3).

EXAMPLE 4: ZEROS OF THE RIEMANN ZETA FUNCTION

We deal with the function $\xi(s) = \sum_{k=0}^{\infty} a_{2k}(s-1/2)^{2k}$ [3], where

$$a_{2k} = \frac{1}{2^{2k-2}(2k)!}\int_1^{\infty} \frac{\mathrm{d}[x^{3/2}\psi(x)]}{\mathrm{d}x} x^{-1/4}(\log x)^{2k}\mathrm{d}x\,.$$

Table 2
Results for example 2.

Root	Disc	m_1/m	Multiplicity	Separation
[1,0]	[1.0040,0.0025,0.0164]	17.5	1	0.1475
[0.809,0.587]	[0.8119,0.5916,0.0194]	17.4	1	0.1475
[0.3090,0.9510]	[0.3073,0.9562,0.0172]	15.5	1	0.1475
[−0.3090, 0.9510]	[−0.3098, 0.9553, 0.0171]	19.2	1	0.1475
[−0.8090, 0.5877]	[−0.8120, 0.5890, 0.0192]	25.0	1	0.1473
[−1, 0]	[−1.0024, 0.0003, 0.0205]	33.2	1	0.1472
[−0.8090, −0.5877]	[−0.8148, −0.5908, 0.0168]	12.8	1	0.1478
[−0.3090, −0.9510]	[−0.3120, −0.9573, 0.0169]	12.3	1	0.1479
[−0.3090, −0.9510]	[0.3120, −0.9551, 0.0194]	16.5	1	0.1476
[0.8090, −0.5877]	[0.8125, −0.5882, 0.0190]	23.2	1	0.1473

Table 3
Results for example 3.

Root	Disc	m_1/m	Multiplicity	Separation
$1+I$	[1.0133,1.010,0.0627]	1.54	3	0.2141
$-1/2-I$	[-.5042, -1.0077, 0.0380]	2.28	2	0.2538
$1/2+I/2$	[0.5005,0.4966,0.0149]	34.72	1	0.2064
$1/2$	[.0054,.5006,.04020]	2.38	2	0.1827

The zeros of the function ξ are the zeros of the Riemann zeta function. The first zeros are 14.1, 21.0, 25.0, 30.4, 32.9, 37.5, 40.0. We search for the zeros on the line $s = 1/2 + Iu$. In this case, the a_{2k} are the result of a computation. This has a great influence on the computation of $m(u)$. We give the evolution of the exclusion algorithm on the real axis with the function $\xi(u) = \sum_{k=0}^{\infty}(-1)^k a_{2k} u^{2k}$ as described in [5] (see table 4). A forthcoming paper will be devoted especially to computation of the zeros of the zeta function by this algorithm.
The zeros 14.135 and 21.002 are localized.

EXAMPLE 5: ZEROS OF THE BERNOULLI POLYNOMIALS

We compute those polynomials by the formula

$$\sum_{k=0}^{n} \binom{n}{k} B_k(x) = nx^{n-1} .$$

We deal with $n = 30$, i.e.

Table 4
Evolution of the exclusion algorithm in example 4.

u	$m(u)$	u	$m(u)$	u	$m(u)$
0	5.541094925	14.13473433	0.0000082519	14.67546328	0.540734211
5.541094925	2.961907860	14.13474258	0.0000163520	15.21619749	1.081354598
8.503002785	1.725121011	14.13475893	0.0000329140	16.29755209	1.487160247
10.22812380	1.250354000	14.13479184	0.0000664007	17.78471234	1.182604074
11.47847780	0.962713885	14.13485824	0.0001327270	18.96731641	0.793887543
12.44119169	0.732775727	14.13499097	0.0002638764	19.76120395	0.568611853
13.17396742	0.519239632	14.13525485	0.0005274647	20.32981580	0.388665300
13.69320705	0.306589983	14.13578232	0.0010556782	20.71848110	0.215699989
13.99979703	0.117538907	14.13683800	0.0021124943	20.93418109	0.078586404
14.11733594	0.017048375	14.13895049	0.0042245799	21.01276750	0.007327349
14.13438432	0.000341435	14.14317507	0.0084493859	21.02009485	0.003309674
14.13472576	0.000000658	14.15162446	0.0168978044	21.02340453	0.011928911
14.13472642	0.000000137	14.16852226	0.0337959889	21.03533344	0.017290093
14.13472656	0.000002104	14.20231825	0.0675919288	21.05262353	0.032761343
14.13472867	0.000001670	14.26991018	0.1351844911	21.08538487	0.064820241
14.13473034	0.000003984	14.40509467	0.2703686046	21.15020511	0.132753562

