

A UNIVERSAL HKR THEOREM

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ABSTRACT. In this work we study the failure of the HKR theorem over rings of positive and mixed characteristic. For this we construct a *filtered circle* interpolating between the usual topological circle and a formal version of it. By mapping to schemes we produce this way a natural interpolation, realized in practice by the existence of a natural filtration, from Hochschild and cyclic homology to derived de Rham cohomology. The construction our filtered circle is based upon the theory of affine stacks and affinization introduced by the third author, together with some facts about schemes of Witt vectors.

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1. INTRODUCTION

The purpose of the present paper is to investigate the failure of the Hochschild-Kostant-Rosenberg in positive and mixed characteristic situations. For this, we construct a *filtered circle* S_{Fil}^1 , an object of algebro-homotopical nature, which interpolates between the usual homotopy type of a topological circle and a degenerate version of it called the *formal circle*. Using mapping spaces from S_{Fil}^1 to schemes, in the sense of derived algebraic geometry, this provides an interpolation between Hochschild and cyclic homology and derived de Rham cohomology. The existence of such interpolations, realized concretely in terms of a filtration, is the main content of this work.

1.1. Background. Over any commutative ring k , the HKR theorem [HKR62] identifies $\Omega_{\mathrm{dR}}^*(X)$ - the graded commutative algebra of differential forms on a smooth k -scheme $X = \mathrm{Spec} A$, with $\mathrm{HH}_*(A)$ - the graded algebra of Hochschild homology. When k is of characteristic zero, this lifts to the level of chain complexes, identifying de Rham complex of differential forms with the Hochschild complex $\mathrm{HH}(A)$. The de Rham differential on forms emerges from the the natural circle action on $\mathrm{HH}(A)$ given by the Connes category. A

precise implementation of this fact requires a further enhanced version of the HKR theorem, that combines the intervention of the homotopical circle action and the multiplicative structure on the Hochschild complex on one side, and the full derived de Rham algebra with its natural grading and de Rham differential on the other. This enhanced version has been established in [TV11], where it is stated as a multiplicative equivalence

$$\mathrm{HH}(A) = A \otimes_k S^1 \simeq \mathrm{Sym}_A(\mathbb{L}_{A/k}[1]) = \mathrm{DR}(A) \quad (1)$$

with the circle action on the left matching to the de Rham differential on the right. Geometrically [BZN12], this can also be interpreted as an identification of the derived stack of free loops on an affine k -scheme $X = \mathrm{Spec}(A)$ with the shifted cotangent stack

$$\mathrm{LX} := \mathrm{Map}(S^1, X) \simeq \mathrm{T}^* X[-1] := \mathrm{Map}(\mathrm{Spec}(k \oplus k[-1]), X) \quad (2)$$

Here $k \oplus k[-1]$ denotes the split square zero extension. Passing to global functions, this recovers the isomorphism in (1).

The starting point for this paper is the observation that the equivalence (2) no longer holds when we abandon the hypothesis of k being a field of characteristic zero; indeed, the proof uses two essential facts about $\mathrm{BG}_{\mathbf{a}k}$ - the classifying stack of the group $\mathbb{G}_{\mathbf{a}k}$:

A) In any characteristic, the stack $\mathrm{BG}_{\mathbf{a}k}$ is equivalent to $\mathrm{Spec}^\Delta(\mathrm{Sym}_k^{\mathrm{co}\Delta}(k[-1]))$ where $\mathrm{Sym}_k^{\mathrm{co}\Delta}(k[-1])$ is the free cosimplicial commutative k -algebra over one generator in degree 1 (see Notation 1.8 below). This can be checked at the level of the functor of points. But when k is a field of characteristic zero, since the cohomology of the symmetric groups with coefficients in k vanishes, we recover an equivalence of commutative differential graded algebras

$$\mathrm{Sym}_k^{\mathrm{co}\Delta}(k[-1]) \simeq k \oplus k[-1]$$

where on the the r.h.s we have the split square zero extension. In particular, we have

$$\mathrm{Map}(\mathrm{BG}_{\mathbf{a}k}, X) \simeq \mathrm{Map}(\mathrm{Spec}(k \oplus k[-1]), X) =: \mathrm{T}^* X[-1]$$

B) Notice that for any ring k the complex of singular cochains $\mathrm{C}^*(S^1, k)$ is given by $k \oplus k[-1]$. The canonical map of groups $\mathbb{Z} \rightarrow \mathbb{G}_{\mathbf{a}k}$ produces a map of group stacks $S^1 := \mathrm{B}\mathbb{Z} \rightarrow \mathrm{BG}_{\mathbf{a}k}$. As in A), because the cohomology of symmetric groups with coefficients in a field of characteristic zero vanishes, the pullback map in cohomology $\mathrm{C}^*(\mathrm{BG}_{\mathbf{a}k}, \mathcal{O}) \rightarrow \mathrm{C}^*(S^1, k)$ is an equivalence. This fact exhibits the abelian group stack $\mathrm{BG}_{\mathbf{a}k}$ as the *affinization* of the constant group stack S^1 in the sense of [Toe06] (see also [Lur11], [BZN12, Lemma 3.13] and our Review 4.3). It follows from the universal property of being an affinization that

$$\mathrm{Map}(S^1, X) \simeq \mathrm{Map}(\mathrm{BG}_{\mathbf{a}k}, X)$$

The accident that allows A) and B) in characteristic zero also allows an interpretation of the circle action as de Rham differential, since in this case the equivalences

$$\text{Aff}(S^1) \underbrace{\simeq}_{B)} \text{BG}_a \underbrace{\simeq}_{A)} \text{Spec}(k \oplus k[-1])$$

are compatible with the group structures.

1.2. In this paper. Our main goal in this paper is to provide a generalization of the HKR theorem removing the hypothesis of characteristic zero. The starting point is the remark that the two copies of BG_a appearing in A) and B) play distinct roles. What we propose, working over $\mathbb{Z}_{(p)}$, is a construction that interpolates between the two. It is inspired by an idea of [Toe06] of using the group scheme W_{p^∞} of p -typical Witt vectors as a natural extension of the additive group \mathbb{G}_a . The group W_{p^∞} is an abutment of infinitely many of copies of \mathbb{G}_a and comes canonically equipped with a Frobenius map Frob_p . The abelian subgroup Fix of fixed points of the Frobenius map has a natural filtration whose associated graded is the kernel of the Frobenius, Ker . After base change from $\mathbb{Z}_{(p)}$ to \mathbb{Q} both Fix and Ker are isomorphic to \mathbb{G}_a (see the Remark 3.1 below) but over $\mathbb{Z}_{(p)}$ they are very different. Without further ado, our first main theorem is the following:

Theorem 1.1. *(i) The abelian group stack BFix is the affinization of S^1 over $\mathbb{Z}_{(p)}$. In particular, for any derived scheme over $\mathbb{Z}_{(p)}$ we have an equivalence of $\mathbb{Z}_{(p)}$ derived mapping stacks*

$$\text{Map}(S^1, X) \simeq \text{Map}(\text{BFix}, X)$$

(ii) The abelian group stack BKer has cohomology ring given the (cosimplicial) split square zero extension (see Notation 4.25)

$$\mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}[-1]$$

In particular, we have

$$\text{Map}(\text{BKer}, X) \simeq T^*X[-1]$$

(iii) The group stack BFix is equipped with a filtration, compatible with the group structure, whose associated graded stack is BKer .

(iv) After base-change along $\text{Spec}(\mathbb{Q}) \rightarrow \text{Spec}(\mathbb{Z}_{(p)})$, we have

$$\text{BFix} \otimes \mathbb{Q} \simeq \text{BG}_a, \mathbb{Q}$$

Moreover, the filtration splits and we have

$$(\text{BFix})_{\mathbb{Q}}^{\text{gr}} \simeq \text{BKer} \otimes \mathbb{Q} \simeq \text{BG}_a, \mathbb{Q}$$

Definition 1.2. The group stack BFix , equipped with the *filtration* of Theorem 1.1 - (iii), will be called the *filtered circle* and denoted as S_{Fil}^1 .

In order to define filtrations on stacks, we will follow the point of view of C. Simpson [Sim91, Lemma 19] and [Sim97a] which identifies filtered objects with objects over the stack $\mathbb{A}^1/\mathbb{G}_m$. The content of Theorem 1.1 and Definition 1.2 can then be reformulated as the construction of an abelian group stack $\mathbf{S}_{\text{Fil}}^1$ over $\mathbb{A}^1/\mathbb{G}_m$ whose fiber at 0 has the property in A); at 1 has the property in B); and whose pullback to \mathbb{Q} is the constant family with values $\mathbf{B}\mathbb{G}_{a, \mathbb{Q}}$. The construction of $\mathbf{S}_{\text{Fil}}^1$ is the subject of Section 3, after reviewing the basics of Witt vectors in Section 2. The proof that $\mathbf{S}_{\text{Fil}}^1$ satisfies (i) and (ii) will be discussed later in Section 4.3 and Section 4.2. The proof of (iv) is explained in the Remark 3.1.

The following consequence of our main theorem follows from the well known interpretation of cyclic homology in terms of derived loop spaces.

Theorem 1.3 (See Theorem 6.6). *Let $X = \text{Spec}(A)$ be a derived affine scheme over $\mathbb{Z}_{(p)}$. Then:*

- (i) *The derived mapping stack $\text{Map}(\mathbf{S}^1, X)$ admits a filtration compatible with the circle action whose associated graded is $\mathbf{T}^*X[-1]$;*
- (ii) *Passing to global functions, (i) produces a filtration on $\mathbf{HH}(A)$, compatible with the circle action and the multiplicative structure, and whose associated graded is the de Rham algebra $\mathbf{DR}(A)$.*
- (iii) *Being compatible with the circle action, the filtration descends to fixed points and makes*

$$\mathbf{HC}(A)^- = \mathbf{HH}(A)^{\mathbf{S}^1}$$

a filtered algebra, whose associated graded pieces are the truncated complete derived de Rham complexes $\widehat{\mathbb{L}\mathbf{DR}}^{\geq p}(A/k)$.

The proof of Theorem 1.3 will be discussed in Section 6 (see Theorem 6.6). The point (iii) does not seem new, such filtrations have already been constructed by different methods by B. Antieau in [Ant18] and Barghav-Morrow-Scholze in [BMS19]. We believe these two filtrations to be the same but the comparison questions seems non-obvious.

In Section 7 we will discuss several applications of Theorem 1.1 and Theorem 1.3.

Application 1.4 (Shifted Symplectic Structures in positive characteristic). In Section 7.1 we discuss the extension of the notion of shifted symplectic structures of [PTVV13] to derived stacks in positive characteristic. For this we use our HKR theorem in order to produce certain classes in the second layer of the filtration induced on negative cyclic homology. This is achieved by analyzing the Chern character map at the first two graded pieces of the HKR filtration. We also suggest a possible definition of n -shifted symplectic structures and show that the universal 2-shifted symplectic structure on \mathbf{BG} exists essentially over

any base ring k . By the techniques developed in [PTVV13] we obtain this way extensions of the previously known n -shifted symplectic structures over non-zero characteristic bases.

Application 1.5 (Generalized Cyclic Homology and Formal groups). The application discussed in Section 7.3 comes from the observation that the degeneration from Fix to Ker of Theorem 1.1-(iii) is Cartier dual to the degeneration of the multiplicative formal group $\widehat{\mathbb{G}}_m$ to the additive formal group $\widehat{\mathbb{G}}_a$ (see Proposition 7.7). In Section 7.3 we will discuss how to generalize Theorem 1.1 and Theorem 1.3 replacing $\widehat{\mathbb{G}}_m$ by a more general formal group law E , in particular, one associated to an elliptic curve. These ideas will be developed in detail in a future work.

Application 1.6 (Topological and q -analogues). In Section 7.4 we briefly present topological and q -deformed possible generalizations of our filtered circle. We investigate two related ideas, a first one that predicts the existence of a topological version of S_{Fil}^1 as an object over the ring spectrum, at least as an object of non-commutative nature. A second one, along the same spirit, predicting the existence of a q -deformed filtered circle $S_{\text{Fil}}^1(q)$ related to q -deformed de Rham complex (see for instance [Sch17]) in a similar fashion that S_{Fil}^1 is related to de Rham theory. Again, such a *quantum circle* can only exist if one admits non-commutative objects in some sense.

Related and future works: The object S_{Fil}^1 and the general constructions behind it, seem related to several other subjects. First of all homotopy theory, as the underlying object of S_{Fil}^1 is the affinization of the topological circle over $\mathbb{Z}_{(p)}$ in the sense of [Toe06]. We believe that our construction is much more general and that for any finite CW homotopy type X the affinization $(X \otimes \mathbb{Z}) = \text{Spec } \mathbb{C}^*(X, \mathbb{Z})$ comes equipped with canonical filtration whose associated graded is $\text{Spec } H^*(X, \mathbb{Z})$ (at least when X has torsion free cohomology groups). This is in a way the canonical filtration that degenerates a homotopy type over \mathbb{Z} to a formal homotopy type. The object S_{Fil}^1 is thus, in a way, part of A. Grothendieck's pursuing stacks program, and it is interesting to note that the schematization problem has already previously been related to integer valued polynomials in [Eke02].

In a different direction, while writing this paper, the authors have realized the strong interaction between S_{Fil}^1 and the theory of abelian formal groups. This is explained in more details in our Section 7.3, but let us mention here that S_{Fil}^1 is specifically related to the formal multiplicative and additive groups, and that similar constructions continue to make sense for a general formal group law E . This suggests the existence of a *generalized Hochschild and cyclic homology* associated to any formal group law E , that might be thought as algebraic analogues of the relations between formal groups and generalized homology theories in topology.

Finally, there are interactions with the world of quantum mathematics, and more particularly with quantum groups and Ringel-Hall algebras, as well as q -analogues of differential calculus. This goes via the fact that formal \mathbb{G}_m as well as formal \mathbb{G}_a do possess quantum

analogue, incarnated for instance in q -deformed integer valued polynomial algebras (see [HH17]) or quantum divided power algebras (or Hall algebras of the punctual Quiver). This suggest quantum versions of our filtered circle, and a notion of q -deformed Hochschild and cyclic homology, related by means of an HKR filtration to the q -differential equation and q -differential calculus.

Acknowledgements 1.7. The idea of a filtered circle has been inspired by the construction of the Hodge filtration in Carlos Simpson's non-abelian Hodge theory, and we thank Carlos Simpson, Tony Pantev, Gabriele Vezzosi and Mauro Porta for several conversations on these themes along the years. We also thank Arpon Raksit for discussions related to his approach to the filtered circle. We are very thankful to Pavel Etingof for having pointed to us the possible existence of the q -deformed circle (presented in Section 7.4), this has also lead the authors to realize the existence of various circle associated to any commutative formal group law.

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Notation 1.8. Unless mentioned otherwise, all higher categorical notations are borrowed from [Lur17b, Lur09]. Let k be a discrete commutative ring. Throughout the paper we will denote by

- a) Mod_k the ∞ -category of chain complexes of k -modules; $\mathrm{Mod}_k^{\geq 0}$, resp. $\mathrm{Mod}_k^{\leq 0}$ the categories of connective and coconnective complexes.
- b) The notation CAlg will always be used to denote $\mathbf{E}_{\infty}^{\otimes}$ -algebras. In particular CAlg_k will denote $\mathbf{E}_{\infty}^{\otimes}$ -algebras Mod_k ; $\mathrm{CAlg}_k^{\mathrm{cn}} \simeq \mathrm{CAlg}(\mathrm{Mod}_k^{\geq 0})$ the full subcategory of connective algebras [Lur17b, 2.2.1.3, 2.2.1.8, 7.1.3.10] and $\mathrm{CAlg}_k^{\mathrm{ccn}}$ the category of $\mathbf{E}_{\infty}^{\otimes}$ -algebras in $\mathrm{Mod}_k^{\leq 0}$ for the symmetric monoidal structure induced from the fact $\tau_{\leq 0}$ is a monoidal localization. In particular, as the inclusion $\mathrm{Mod}_k^{\leq 0} \subseteq \mathrm{Mod}_k$ is lax monoidal, we have an induced map at the level of algebras $\mathrm{CAlg}_k^{\mathrm{ccn}} \rightarrow \mathrm{CAlg}_k$
- c) SCR_k the ∞ -category of simplicial commutative rings over k . This is the sifted completion of the discrete category of polynomial algebras. See [Lur17a, 25.1.1.5] and [Lur09, 5.5.9.3]. The universal property of sifted completion gives us the normalized Dold-Kan functor $\theta : \mathrm{SCR}_k \rightarrow \mathrm{CAlg}_k^{\mathrm{cn}}$. By [Lur17a, 25.1.2.2, 25.1.2.4] this is both monadic and comonadic and if k is of characteristic zero it is an equivalence. Given $A \in \mathrm{SCR}_k$, $\theta(A)$ will be called the *underlying $\mathbf{E}_{\infty}^{\otimes}$ -algebra of A* .
- d) By Sym we will always mean the simplicial version Sym^{Δ} as a monad in $\mathrm{Mod}_k^{\geq 0}$;

- e) \mathbf{coSCR}_k the ∞ -category of cosimplicial commutative rings over k (see [Toe06, 2.1.2]). We also denote by $\theta : \mathbf{coSCR}_k \rightarrow \mathbf{CAlg}_k$ the conormalized Dold-Kan construction (see [Toe06, §2.1]). This functor is conservative and can be identified with the totalization of cosimplicial objects and therefore preserves limits. It can be factored by a functor θ^{ccn}

$$\mathbf{coSCR}_k \xrightarrow{\theta^{\text{ccn}}} \mathbf{CAlg}_k^{\text{ccn}} \subseteq \mathbf{CAlg}_k$$

where θ^{ccn} is the co-dual Dold-Kan construction of [Kř93]. In particular, θ^{ccn} commutes with tensor products and therefore with finite colimits.

- f) By $\mathbf{Sym}^{\text{co}\Delta}$ we will mean the free cosimplicial commutative algebra on $\mathbf{Mod}_k^{\text{ccn}}$.
- g) \mathbf{St}_k the ∞ -category of stacks over the site of discrete commutative k -algebras and \mathbf{dSt}_k the ∞ -category of derived stacks, ie, stacks over \mathbf{SCR}_k .
- h) $\mathbf{Spec}^\Delta : \mathbf{coSCR}_k^{\text{op}} \rightarrow \mathbf{St}_k$ the ∞ -functor sending an object $A \in \mathbf{coSCR}_k$ to the (higher) stack which sends a classical commutative ring B to the mapping space $\mathbf{Map}_{\mathbf{coSCR}_k}(A, B)$. See also Review 4.3
- i) \mathbb{G}_{mk} and \mathbb{A}_k^1 will always denote the flat versions of the multiplicative group and affine line over k .
- j) \mathbf{QCoh} will always denote the ∞ -category of quasi-coherent sheaves.

2. REMINDERS ON WITT VECTORS.

In this section we review some classical materials concerning Witt vectors. We follow [Hes08, HL13, KN, Mum66]. More standard references are [Bou06, Haz12, Ill79, DG70]

2.1. Witt vectors. Let $W : \mathbf{CRings} \rightarrow \mathbf{AbGrp}$ denote the abelian affine group scheme over \mathbb{Z} , of big Witt vectors. For a commutative ring A , $W(A)$ is the submonoid of the ring of formal power series $A[[t]]$ spanned by invertible power series P with $P(0) = 1$. As an affine scheme, W is isomorphic to an infinite product $\prod_{i \geq 1} \mathbb{A}^1$ by sending a power series to its list of coefficients.

