Algebraic geometry, categories and trace formula

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Topology of algebraic varieties

Topology of algebraic varieties

Homogeneous polynomials $F_1, \ldots, F_p \in \mathbb{C}[X_0, \ldots, X_n]$

$$X := \{(x_0, \ldots, x_n)/F_i(x) = 0\} \subset \mathbb{P}^n_{\mathbb{C}}.$$

Problem: read the topology of X in terms of the F_i's.

Topology of algebraic varieties

Typical answers in low dimension

- $(n = 1, p = 1)$: X finite set of cardinality $deg(F_1)$ counted with multiplicities.
- $(n = 2$ and $p = 1)$: X is a compact Riemann surface and

$$g(X) = \frac{(d-1)(d-2)}{2} \qquad d = deg(F_1)$$

($g(X)$ is the arithmetic genus if X not smooth).

Euler characteristic of algebraic varieties

Simple topological invariant: Euler characteristic $\chi(X)$

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Here: Ω_X^q is the sheaf of holomorphic differential q-forms on X. The right hand side can be determined purely in terms of the F_i ("GAGA" theorem).

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topological invariant $=$ algebraic invariant

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The theorem follows from the existence of the Hodge decomposition

$$H^i(X,\mathbb{Q})\otimes\mathbb{C}\simeq\bigoplus_{p+q=i}H^p(X,\Omega^q_X).$$

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$$H^i(X,\mathbb{Q})\otimes\mathbb{C}\simeq\bigoplus_{p+q=i}H^p(X,\Omega_X^q).$$

But, it has also an independent proof:

$$(GB)\quad \chi(X)=\int_X C_{top}(X)$$

$$(HRR)\quad \int_X C_{top}(X)=\sum_{p,q}(-1)^{p+q}dim\,H^p(X,\Omega_X^q).$$

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Homogeneous polynomials $F_1, \ldots, F_p \in k[X_0, \ldots, X_n]$ (k an algebraically closed field) $X := \{(x_0, \ldots, x_n)/F_i(x) = 0\} \subset \mathbb{P}^n_k$.

$$\chi(X) := \sum_i (-1)^i dim\, H^i_{et}(X, \mathbb{Q}_\ell) = \sum_{p,q} (-1)^{p+q} dim\, H^p(X, \Omega^q_X)$$

$H^i_{et}(X, \mathbb{Q}_\ell)$ are the ℓ-adic cohomology groups introduced by Grothendieck. $H^p(X, \Omega^q_X)$ sheaf cohomology for the Zariski topology.

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Warning: In general $H^i_{et}(X, \mathbb{Q}_\ell)$ and $\oplus_{p+q=i} H^p(X, \Omega^q_X)$ dont have the same dimension !

The trace formula for algebraic varieties

The formula is a special case of the trace formula: $f \circlearrowright X$ algebraic endomorphism of X.

$$\sum_i (-1)^i Trace(f : H^i_{et}(X, \mathbb{Q}_\ell)) = [\Gamma_f . \Delta_X]$$

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$[\Gamma_f.\Delta_X]$ is the intersection number of the graph of f with the diagonal inside $X \times X$. The case $f = id$ recovers the formula for $\chi(X)$.

Topology of degeneration of algebraic varieties

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- Moduli theory: moduli of varieties are non-compact. Compactification by adding singular varieties at infinity (e.g. $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$).
- Arithmetic geometry: algebraic varieties defined over rings of integers in number fields. Bad reduction at some prime.

Topology of degeneration of algebraic varieties

Homogeneous polynomials $F_1, \ldots, F_p \in A[X_0, \ldots, X_n]$ where $A = \mathcal{O}(S)$=ring of functions on the parameter space S.

$$X := \{(x_0, \ldots, x_n, s)/F_i(x,s) = 0\} \subset \mathbb{P}^n \times S.$$

Family of algebraic varieties parametrized by S. For a point $s \in S$, we have $X_s \subset \mathbb{P}^n_{k(s)}$ an algebraic variety defined over $k(s)$=residue field of s.

Topology of degeneration of algebraic varieties

S is taken to be a "small disk" $:=$ *henselian trait*. Typical examples:

- $S = Spec\,\mathbb{C}[[t]]$ formal holomorphic disk.
- $S = Spec\,k[[t]]$ formal disk over an algebraically closed field.
- $S = Spec\,\mathbb{Z}_p$ formal p-adic disk.

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$S = Spec\,A$ for A a complete d.v.r. Two points in S:

- The **special** point $o \in S$ ($k := k(s) = A/m$ is the residue field).
- The **generic** point $\eta \in S$ ($K := k(\eta) = Frac(A)$ the fraction field).

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Variational problem: understand the change of topology between $X_{\bar{K}}$ and $X_{\bar{k}}$.