$$B_{30}(x) = \frac{8615841276005}{14322} - \frac{23749461029x^2}{2} + \frac{78132595905x^4}{2} - \frac{102818379585x^6}{2} + \frac{72484065225x^8}{2} - \frac{31795091601x^{10}}{2} + \frac{9509268925x^{12}}{2} - \frac{2062720845x^{14}}{2} + \frac{339319575x^{16}}{2} - \frac{43785215x^{18}}{2} + \frac{4552275x^{20}}{2} - \frac{390195x^{22}}{2} + \frac{28275x^{24}}{2} - \frac{1827x^{26}}{2} + \frac{145x^{28}}{2} - 15x^{29} + x^{30}.$$

Figure 9 gives the zeros with the separation balls.

EXAMPLE 6: ZEROS OF THE CURTZ POLYNOMIALS

These polynomials are studied in [7] and are defined by

$$P_0(x) = 0, \quad P_n(x) = x \sum_{i=0}^{n-1} \frac{(-1)^i}{i+1} P_{n-i-1}(x) + \frac{(-1)^n}{n+1}.$$

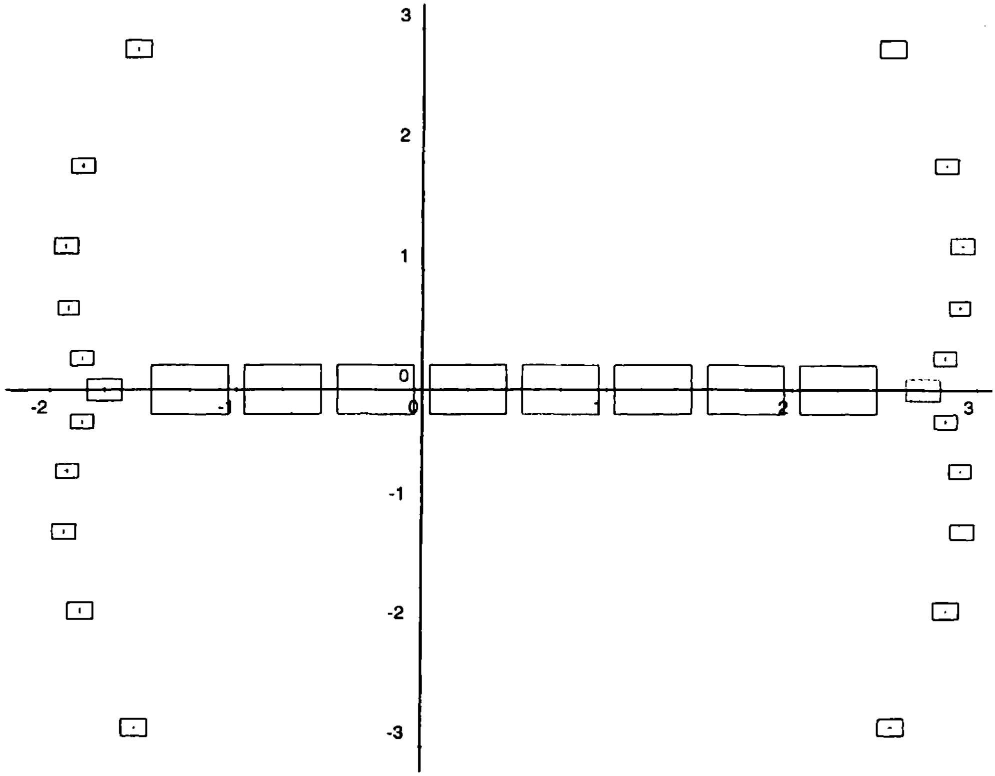


Fig. 9.