The group scheme W behaves like an abutement of copies of the additive algebraic group scheme \mathbb{G}_a , in two different senses. The first is the natural pro-group structure. Let $m \geq 1$. The abelian group of big Witt vectors of length m , denoted by $W_m(A)$, is the quotient of $W(A)$ by the subgroup of all invertible power series of the form $1 + t^{m+1} \cdot g$. In particular, elements in $W_m(A)$ can be written as polynomials of degree $\leq m$ and the maps forgetting the last coefficient, $W_m(A) \rightarrow W_{m-1}(A)$, are compatible with the abelian group structure. Scheme-theoretically these maps correspond to the canonical projection $W_{m-1}(A) \times \mathbb{A}^1 \rightarrow W_{m-1}(A)$. The collection of restriction maps provides a natural limit decomposition as a pro-group scheme $W \simeq \lim_m W_m$.

The other sense under which W is built out of copies of \mathbb{G}_a is implemented by the Ghost map. Let $f \in W(A)$. Then, as in [KN, B.5], there exists a unique decomposition of the

form ^(*) $P_f(t) = \prod_{n \geq 1} (1 - \lambda_n t^n)^{-1}$. In these coordinates (called Teichmüller coordinates) the sum of two Witt vectors f and f' is given by the multiplication of the two power series $P_f.P_{f'}$. We define the *Ghost* power series (not necessarily invertible) of P_f , denoted by $\text{Ghost}(f)$, by the formula $\text{Ghost}(f)(t) := t \cdot \frac{d}{dt} \log(P_f(t))$. The n -th ghost component of $f \in W(A)$ is the n -th coefficient of the power series development $\text{Ghost}(f)(t) = \sum \omega_n t^n$. A simple computation shows that $\omega_n = \sum_{(d,i): d \cdot i = n} i \cdot \lambda_i^d$. The logarithmic definition implies that the map

$$\text{Ghost} : W(A) \mapsto \prod_{i \geq 1} A \quad \text{sending } P_f \mapsto (\omega_1, \omega_2, \dots)$$

is in fact a map of abelian groups, on the l.h.s with the multiplication of invertible power series and on the r.h.s the levelwise addition. This construction is functorial in A and defines a map of groups schemes over $\text{Spec}(\mathbb{Z})$

$$\text{Ghost} : W \rightarrow \prod_{i \geq 1} \mathbb{G}_a$$

whose base-change to $\text{Spec}(\mathbb{Q})$ is an isomorphism [KN, B.3(2)].

In terms of the Teichmüller coordinates, the groups $W_m(A)$ can be recovered as the quotient of $W(A)$ by the ideal generated by the Witt vectors of the form $\prod_{i \notin \{1, 2, \dots, m\}} (1 - \lambda_i t^i)^{-1}$. See [Hes08, Example 16]. This description implies that the composition of the Ghost map with the projection to the first m -coordinates

$$\text{Ghost} : W \rightarrow \prod_{i \geq 1} \mathbb{G}_a \rightarrow \prod_{i=1}^m \mathbb{G}_a \tag{3}$$

factors through $W_m(A)$.

$$W_m(A) \rightarrow \prod_{i=1}^m \mathbb{G}_a$$

Again, after base-change to \mathbb{Q} , the induced Ghost map of finite length is an isomorphism [KN, B.11].

2.2. Frobenius. The group scheme W comes naturally equipped with a collection of Frobenius endomorphisms Frob_n . These can first be defined at the level of the Ghost coefficients, by the formula $F_n : (\omega_1, \omega_2, \dots) \mapsto (\omega_n, \omega_{2n}, \dots)$. The Frobenius operation on Witt-vectors $\text{Frob}_n : W(A) \rightarrow W(A)$ is defined by translating the rule F_n so that the diagram

(*) In fact we have four different choices of coordinates, corresponding to $(1 \pm t)^{\pm 1}$. These choices serve different purposes. For instance, the choice $(-1)^{+1}$ is more naturally understood from the viewpoint of K-theory of endomorphisms [Gra78, Alm78]: given a square matrix with coefficients in A , its characteristic polynomial seen as a formal power series is a Witt-vector. The other choices are related with the theory of chern classes (see [Ram14, Remark 1.1]) or with residues [Kal12]. See [Hes15, Remark 1.15] for a detailed discussion.

$$\begin{array}{ccc}
W(A) & \xrightarrow{\text{Frob}_n} & W(A) \\
\text{Ghost} \downarrow & & \downarrow \text{Ghost} \\
\prod_{i \geq 1} A & \xrightarrow{F_n} & \prod_{i \geq 1} A
\end{array} \tag{4}$$

commutes. Let $d = \gcd(n, k)$. It is an exercise using the additivity of the Ghost maps, to check that the Frobenius on Witt-vectors as a map of abelian groups is uniquely determined by the formula

$$\text{Frob}_k((1 - \lambda t^n)^{-1}) := (1 - \lambda^{\frac{k}{d}} t^{\frac{n}{d}})^{-d} \tag{5}$$

Moreover, we have canonical identifications $\text{Frob}_{n \cdot m} = \text{Frob}_n \circ \text{Frob}_m$. This follows directly from the formula $F_{nm} = F_n \circ F_m$ on Ghost components.

The Frobenius maps are functorial in A and provide maps of abelian group schemes

$$\text{Frob}_n : W \rightarrow W$$

2.3. p -typical Witt vectors. For the applications in this paper instead of looking at all big Witt-vectors, we will focus on the group of p -typical Witt vectors $W_{p^\infty}(A)$ defined as the quotient of $W(A)$ by the subgroup of all Witt-vectors of the form $\prod_{i \notin \{1, p, p^2, p^3, \dots\}} (1 - \lambda_i t^i)^{-1}$. See [Hes08, Addendum 15] or [KN, B.9].

We also restrict our attention to $\mathbb{Z}_{(p)}$ -algebras. Under this restriction, the abelian group $W(A)$ admits an idempotent decomposition as abelian groups $W(A) \simeq \prod_{k: p \nmid k} W_{p^\infty}(A)$ [Hes08, Proposition 10] and the quotient map $W(A) \rightarrow W_{p^\infty}(A)$ admits a natural right inverse which identifies $W_{p^\infty}(A)$ with the subgroup of $W(A)$ spanned by all Witt vectors of the form $\prod_{i \in \mathbb{N}} (1 - \lambda_i t^{p^i})^{-1}$. Using this fact, over $\text{Spec}(\mathbb{Z}_{(p)})$, the assignment

$$W_{p^\infty} : \text{CRings}_{\mathbb{Z}_{(p)}} \rightarrow \text{AbGrp}$$

can be presented as an abelian sub-group scheme of W . The restriction of the Ghost coordinates to p -typical Witt vectors is well-defined as a map of $\text{Spec}(\mathbb{Z}_{(p)})$ -group schemes

$$\text{Ghost}_p : W_{p^\infty}(A) \rightarrow \prod_{p^i: i \geq 0} A$$

by the rule

$$\prod_{n \geq 1} (1 - \lambda_n t^{p^n})^{-1} \mapsto (\omega_1, \omega_p, \omega_{p^2}, \dots)$$

and as before, the base-change of Ghost_p to $\text{Spec}(\mathbb{Q})$ is an isomorphism.

To conclude, let us mention that as Section 2.1, there is a version of p -typical Witt vectors of finite length, $W_{p^\infty}^{(m)}$, obtained from W_{p^∞} by taking the quotient with respect to the ideal spanned by p -typical Witt vectors of the form $\prod_{i > m} (1 - \lambda_i t^{p^i})^{-1}$. The restriction maps $W_{p^\infty}^{(m)} \rightarrow W_{p^\infty}^{(m-1)}$ have fiber isomorphic to \mathbb{A}^1 and W_{p^∞} can be exhibited as the limit of this tower. As before, the restriction of Ghost coordinates to truncated Witt vectors induces an equivalence rationally

$$W_{p^\infty}^{(m)} \otimes \mathbb{Q} \simeq \prod_{\{p^0, p^1, \dots, p^m\}} \mathbb{G}_a \mathbb{Q} \quad (6)$$

The p -Frobenius Frob_p descends to W_{p^∞} as a map of group schemes. For a $\mathbb{Z}_{(p)}$ -algebra A , it is defined on Ghost coordinates by $(\omega_1, \omega_p, \omega_{p^2}, \dots) \mapsto (\omega_p, \omega_{p^2}, \dots)$ and in terms of the pro-structure it decomposes as maps

$$\text{Frob}_p : W_{p^\infty}^{(m)} \rightarrow W_{p^\infty}^{(m-1)}$$

Notation 2.1. We will denote by Fix the group scheme given by the kernel of the map $\text{Frob}_p - \text{id} : W_{p^\infty} \rightarrow W_{p^\infty}$. We will write Ker for the kernel of $\text{Frob}_p : W_{p^\infty} \rightarrow W_{p^\infty}$.

3. FILTRATIONS, FIXED POINTS AND KERNEL OF FROBENIUS

To illustrate how Witt vectors will be used in our HKR theorem, let us start with the following observation of what happens in characteristic zero:

Remark 3.1. Let A be a \mathbb{Q} -algebra. Then the explicit formula for Frob_p on $W_{p^\infty}(A)$ in terms of the Ghost coordinates tells us that the fixed points for the Frobenius are given by

$$\text{Fix}(A) \simeq \Delta \subseteq \prod_{i \geq 0} \mathbb{G}_a(A)$$

Another easy computation in Ghost coordinates, also tells us that the Kernel of the Frobenius is given by

$$\text{Ker}(A) \simeq (\mathbb{G}_a(A), 0, 0, 0, 0, \dots) \subseteq \prod_{i \geq 0} \mathbb{G}_a(A)$$

In other words, as group-schemes we obtain

$$\text{Fix}_{|\mathbb{Q}} \simeq \mathbb{G}_a \quad \text{and} \quad \text{Ker}_{|\mathbb{Q}} \simeq \mathbb{G}_a$$

The Remark 3.1 shows that for \mathbb{Q} -algebras, the additive group scheme \mathbb{G}_a can be defined abstractly via Witt vectors, either as Frobenius fixed points or as the kernel. In this paper we utilize this feature to understand the HKR theorem in positive characteristic. Outside \mathbb{Q} -algebras, the fixed points and the kernel of Frob_p on W_{p^∞} do not agree. However, we shall see that there is a natural degeneration from the first to the second, or more precisely, a filtration on Fix whose associated graded is Ker . The delooping of this filtration will be, by definition, our filtered circle. For this purpose we will need to explain what is a filtration on a stack. Before addressing that question, let us be precise what are the linear versions of filtrations and gradings used in this paper:

Construction 3.2. [Lur15] Let \mathcal{C} be a cocomplete stable ∞ -category. The category of filtered objects in \mathcal{C} is the ∞ -category of diagrams $\mathrm{Fil}(\mathcal{C}) := \mathrm{Fun}(\mathrm{N}(\mathbb{Z})^{\mathrm{op}}, \mathcal{C})$, with $\mathrm{N}(\mathbb{Z})$ the nerve of the category associated to the poset (\mathbb{Z}, \leq) . The category of \mathbb{Z} -graded objects in \mathcal{C} is the ∞ -category of diagrams $\mathcal{C}^{\mathbb{Z}\text{-gr}} := \mathrm{Fun}(\mathbb{Z}^{\mathrm{disc}}, \mathcal{C})$ where $\mathbb{Z}^{\mathrm{disc}}$ is the \mathbb{Z} seen as a discrete category. Both categories are endowed with symmetric monoidal structures given by Day convolution. Following [Lur15, §3.1 and 3.2], the construction of the associated graded object, respectively, the underlying object, are implemented by symmetric monoidal functors

$$\mathrm{gr} : \mathrm{Fil}(\mathcal{C}) \rightarrow \mathcal{C}^{\mathbb{Z}\text{-gr}} \quad \text{and} \quad \mathrm{colim} : \mathrm{Fil}(\mathcal{C}) \rightarrow \mathcal{C}.$$

Definition 3.3. We define a *graded category* to be a stable presentable ∞ -category endowed with a structure of object in $\mathrm{Mod}_{\mathrm{Sp}^{\mathbb{Z}\text{-gr}, \otimes}}(\mathrm{Pr}^{\mathrm{L}})$. Similarly, a *filtered category* is an object in $\mathrm{Mod}_{\mathrm{Fil}(\mathcal{C})^{\otimes}}(\mathrm{Pr}^{\mathrm{L}})$.

We will need to use the point of view on filtrations and gradings given by the Rees construction of Simpson [Sim91, Lemma 19] where the following geometric objects play a central role:

Construction 3.4. Let $\mathrm{BG}_{\mathrm{m}\mathbb{S}}$ be the classifying stack of the flat multiplicative abelian group scheme over the sphere spectrum and $e : \mathrm{Spec}(\mathbb{S}) \rightarrow \mathrm{BG}_{\mathrm{m}\mathbb{S}}$ the canonical atlas. Consider also the canonical action of $\mathbb{G}_{\mathrm{m}\mathbb{S}}$ on the flat affine line $\mathbb{A}_{\mathbb{S}}^1$ and form the stacky quotient $\mathbb{A}_{\mathbb{S}}^1/\mathbb{G}_{\mathrm{m}\mathbb{S}}$. This lives canonically as a stack over $\mathrm{BG}_{\mathrm{m}\mathbb{S}}$ via a map

$$\pi : \mathbb{A}_{\mathbb{S}}^1/\mathbb{G}_{\mathrm{m}\mathbb{S}} \rightarrow \mathrm{BG}_{\mathrm{m}\mathbb{S}}$$

The inclusion of the zero point $0 : \mathrm{Spec}(\mathbb{S}) \rightarrow \mathbb{A}_{\mathbb{S}}^1$ provides a section of the canonical projection $\mathbb{A}_{\mathbb{S}}^1/\mathbb{G}_{\mathrm{m}\mathbb{S}} \rightarrow \mathrm{BG}_{\mathrm{m}\mathbb{S}}$

$$0 : \mathrm{BG}_{\mathrm{m}\mathbb{S}} \rightarrow \mathbb{A}_{\mathbb{S}}^1/\mathbb{G}_{\mathrm{m}\mathbb{S}}$$

The inclusion of $\mathbb{G}_{\mathrm{m}\mathbb{S}}$ in $\mathbb{A}_{\mathbb{S}}^1$ also passes to the quotient and provides a map

$$\mathbb{A}_{\mathbb{S}}^1/\mathbb{G}_{\mathrm{m}\mathbb{S}} \leftarrow \mathbb{G}_{\mathrm{m}\mathbb{S}}/\mathbb{G}_{\mathrm{m}\mathbb{S}} \simeq * : 1$$

The Rees construction gives us a geometric interpretation of filtered objects and gradings when $\mathcal{C} = \mathrm{Sp}$ is the ∞ -category of spectra, in terms of objects over $\mathbb{A}_{\mathbb{S}}^1/\mathbb{G}_{\mathrm{m}\mathbb{S}}$ and $\mathrm{BG}_{\mathrm{m}\mathbb{S}}$. The proof of the following result will appear in [Tas]:

Theorem 3.5. *There exists symmetric monoidal equivalences*

$$\mathrm{Sp}^{\mathbb{Z}\text{-gr}, \otimes} \simeq \mathrm{QCoh}(\mathrm{BG}_{\mathrm{m}\mathbb{S}})^{\otimes} \tag{7}$$

$$\mathrm{Rees} : \mathrm{Fil}(\mathrm{Sp})^{\otimes} \rightarrow \mathrm{QCoh}(\mathbb{A}_{\mathbb{S}}^1/\mathbb{G}_{\mathrm{m}\mathbb{S}})^{\otimes} \tag{8}$$

such that the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{QCoh}(\mathrm{BG}_m \mathbb{S})^\otimes & \xleftarrow{0^*} & \mathrm{QCoh}(\mathbb{A}_\mathbb{S}^1/\mathrm{G}_m \mathbb{S})^\otimes & \xrightarrow{1^*} & \mathrm{QCoh}(\mathrm{Spec}(\mathbb{S}))^\otimes = \mathrm{Sp}^\otimes \\
\sim \downarrow & & \sim \downarrow & & \parallel \\
\mathrm{Sp}^{\mathbb{Z}-\mathrm{gr}, \otimes} & \xleftarrow{\mathrm{gr}} & \mathrm{Fil}(\mathrm{Sp})^\otimes & \xrightarrow{\mathrm{colim}} & \mathrm{Sp}^\otimes
\end{array}$$

Moreover, after base change along $\mathrm{Spec}(\mathbb{Z}) \rightarrow \mathrm{Spec}(\mathbb{S})$ we recover analogues of these comparisons for filtered and graded objects in $\mathrm{Mod}_\mathbb{Z}$ the ∞ -category derived category of abelian groups and quasi-coherent sheaves on $\mathrm{BG}_m \mathbb{Z}$ and $\mathbb{A}_\mathbb{Z}^1/\mathrm{G}_m \mathbb{Z}$ via the Rees construction (see for instance [Sim97b]).

In view of the Theorem 3.5 the following definition becomes natural.

Definition 3.6. We define a graded stack to be a stack over BG_m and a filtered stack to be a stack over $\mathbb{A}^1/\mathrm{G}_m$. Let $X \rightarrow \mathbb{A}^1/\mathrm{G}_m$ be a filtered stack. The *associated graded* of X , denoted X^{gr} , is the base change of X along the map $0 : \mathrm{BG}_m \rightarrow \mathbb{A}^1/\mathrm{G}_m$. By abuse of notation we will also write X^{gr} to denote the further pullback along the atlas $* \rightarrow \mathrm{BG}_m$, endowed with its canonical G_m -action.

The *underlying stack*, X^u , is the base-change along $1 : * \rightarrow \mathbb{A}^1/\mathrm{G}_m$.

Remark 3.7. Let X be a filtered (resp. graded) stack. Then $\mathrm{QCoh}(X)$ is a filtered (resp. graded) category in the sense of Definition 3.3 via the symmetric monoidal pullback along the structure map to $\mathbb{A}^1/\mathrm{G}_m$ (resp. BG_m).

Remark 3.8. Let $\pi : X \rightarrow \mathbb{A}^1/\mathrm{G}_m$ be a filtered stack and consider the cartesian diagrams

$$\begin{array}{ccccc}
X^{\mathrm{gr}} & \longrightarrow & X & \longleftarrow & X^u \\
\pi^{\mathrm{gr}} \downarrow & & \downarrow \pi & & \downarrow \pi^u \\
\mathrm{BG}_m & \xrightarrow{0} & \mathbb{A}^1/\mathrm{G}_m & \xleftarrow{1} & *
\end{array}$$

Then taking fiber at 0 and 1 commute with push-forward of quasi-coherent sheaves. Indeed, for what concerns the map $1 : * \simeq \mathrm{G}_m/\mathrm{G}_m \rightarrow \mathbb{A}^1/\mathrm{G}_m$, this follows directly from the characterizations of $\mathrm{QCoh}(\mathbb{A}^1/\mathrm{G}_m)$ and $\mathrm{QCoh}(\mathrm{BG}_m)$ via descent along the canonical atlases $\mathbb{A}^1 \rightarrow \mathbb{A}^1/\mathrm{G}_m$ and $\mathrm{G}_m \rightarrow \mathrm{G}_m/\mathrm{G}_m$ and the fact that pushforwards for QCoh are defined on atlases via descent together with the observation that $\mathrm{G}_m \rightarrow \mathbb{A}^1$ is a Zariski open immersion. For what concerns the map $0 : \mathrm{BG}_m \rightarrow \mathbb{A}^1/\mathrm{G}_m$ we can again test the statement by pulling back along the atlases $e : * \rightarrow \mathrm{BG}_m$ and $\mathbb{A}^1 \rightarrow \mathbb{A}^1/\mathrm{G}_m$. It is therefore enough to test that

the induced commutative diagram of pullbacks along the cartesian diagram

$$\begin{array}{ccc} X^{\text{gr}} \times_{\mathcal{B}\mathbb{G}_m} * & \longrightarrow & X \times_{\mathbb{A}^1/\mathbb{G}_m} \mathbb{A}^1 \\ \downarrow & & \downarrow \\ * & \xrightarrow{0} & \mathbb{A}^1 \end{array}$$

is right-adjointable. But this follows because the inclusion $0 : * \rightarrow \mathbb{A}^1$ is lci.