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When $X \to S$ is a submersion, $X_{\bar{k}}$ and $X_{\bar{K}}$ smooth and the topology is constant:

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We assume $X_{\bar{K}}$ is smooth and $X_{\bar{k}}$ possibly singular.

Euler characteristic of degeneration of algebraic varieties: charac. 0

Easy solution for characteristic zero case ($A = \mathbb{C}[[t]]$): twisted de Rham complex. t defines a function on X and we have

Theorem (Milnor, Kapranov, Saito, ...)

$$\chi(X_{\bar{k}}) - \chi(X_{\bar{K}}) = \sum_i (-1)^i dim\, H^i(X, (\Omega_X^*, \wedge df)).$$

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$$(\Omega_X^*, \wedge df): \ \mathcal{O}_X \xrightarrow{df} \Omega_X^1 \xrightarrow{\wedge df} \Omega_X^2 \xrightarrow{\wedge df} \dots$$

Case where $X_{\bar{k}}$ has only isolated singularities: Milnor formula (dimension of Jacobian ring).

Euler characteristic of degeneration of algebraic varieties: Bloch's formula

Without characteristic zero: more complicated due to arithmetic aspects.

Conjecture (Deligne 67, Bloch 85)

If the scheme X is regular and k perfect.

$$\chi(X_{\bar{k}}) - \chi(X_{\bar{K}}) = [\Delta_X.\Delta_X]_0 + Sw(X_{\bar{K}}).$$

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- $[\Delta_X.\Delta_X]_0$ is the degree of a 0-cycle on $X_{\bar{k}}$ that measures the singularities (Bloch's localized intersection number). Generalization of the different in algebraic number theory.
- $Sw(X_{\bar{K}})$ is the Swan conductor which measures wild ramifications: action of $Gal(\bar{K}/K)$ on $H^*_{et}(X_{\bar{K}})$.

Euler characteristic of degeneration of algebraic varieties: Bloch's formula

The Bloch's formula is a theorem when:

- Characteristic 0: Milnor, Kapranov, M. Saito.
- Equicharacteristic $p > 0$ ($A = k[[t]]$): T. Saito (2017).
- Semi-stable case: $X_{\bar{k}} \subset X$ is (supported by) a simple normal crossing divisor (Kato-Saito 2001).
- Degenerate case: $X_{\bar{\eta}} = \emptyset$ (recovers formula for $\chi(X)$).
- Family of curves (Bloch), and finite ramified cover $X \to S$ (standard formula for the different).

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Mixed characteristic case (e.g. $A = \mathbb{Z}_p$) is open: in particular isolated singularities (Deligne-Milnor formula).

Bloch's conductor formula

Conjecture : $\chi(X_{\bar{k}}) - \chi(X_{\bar{K}}) = [\Delta_X.\Delta_X]_0 + Sw(X_{\bar{K}})$.

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We want to make progress on this conjecture by introducing a new point of view: non-commutative geometry.

- Degenerate case $X_{\bar{\eta}} = \emptyset$ is the commutative case.
- General case involves a quantum parameter: the function $\pi \in A$, a choice of uniformizer on S.

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- Degenerate case $X_{\bar{\eta}} = \emptyset$ is the commutative case.
- General case involves a quantum parameter: the function $\pi \in A$, a choice of uniformizer on S.

For this: find a non-commutative variety $\mathcal{X}_\pi$ such that $\chi(\mathcal{X}_\pi) = \chi(X_{\bar{k}}) - \chi(X_{\bar{K}})$ and apply a non-commutative trace formula for $\mathcal{X}_\pi$.

Non-commutative varieties

Definition

A **non-commutative variety over some base commutative ring** k *is a* k*-linear (dg-)category.*

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Reminder: a (dg-)category is

- a set of objects
- for two objects x and y a (complex of) k-module $Hom(x, y)$
- compositions $Hom(x, y) \otimes_k Hom(y, z) \to Hom(x, z)$

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Dg-categories are considered up to Morita equivalences ("generate the same triangulated categories").

Non-commutative varieties: examples

Very weak definition $\Rightarrow$ plenty of examples !

1. Algebras: A a k-algebra, $\mathcal{D}(A) =$ dg-category of complexes of A-modules.
2. Schemes over k: as above + gluing $\Rightarrow \mathcal{D}(X)$.
3. Other examples: Quivers, topology, symplectic manifolds.

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Good surprise: non-commutative schemes have reasonable notions of

- differential forms (Hochschild homology)
- ℓ-adic cohomology (recent construction, see below).