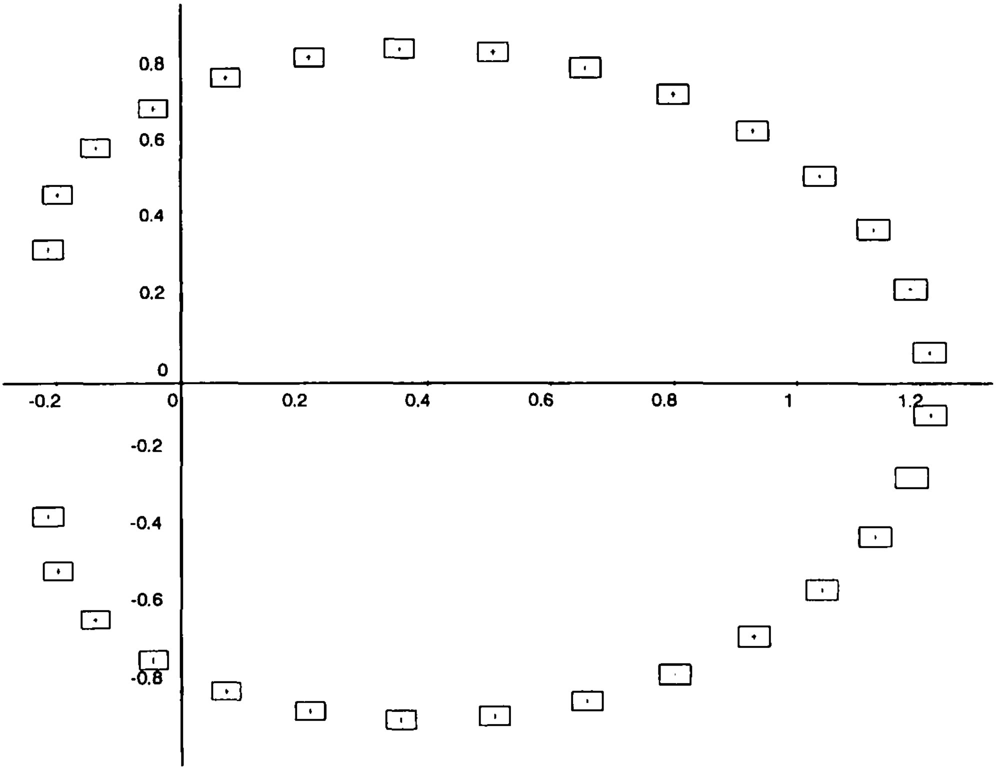


Fig. 10.

The zeros of

$$
\begin{aligned}
P_{30}(x) =\, & 0.032258 - 0.257741x + 1.388494x^2 - 6.044021x^3 + 22.587747x^4 \\
& - 74.593209x^5 + 221.224233x^6 - 595.104901x^7 + 1461.373299x^8 \\
& - 3289.577029x^9 + 6805.540776x^{10} - 12958.719750x^{11} \\
& + 22723.440458x^{12} - 36685.638536x^{13} + 54477.738916x^{14} \\
& - 74291.510330x^{15} + 92822.409152x^{16} - 105934.236198x^{17} \\
& + 110004.087856x^{18} - 103436.769937x^{19} + 87546.960682x^{20} \\
& - 66205.485475x^{21} + 44322.453992x^{22} - 25963.268849x^{23} \\
& + 13108.652447x^{24} - 5591.625x^{25} + 1960.275x^{26} - 542.5x^{27} \\
& + 111.166667x^{28} - 15x^{29} + x^{30}
\end{aligned}
$$

are given with their separation balls in fig. 10.

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