It follows that for a filtered stack $\pi : X \rightarrow \mathbb{A}^1/\mathbb{G}_m$, its derived global sections $\pi_*(\mathcal{O}) = \mathcal{C}^*(X, \mathcal{O})$ admit a structure of $\mathbf{E}_\infty^\otimes$ -algebra in $\mathbf{Fil}(\mathbf{Sp})$ which can be interpreted as a filtration on the $\mathbf{E}_\infty^\otimes$ -algebra $(\pi^u)_*\mathcal{O} = \mathcal{C}^*(X^u, \mathcal{O})$, whose associated graded is $(\pi^{\text{gr}})_*\mathcal{O} = \mathcal{C}^*(X^{\text{gr}}, \mathcal{O})$. By the same mechanics, given a graded stack, $Y \rightarrow \mathcal{B}\mathbb{G}_m$, the $\mathbf{E}_\infty^\otimes$ -algebra $\mathcal{C}^*(Y, \mathcal{O})$ carries a grading compatible with the $\mathbf{E}_\infty^\otimes$ -structure.

Construction 3.9. Let X be a stack with a \mathbb{G}_m -action and consider the stacky quotient X/\mathbb{G}_m which lives canonically over $\mathcal{B}\mathbb{G}_m$. We defined the filtered stack associated to X with the \mathbb{G}_m -action to be the fiber product of stacks

$$\begin{array}{ccc} X^{\text{Fil}} := X/\mathbb{G}_m \times_{\mathcal{B}\mathbb{G}_m} \mathbb{A}^1/\mathbb{G}_m & \longrightarrow & X/\mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathbb{A}^1/\mathbb{G}_m & \longrightarrow & \mathcal{B}\mathbb{G}_m \end{array}$$

We have cartesian squares

$$\begin{array}{ccccc} (X^{\text{Fil}})^{\text{gr}} = X/\mathbb{G}_m & \longrightarrow & X^{\text{Fil}} & \longleftarrow & X =: (X^{\text{Fil}})^u \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B}\mathbb{G}_m & \xrightarrow{0} & \mathbb{A}^1/\mathbb{G}_m & \xleftarrow{1} & * \end{array}$$

Moreover, the cartesian diagram

$$\begin{array}{ccc} \mathbb{A}^1 & \xrightarrow{h} & * \\ p \downarrow & & \downarrow e \\ \mathbb{A}^1/\mathbb{G}_m & \xrightarrow{\pi} & \mathcal{B}\mathbb{G}_m \end{array} \tag{9}$$

tells us that the pullback of X^{Fil} to \mathbb{A}^1 is isomorphic to $\mathbb{A}^1 \times X$ and exhibits X^{Fil} as the quotient $(X \times \mathbb{A}^1)/\mathbb{G}_m$ for the product of the \mathbb{G}_m -action on X and the canonical action of \mathbb{G}_m on \mathbb{A}^1 .

The filtered stacks obtained from \mathbb{G}_m -equivariant stacks using the previous construction are precisely the *split filtered stacks*.

We now turn back to Witt vectors and the interpolation between fixed points and kernel of the Frobenius. The starting ingredient is the following natural grading:

Construction 3.10. The abelian group of Witt vectors $W(A)$ carries an action of the underlying multiplicative monoid of A : for a given $a \in A$, we send the invertible power series $f(t)$ to $f(at)$ which is again in $W(A)$. The formula $[a.f(t)].[a.g(t)] = [a.(f.g)(t)]$ implies that a acts as a map of groups. This defines, when restricted to units in A , an action of the multiplicative group scheme \mathbb{G}_m on the group scheme W and makes it a graded group scheme. It follows from this definitions that this action descends to W_{p^∞} . We consider the quotient stack $W_{p^\infty}/\mathbb{G}_m$ which lives canonically as an abelian group stack over $B\mathbb{G}_m$.

Remark 3.11. The operations Frob_n are compatible with the action of the multiplicative monoid of \mathbb{A}^1 on W in the following sense: given $a \in A$ and denoting by $[a] : W(A) \rightarrow W(A)$ the action by a , we have

$$\text{Frob}_n \circ [a] = [a^n] \circ \text{Frob}_n$$

Construction 3.12. We let $W_{p^\infty}^{\text{Fil}}$ be the output of the Construction 3.9 applied to the action of \mathbb{G}_m on W_{p^∞} of the Construction 3.10. As $W_{p^\infty}/\mathbb{G}_m$ is a group stack over $B\mathbb{G}_m$, it follows that $W_{p^\infty}^{\text{Fil}}$ is a group stack over $\mathbb{A}^1/\mathbb{G}_m$. Explicitly,

$$W_{p^\infty}^{\text{Fil}} \simeq (W_{p^\infty} \times \mathbb{A}^1)/\mathbb{G}_m$$

the quotient of trivial family $W_{p^\infty} \times \mathbb{A}^1$ by the diagonal action of \mathbb{G}_m .

We will now explain how to use the trivial family of the Construction 3.12 to construct a new family that interpolates between Frobenius fixed points and the kernel.

Construction 3.13. Consider the trivial group scheme $W_{p^\infty} \times \mathbb{A}^1$ over \mathbb{A}^1 . For each A over $\mathbb{Z}_{(p)}$, consider the endomorphism of abelian groups

$$\mathcal{E}_p : W_{p^\infty}(A) \times A \rightarrow W_{p^\infty}(A) \times A \quad \text{given by} \quad (f, a) \mapsto (\text{Frob}_p(f) - [a^{p-1}](f), a)$$

This is functorial in A and defines a morphism of abelian group schemes over \mathbb{A}^1

$$\mathcal{E}_p : W_{p^\infty} \times \mathbb{A}^1 \rightarrow W_{p^\infty} \times \mathbb{A}^1$$

Remark 3.14. The Remark 3.11 is equivalent to the statement that \mathcal{E}_p is \mathbb{G}_m -equivariant with respect to the diagonal action of \mathbb{G}_m on the source and the twist by $(-)^p : \mathbb{G}_m \rightarrow \mathbb{G}_m$

on the target. This implies that the inclusion

$$\ker \mathcal{G}_p \subseteq W_{p^\infty} \times \mathbb{A}^1$$

is \mathbb{G}_m -equivariant. The fiber of $\ker \mathcal{G}_p$ over 0 is Ker (Notation 2.1) and is closed under the \mathbb{G}_m -action. The fiber over any $\lambda \in \mathbb{A}^\times$, $(\ker \mathcal{G}_p)_\lambda$ is isomorphic to $(\ker \mathcal{G}_p)_1 \simeq \text{Fix}$ via the isomorphism sending $(P, \lambda) \mapsto ([\frac{1}{\lambda}](P), 1)$.

Lemma 3.15. *The morphism $\mathcal{G}_p : W_{p^\infty} \times \mathbb{A}^1 \rightarrow W_{p^\infty} \times \mathbb{A}^1$ is a cover for the fpqc topology. In particular, it is fpqc-locally surjective and we have a short exact sequence of abelian group-stacks over \mathbb{A}^1*

$$0 \longrightarrow \ker \mathcal{G}_p \longrightarrow W_{p^\infty} \times \mathbb{A}^1 \xrightarrow{\mathcal{G}_p} W_{p^\infty} \times \mathbb{A}^1 \longrightarrow 0$$

Proof. As discussed in Section 2.3, the group scheme W_{p^∞} is the inverse limit of the system $W_{p^\infty}^{(m)}$ of m -truncated p -typical Witt vectors and each restriction map $W_{p^\infty}^{(m)} \rightarrow W_{p^\infty}^{(m-1)}$ is isomorphic to a projection $W_{p^\infty}^{(m-1)} \times \mathbb{A}^1 \rightarrow W_{p^\infty}^{(m-1)}$. Each restriction map is a flat surjection between affine schemes, and therefore, an fpqc cover. In this case (see for instance [Sta19, Lemma 05UU]), in order to show that \mathcal{G}_p is an fpqc cover, it is enough to show that each composition

$$W_{p^\infty} \times \mathbb{A}^1 \xrightarrow{\mathcal{G}_p} W_{p^\infty} \times \mathbb{A}^1 \longrightarrow W_{p^\infty}^{(m)} \times \mathbb{A}^1 \quad (10)$$

is an fpqc cover.

As remarked in Section 2.3, the Frobenius map on m -truncated Witt vectors factors as $W_{p^\infty}^{(m)} \rightarrow W_{p^\infty}^{(m-1)}$. The \mathbb{G}_m -action on the contrary is defined levelwise $[x] : W_{p^\infty}^{(m)} \rightarrow W_{p^\infty}^{(m)}$. By composing with the truncation maps $[x] : W_{p^\infty}^{(m)} \rightarrow W_{p^\infty}^{(m)} \rightarrow W_{p^\infty}^{(m-1)}$ we obtain a system of maps that after passing to the inverse limit, it recovers the \mathbb{G}_m -action on W_{p^∞} . In this case, the composition (10) factors as

$$\begin{array}{ccc} W_{p^\infty} \times \mathbb{A}^1 & \xrightarrow{\mathcal{G}_p} & W_{p^\infty} \times \mathbb{A}^1 \\ \downarrow & & \downarrow \\ W_{p^\infty}^{(m+1)} \times \mathbb{A}^1 & \longrightarrow & W_{p^\infty}^{(m)} \times \mathbb{A}^1 \end{array} \quad (11)$$

So that (as in [Sta19, Lemma 090N]) to show that each composition (10) is an fpqc cover, it is enough to show that each truncated map is an fpqc cover

$$W_{p^\infty}^{(m+1)} \times \mathbb{A}^1 \longrightarrow W_{p^\infty}^{(m)} \times \mathbb{A}^1 \quad (12)$$

This morphism is a map of smooth group schemes that commutes with the projections to $\mathbb{A}^1 := \mathbb{A}_{\mathbb{Z}(p)}^1 \simeq \mathbb{A}_{\mathbb{Z}}^1 \times \text{Spec}(\mathbb{Z}_{(p)})$ and therefore, as each is now of finite presentation, to check that it is a flat cover, it is enough by a local criterion for flatness [Sta19, Lemma 039D] to check that it is so after base change to any field valued point $\text{Spec } K \rightarrow \mathbb{A}_{\mathbb{Z}}^1 \times \text{Spec}(\mathbb{Z}_{(p)})$.

As both projections to $\mathbb{A}_{\mathbb{Z}(p)}^1$ are compatible with the \mathbb{G}_m -action, it is enough to test the statement for the four different points

$$(0, \mathbb{Q}), (1, \mathbb{Q}), (0, \mathbb{F}_p), (1, \mathbb{F}_p)$$

ie, the four maps

$$\begin{aligned} \mathbb{W}_{p^\infty}^{(m+1)} \times \mathbb{Q} &\xrightarrow{\text{Frob}_p} \mathbb{W}_{p^\infty}^{(m)} \times \mathbb{Q} & \mathbb{W}_{p^\infty}^{(m+1)} \times \mathbb{Q} &\xrightarrow{\text{Frob}_p - \text{id}} \mathbb{W}_{p^\infty}^{(m)} \times \mathbb{Q} \\ \mathbb{W}_{p^\infty}^{(m+1)} \times \mathbb{F}_p &\xrightarrow{\text{Frob}_p} \mathbb{W}_{p^\infty}^{(m)} \times \mathbb{F}_p & \mathbb{W}_{p^\infty}^{(m+1)} \times \mathbb{F}_p &\xrightarrow{\text{Frob}_p - \text{id}} \mathbb{W}_{p^\infty}^{(m)} \times \mathbb{F}_p \end{aligned}$$

Using the Ghost components for truncated p -typical Witt vectors of (6), the first two maps becomes isomorphic to, respectively, the projection away from the first coordinate and a linear projection

$$\prod_{i=0}^{m+1} \mathbb{G}_a \mathbb{Q} \longrightarrow \prod_{i=0}^m \mathbb{G}_a \mathbb{Q} \qquad \prod_{i=0}^{m+1} \mathbb{G}_a \mathbb{Q} \longrightarrow \prod_{i=0}^m \mathbb{G}_a \mathbb{Q}$$

both being clearly surjective and flat.

Remark that the Teichmuller coordinates provide an isomorphism as schemes $\mathbb{W}_{p^\infty}^{(m)} \simeq \prod_{i=0}^m \mathbb{A}^1$. After base-change to \mathbb{F}_p , this identification exhibits Frob_p as the standard power p Frobenius on \mathbb{A}^1 and the morphisms over \mathbb{F}_p become, respectively, the compositions

$$\begin{aligned} \text{Frob}_p : \prod_{i=0}^{m+1} \mathbb{A}^1 &\rightarrow \prod_{i=0}^{m+1} \mathbb{A}^1 \rightarrow \prod_{i=0}^m \mathbb{A}^1 \\ (\lambda_1, \lambda_p, \dots, \lambda_{p^m}, \lambda_{p^{m+1}}) &\mapsto (\lambda_1^p, \lambda_p^p, \dots, \lambda_{p^m}^p, \lambda_{p^{m+1}}^p) \mapsto (\lambda_1^p, \dots, \lambda_{p^m}^p) \end{aligned}$$

$$\text{Frob}_p - \text{id} : \prod_{i=0}^{m+1} \mathbb{A}^1 \rightarrow \prod_{i=0}^{m+1} \mathbb{A}^1 \rightarrow \prod_{i=0}^m \mathbb{A}^1$$

$$(\lambda_1, \lambda_p, \dots, \lambda_{p^m}, \lambda_{p^{m+1}}) \mapsto (\lambda_1^p - \lambda_1, \dots, \lambda_{p^m}^p - \lambda_{p^m}, \lambda_{p^{m+1}}^p - \lambda_{p^{m+1}}) \mapsto (\lambda_1^p - \lambda_1, \dots, \lambda_{p^m}^p - \lambda_{p^m})$$

Both projections are fpqc covers. We conclude using the fact that the standard power p Frobenius on \mathbb{A}^1 , $(-)^p$, is fpqc in our situation (see for instance [Liu02, Exercice 3.13]) and each $(-)^p - \text{id}$ is the Artin-Schreier isogeny well known to be an étale cover in characteristic p . □

Definition 3.16. We define a filtered stack $\mathbb{H}_{p^\infty} \rightarrow \mathbb{A}^1/\mathbb{G}_m$ to be stack over $\mathbb{A}^1/\mathbb{G}_m$ given by the quotient

$$\mathbb{H}_{p^\infty} := (\ker \mathcal{E}_p)/\mathbb{G}_m \rightarrow \mathbb{A}^1/\mathbb{G}_m$$

Remark 3.17. It follows from Remark 3.14 that the underlying stack $H_{p^\infty}^u$ is Fix and the associated graded $H_{p^\infty}^{\text{gr}}$ is the quotient Ker/\mathbb{G}_m under the \mathbb{G}_m -action of the Construction 3.10 and Remark 3.14.

4. THE FILTERED CIRCLE

4.1. Affinization and Cohomology of stacks. At this point, after the constructions in the previous section, we have in fact proved Theorem 1.1-(iii). We absorb it in the following definition:

Definition 4.1. The *filtered circle* (local at p) is the filtered stack given by the classifying stack of the filtered abelian group stack H_{p^∞} :

$$S_{\text{Fil}}^1 := \text{BH}_{p^\infty} \rightarrow \mathbb{A}^1/\mathbb{G}_m$$

Remark 4.2. The filtered stack S_{Fil}^1 , being the classifying stack of a filtered abelian group stack is again a filtered abelian group stack. In other words, it carries a canonical abelian group structure compatible with the filtration. In particular, we can take its classifying stack $\text{BS}_{\text{Fil}}^1 \simeq K(H_{p^\infty}, 2)$.

One of the claims in Theorem 1.1-(i) is that our filtered circle S_{Fil}^1 is related to the topological circle S^1 via the notion of *affinization* of [Toe06]. *Affine stacks* were introduced in [Toe06, Def. 2.2.4] (see also [Lur11] where these are called *Coaffine*). Informally, an (higher) stack X is affine if it can be recovered from its cohomology of global sections $\mathbb{R}\Gamma(X, \mathcal{O})$. More precisely:

Review 4.3. See Notation 1.8. By [Toe06, 2.2.3] the ∞ -functor $\text{Spec}^\Delta : \text{coSCR}_{\mathbb{Z}_{(p)}} \rightarrow \text{St}_{\mathbb{Z}_{(p)}}$ is fully faithful and admits a left adjoint $C_\Delta^*(-, \mathcal{O})$ that enhances the standard E_∞^\otimes -algebra structure of cohomology of global sections $C^*(-, \mathcal{O})$ with a structure of cosimplicial commutative algebra, namely, it provides a lifting

$$\begin{array}{ccc} & & \text{coSCR}_{\mathbb{Z}_{(p)}} \\ & \nearrow C_\Delta^*(-, \mathcal{O}) & \downarrow \theta \\ \text{St}_{\mathbb{Z}_{(p)}} & \xrightarrow{C^*(-, \mathcal{O})} & \text{CAlg}_{\mathbb{Z}_{(p)}} \end{array}$$

We say that X is affine if it lives in the essential image of Spec^Δ . Given $X \in \text{St}_{\mathbb{Z}_{(p)}}$, its affinization is the stack $\text{Spec}^\Delta(C_\Delta^*(X, \mathcal{O}))$ [Toe06, 2.3.2].

Proposition 4.4. Both S_{Fil}^1 and BS_{Fil}^1 are relatively affine stacks over $\mathbb{A}^1/\mathbb{G}_m$ in the sense of [Toe06]. By stability of affine stacks under base-change ^(†) so are all the stacks $(S_{\text{Fil}}^1)^u$, $(S_{\text{Fil}}^1)^{\text{gr}}$, $\text{B}(S_{\text{Fil}}^1)^u$ and $\text{B}(S_{\text{Fil}}^1)^{\text{gr}}$.

^(†)see [Toe06, 2.2.7, 2.2.9, Remarque p.49]

Proof. Let us start by showing that S_{Fil}^1 is relatively affine over $\mathbb{A}^1/\mathbb{G}_m$. Using the atlas $\mathbb{A}^1 \rightarrow \mathbb{A}^1/\mathbb{G}_m$ this is equivalent to show that $B \ker \mathcal{G}_p$ is an affine stack over $\mathbb{A}^1 = \text{Spec}(\mathbb{Z}_{(p)}[T])$. Applying B to the short exact sequence in the Lemma 3.15, we get a fiber sequence of fpqc sheaves over \mathbb{A}^1

$$\begin{array}{ccc} B \ker \mathcal{G}_p & \longrightarrow & BW_{p^\infty} \times \mathbb{A}^1 \\ \downarrow & & \downarrow B\mathcal{G}_p \times \text{id} \\ \mathbb{A}^1 & \longrightarrow & BW_{p^\infty} \times \mathbb{A}^1 \end{array}$$

By [Toe06, 2.2.7] the class of affine stacks over any commutative ring is closed under limits. Therefore, to conclude that $B \ker \mathcal{G}_p$ is affine it is enough to show that $BW_{p^\infty} \times \mathbb{A}^1$ is affine. But the group scheme W_{p^∞} can be written as a limit $\lim W_{p^\infty}^{(m)}$ where each projection $W_{p^\infty}^{(m+1)} \rightarrow W_{p^\infty}^{(m)}$ is a smooth epimorphism of affine groups with fiber \mathbb{G}_a . We claim that this fact together with the fact with the Witt schemes are truncated, implies that the limit decomposition of W_{p^∞} induces a limit decomposition

$$BW_{p^\infty} \simeq \lim BW_{p^\infty}^{(m)}. \quad (13)$$

Indeed, the Milnor sequences (see for instance [GJ09, 2.2.9]) tell us that the obstructions for the limit decomposition (13) are given by the groups $\lim^1 \pi_i \text{Map}_{\text{fpqc}}(X, W_{p^\infty}^{(m)})$ for $i \geq 0$ and X affine classical. For $i \geq 1$ these groups vanish because the mapping spaces are discrete. For $i \geq 0$ we have $\pi_i \text{Map}_{\text{fpqc}}(X, W_{p^\infty}^{(m)}) \simeq H_{\text{fpqc}}^i(X, W_{p^\infty}^{(m)})$, which vanishes for $i > 0$ because X is affine.

In fact, more generally, we also have the same decomposition for the iterated construction

$$B^j W_{p^\infty} \simeq \lim B^j W_{p^\infty}^{(m)}. \quad (14)$$

This follows again because the mapping spaces are discrete and because of the vanishing of the higher cohomology groups

$$\pi_0 \text{Map}_{\text{fpqc}}(X, B^j W_{p^\infty}^{(m)}) = H_{\text{fpqc}}^j(X, W_{p^\infty}^{(m)}) = 0 \quad j \geq 1 \quad (15)$$

for X affine classical. One can see this by induction using the long exact sequences extracted from the fact $W_{p^\infty}^{(m)}$ is an extension of $W_{p^\infty}^{(m-1)}$ by \mathbb{G}_a ,

$$0 \rightarrow \mathbb{G}_a \rightarrow W_{p^\infty}^{(m+1)} \rightarrow W_{p^\infty}^{(m)} \rightarrow 0 \quad (16)$$

The result is true for \mathbb{G}_a and as $W_{p^\infty}^{(1)} = \mathbb{G}_a$, by induction, it is true all for m .

Finally, knowing (13), by [Toe06, 2.2.7] it becomes enough to show that each $BW_{p^\infty}^{(m)}$ is affine. But now, each of the group extensions (16) is classified by a map of group stacks $W_{p^\infty}^{(m)} \rightarrow K(\mathbb{G}_a, 1)$, which we can write as a map $BW_{p^\infty}^{(m)} \rightarrow K(\mathbb{G}_a, 2)$. By definition of this map, we have a pullback square

$$\begin{array}{ccc}
\mathrm{BW}_{p^\infty}^{(m+1)} & \longrightarrow & * \\
\downarrow & & \downarrow \\
\mathrm{BW}_{p^\infty}^{(m)} & \longrightarrow & \mathrm{K}(\mathbb{G}_a, 2)
\end{array}$$

Each $\mathrm{K}(\mathbb{G}_a, n)$ is known to be affine [Toe06, 2.2.5]. An induction argument concludes the proof.

To prove the claim for $\mathrm{BS}_{\mathrm{Fil}}^1$ it is enough to show that $\mathrm{B}(\mathrm{B} \ker \mathcal{G}_p)$ is affine over \mathbb{A}^1 . The argument runs the same, using the iterated formula (14). \square

4.2. The Underlying Stack of $\mathrm{S}_{\mathrm{Fil}}^1$. As we now know, by the Proposition 4.4, the underlying stack $(\mathrm{S}_{\mathrm{Fil}}^1)^u$ is affine. We would like, in order to establish Theorem 1.1-(i), to identify it with the affinization of S^1 over $\mathbb{Z}_{(p)}$.

Construction 4.5. Recall that $\mathrm{S}_{\mathrm{Fil}}^1 = \mathrm{BH}_{p^\infty}$ (Definition 3.16). Let \mathbb{Z} denote the constant group scheme with value \mathbb{Z} . There is a canonical morphism of group schemes $\mathbb{Z} \rightarrow \mathrm{W}_{p^\infty}$ given by $1 \mapsto (1-t)^{-1} \in \mathrm{W}_{p^\infty}(R)$. This Witt vector is fixed by the Frobenius so clearly the map factors through $\mathrm{H}_{p^\infty}^u = \mathrm{Fix}$. By passing to classifying stacks we obtain a morphism of stacks

$$\mathrm{S}^1 = \mathrm{B}\mathbb{Z} \rightarrow (\mathrm{S}_{\mathrm{Fil}}^1)^u = \mathrm{B}\mathrm{Fix} \quad (17)$$

with affine target.

The main result of this section is the following:

Proposition 4.6. *The map (17) displays $(\mathrm{S}_{\mathrm{Fil}}^1)^u = \mathrm{B}\mathrm{Fix}$ as the affinization of S^1 over $\mathrm{Spec} \mathbb{Z}_{(p)}$. By [Toe06, Corollaire 2.3.3], this is equivalent to say that (17) induces an equivalence on cochain algebras*

$$\mathrm{C}_\Delta^*(\mathrm{B}\mathrm{Fix}, \mathcal{O}) \simeq \mathrm{C}_\Delta^*(\mathrm{S}^1, \mathbb{Z}_{(p)}) \quad (18)$$

We will establish below in the Lemma 4.12 a local criterion for affinization: in order to prove that the map (17) is the affinization of S^1 over $\mathbb{Z}_{(p)}$ it is enough to know that when base-changed to \mathbb{Q} and \mathbb{F}_p , the maps

$$\mathrm{S}^1 \rightarrow \mathrm{B}\mathrm{Fix}_{|\mathbb{Q}} := \mathrm{B}\mathrm{Fix} \times_{\mathrm{Spec}(\mathbb{Z}_{(p)})} \mathrm{Spec}(\mathbb{Q}) \quad (19)$$

$$\mathrm{S}^1 \rightarrow \mathrm{B}\mathrm{Fix}_{|\mathbb{F}_p} := \mathrm{B}\mathrm{Fix} \times_{\mathrm{Spec}(\mathbb{Z}_{(p)})} \mathrm{Spec}(\mathbb{F}_p) \quad (20)$$

are affinizations of S^1 , respectively, over \mathbb{Q} and \mathbb{F}_p .

For the moment let us describe the targets of the maps (19) and (20). By the Remark 3.1, we already know that $\mathbf{B}\mathrm{Fix}_{|\mathbb{Q}} \simeq \mathbf{B}\mathbb{G}_{\mathbf{a}, \mathbb{Q}}$. It remains to work over \mathbb{F}_p :

Lemma 4.7. *There is an equivalence $(S_{\mathrm{Fil}}^1)_{|\mathbb{F}_p}^u \simeq \mathbf{B}\mathbb{Z}_p$ where $\mathbf{B}\mathbb{Z}_p$ is the classifying stack of the proconstant group scheme with values in the p -adic integers.*

Proof. Using the formula (5) one sees that for an \mathbb{F}_p -algebra A the map Frob_p on $W_{p^\infty}(A)$ takes the form

$$\mathrm{Frob}_p(\lambda_1, \lambda_p, \dots, \lambda_{p^k}, \dots) = (\lambda_1^p, \lambda_p^p, \dots, \lambda_{p^k}^p, \dots)$$

so that levelwise it coincides with the standard Frobenius of A . Now, for each n we have Artin-Schreier-Witt exact sequences [Ill79, Proposition 3.28]

$$0 \longrightarrow (\mathbb{Z}/p^n\mathbb{Z}) \longrightarrow W_{p^\infty}^{(m)} \xrightarrow{\mathrm{Frob}_p - \mathrm{id}} W_{p^\infty}^{(m)} \longrightarrow 0 \quad (21)$$

where we consider $(\mathbb{Z}/p^n\mathbb{Z})$ as the constant-valued group scheme over \mathbb{F}_p . By passing to the limit over n we obtain exact sequences

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow W_{p^\infty} \xrightarrow{\mathrm{Frob}_p - \mathrm{id}} W_{p^\infty} \longrightarrow 0 \quad (22)$$

from where we can conclude the identification $\mathrm{Fix} \simeq \mathbb{Z}_p$. \square

The following is a key computation:

Proposition 4.8. [Toe06, Corollaire 2.5.3] *The affinization of S^1 over \mathbb{Q} is $\mathbf{B}\mathbb{G}_{\mathbf{a}}$ and over \mathbb{F}_p is $\mathbf{B}\mathbb{Z}_p$.*

Let us now work our local criterion for affinization over $\mathbb{Z}_{(p)}$. We start with a simple remark:

Remark 4.9. Let M be an object in $\mathrm{Mod}_{\mathbb{Z}_{(p)}}(\mathbf{Sp})$. Suppose that both base changes $M \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ and $M \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p$ are zero. Then $M \simeq 0$. Indeed, as \mathbb{Q} is obtained from $\mathbb{Z}_{(p)}$ by inverting p , $M \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ is obtained via the filtered colimit of the diagram given by multiplication by p

$$\dots \xrightarrow{\cdot p} M \xrightarrow{\cdot p} M \xrightarrow{\cdot p} M \xrightarrow{\cdot p} \dots \quad (23)$$

At the same time, we know that \mathbb{F}_p is obtained from $\mathbb{Z}_{(p)}$ via an exact sequence

$$0 \longrightarrow \mathbb{Z}_{(p)} \xrightarrow{\cdot p} \mathbb{Z}_{(p)} \longrightarrow \mathbb{F}_p \longrightarrow 0$$

In particular, we have a cofiber-fiber sequence

$$\begin{array}{ccc} M \simeq M \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)} & \xrightarrow{\cdot p} & M \simeq M \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & M \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \end{array}$$

It follows that $M \otimes_{\mathbb{Z}(p)} \mathbb{F}_p \simeq 0$ if and only if the multiplication by p is an equivalence of M . In that case, the colimit of the diagram (23), meaning $M \otimes_{\mathbb{Z}(p)} \mathbb{Q}$, is equivalent to M . But the assumption $M \otimes_{\mathbb{Z}(p)} \mathbb{Q} \simeq 0$ concludes that $M \simeq 0$.

In particular, given $f : E \rightarrow F$ a morphism of chain complexes over $\mathbb{Z}(p)$, if the two base changes to \mathbb{Q} and \mathbb{F}_p are equivalences, then so is f .

Lemma 4.10. *Consider the pullback diagrams:*

$$\begin{array}{ccccc} \mathrm{B}\mathbb{G}_a \mathbb{Q} \simeq \mathrm{B}\mathrm{Fix}|_{\mathbb{Q}} & \xrightarrow{J} & \mathrm{B}\mathrm{Fix} & \xleftarrow{I} & \mathrm{B}\mathrm{Fix}|_{\mathbb{F}_p} \simeq \mathrm{B}\mathbb{Z}_p \\ \downarrow f_{\mathbb{Q}} & & \downarrow f & & \downarrow f_p \\ \mathrm{Spec}(\mathbb{Q}) & \xrightarrow{j} & \mathrm{Spec}(\mathbb{Z}(p)) & \xleftarrow{i} & \mathrm{Spec}(\mathbb{F}_p) \end{array}$$

Then we have equivalences of cosimplicial commutative k -algebras

$$\mathrm{C}_{\Delta}^*(\mathrm{B}\mathrm{Fix}, \mathcal{O}) \otimes_{\mathbb{Z}(p)} \mathbb{Q} \simeq \mathrm{C}_{\Delta}^*(\mathrm{B}\mathrm{Fix}|_{\mathbb{Q}}, \mathcal{O}) \quad \text{and} \quad \mathrm{C}_{\Delta}^*(\mathrm{B}\mathrm{Fix}, \mathcal{O}) \otimes_{\mathbb{Z}(p)} \mathbb{F}_p \simeq \mathrm{C}_{\Delta}^*(\mathrm{B}\mathrm{Fix}|_{\mathbb{F}_p}, \mathcal{O}) \quad (24)$$

Proof. The rational equivalence follows because the Beck-Chevalley transformation

$$j^* f_* \rightarrow f_{\mathbb{Q}} J^*$$

for QCoh is an equivalence. This follows from the local definition of pushforwards defined directly at the level of QCoh [Lur17a, §6.2, Chapter 6], in this case, through descent along Zariski open immersions (j being a Zariski open). We obtain (24) by evaluating the Beck-Chevalley transformation on the structure sheaf.

Let us now show the equivalence over \mathbb{F}_p . The Lemma 3.15 gives us a short exact sequence of fpqc sheaves of groups

$$0 \longrightarrow \mathrm{Fix} \longrightarrow \mathrm{W}_{p^\infty} \xrightarrow{\mathrm{Frob}_p - \mathrm{id}} \mathrm{W}_{p^\infty} \longrightarrow 0$$

where the map $\mathrm{Frob}_p - \mathrm{id}$ is flat. This implies that Fix is a flat group scheme over $\mathrm{Spec}(\mathbb{Z}(p))$ and that the square

$$\begin{array}{ccc} \mathrm{Fix} & \longrightarrow & \mathrm{W}_{p^\infty} \\ \downarrow & & \downarrow \mathrm{Frob}_p - \mathrm{id} \\ \mathrm{Spec}(\mathbb{Z}(p)) & \xrightarrow{0} & \mathrm{W}_{p^\infty} \end{array} \quad (25)$$

is actually a derived fiber product. The derived base change [Lur17a, Proposition 2.5.4.5] formula tells us that

$$\mathrm{C}^*(\mathrm{Fix}, \mathcal{O}) \otimes_{\mathbb{Z}(p)} \mathbb{F}_p \simeq \mathrm{C}^*(\mathrm{Fix}|_{\mathbb{F}_p}, \mathcal{O}) \quad (26)$$

where $\mathrm{C}^*(\mathrm{Fix}, \mathcal{O})$ is the Hopf-algebra of functions on the affine group scheme Fix . To show that formula (26) implies formula (24) over \mathbb{F}_p we use the description of the classifying

stack $\mathbf{B}\mathrm{Fix}$ as the geometric realization of the simplicial object

$$\cdots \quad \mathrm{Fix} \times \mathrm{Fix} \quad \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \quad \mathrm{Fix} \quad \longrightarrow \quad \mathrm{Spec}(\mathbb{Z}_{(p)})$$

which exhibits

$$\mathbf{C}^*(\mathbf{B}\mathrm{Fix}, \mathcal{O}) \simeq \lim_{[n] \in \Delta} \mathbf{C}^*(\mathrm{Fix}, \mathcal{O})^{\otimes n} \quad (27)$$

Here, because Fix is affine, the Kunneth formulas for the cohomology of its cartesian powers are automatic. The same argument also tells us that

$$\mathbf{C}^*(\mathbf{B}\mathrm{Fix}_{|\mathbb{F}_p}, \mathcal{O}) \simeq \lim_{[n] \in \Delta} \mathbf{C}^*(\mathrm{Fix}_{|\mathbb{F}_p}, \mathcal{O})^{\otimes n} \quad (28)$$

But now we know that, as in the Remark 4.9, $\mathbf{C}^*(\mathbf{B}\mathrm{Fix}, \mathcal{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p$ is the cofiber of multiplication by p

$$\begin{array}{ccc} \mathbf{C}^*(\mathbf{B}\mathrm{Fix}, \mathcal{O}) & \xrightarrow{\cdot p} & \mathbf{C}^*(\mathbf{B}\mathrm{Fix}, \mathcal{O}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbf{C}^*(\mathbf{B}\mathrm{Fix}, \mathcal{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \end{array} \quad (29)$$

As multiplication by p is actually happening levelwise in (27), we deduce that the square (29) is obtained from the squares

$$\begin{array}{ccc} \mathbf{C}^*(\mathrm{Fix}, \mathcal{O})^{\otimes n} & \xrightarrow{\cdot p} & \mathbf{C}^*(\mathrm{Fix}, \mathcal{O})^{\otimes n} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbf{C}^*(\mathrm{Fix}, \mathcal{O})^{\otimes n} \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \end{array} \quad (30)$$

by passing to the limit in Δ . The formula formula (24) over \mathbb{F}_p follows from the comparison (26) applied to each entry of the cartesian square (30), passing to the limit and using formula (28).

□

Remark 4.11. For a fixed ring R , the \mathbf{E}_∞ -algebra of singular cochains $\mathbf{C}^*(S^1, R)$ can be defined as the co-tensorisation R^{S^1} in the ∞ -category category $\mathbf{CAlg}(\mathrm{Mod}_R)$ which can be explicitly described as the limit of the constant diagram with value R , $\lim_{S^1} R$. Because S^1 has a finite model as a simplicial set, this limit is finite. It follows that for any map of rings $R \rightarrow R'$, the derived base change $\mathbf{C}^*(S^1, R) \otimes_R R' \rightarrow \mathbf{C}^*(S^1, R')$ is an equivalence of algebras. In particular, we have equivalences

$$\mathbf{C}^*(S^1, \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \simeq \mathbf{C}^*(S^1, \mathbb{F}_p) \quad \text{and} \quad \mathbf{C}^*(S^1, \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \simeq \mathbf{C}^*(S^1, \mathbb{Q})$$

We are now ready to prove our local criterion for affinization:

Lemma 4.12. *If the two maps (19) and (20) are affinizations of S^1 , respectively over \mathbb{Q} and \mathbb{F}_p , then the map (17) is an affinization over $\mathbb{Z}_{(p)}$.*

Proof. The combination of the Remark 4.11 and the Lemma 4.10 with [Toe06, Corollaire 2.3.3] tells us that the statement in the lemma is equivalent to the following: to deduce that the map (18) is an equivalence, it is enough to check that both maps

$$C^*(\text{BFix}, \mathcal{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \rightarrow C^*(S^1, \mathbb{Q}) \quad \text{and} \quad C^*(\text{BFix}, \mathcal{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \rightarrow C^*(S^1, \mathbb{F}_p) \quad (31)$$

are equivalences. Formulated this way, the lemma is immediate from the Remark 4.9. \square

This concludes the proof of Proposition 4.6.

4.3. The associated graded of S_{Fil}^1 . Our next order of business is to prove Theorem 1.1-(ii) concerning the associated graded $(S_{\text{Fil}}^1)^{\text{gr}} \simeq \text{BKer}$. As in the previous section, we reduce the problem to computations over \mathbb{Q} and \mathbb{F}_p .

Lemma 4.13. *We have canonical equivalences of commutative cosimplicial k -algebras*

$$C_{\Delta}^*(\text{BKer}, \mathcal{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \simeq C_{\Delta}^*(\text{BKer}_{|\mathbb{Q}}, \mathcal{O}) \quad \text{and} \quad C_{\Delta}^*(\text{BKer}, \mathcal{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \simeq C_{\Delta}^*(\text{BKer}_{|\mathbb{F}_p}, \mathcal{O}) \quad (32)$$

Proof. As in the proof of Lemma 4.10, we use base change together with the observation that the exact sequence

$$0 \longrightarrow \text{Ker} \longrightarrow W_{p^\infty} \xrightarrow{\text{Frob}_p} W_{p^\infty} \longrightarrow 0$$

exhibits Ker as a flat group scheme over $\mathbb{Z}_{(p)}$. From here the proof goes as in Lemma 4.10. \square

Construction 4.14. Consider the composition

$$\text{Ker} \subseteq W_{p^\infty} \xrightarrow{\text{Ghost}} \prod_{i=0}^{\infty} \mathbb{G}_a \xrightarrow{\text{proj}_0} \mathbb{G}_a \quad (33)$$

and the induced map

$$u : \text{BKer} \rightarrow \text{B}\mathbb{G}_a \quad (34)$$

By definition of $\text{B}\mathbb{G}_a$, the map u (34) corresponds to an element $u \in H^1(C^*(\text{BKer}, \mathcal{O}))$, $u : \mathbb{Z}_{(p)}[-1] \rightarrow C^*(\text{BKer}, \mathcal{O})$. One can check using explicit formulas for the Ghost map that the composition (33) is compatible with the \mathbb{G}_m -actions, where on the l.h.s we have the action of the Construction 3.10 and Remark 3.14 and on the r.h.s we have the standard \mathbb{G}_m -action on \mathbb{G}_a . In particular, (34) is \mathbb{G}_m -equivariant and the element u is defined in $\text{Mod}_{\mathbb{Z}_{(p)}}^{\mathbb{Z}\text{-gr}}$ where $\mathbb{Z}_{(p)}[-1]$ is pure of weight 1. At the same time we consider the canonical element $1 : \mathbb{Z}_{(p)} \rightarrow C^*(\text{BKer}, \mathcal{O})$ in $H^0(C^*(\text{BKer}, \mathcal{O}))$. Because the structure map $\text{BKer} \rightarrow \text{Spec}(\mathbb{Z}_{(p)})$

is \mathbb{G}_m -equivariant for the trivial action on the target, 1 also defines a graded map, with $\mathbb{Z}_{(p)}$ sitting in weight 0.

The sum of the graded maps u and 1 give us a map

$$\mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}[-1] \rightarrow \mathbf{C}^*(\mathbf{BKer}, \mathcal{O}) \quad \text{in} \quad \text{Mod}_{\mathbb{Z}_{(p)}}^{\mathbb{Z}\text{-gr}} \quad (35)$$

This map becomes an equivalence after base change to \mathbb{Q} (Remark 3.1).

Proposition 4.15. *The map of graded complexes (35) is an equivalence after tensoring with \mathbb{F}_p . By the Lemma 4.13 and the Remark 4.9, it is also an equivalence over $\mathbb{Z}_{(p)}$. In particular, the grading on $\mathbf{C}^*(\mathbf{BKer}, \mathcal{O})$ coincides with the cohomological grading.*

We will establish the proof of Proposition 4.15 by computing the underlying complex of global sections of the structure sheaf of $\mathbf{BKer}_{|\mathbb{F}_p}$.

Remark 4.16. Notice that $\text{Ker}_{|\mathbb{F}_p} = \text{Spec}(\mathbf{C}^*(\text{Ker}_{|\mathbb{F}_p}, \mathcal{O}))$ is an affine group scheme over \mathbb{F}_p . Both \mathbf{BKer} and Ker are graded and we have an equivalence of cosimplicial graded Hopf algebras

$$\mathbf{C}_\Delta^*(\mathbf{BKer}_{|\mathbb{F}_p}, \mathcal{O}) \simeq \lim_\Delta \mathbf{C}_\Delta^*(\text{Ker}_{|\mathbb{F}_p}, \mathcal{O})^{\otimes n} \quad (36)$$

In order to understand the underlying complex of $\mathbf{C}_\Delta^*(\text{Ker}_{|\mathbb{F}_p}, \mathcal{O})$ we will characterize its category of representations as an Hopf algebra.

Remark 4.17. By descent for QCoh , the pullback along the atlas $\text{Spec}(\mathbb{F}_p) \rightarrow \mathbf{BKer}_{|\mathbb{F}_p}$ makes $\text{QCoh}(\mathbf{BKer}_{|\mathbb{F}_p})$ comonadic over $\text{Mod}_{\mathbb{F}_p}$ and the Barr-Beck theorem provides an equivalence of ∞ -categories

$$\text{QCoh}(\mathbf{BKer}_{|\mathbb{F}_p}) \simeq \text{CoMod}_{\mathbf{C}^*(\text{Ker}_{|\mathbb{F}_p}, \mathcal{O})}(\text{Mod}_{\mathbb{F}_p})$$

Construction 4.18. The group scheme $\text{Ker}_{|\mathbb{F}_p}$ has a natural pro-group structure induced from the decomposition $W_{p^\infty} \simeq \lim W_{p^\infty}^{(m)}$ by defining $\text{Ker}_{|\mathbb{F}_p}$ to be the kernel of the exact sequence

$$0 \longrightarrow \text{Ker}_{|\mathbb{F}_p}^{(m)} \longrightarrow (W_{p^\infty}^{(m)})_{|\mathbb{F}_p} \xrightarrow{\text{Frob}_p} (W_{p^\infty}^{(m)})_{|\mathbb{F}_p} \longrightarrow 0$$

We obtain $\text{Ker}_{|\mathbb{F}_p} \simeq \lim \text{Ker}_{|\mathbb{F}_p}^{(m)}$ and therefore a colimit of Hopf algebras

$$\mathbf{C}^*(\text{Ker}_{|\mathbb{F}_p}, \mathcal{O}) \simeq \text{colim}_m \mathbf{C}^*(\text{Ker}_{|\mathbb{F}_p}^{(m)}, \mathcal{O}) \quad (37)$$

Definition 4.19. Let us denote by α_{p^m} the affine scheme over \mathbb{F}_p given by $\text{Spec}(\mathbb{F}_p[T]/(T^{p^m}))$. Its functor of points is given by $R \mapsto \{r \in R : r^{p^m} = 0\}$ classifying p^m -roots of zero. This is an abelian affine group scheme under the additive law over \mathbb{F}_p .

Lemma 4.20. *For each $m \geq 1$, the Cartier dual of the group scheme α_{p^m} is the algebraic group $\text{Ker}_{|\mathbb{F}_p}^{(m)}$. In particular, it follows from Cartier duality that we have an equivalence of ∞ -categories*

$$\text{CoMod}_{\mathbf{C}^*(\text{Ker}_{|\mathbb{F}_p}^{(m)}, \mathcal{O})} \simeq \text{Mod}_{\mathbb{F}_p[T]/(T^{p^m})} \quad (38)$$

Proof. This is [Oor66, II.10.3, Remark]^(†) (see also [Dem86, III §4]). See also [Sul78] for the equivalence of module categories. \square

Remark 4.21. If we forget the group structures, the affine scheme α_{p^m} is isomorphic to the underlying scheme of $\mu_{p^m} = \text{Spec}(\mathbb{F}_p[U]/(U^{p^m} - 1))$ of p^m -roots of unity under the change of coordinates $T \mapsto (U - 1)$. This induces an equivalence of categories

$$\text{Mod}_{\mathbb{F}_p[U]/(U^{p^m} - 1)} \simeq \text{Mod}_{\mathbb{F}_p[T]/(T^{p^m})} \quad (39)$$

Furthermore, we know that μ_{p^m} is Cartier dual to the group scheme $\mathbb{Z}/p^m\mathbb{Z}$ [Oor66, I.2.12, Lemma 2.15] and this gives us an equivalence

$$\text{Mod}_{\mathbb{F}_p[T]/(T^{p^m})} \simeq \text{CoMod}_{\mathbf{C}^*(\mathbb{Z}/p^m\mathbb{Z}, \mathcal{O})} \quad (40)$$

Notice moreover that the isomorphisms of schemes $\mu_{p^m} \simeq \alpha_{p^m}$ are compatible for different m 's.

Proof of Proposition 4.15. Composing the equivalences (38), (39) and (40), the Barr-Beck theorem gives us an equivalence of coalgebras

$$\mathbf{C}^*(\text{Ker}_{|\mathbb{F}_p}^{(m)}, \mathcal{O}) \simeq \mathbf{C}^*(\mathbb{Z}/p^m\mathbb{Z}, \mathcal{O}) \quad (41)$$

Because Cartier duality is functorial, these equivalences are compatible under the restriction maps and therefore, the equivalence extends to the limit

$$\mathbf{C}^*(\text{Ker}_{|\mathbb{F}_p}, \mathcal{O}) \simeq \lim_m \mathbf{C}^*(\text{Ker}_{|\mathbb{F}_p}^{(m)}, \mathcal{O}) \simeq \lim_m \mathbf{C}^*(\mathbb{Z}/p^m\mathbb{Z}, \mathcal{O}) \quad (42)$$

But now the r.h.s is by definition the coalgebra of the group scheme \mathbb{Z}_p , and we get an equivalence of coalgebras

$$\mathbf{C}^*(\text{Ker}_{|\mathbb{F}_p}, \mathcal{O}) \simeq \mathbf{C}^*(\mathbb{Z}_p, \mathcal{O}) \quad (43)$$

Finally, using the formula (36), we find

$$\mathbf{C}^*(\text{BKer}_{|\mathbb{F}_p}, \mathcal{O}) \simeq \lim_{\Delta} \mathbf{C}^*(\text{Ker}_{|\mathbb{F}_p}, \mathcal{O})^{\otimes n} \simeq \lim_{\Delta} \mathbf{C}^*(\mathbb{Z}_p, \mathcal{O})^{\otimes n} =: \mathbf{C}^*(\text{B}\mathbb{Z}_p, \mathcal{O}) \quad (44)$$

^(†)Formula $L_{m,n}^D = L_{n,m}$

Finally, from the Proposition 4.8 we deduce an equivalence of \mathbb{F}_p -modules

$$\mathbf{C}^*(\mathrm{BKer}_{|\mathbb{F}_p}, \mathcal{O}) \simeq \mathbb{F}_p \oplus \mathbb{F}_p[-1] \quad (45)$$

□

We now discuss the algebra structure on $\mathbf{C}^*(\mathrm{BKer}, \mathcal{O})$. We start by the $\mathbf{E}_\infty^\otimes$ -structure:

Lemma 4.22. *Let $M := \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}[-1] \in \mathrm{Mod}_{\mathbb{Z}_{(p)}}^{\mathbb{Z}\text{-gr}}$ denote the graded complex of the Construction 4.14. Then, the space of $\mathbf{E}_\infty^\otimes$ -algebra structures on M compatible with the grading is equivalent to the set of classical commutative graded algebra structures on its cohomology $H^*(M)$. In particular, it is homotopically discrete.*

Proof. The data of an $\mathbf{E}_\infty^\otimes$ -algebra structure on a given object $M \in \mathrm{Mod}_{\mathbb{Z}_{(p)}}$ is the data of a map of ∞ -operads from $\mathbf{E}_\infty^\otimes$ to the ∞ -operad of endomorphisms of M , for which we shall write $\mathrm{End}(M)^\otimes$. In our case, the object $M := \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}[-1]$ is endowed with a structure of object in $\mathrm{Mod}_{\mathbb{Z}_{(p)}}^{\mathbb{Z}\text{-gr}}$ of the Construction 4.14, so in fact, we are interested in the ∞ -operad of endomorphisms of M in this ∞ -category $\mathrm{End}_{\mathrm{gr}}(M)^\otimes$. Its space of n -ary operations is given by the mapping space $\mathrm{Map}_{\mathrm{Mod}_{\mathbb{Z}_{(p)}}^{\mathbb{Z}\text{-gr}}}(M^{\otimes n}, M)$ where the powers $M^{\otimes n}$ are taken with respect to the graded tensor product. It follows from the formula for the Day convolution (see Construction 3.2 and the references to [Lur15]) that for every $n \geq 1$ the piece of weight 0 in $M^{\otimes n}$ is $\mathbb{Z}_{(p)}$ and the piece in weight 1 is given by $\bigoplus_{i=1}^n \mathbb{Z}_{(p)}[-1]$. In particular, we get

$$\mathrm{End}_{\mathrm{gr}}(M)^\otimes(n) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbb{Z}_{(p)}}}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \times \mathrm{Map}_{\mathrm{Mod}_{\mathbb{Z}_{(p)}}}(\bigoplus_{i=1}^n \mathbb{Z}_{(p)}[-1], \mathbb{Z}_{(p)}[-1]) \simeq \bigoplus_{i=0}^n \mathbb{Z}_{(p)}$$

which is a discrete space. This implies that the space of maps of ∞ -operads

$$\mathrm{Map}_{\mathrm{Op}_\infty}(\mathbf{E}_\infty^\otimes, \mathrm{End}_{\mathrm{gr}}(M)^\otimes)$$

is discrete. But more is true: consider the cohomology $H^*(M)$ as a (classical) graded $\mathbb{Z}_{(p)}$ -vector space and $\mathrm{End}_{\mathrm{gr}, \mathrm{cl}}(H^*(M))^\otimes$ its classical operad of (graded) endomorphisms. As H^* is lax monoidal, we get a map

$$\mathrm{Map}_{\mathrm{Op}_\infty}(\mathbf{E}_\infty^\otimes, \mathrm{End}_{\mathrm{gr}}(M)^\otimes) \rightarrow \mathrm{Map}_{\mathrm{Op}_\infty}(\mathbf{E}_\infty^\otimes, \mathrm{End}_{\mathrm{gr}, \mathrm{cl}}(H^*(M))^\otimes)$$

The computation above applied to the classical graded version shows that this map is actually an equivalence of spaces.

□

Remark 4.23. The argument in the proof of the Lemma 4.22 also shows that there exists a unique commutative $\mathbf{E}_\infty^\otimes$ -algebra structure on the object $M := \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}[-1]$ seen as an object of $\mathrm{Mod}_{\mathbb{Z}_{(p)}}^{\leq 0, \mathrm{gr}}$ with the symmetric monoidal structure of Notation 1.8.

We now discuss the co-simplicial multiplicative structure.

Notation 4.24. Let us consider graded versions $\text{coSCR}_{\mathbb{Z}(p)}^{\text{gr}}$ and $\text{CAlg}_{\mathbb{Z}(p)}^{\text{ccn,gr}} := \text{CAlg}(\text{Mod}_{\mathbb{Z}(p)}^{\leq 0, \text{gr}})$ of respectively, cosimplicial and E_{∞}^{\otimes} -algebras and $\theta^{\text{ccn}} : \text{coSCR}_{\mathbb{Z}(p)}^{\text{gr}} \rightarrow \text{CAlg}_{\mathbb{Z}(p)}^{\text{ccn,gr}}$ denote the graded version of the dual Dold-Kan construction (see Notation 1.8).

Notation 4.25. We denote by $\mathbb{Z}_{(p)}[\epsilon_{-1}]$ the trivial square zero extension structure on the complex $\mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}[-1]$ as an object in $\text{coSCR}_{\mathbb{Z}(p)}$. We will use the same notation for its underlying E_{∞}^{\otimes} -algebra under the functor θ of Notation 1.8.

Corollary 4.26 (of Lemma 4.22 and Remark 4.23). *The map (35) extends as an equivalence of graded E_{∞}^{\otimes} -algebras in $\text{Mod}_{\mathbb{Z}(p)}^{\leq 0}$ compatible with the augmentations to $\mathbb{Z}_{(p)}$:*

$$\mathbb{Z}_{(p)}[\epsilon_{-1}] \rightarrow \text{C}^*(\text{BKer}, \mathcal{O}) \quad (46)$$

We now claim that (46) actually lifts to an equivalence of graded cosimplicial commutative algebras. In order to prove this we use the limit formula (36) to reduce to an argument about the discrete graded group scheme Ker . We will need some preliminaries:

Construction 4.27. Let \mathcal{C} be a presentable ∞ -category endowed with the cartesian symmetric monoidal structure. We denote by $\text{Gr}(\mathcal{C}) := \text{Mon}_{E_1^{\text{gp}}}^{\text{gp}}(\mathcal{C})$ the category of group objects in \mathcal{C} . Following [Lur17b, 5.2.6.6, 4.1.2.11, 2.4.2.5.], an explicit model is given by category of diagrams $\text{Fun}(\Delta^{\text{op}}, \mathcal{C})$ satisfying the Segal conditions. The colimit functor

$$\text{B} : \text{Gr}(\mathcal{C}) \rightarrow \mathcal{C}_* \quad (47)$$

lands in the category of pointed objects in \mathcal{C} and admits a right adjoint, sending a pointed object $* \rightarrow X$ to its nerve.

Construction 4.28. We apply a dual version of the Construction 4.27 to the categories $\mathcal{C} = \text{coSCR}_{\mathbb{Z}(p)}^{\text{gr}}$ and $\mathcal{C} = \text{CAlg}_{\mathbb{Z}(p)}^{\text{ccn,gr}}$. Namely, instead of considering group objects with consider cogroup-objects with the limit functor:

$$\begin{array}{ccc} \text{coGr}(\text{coSCR}_{\mathbb{Z}(p)}^{\text{gr}}) & \xrightarrow[\text{CoNerve}]{\lim_{\Delta}} & (\text{coSCR}_{\mathbb{Z}(p)}^{\text{gr}})_{./\mathbb{Z}(p)} \\ \text{coGr}(\text{CAlg}_{\mathbb{Z}(p)}^{\text{ccn,gr}}) & \xrightarrow[\text{CoNerve}]{\lim_{\Delta}} & (\text{CAlg}_{\mathbb{Z}(p)}^{\text{ccn,gr}})_{./\mathbb{Z}(p)} \end{array} \quad (48)$$

Both adjunctions commute with $\theta^{\text{ccn}} : \text{coSCR}_{\mathbb{Z}(p)}^{\text{gr}} \rightarrow \text{CAlg}_{\mathbb{Z}(p)}^{\text{ccn,gr}}$ (Notation 1.8):

$$\begin{array}{ccc}
\mathrm{coGr}(\mathrm{coSCR}_{\mathbb{Z}(p)}^{\mathrm{gr}}) & \xrightleftharpoons[\mathrm{CoNerve}]{\lim_{\Delta}} & (\mathrm{coSCR}_{\mathbb{Z}(p)}^{\mathrm{gr}})_{./\mathbb{Z}(p)} \\
\downarrow \theta^{\mathrm{ccn}} & & \downarrow \theta^{\mathrm{ccn}} \\
\mathrm{coGr}(\mathrm{CAlg}_{\mathbb{Z}(p)}^{\mathrm{ccn}, \mathrm{gr}}) & \xrightleftharpoons[\mathrm{CoNerve}]{\lim_{\Delta}} & (\mathrm{CAlg}_{\mathbb{Z}(p)}^{\mathrm{ccn}, \mathrm{gr}})_{./\mathbb{Z}(p)}
\end{array} \tag{49}$$

Our main computation is the following:

Theorem 4.29. *The equivalence of graded $\mathbf{E}_{\infty}^{\otimes}$ -algebras of the Corollary 4.26 can be promoted to an equivalence of graded commutative cosimplicial algebras*

$$\mathbb{Z}_{(p)}[\epsilon_{-1}] \simeq \mathbf{C}_{\Delta}^*(\mathbf{BKer}, \mathcal{O}) \tag{50}$$

In particular, as \mathbf{BKer} is an affine stack (Proposition 4.4) we can write

$$\mathbf{BKer} \simeq \mathrm{Spec}^{\Delta}(\mathbb{Z}_{(p)}[\epsilon_{-1}])$$

Proof. By construction, the cosimplicial object $\mathbf{C}_{\Delta}^*(\mathbf{Ker}, \mathcal{O})^{\bullet}$ in the formula (36) defines an object in $\mathrm{coGr}(\mathrm{coSCR}^{\mathrm{gr}})$ and in the terminology of the Construction 4.28 the equivalence (36) reads as an equivalence in $\mathrm{coSCR}_{./\mathbb{Z}(p)}^{\mathrm{gr}}$:

$$\mathbf{C}_{\Delta}^*(\mathbf{BKer}, \mathcal{O}) \simeq \lim_{\Delta} [\mathbf{C}_{\Delta}^*(\mathbf{Ker}, \mathcal{O})^{\bullet}]$$

Because the stack \mathbf{BKer} is affine (Proposition 4.4) we know that $\mathbf{BKer} \simeq \mathrm{Spec}^{\Delta}(\mathbf{C}_{\Delta}^*(\mathbf{BKer}, \mathcal{O}))$. Now, using the fact Spec^{Δ} is a right adjoint, we deduce an equivalence in $\mathrm{coGr}(\mathrm{coSCR}^{\mathrm{gr}})$

$$\mathbf{C}_{\Delta}^*(\mathbf{Ker}, \mathcal{O})^{\bullet} \simeq \mathrm{coNerve}[\mathbf{C}_{\Delta}^*(\mathbf{BKer}, \mathcal{O})]$$

where we see $\mathbf{C}_{\Delta}^*(\mathbf{BKer}, \mathcal{O})$ augmented over $\mathbb{Z}_{(p)}$ via the atlas. We deduce that

$$\mathbf{C}_{\Delta}^*(\mathbf{BKer}, \mathcal{O}) \simeq \lim_{\Delta} \circ \mathrm{coNerve} [\mathbf{C}_{\Delta}^*(\mathbf{BKer}, \mathcal{O})]$$

in $\mathrm{coSCR}_{./\mathbb{Z}(p)}^{\mathrm{gr}}$.

The commutativity of the diagram (49) tells us that a similar formula holds for the graded coconnective $\mathbf{E}_{\infty}^{\otimes}$ -version. Therefore, the equivalence of the Corollary 4.26 tells us that a similar formula holds for the graded coconnective $\mathbf{E}_{\infty}^{\otimes}$ -version of $\mathbb{Z}_{(p)}[\epsilon_{-1}]$, ie, an equivalence in $\mathrm{CAlg}_{./\mathbb{Z}(p)}^{\mathrm{ccn}, \mathrm{gr}}$

$$\theta^{\mathrm{ccn}}(\mathbb{Z}_{(p)}[\epsilon_{-1}]) \simeq \lim_{\Delta} \circ \mathrm{coNerve} [\theta^{\mathrm{ccn}}(\mathbb{Z}_{(p)}[\epsilon_{-1}])]$$

In particular, we have an equivalence in $\mathrm{coGr}(\mathrm{CAlg}^{\mathrm{ccn}, \mathrm{gr}})$

$$\mathrm{coNerve} [\theta^{\mathrm{ccn}}((\mathbb{Z}_{(p)}[\epsilon_{-1}])) \simeq \mathrm{coNerve} [\theta^{\mathrm{ccn}}((\mathbf{C}_{\Delta}^*(\mathbf{BKer}, \mathcal{O})))] \simeq \theta^{\mathrm{ccn}}((\mathbf{C}^*(\mathbf{Ker}, \mathcal{O})^{\bullet})) \tag{51}$$

Finally, we remark that \mathbf{Ker} is a classical affine scheme, so that its global sections are discrete in the co-simplicial direction. In particular, as the functor $\theta^{\mathrm{ccn}} : \mathrm{coSCR}_{\mathbb{Z}(p)}^{\mathrm{gr}} \rightarrow$

$\mathrm{CAlg}_{\mathbb{Z}_{(p)}}^{\mathrm{cn}, \mathrm{gr}}$ induces an equivalence on discrete algebras, the equivalence (51) lifts to an equivalence in $\mathrm{coGr}(\mathrm{coSCR}^{\mathrm{gr}})$

$$\mathrm{coNerve} [\mathbb{Z}_{(p)}[\epsilon_{-1}]] \simeq \mathrm{coNerve} [C_{\Delta}^*(\mathrm{BKer}, \mathcal{O})] \quad (52)$$

and therefore an equivalence in $\mathrm{coSCR}_{\cdot/\mathbb{Z}_{(p)}}^{\mathrm{gr}}$

$$\mathbb{Z}_{(p)}[\epsilon_{-1}] \simeq \lim_{\Delta} \circ \mathrm{coNerve} [\mathbb{Z}_{(p)}[\epsilon_{-1}]] \simeq \lim_{\Delta} \circ \mathrm{coNerve} [C_{\Delta}^*(\mathrm{BKer}, \mathcal{O})] \simeq C_{\Delta}^*(\mathrm{BKer}, \mathcal{O}) \quad (53)$$

□

5. REPRESENTATIONS OF THE FILTERED CIRCLE

We now turn our attention to the classifying stack $\mathrm{BS}_{\mathrm{Fil}}^1$; note that by Proposition 4.4 this is an affine stack relatively to $\mathbb{A}^1/\mathbb{G}_m$. Moreover, by the Remark 3.8, the cohomology ring $C^*(\mathrm{BS}_{\mathrm{Fil}}^1, \mathcal{O})$ admits a natural structure as a filtered E_{∞}^{\otimes} -algebra.

As we shall see, the ∞ -category of $\mathrm{S}_{\mathrm{Fil}}^1$ -representations will coincide with a notion of mixed complex; but this identification will not preserve the relevant symmetric monoidal structures. Our first evidence of this is the following

Proposition 5.1. *There is an equivalence of filtered E_1^{\otimes} -algebras*

$$C^*(\mathrm{BS}_{\mathrm{Fil}}^1, \mathcal{O}) \simeq \mathbb{Z}_{(p)}[u]$$

where u sits in degree 2 and has a split filtration induced by the canonical grading for which u is of weight -1 .

Proof. We construct a map of filtered E_1^{\otimes} -algebras $\mathbb{Z}_{(p)}[u] \rightarrow C^*(\mathrm{BS}_{\mathrm{Fil}}^1, \mathcal{O})$ which realizes this equivalence. For this we first define a class corresponding to the image of the generator u . Algebraically, this corresponds to a certain morphism in the ∞ -category of filtered complexes. Geometrically, this will correspond to a morphism of stacks $\mathrm{BS}_{\mathrm{Fil}}^1 \rightarrow \mathrm{B}^2\mathbb{G}_a(1)$ where $\mathbb{G}_a(1)$ denotes the filtered group scheme with filtration induced by the pure weight one action of \mathbb{G}_m on \mathbb{G}_a . By delooping, this corresponds to a map of filtered groups

$$\mathrm{H}_{p^{\infty}} \rightarrow \mathbb{G}_a(1)$$

We take the map given by the composition of the inclusion $\mathrm{H}_{p^{\infty}} \rightarrow \mathrm{W}_{p^{\infty}}$ with projection on the weight one factor

$$\pi_1 : \mathrm{W}_{p^{\infty}} \simeq \prod_{i \geq 0} \mathbb{G}_a \rightarrow \mathbb{G}_a.$$

The result is a class $u \in \mathrm{H}^2(\mathrm{BS}_{\mathrm{Fil}}^1, \mathcal{O}(1))$ where $\mathcal{O}(1)$ is the line bundle of weight 1 over $\mathbb{A}^1/\mathbb{G}_m$ (this is the pullback along the structure map $\mathbb{A}^1/\mathbb{G}_m \rightarrow \mathrm{B}\mathbb{G}_m$ of the line bundle on $\mathrm{B}\mathbb{G}_m$ corresponding to the graded $\mathbb{Z}_{(p)}$ -module concentrated in weight 1). This map has an incarnation as a map $\mathcal{O}(1)[-2] \rightarrow C^*(\mathrm{BS}_{\mathrm{Fil}}^1, \mathcal{O})$ in the ∞ -category $\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m)$. As

$C^*(BS_{\text{Fil}}^1, \mathcal{O})$ may be viewed as a filtered E_∞^\otimes -algebra, and in particular a filtered E_1^\otimes -algebra, there is a morphism of filtered E_1^\otimes -algebras induced by the universal property

$$\mathbb{Z}_{(p)}[u] \rightarrow C^*(BS_{\text{Fil}}^1, \mathcal{O}) \quad \text{in} \quad \text{Alg}_{E_1^\otimes}(\text{QCoh}(\mathbb{A}_{\mathbb{Z}_{(p)}}^1/\mathbb{G}_m \mathbb{Z}_{(p)})) \quad (54)$$

where $\mathbb{Z}_{(p)}[u]$ is the free E_1^\otimes -algebra on $\mathcal{O}(1)$. We now show that (54) is an equivalence. As in the Lemma 3.15, it will be enough to show that (54) is an equivalence after evaluating at the stalks corresponding to the field-value points

$$(0, \mathbb{Q}), (1, \mathbb{Q}), (0, \mathbb{F}_p), (1, \mathbb{F}_p) \quad \text{in} \quad \mathbb{A}_{\mathbb{Z}_{(p)}}^1/\mathbb{G}_m \mathbb{Z}_{(p)}$$

As explained in the Remark 3.8, the fiber of (54) at 0 gives

$$(\mathbb{Z}_{(p)}[u])^{\text{ass-gr}} \rightarrow C^*(B^2\text{Ker}, \mathcal{O}) \quad \text{in} \quad \text{Alg}_{E_1^\otimes}(\text{QCoh}(B\mathbb{G}_m \mathbb{Z}_{(p)})) \quad (55)$$

and the fiber at 1

$$(\mathbb{Z}_{(p)}[u])^u \rightarrow C^*(B^2\text{Fix}, \mathcal{O}) \quad \text{in} \quad \text{Alg}_{E_1^\otimes}(\text{Mod}_{\mathbb{Z}_{(p)}}) \quad (56)$$

One now observes that the proofs of Lemma 4.10 and Lemma 4.13 also work for B^2 . In this case, after passing to rational coefficients, the Remark 3.1 and the [Toe06, Cor 2.5.3] tell us that the maps (55) and (56) become, respectively:

$$(\mathbb{Q}[u])^{\text{ass-gr}} \rightarrow C^*(B^2\mathbb{G}_{a\mathbb{Q}}, \mathcal{O}) \simeq C^*(K(\mathbb{Z}, 2), \mathbb{Q}) \quad (57)$$

and

$$(\mathbb{Q}[u])^u \rightarrow C^*(B^2\mathbb{G}_{a\mathbb{Q}}, \mathcal{O}) \simeq C^*(K(\mathbb{Z}, 2), \mathbb{Q}) \quad (58)$$

These maps are equivalences since after passing to cohomology groups we get in both cases the isomorphism of commutative graded algebras $\mathbb{Q}[u] \rightarrow H^*(K(\mathbb{Z}, 2), \mathbb{Q})$.

Concerning the base change to \mathbb{F}_p , we get:

$$(\mathbb{F}_p[u])^u \rightarrow C^*(B^2\text{Fix}_{|\mathbb{F}_p}, \mathcal{O}) \simeq C^*(K(\mathbb{Z}_p, 2), \mathcal{O}) \underbrace{\simeq}_{(18)} C^*(K(\mathbb{Z}, 2), \mathbb{F}_p) \quad (59)$$

and because of (43), we know that $C^*(\text{Ker}_{|\mathbb{F}_p}, \mathcal{O})$ and $C^*(\mathbb{Z}_p, \mathcal{O})$ are equivalent as coalgebras. But then using limit formula similar to (36) we deduce that that $C^*(B\text{Ker}_{|\mathbb{F}_p}, \mathcal{O})$ and $C^*(B\mathbb{Z}_p, \mathcal{O})$ are equivalent as coalgebras. In particular, their categories of representations are equivalent, ie., $\text{QCoh}(B^2(\text{Ker}_{|\mathbb{F}_p}))$ and $\text{QCoh}(B^2(\mathbb{Z}_{(p)}))$ are equivalent. But then, their cohomology of global sections are necessarily equivalent

$$C^*(K(\text{Ker}_{|\mathbb{F}_p}, 2), \mathcal{O}) \simeq C^*(K(\mathbb{Z}_p, 2), \mathcal{O})$$

Finally, we get

$$(\mathbb{F}_p[u])^{\text{ass-gr}} \rightarrow C^*(K(\text{Ker}_{|\mathbb{F}_p}, 2), \mathcal{O}) \simeq C^*(K(\mathbb{Z}_p, 2), \mathcal{O}) \underset{(18)}{\simeq} C^*(K(\mathbb{Z}, 2), \mathbb{F}_p) \quad (60)$$

It is clear that both (60) and (59) produce the same isomorphism after passing to cohomology

$$\mathbb{F}_p[u] \simeq H^*(K(\mathbb{Z}, 2), \mathbb{F}_p)$$

□

Before we proceed with the ramifications of Proposition 5.1, we recall in a bit more detail the notion of mixed complexes.

Definition 5.2. Let $\Lambda := \mathbb{Z}_{(p)}[\epsilon] = H_*(S^1, \mathbb{Z}_{(p)})$ be the graded associative algebra freely generated with ϵ in degree -1 with $\epsilon^2 = 0$. The dga Λ is graded, with ϵ sitting in weight 1. By giving it the split filtration corresponding to this grading, we may consider Λ as a filtered E_1 -algebra.

We let Mod_Λ denote the ∞ -category of Λ -modules; note that, as a category of modules over a filtered algebra it acquires the structure of a *filtered* ∞ -category, equivalently a sheaf of E_∞^\otimes -categories over $\mathbb{A}^1/\mathbb{G}_m$.

Proposition 5.3. (i) *The pullback along the map (17) : $S^1 \rightarrow (S_{\text{Fil}}^1)^u$ induces a symmetric monoidal equivalence*

$$\text{QCoh}(B(S_{\text{Fil}}^1)^u) \simeq \text{QCoh}(BS^1)$$

(ii) *There exist a natural symmetric monoidal equivalence*

$$\text{Mod}_\Lambda \simeq \text{QCoh}(B(S_{\text{Fil}}^1)^{\text{gr}})$$

compatible with the graduations on both sides.

Proof. (1) The fact that the pullback along (17) is equivalence is a consequence of proposition Proposition 5.1. Indeed, both of these categories are generated by the (non-compact) generator which is the unit. In particular the above functor is essentially surjective. Moreover, Proposition 5.1 implies that the above functor is also fully faithful.

(2) The proof is essentially the same as for (1), except that we have to produce a symmetric monoidal ∞ -functor

$$\text{Mod}_\Lambda \longrightarrow \text{QCoh}(\text{BKer}).$$

Such a functor is obtained as follows. We can consider $\Lambda' = H^*(\text{Ker}, \mathcal{O})$ as a strict commutative dg-Hopf algebra. It then has a strict symmetric monoidal category of comodules $\text{Comod}_{\Lambda'}^{\text{strict}}$, which can be localized along quasi-isomorphism to produce a symmetric

monoidal ∞ -category equivalent to Mod_Λ . On the other hand, the symmetric monoidal ∞ -category $\mathrm{QCoh}(\mathrm{BKer})$ is obtained as a limit (by descent from $*$ to BKer)

$$\mathrm{QCoh}(\mathrm{BKer}) \simeq \lim_{n \in \Delta} \mathrm{QCoh}((\Lambda')^{\otimes n}).$$

Note that $\mathrm{Comod}_{\Lambda'}^{\mathrm{strict}}$ is itself also the limit of $\mathrm{Mod}_{(\Lambda')^{\otimes n}}^{\mathrm{strict}}$ by now as strict symmetric monoidal 1-categories. After inverting the quasi-isomorphism inside this limit we get a naturally defined symmetric monoidal ∞ -functor

$$(\lim_{n \in \Delta} \mathrm{Mod}_{(\Lambda')^{\otimes n}})[q - \mathrm{iso}^{-1}] \longrightarrow \lim_{n \in \Delta} \mathrm{QCoh}((\Lambda')^{\otimes n}).$$

We get this the desired symmetric monoidal ∞ -functor

$$\mathrm{Mod}_\Lambda \longrightarrow \mathrm{QCoh}(\mathrm{BKer}).$$

The fact that this is an equivalence on the underlying categories is a similar argument based again on the fact that the unit generates these categories and Proposition 5.1. By construction this equivalence is clearly compatible with the action of \mathbb{G}_m on both sides. \square

6. THE HKR THEOREM

We are now ready to state and prove the HKR theorem and Theorem 1.3 for a general simplicial commutative k -algebra $A \in \mathrm{SCR}_k$ (and more generally for derived schemes or stacks by gluing) where k is a fixed commutative $\mathbb{Z}_{(p)}$ -algebra.

The filtered circle $\mathbf{S}_{\mathrm{Fil}}^1$ sits over $\mathbb{Z}_{(p)}$ and we pull-it back over $\mathrm{Spec} k$ to make it a filtered group stack over k . We will continue to denote it by $\mathbf{S}_{\mathrm{Fil}}^1$.

Construction 6.1. Recall that any $A \in \mathrm{SCR}_k$ possesses a derived de Rham algebra $\mathrm{DR}(A/k)$. For us this is a graded mixed $\mathbf{E}_\infty^\otimes$ -algebra constructed as follows. When A is smooth over k , $\mathrm{DR}(A/k)$ simply is the strictly commutative dg-algebra $\mathrm{Sym}_A(\Omega_{A/k}^1)$ endowed with its de Rham differential. This is a strictly commutative monoid inside the strict category of graded mixed complexes, and thus can be considered as a $\mathbf{E}_\infty^\otimes$ -algebra object inside $\mathrm{QCoh}(\mathrm{B}(\mathbf{S}_{\mathrm{Fil}}^1)^{\mathrm{gr}})$, the symmetric monoidal ∞ -category of graded mixed complexes (here over k).

This produces an ∞ -functor from smooth algebras over k , and thus from polynomial k -algebras, to graded mixed $\mathbf{E}_\infty^\otimes$ -algebras. By left Kan extension we get the de Rham functor

$$\mathrm{DR} : \mathrm{SCR}_k \longrightarrow \mathrm{CAlg}(\mathrm{QCoh}(\mathrm{B}(\mathbf{S}_{\mathrm{Fil}}^1)^{\mathrm{gr}})).$$

Remark 6.2. We will see below that the functor DR as defined above with values in $\mathbf{E}_\infty^\otimes$ -algebras in $\mathrm{QCoh}(\mathrm{B}(\mathbf{S}_{\mathrm{Fil}}^1)^{\mathrm{gr}})$ can be refined to take in values "simplicial commutative mixed graded algebras". More precisely, what this means is that we can exhibit it as a

functor with values in derived affine schemes over $(S_{\text{Fil}}^1)^{\text{gr}}$, which contains slightly more information.

Construction 6.3. The functor of global sections produces a symmetric lax-monoidal ∞ -functor

$$\text{QCoh}(\mathcal{B}(S_{\text{Fil}}^1)^{\text{gr}}) \longrightarrow \text{QCoh}(\mathcal{B}\mathbb{G}_m).$$

which formally corresponds to taking homotopy fixed points with respect to the action of the graded group stack $(S_{\text{Fil}}^1)^{\text{gr}}$. We will denote it by $(-)^{\text{h}(S_{\text{Fil}}^1)^{\text{gr}}}$. The composition

$$\text{SCR}_k \xrightarrow{\text{DR}} \text{QCoh}(\mathcal{B}(S_{\text{Fil}}^1)^{\text{gr}}) \longrightarrow \text{QCoh}(\mathcal{B}\mathbb{G}_m)$$

sends A to the the derived de Rham complex of A

$$\widehat{\mathbb{L}\text{DR}}(A/k) := \text{DR}(A/k)^{\text{h}(S_{\text{Fil}}^1)^{\text{gr}}} \in \text{CAlg}(\text{QCoh}(\mathcal{B}\mathbb{G}_m)).$$

This is a graded $\mathbf{E}_{\infty}^{\otimes}$ -algebra whose piece of weight i will be denoted by $\widehat{\mathbb{L}\text{DR}}^{\geq i}(A/k)$. Note that the weight i pieces vanish when $i < 0$, and morally speaking $\widehat{\mathbb{L}\text{DR}}^{\geq i}(A/k)$ should be understood as the complex $\prod_{q \geq i} (\wedge^q \mathbb{L}_{A/k}[-q])$ endowed with the total differential $d + \mathbf{d}_{\text{dR}}$, where d is the cohomological differential induced from A and \mathbf{d}_{dR} is the de Rham differential.

Definition 6.4. Let $X = \text{Spec } A$ be an affine derived scheme over k . The *filtered loop space* of X (over k) is defined by

$$\mathbf{L}_{\text{Fil}} X := \text{Map}_{/\mathbb{A}^1/\mathbb{G}_m}(S_{\text{Fil}}^1, X \times \mathbb{A}^1/\mathbb{G}_m).$$

This is a derived scheme over $\mathbb{A}^1/\mathbb{G}_m$ equipped with a canonical action of the group stack S_{Fil}^1 .

Construction 6.5. Let $\pi : Y \rightarrow \mathbb{A}^1/\mathbb{G}_m$ be a (derived) stack over $\mathbb{A}^1/\mathbb{G}_m$. We use the notation

$$\mathcal{O}^{\text{fil}}(Y) := \pi_* \mathcal{O}_Y$$

for the push-forward of the structure sheaf; this is an object of the ∞ -category $\text{QCoh}(\mathbb{A}^1/\mathbb{G}_m)$ of filtered k -modules. For $Y = \mathbf{L}_{\text{Fil}} X$ as above, $\mathcal{O}^{\text{fil}}(\mathbf{L}_{\text{Fil}} X)$ will carry a canonical action of S_{Fil}^1 as the structure map

$$\pi : \mathbf{L}_{\text{Fil}} X \rightarrow \mathbb{A}^1/\mathbb{G}_m$$

is S_{Fil}^1 -equivariant. Hence we view $\mathcal{O}^{\text{fil}}(\mathbf{L}_{\text{Fil}} X)$ as an object in $\text{QCoh}(\mathcal{B}S_{\text{Fil}}^1)$; by Proposition 5.3-(i) its underlying object can be identified with an S^1 -equivariant $\mathbf{E}_{\infty}^{\otimes}$ -algebra, and by Proposition 5.3-(ii) its associated graded with a graded mixed $\mathbf{E}_{\infty}^{\otimes}$ -algebra. See also Remark 3.8.

We are now ready to prove our HKR theorem:

Theorem 6.6. *Let $X = \mathrm{Spec} A$ be an affine derived scheme over k and $\mathbf{L}_{\mathrm{Fil}}X$ its filtered loop space.*

- (1) *The derived stack $\mathbf{L}_{\mathrm{Fil}}X$ is representable by an affine derived scheme relative to $\mathbb{A}^1/\mathbb{G}_m$.*
- (2) *The composition with the natural map (17) : $S^1 \rightarrow (\mathbf{S}_{\mathrm{Fil}}^1)_k^u = \mathbf{B}\mathrm{Fix}_k$. induces an isomorphism of derived schemes:*

$$\mathrm{Map}_k((\mathbf{S}_{\mathrm{Fil}}^1)_k^u, X) \rightarrow \mathrm{Map}_k(S^1, X) \quad (61)$$

- (3) *There is a natural equivalence of graded derived schemes*

$$\mathrm{Map}_k((\mathbf{S}_{\mathrm{Fil}}^1)_k^{\mathrm{gr}}, X) \simeq \mathrm{Spec} \mathrm{Sym}_A(\mathbb{L}_{A/k}[1]). \quad (62)$$

- (4) *The cohomology $\mathbf{E}_{\infty}^{\otimes}$ -algebra $\mathcal{O}(\mathbf{L}_{\mathrm{Fil}}X)$ endowed with its natural $\mathbf{S}_{\mathrm{Fil}}^1$ -action is such that*
- (a) *Its underlying object is naturally equivalent to $\mathrm{HH}(A/k)$, the Hochschild homology of A over k , together with its natural S^1 -action.*
 - (b) *Its associated graded is naturally equivalent to $\mathrm{DR}(A/k)$ as a graded mixed $\mathbf{E}_{\infty}^{\otimes}$ -algebra.*
 - (c) *Being compatible with the circle action, the filtration descends to fixed points and makes*

$$\mathrm{HC}(A)^- = \mathrm{HH}(A)^{S^1}$$

a filtered algebra, whose associated graded pieces are the truncated complete derived de Rham complexes $\widehat{\mathbb{L}\mathrm{DR}}^{\geq p}(A/k)$.

Proof. We start by the fact (1) that $\mathbf{L}_{\mathrm{Fil}}X$ is relatively affine. For this we first notice that its truncation is simply the truncation of X , and thus that it is an affine scheme. In order to prove that $\mathbf{L}_{\mathrm{Fil}}X$ is affine it just remains to check that $\mathbf{L}_{\mathrm{Fil}}X$ has a global cotangent complex, admits an obstruction theory and is nil-complete (see [TV08] Appendix C). But this follows easily from the fact that it is a mapping derived stack from $\mathbf{B}\mathrm{H}_{p^{\infty}}$, the classifying stack of a filtered group scheme.

We now analyse the statement (2). The map (61) is obviously an isomorphism on the truncations, as these will just be truncations of X . We claim that it also an equivalence of derived schemes. Indeed, we check that it induces an equivalence on cotangent complexes, because, thanks to the Proposition 5.3-(i), S^1 and $(\mathbf{S}_{\mathrm{Fil}}^1)_k^u$ have the same quasi-coherent cohomologies. In more detail, we let B be an arbitrary simplicial commutative k -algebra, and let $u : \mathrm{Spec}(B) \times \mathbf{B}\mathrm{Fix}_k \rightarrow X$ be a B -point of the mapping stack $\mathrm{Map}((\mathbf{S}_{\mathrm{Fil}}^1)_k^u, X)$. The cotangent complex of $\mathrm{Map}_k(\mathbf{B}(\mathbf{S}_{\mathrm{Fil}}^1)_k^u, X)$ at u is given by

$$\mathbb{L}_{\mathrm{Map}_k(\mathbf{B}(\mathbf{S}_{\mathrm{Fil}}^1)_k^u, X), u} \simeq p^! u^* \mathbb{L}_X,$$

where $p_! : \mathrm{QCoh}(\mathrm{Spec} B \times \mathrm{BFix}_k) \rightarrow \mathrm{Mod}_B$ is the left adjoint of the pullback functor along the canonical projection to $\mathrm{Spec} B$. The existence of this left-adjoint is specific to our situation: as S^1 is a finite CW-complex, the base change $(17)_B := (17) \times \mathrm{Spec} B$ exhibits $\mathrm{Spec} B \times \mathrm{BFix}_k$ as the affinization of S^1_B . It then follows from an argument similar to the one of the Proposition 5.3-(i) that the pullback along $(17)_B$ induces a monoidal equivalence of categories

$$\mathrm{QCoh}(\mathrm{Spec} B \times \mathrm{BFix}_k) \xrightarrow{\sim} \mathrm{QCoh}(S^1_B) \simeq \mathrm{Fun}(S^1, \mathrm{Mod}_B) \quad (63)$$

In particular, the pullback along $(17)_B$ preserves all limits, and commutes with the pullback along the projections to $\mathrm{Spec} B$. This pullback p^* assigns to a object in Mod_B the trivial S^1 -action. In particular, it commutes with all limits and therefore admits a left adjoint $p_!$. It follows that these left adjoints coincide on the l.h.s and r.h.s of (63). In particular, in the r.h.s it takes on the role of the "homology" push-forward functor of local systems of B -modules on S^1

$$\mathrm{colim}_{S^1} : \mathrm{Fun}(S^1, \mathrm{Mod}_B) \rightarrow \mathrm{Mod}_B$$

Finally, this implies that (61) is an isomorphism of derived schemes. The statement about the Hochschild complex follows.

We now arrive at the statement (3), which is the true content of the theorem. We start by noticing the existence of commutative square of graded affine stacks

$$\begin{array}{ccc} \mathrm{Spec} k[\epsilon] & \longrightarrow & \mathrm{Spec} k \\ \downarrow & & \downarrow \\ \mathrm{Spec} k & \longrightarrow & (S^1_{\mathrm{Fil}})_k^{\mathrm{gr}}. \end{array}$$

Such a commutative square of stacks is given by an element of $\mathrm{Ker}(k[\epsilon])$ interpreted as a map

$$\mathrm{Spec} k[\epsilon] \rightarrow \mathrm{Ker} \simeq \Omega(S^1_{\mathrm{Fil}})^{\mathrm{gr}}$$

which simply is the Witt vector $(1 - \epsilon.t)^{-1}$. This is in the kernel of the Frobenius because $\epsilon^p = 0$.

This square induces a commutative diagram of graded derived affine schemes, obtained by mapping to X

$$\begin{array}{ccc} \mathrm{Sym}_A(\mathbb{L}_A) & \longleftarrow & X \\ \uparrow & & \uparrow \\ X & \longleftarrow & \mathrm{Map}_k((S^1_{\mathrm{Fil}})_k^{\mathrm{gr}}, X). \end{array}$$

This in turn produces a natural morphism of derived stacks

$$\mathrm{Map}_k((S^1_{\mathrm{Fil}})_k^{\mathrm{gr}}, X) \longrightarrow X \times_{\mathrm{Spec} \mathrm{Sym}_A(\mathbb{L}_A)} X \simeq \mathrm{Spec} \mathrm{Sym}_A(\mathbb{L}_A[1]).$$

To check its an equivalence we proceed as usual, we prove its is isomorphism on truncations as well as on cotangent complexes. This second assertion is a consequence of the fact that the cohomology of BKer_k is the split square zero extension $\mathbb{Z}_{(p)}[\epsilon_{-1}]$ (see Theorem 4.29).

To finish the proof of the theorem, we have to compare the two ∞ -functors

$$\mathrm{SCR}_k \longrightarrow \mathrm{CAlg}(\mathrm{QCoh}(\mathrm{B}(\mathrm{S}_{\mathrm{Fil}}^1)_k^{\mathrm{gr}}))$$

The first one given by $\mathrm{DR}(-/k)$, the second given by

$$A \mapsto \mathcal{O}(\mathrm{Map}_k((\mathrm{S}_{\mathrm{Fil}}^1)_k^{\mathrm{gr}}, \mathrm{Spec} A))$$

. For this, we first notice that if we forget the action of the group $(\mathrm{S}_{\mathrm{Fil}}^1)^{\mathrm{gr}}$ (i.e. the mixed structure), then these two ∞ -functors are equivalent and given by $A \mapsto \mathrm{Sym}_A(\mathbb{L}_A[1])$. As the functor $A \mapsto \mathbb{L}_A$ is obtained by left Kan extension from polynomial rings (see Notation 1.8), this shows that the same is true for the two functors to be shown to be equivalent. In other words, we can restrict these to the category of polynomial k -algebras.

We then observe that for any polynomial k -algebra A the space of graded mixed structures on the graded $\mathrm{E}_{\infty}^{\otimes}$ -algebra $\mathrm{Sym}_A(\Omega_{A/k}^1[1])$ is a discrete space. Indeed, this follows from the fact that the space of graded $\mathrm{E}_{\infty}^{\otimes}$ -endomorphisms is itself discrete, because the weight grading coincide with the cohomological grading (as in the proof of Lemma 4.22). As a consequence, in order to show that the two above ∞ -functors are equivalent it is enough to show that for a fixed polynomial k -algebra A , the natural isomorphism of graded algebras

$$\mathrm{DR}(A/k) \simeq \mathcal{O}(\mathrm{L}_{\mathrm{Fil}}(X))$$

intertwine the two graded mixed structures. We can even be more precise, the compatible graded mixed structures on the graded $\mathrm{E}_{\infty}^{\otimes}$ -algebra $\mathrm{DR}(A/k)$ form a discrete space which embeds into the set of k -linear derivations $A \longrightarrow \Omega_A^1$.

As a result, we are reduced to prove that, by the above identification, the differential obtain from the $(\mathrm{S}_{\mathrm{Fil}}^1)^{\mathrm{gr}}$ -action on the right hand side

$$d : \pi_0(\mathcal{O}(\mathrm{L}_{\mathrm{Fil}}(X))) \simeq A \longrightarrow \pi_1(\mathcal{O}(\mathrm{L}_{\mathrm{Fil}}(X))) \simeq \Omega_{A/k}^1$$

is indeed equal to the standard de Rham differential. For this, we can of course assume that $k = \mathbb{Z}_{(p)}$, as the general case would be obtained by base change. But in this case all complexes involved are torsion free; one may then simply base change to \mathbb{Q} to check the the mixed structure above is the de Rham differential. But the result is well known in characteristic zero (see [TV11]). \square

7. APPLICATIONS AND COMPLEMENTS

7.1. Towards shifted symplectic structures in non-zero characteristics. Let k be a commutative $\mathbb{Z}_{(p)}$ -algebra. We assume that $p \neq 2$.

For a commutative simplicial k -algebra A , Theorem 6.6 provides a filtration F^* on the negative cyclic homology complex $\mathrm{HC}^-(A/k)$ and tells us that the graded pieces are canonically given by

$$Gr_F^i \mathrm{HC}^-(A/k) \simeq \widehat{\mathrm{LDR}}^{\geq i}(A/k).$$

The ∞ -functors $A \mapsto \mathrm{HC}^-(A/k)$ and $A \mapsto \widehat{\mathrm{LDR}}^{\geq i}(A/k)$ are extended by descent to all derived stacks \mathcal{Y} :

$$\mathrm{HC}^-(\mathcal{Y}/k) = \lim_{\mathrm{Spec} A \rightarrow \mathcal{Y}} \mathrm{HC}^-(A/k)$$

and this comes equipped with a canonical filtration whose graded pieces are $\mathbb{L}\widehat{\mathrm{DR}}^{\geq i}(\mathcal{Y}/k)$, also defined by left Kan extension. The natural generalization of the notion of shifted symplectic structures of [PTVV13] is the following definition.

Definition 7.1. (i) For a derived stack \mathcal{Y} over k , we define the *complex of closed q -forms on \mathcal{Y}* to be

$$\mathcal{A}^{cl,q}(\mathcal{Y}/k) := Gr_F^q \mathrm{HC}^-(\mathcal{Y}/k)[-q] \simeq \mathbb{L}\widehat{\mathrm{DR}}^{\geq q}(\mathcal{Y}/k).$$

(ii) A closed 2-form ω of degree n on a derived stack \mathcal{Y} is *non-degenerate* if the underlying element in $H^n(\mathcal{Y}, \wedge^2 \mathbb{L}_{\mathcal{Y}/k})$ is non-degenerate in the sense of [PTVV13].

The above definition is a rather naive notion, we believe that there exists more subtle versions. For instance, it is very natural to ask for a shifted symplectic structure to lift to an element in $F^2 \mathrm{HC}^-(\mathcal{Y}/k)$. In characteristic zero, the HKR theorem implies that there is always a canonical lift, but outside of this case lifts might not even exist (and if they do might not be canonical). The data of such lifting seems of some importance to us, in particular for questions concerning quantization.

This will be investigated in a further work, and as a first example of existence of such lifting we show below that most shifted symplectic structures constructed in nature do possess such a lifting, by means of the Chern character construction.

Recall the existence of a Chern character $\mathrm{Ch} : K^c(A) \rightarrow \mathrm{HC}^-(A/k)$, which is here considered as a map of spectra (see for instance [TV15]). Note also that K^c stands here for the space of connective K -theory of A . This map can be enhanced into a morphism of stacks of spaces on the site of derived affine schemes over k

$$\mathrm{Ch} : \underline{K}^c \rightarrow \mathrm{HC}^-$$

(\underline{K}^c is the stack associated to $A \mapsto K^c(A)$, note that $A \mapsto \mathrm{HC}^-(A/k)$ is itself already a stack because HH itself is a stack in the étale topology [WG91]).

There is a graded determinant map

$$\mathrm{det}^{\mathrm{gr}} : \underline{K}^c \rightarrow \mathbb{Z} \times \mathrm{BG}_m$$

sending a perfect complex E to $(\mathrm{rank}(E), \mathrm{det}(E))$, where $\mathrm{rank}(E) \in \mathbb{Z}$ is the Euler characteristic of E and $\mathrm{det}(E)$ is its determinant line bundle.

Proposition 7.2. *The composition $\mathrm{Ch}_{\leq 2}$*

$$\underline{K}^c \xrightarrow{\mathrm{Ch}} \mathrm{HC}^-(A/k) \longrightarrow \mathrm{HC}^-(A/k)/F^2$$

factors naturally through the graded determinant map

$$\underline{K}^c \xrightarrow{\mathrm{det}^{\mathrm{gr}}} \mathbb{Z} \times \mathrm{BG}_m \longrightarrow \mathrm{HC}^-(A/k)/F^2.$$

Proof. As explained in [Gil81] the chern character map is completely determined by a morphism of underived stacks of spaces

$$\mathbb{Z} \times \mathrm{BGl}_\infty \longrightarrow \mathrm{HC}^-,$$

or equivalently by a series of morphisms

$$\mathbb{Z} \times \mathrm{BGl}_n \longrightarrow \mathrm{HC}^-$$

together with compatibility data for the standard inclusions $\mathrm{Gl}_n \subset \mathrm{Gl}_{n+1}$. This essentially reduces the proposition to a statement about the Chern character of the universal bundle on $\mathbb{Z} \times \mathrm{BGl}_n$ as a class in $\pi_0(\mathrm{HC}^-(\mathbb{Z} \times \mathrm{BGl}_n)/F^2)$. We can use the HKR filtration to see that $\mathrm{HH}(\mathrm{BGl}_n) \simeq \mathrm{H}^*(\mathrm{Gl}_n, A)$, where A is the ring of formal functions on Gl_n near the identity, and $\mathrm{H}^*(\mathrm{Gl}_n, A)$ is the cohomology of the group scheme Gl_n acting on A by conjugation. In particular, $\mathrm{HH}(\mathrm{BGl}_n)$ is positively graded. The associated graded of $\mathrm{HH}(\mathrm{BGl}_n)$ can be identified with $\mathrm{H}^*(\mathrm{Gl}_n, \mathrm{Sym}_k(\mathfrak{gl}_n^*))$, where \mathfrak{gl}_n is the lie algebra of Gl_n . In particular, we have that the graded circle BKer acts trivially on $\mathrm{H}^*(\mathrm{Gl}_n, \mathrm{Sym}_k(\mathfrak{gl}_n^*))$ in such a manner that we have

$$\pi_0(Gr_F^0(\mathrm{HC}^-(\mathrm{BGl}_n))) \simeq k \quad \pi_i(Gr_F^0(\mathrm{HC}^-(\mathrm{BGl}_n))) \simeq 0 \quad \forall i > 0.$$

In the same manner, we have that

$$\pi_0(Gr_F^1(\mathrm{HC}^-(\mathrm{BGl}_n))) \simeq (\mathfrak{gl}_n^*)^{\mathrm{Gl}_n} \simeq k \quad \pi_i(Gr_F^1(\mathrm{HC}^-(\mathrm{BGl}_n))) \simeq 0 \quad \forall i > 0.$$

From this we deduce that there is a short exact sequence

$$0 \longrightarrow \prod_{\mathbb{Z}} k \longrightarrow \pi_0(\mathrm{HC}^-(\mathbb{Z} \times \mathrm{BGl}_n)/F^2) \longrightarrow \prod_{\mathbb{Z}} k \longrightarrow 0.$$

Consider now the determinant morphism $\mathbb{Z} \times \mathrm{B} \longrightarrow \mathbb{Z} \times \mathrm{BGl}_m$. This morphism clearly induces an isomorphism of invariants covectors on lie algebras

$$k \simeq \mathfrak{gl}_1^* \simeq (\mathfrak{gl}_n^*)^{\mathrm{Gl}_n}.$$

As a result, using the short exact sequence above we see that the determinant map induces an isomorphism

$$\pi_0(\mathrm{HC}^-(\mathbb{Z} \times \mathrm{BGl}_n)/F^2) \simeq \pi_0(\mathrm{HC}^-(\mathbb{Z} \times \mathrm{BGl}_m)/F^2).$$

It can be checked that this identification matches the image of the chern character of the universal bundle on $\mathbb{Z} \times \mathrm{BGl}_n$ with the one on BGl_m . The results follows. \square

As a consequence of the proposition we can consider the fiber of the graded determinant map

$$\underline{K}^{c,0} \longrightarrow \underline{K}^c \longrightarrow \mathbb{Z} \times \mathrm{BGl}_m,$$

in order to obtain a reduced chern character

$$\mathrm{Ch}_{\geq 2} : \underline{K}^{c,0} \longrightarrow F^2 \mathrm{HC}^-.$$

Moreover, there is a canonical section of the graded determinant map $\mathbb{Z} \times \mathrm{BGl}_m \longrightarrow \underline{K}^c$ which simply sends a pair (n, L) , of an integer and a line bundle, to the perfect complex $L[n]$. This section defines a canonical retraction

$$r : \underline{K}^c \longrightarrow \underline{K}^{c,0}$$

which morally sends a vector bundle V on $V - \text{rank}(V) - \det(V)$. Note that as the section is not a map of spectra the retraction r is merely a morphism of stacks of spaces which is not additive in any sense.

As a result we have defined a reduced chern character map

$$\text{Ch}_{\geq 2} : \underline{K}^c \longrightarrow F^2\text{HC}^-.$$

For example, we can apply this reduced chern character on the stack BSl_n for the universal vector bundle. The resulting chern character is an element in $\pi_0(F^2\text{HC}^-(\text{BSl}_n))$ whose projection in $\pi_0(\text{Gr}_F^2\text{HC}^-(\text{BSl}_n)) \simeq (\text{Sym}^2(\mathfrak{sl}_n^*))^{\text{Sl}_n}$ is the trace quadratic form, and thus is non-degenerate.

Similarly, we can consider the derived stack Perf^0 of perfect complexes with fixed determinant. The reduced chern character of the universal object will produce an element in $F^2\text{HC}^-(\text{Perf}^0/k)$ which is a natural lift of the closed 2-form defining the 2-shifted symplectic structure on Perf^0 .

7.2. Filtration on Hochschild cohomology. As a second example of possible applications of the filtered circle, we explain here how it can also provide interesting filtrations on Hochschild cohomology. For this, we will have to consider S_{Fil}^1 not as a group anymore but as a cogroup object inside filtered stacks. It is even more, as it carries a E_2^\otimes -cogroupoid structure over $\mathbb{A}^1/\mathbb{G}_m$ which can be exploited to get a filtration on Hochschild cohomology compatible with its natural E_2 -structure. Moreover, all the constructions in this part make sense over the sphere spectrum, and so provide filtrations on topological Hochschild cohomology as well.

As a start we consider the natural closed embedding of stacks

$$0 : \text{BG}_m \hookrightarrow \mathbb{A}^1/\mathbb{G}_m.$$

The direct image of the structure sheaf defines a commutative cosimplicial algebra over $\mathbb{A}^1/\mathbb{G}_m$. Let us denote it by \mathcal{Z} , and let $\mathcal{O} \longrightarrow \mathcal{Z}$ be the unit map. The nerve of this map produces a groupoid object inside commutative cosimplicial algebras over $\mathbb{A}^1/\mathbb{G}_m$, which is denoted by $\mathcal{Z}^{(1)}$. In the same manner, we can consider the nerve of $\mathcal{O} \longrightarrow \mathcal{Z}^{(1)}$ to get an E_2^\otimes -groupoid object (that is a groupoid object inside groupoid objects) $\mathcal{Z}^{(2)}$ and so on and so forth.

We define this way an E_n^\otimes -groupoid object $\mathcal{Z}^{(n)}$ inside the ∞ -category of commutative cosimplicial algebras over $\mathbb{A}^1/\mathbb{G}_m$.

Definition 7.3. The *filtered n -sphere* is defined to be $\text{Spec}^\Delta \mathcal{Z}^{(n+1)}$. It is an E_{n+1}^\otimes -cogroupoid object in affine stacks over $\mathbb{A}^1/\mathbb{G}_m$. It is denoted by S_{Fil}^n .

Note that S_{Fil}^n possesses an underlying object of $(1, \dots, 1)$ -morphisms. Explicitly, this is given by $\text{Spec } H^*(S^n, \mathbb{Z}_{(p)})$, and is called the formal or graded sphere. When $n = 1$ we recover our filtered circle S_{Fil}^1 as a filtered affine stack, but now it comes equipped with an E_2^\otimes -cogroupoid structure rather than a group structure.

We now consider $\mathcal{L}_{Fil}^{(n)}X = \text{Map}(\mathcal{S}_{Fil}^n, X)$, for a derived affine scheme $X = \text{Spec } A$. The cogroupoid structure on \mathcal{S}_{Fil}^n endows $\mathcal{L}_{Fil}^{(n)}X$ with an \mathbf{E}_{n+1}^\otimes -groupoid structure acting on X . Passing to functions and taking linear dual we get a filtered \mathbf{E}_{n+1}^\otimes -algebra over \mathbb{Z}_p whose underlying object is $\text{HH}_{\mathbf{E}_n^\otimes}^*(A)$, the n -th iterated Hochschild cohomology of A , and the associated graded is $\text{Sym}_A(\mathbb{L}_A[n])^\vee$, the dual of shifted differential forms, which can be defined as shifted polyvector fields over X . In summary, we expect the following proposition:

Proposition 7.4. *Let k be a commutative \mathbb{Z}_p -algebra, and A a commutative simplicial k -algebra. The iterated Hochschild cohomology $\text{HH}_{\mathbf{E}_{n+1}^\otimes}^*(A/k)$, carries a canonical filtration compatible with its \mathbf{E}_{n+1}^\otimes -multiplicative structure, whose associated graded is the $\mathbf{E}_\infty^\otimes$ -algebra of n -shifted polyvectors on X .*

The last proposition can be used, for instance, in order to define singular supports of coherent sheaves, or of sheaves of linear categories, over any base scheme. For instance, in the context of bounded coherent sheaves, this will allow us to extend the notion and construction of [AG15].

7.3. Generalized cyclic homology and formal groups. The filtered circle \mathcal{S}_{Fil}^1 we have constructed in this paper is part of a much more general framework that associates a circle \mathcal{S}_E^1 to any reasonable abelian formal group E . To be more precise:

Construction 7.5. We can start by an abelian formal group E over some base commutative ring k , and assume that E is formally smooth and of relative dimension 1 over k . The Cartier dual G_E of E is a flat abelian group scheme over $\text{Spec } k$, obtained as $\text{Spec } \mathcal{O}(E)^\vee$, where $\mathcal{O}(E)^\vee$ is the commutative and cocommutative Hopf algebra of distributions on E . Because E is smooth and of relative dimension 1, $\mathcal{O}(E)^\vee$ is a flat commutative k -coalgebra which is locally for the Zariski topology on k isomorphic to $k[X]$ with the standard comultiplication $\Delta(X^n) = \sum_{i+j=n} \binom{n}{i} X^i \otimes X^j$. The E -circle is defined as the group stack over k defined by

$$\mathcal{S}_E^1 := \text{BG}_E.$$

Under reasonable assumptions on E the stack \mathcal{S}_E^1 is an affine stack over k . Moreover, its ∞ -category of representations, $\text{QCoh}(\text{BS}_E^1)$ is naturally equivalent to the ∞ -category of mixed complexes over k , at least locally on $\text{Spec } k$. To be more precise, if we denote by ω_E the line bundle of relative 1-forms on G_E , the ∞ -category $\text{QCoh}(\text{BS}_E^1)$ is equivalent to ω_E -twisted mixed complexes, namely comodules over the k -coalgebra $k \oplus \omega_E[-1]$. However, the symmetric monoidal structure on $\text{QCoh}(\text{BS}_E^1)$ corresponds to a non-standard monoidal structure on mixed complexes that depends on the formal group structure on E .

The filtration on $(\mathcal{S}_{Fil}^1)^u$ whose associated graded is $(\mathcal{S}_{Fil}^1)^{\text{gr}}$ seems to also exist in some interesting examples of formal group laws. We will address this in future works.

We recover the results in this paper when E is either the additive or the multiplicative formal group $\widehat{\mathbb{G}}_a$, resp. $\widehat{\mathbb{G}}_m$:

Construction 7.6. Let k be a commutative ring. There exists a filtered group deforming \mathbb{G}_{mk} to \mathbb{G}_{ak} . Namely, given $\lambda \in k$ take $\mathbb{G}_{mk}^\lambda =: \text{Spec}(k[T, \frac{1}{1+\lambda T}])$. This is a group scheme under the multiplicative rule $T \mapsto 1 \otimes T + T \otimes 1 + \lambda T \otimes T$ and unit $T \mapsto 0$. When $\lambda = 0$ we get \mathbb{G}_{ak} and for $\lambda = 1$ we get \mathbb{G}_{mk} . Taking formal completions this deforms $\widehat{\mathbb{G}_{mk}}$ to $\widehat{\mathbb{G}_{ak}}$.

Proposition 7.7. *Let $k = \mathbb{Z}_{(p)}$. Then*

$$S_{\widehat{\mathbb{G}_a}}^1 := \text{BG}_{\widehat{\mathbb{G}_a}} \simeq (S_{\text{Fil}}^1)^{\text{gr}} \quad \text{and} \quad S_{\widehat{\mathbb{G}_m}}^1 := \text{BG}_{\widehat{\mathbb{G}_m}} \simeq (S_{\text{Fil}}^1)^u$$

Moreover, the filtration on Fix is Cartier dual to the filtration on $\widehat{\mathbb{G}_{mk}}$ of Construction 7.6.

Proof. The proposition is equivalent to the claims that:

- (i) Fix is Cartier dual to $\widehat{\mathbb{G}_m}$;
- (ii) Ker is Cartier dual to $\widehat{\mathbb{G}_a}$;
- (iii) the filtrations are Cartier dual

This is precisely the content of [SS01, Theorem]. □

Let E be an abelian formal group over k as before and S_E^1 the corresponding E -circle. For any derived affine k -scheme X we define the E -loop space $\mathcal{L}_E X := \text{Map}(S_E^1, X)$, that comes equipped with an S_E^1 -action. The E -Hochschild homology of X over k is by definition the complex of functions $\mathcal{O}(\mathcal{L}_E X)$. It is denoted by $\text{HH}(X, E)$. The S_E^1 -action on $\text{HH}^E(X)$ induces a mixed structure on $\text{HH}(X, E)$ whose total complex computes the S_E^1 -equivariant cohomology and is called by definition the negative cyclic E -homology $\text{HC}^-(X, E)$. When a filtration exists on E , then there is an HKR-type filtration on $\text{HC}^-(X, E)$ whose associated graded is again derived de Rham cohomology.

Of course, the results of this work are recovered when E is taken to be the multiplicative formal group law and we recover an isomorphism of filtered group schemes $\mathbf{H}_{p^\infty} = G_{E_T}$. See [SS01].

An example of particular interest is when E comes, by completion, from an elliptic curve. The corresponding Hochschild and cyclic homology can be called *elliptic Hochschild and cyclic homology* and its features will be studied in future works.

7.4. Topological and q-analogues. The filtered circle S_{Fil}^1 constructed in this work possesses at least two extensions, both of quantum /non-commutative nature: one as a non-commutative group stack over the sphere spectrum and a second extension as filtered group stack over $\mathbb{Z}[q, q^{-1}]$.

7.4.1. q-analogue. As a start, working around the prime p can be relaxed and definitions can be done over \mathbb{Z} . A first possibility is simply to use big Witt vectors and define the filtered group scheme H as the intersection of all kernels of the endomorphisms \mathcal{G}_p for all

primes p . There is however a second possible description, which has the merit of showing the natural q -deformed version, which we now describe.

We start by the filtered formal group \mathbb{G} , interpolating between the formal multiplicative and the formal additive group over \mathbb{Z} . The corresponding formal group over \mathbb{A}^1 is given by $X + Y + \lambda XY$ where λ is the coordinate on the affine line. The underlying formal group is $\widehat{\mathbb{G}}_m$ whereas the associated graded is $\widehat{\mathbb{G}}_a$ together with its natural graduation given the natural action of \mathbb{G}_m . The algebra of distributions of the filtered formal scheme \mathbb{G} defines a filtered commutative and cocommutative Hopf algebra \mathcal{R} . This algebra can be described explicitly as being the algebra of integer valued polynomials, that is the subring of $\mathbb{Q}[X]$ formed by all polynomials P such that $P(\mathbb{Z}) \subset \mathbb{Z}$. The filtration is then induced the the degree of polynomials.

The associated graded to this filtration is the ring $Gr\mathcal{R}$ of divided powers over \mathbb{Z} . This is the subring of $\mathbb{Q}[X]$ generated by the $\frac{X^n}{n!}$.

An integral version of the filtered group scheme H_{p^∞} , and of the filtered circle S_{Fil}^1 , can then be defined as $H_{\mathbb{Z}} := \text{Spec}(Rees(\mathcal{R}))$, where $Rees(\mathcal{R})$ is the Rees construction associated to the filtered Hopf algebra \mathcal{R} . The integral version of the filtered circle is then defined to be

$$S_{\text{Fil}, \mathbb{Z}}^1 := BH_{\mathbb{Z}}.$$

It is a pleasant exercise to show that when restricted over $\text{Spec } \mathbb{Z}_p$ this recovers our filtered circle S_{Fil}^1 . We believe that all the statement proved in this work can be extended over \mathbb{Z} , but some of the strategies of proof we use do not obviously extend to the situation where we deal with an infinite number of primes.

One advantage of the above presentation using integer valued polynomial algebras is the striking fact that these admits natural q -deformed versions. The q -deformed version \mathcal{R}_q of the ring \mathcal{R} is introduced and studied in [HH17], and is essentially the Cartan part $U^0(\mathfrak{sl}_2)$ of the divided power quantum group of Lusztig (see [HH17] end of section 4). In particular, we think that the filtered Hopf algebra \mathcal{R} possesses a q -deformed version \mathcal{R}_q , which is a commutative and cocommutative filtered Hopf algebra over $\mathbb{Z}[q, q^{-1}]$, recovering \mathcal{R} when $q = 1$. The spectrum of this provides a q -deformed version of $H_{\mathbb{Z}}$ that we denote by $H_{\mathbb{Z}, q}$. Its classifying stack is by definition the q -deformed filtered circle.

Definition 7.8. The *q -deformed filtered circle* is the filtered stack $S_{\text{Fil}, \mathbb{Z}}^1(q) := BH_{\mathbb{Z}, q}$. It is a stack over $\mathbb{A}^1/\mathbb{G}_m \times \text{Spec } \mathbb{Z}[q, q^{-1}]$.

As in Theorem 6.6, by considering the derived mapping stack $\text{Map}(S_{\text{Fil}, \mathbb{Z}}^1(q), X)$, it is then possible to define q -analogues of Hochschild and cyclic homology of a scheme X , together with a filtration whose associated graded should gives back the notion of q -deformed derived de Rham cohomology of [Sch17].

However, to make the above definition precise requires some extra work. For instance, it seems to us that the associated graded of $S_{\text{Fil}, \mathbb{Z}}^1(q)$ can not truly exist as a naive commutative object and requires to work over some braided monoidal base category associated to

$\mathbb{Z}[q, q^{-1}]$, as this is done for instance in the theory of Ringel-Hall algebras, see for instance [LZ00]. In fact, we expect the associated graded of $\mathbf{S}_{\text{Fil}, \mathbb{Z}}^1(q)$ to be of the form $\mathbf{BK}(q)$, where $K(q)$ is the spectrum of the Ringel-Hall algebra over the one point Quiver. This particular point together with the precise meaning of the q -deformed filtered circle, will be investigated in a future work.

7.4.2. Topological Analogue. Let us mention yet another extension of the filtered circle, now over the sphere spectrum. We do not believe that the filtered stack $\mathbf{S}_{\text{Fil}}^1$ can exist as a spectral stack in any sense, as the associated graded $(\mathbf{S}_{\text{Fil}}^1)^{\text{gr}}$ probably can't exist over the sphere spectrum. However, it is possible to construct a non-commutative version of this object, using the 2-periodic sphere spectrum of [Lur15]. As shown in [Lur15] there exists a filtered \mathbf{E}_2^\otimes -algebra whose underlying object is $\mathbb{S}^{K(\mathbb{Z}, 2)}$ (so is $\mathbf{E}_\infty^\otimes$) but its associated graded is a 2-periodic version of the sphere spectrum $\mathbb{S}[\beta, \beta^{-1}]$. This 2-periodic sphere spectrum is known not to exist as an $\mathbf{E}_\infty^\otimes$ -ring. However, we can consider the natural augmentation

$$\mathbb{S}^{K(\mathbb{Z}, 2)} \longrightarrow \mathbb{S}$$

and consider the spectrum

$$A := \mathbb{S} \otimes_{\mathbb{S}^{K(\mathbb{Z}, 2)}} \mathbb{S}$$

As a mere spectrum, this is equivalent to the group ring over the circle $A \simeq \mathbb{S}[K(\mathbb{Z}, 1)]$. However, the E_2 -filtration on $\mathbb{S}^{K(\mathbb{Z}, 2)}$ induces a structure of a filtered bialgebra on A , which should be considered as a non-commutative analogue of the filtered circle.

More precisely, we would like to consider the dual filtered bialgebra $B = A^*$ and consider $\text{Spec } B$ in some sense to produce a topological version of the filtered circle. We however do not know how to exploit the existence of B .

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