Hochschild homology (differential forms)

For a k-algebra B we have a Hochschild complex

$$HH(B) := \ \dots B^{\otimes n} \longrightarrow B^{\otimes (n-1)} \longrightarrow \dots \longrightarrow B^{\otimes 2} \longrightarrow B$$

with differential

$$d(b_1 \otimes \cdots \otimes b_n) = \sum_i (-1)^{i-1} b_1 \otimes \dots b_{i-1} \otimes b_i b_{i+1} \otimes \cdots \otimes b_n$$

$$+(-1)^n b_n b_1 \otimes b_2 \otimes \cdots \otimes b_{n-1}.$$

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$$+(-1)^n b_n b_1 \otimes b_2 \otimes \cdots \otimes b_{n-1}.$$

This extends to dg-algebras and dg-categories: for any non-commutative scheme T we have a Hochschild complex $HH(T)$.

Hochschild homology (differential forms)

Theorem (Hochschild-Kostant-Rosenberg, Keller)

For X a smooth algebraic variety over k, considered as a non-commutative scheme $\mathcal{D}(X)$ we have

$$HH_n(\mathcal{D}(X)) \simeq \bigoplus_{p-q=n} H^p(X, \Omega_X^q).$$

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In particular we have

$$\chi(HH(\mathcal{D}(X))) = \sum_{p,q} (-1)^{p+q} dim\, H^p(X, \Omega_X^q).$$

Hochschild homology (differential forms)

There is a version with coefficients in $f \circlearrowright T$ $HH(T, f)$. Hochschild-Kostant-Rosenberg becomes

$$HH(\mathcal{D}(X), f) \simeq \mathbb{R}\Gamma(X \times X, \mathcal{O}_{\Gamma_f} \otimes^{\mathbb{L}} \mathcal{O}_{\Delta_X}).$$

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$$\chi(HH(\mathcal{D}(X), f)) = [\Gamma_f . \Delta_X].$$

ℓ-adic cohomology

Theorem (Blanc, Robalo, T., Vezzosi)

For all non-commutative scheme T, and of a prime ℓ invertible in k, it is possible to define a $\mathbb{Q}_\ell$-vector space $H^*(T, \mathbb{Q}_\ell)$. For $T = \mathcal{D}(X)$ we have

$$H^n(T, \mathbb{Q}_\ell) \simeq \bigoplus_i H^{2i+n}(X, \mathbb{Q}_\ell(i)).$$

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Construction: approximation of T by commutative algebraic varieties + homotopy theory of schemes of Voevodsky-Morel. In particular, we can define Euler characteristic in the non-commutative situation

$$\chi(T) := dim H^0(T, \mathbb{Q}_\ell) - dim H^1(T, \mathbb{Q}_\ell) \in \mathbb{Z}.$$

The non-commutative trace formula

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By definition, the dual $T^\vee$ comes equipped with

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$+$ usual properties ($T \longrightarrow T \otimes T^\vee \otimes T \longrightarrow T \;= id$).
$\mathcal{D}(X)$ is saturated $\Longleftrightarrow$ X is proper and smooth. Then $\mathcal{D}(X)^\vee \simeq \mathcal{D}(X)$ (Poincaré duality). ev and $coev$ are non-commutative maps which do not exists in the commutative setting !

The non-commutative trace formula

Theorem (Vezzosi, T.)

Let T be a **saturated** *non-commutative variety T + technical condition called* **admissibility**. *For $f \circlearrowright T$*

$$Tr(f : H^*(T, \mathbb{Q}_\ell)) = \chi(HH_*(T, f)).$$

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$$Tr(f : H^*(T, \mathbb{Q}_\ell)) = \chi(HH_*(T, f)).$$

- For $T = \mathcal{D}(X)$ gives back the trace formula of Grothendieck.
- Also extends to non-commutative schemes T over non-commutative base B.

Bloch's formula and non-commutative schemes

Back to $X \subset \mathbb{P}^n \times S$. We fix a uniformizer π of A, which defines a function π on X.

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Back to $X \subset \mathbb{P}^n \times S$. We fix a uniformizer π of A, which defines a function π on X. A **matrix factorisation for** π consists of

- two vector bundles E_0, E_1 on X
- two morphisms

$$E_0 \xrightarrow{\partial} E_1 \xrightarrow{\partial} E_0$$

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Matrix factorizations form a (dg-)category $MF(X, \pi)$.

Definition

The **non-commutative scheme of singularities of** X/S *is defined to be* $\mathcal{X}_\pi := MF(X, \pi)$.

Two examples

1. X a vector space, $\pi = q$ a quadratic form on X.

$$\mathcal{X}_\pi \simeq \mathcal{D}(\mathit{Ciff}(q))$$

where $\mathit{Ciff}(q)$ is the Clifford algebra of (X, q). $\mathcal{X}_\pi$ sees arithmetic aspects.

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2. Over $\mathbb{C}$, and X_o with an isolated singularity we have (Dyckerhoff)

$$HH_0(\mathcal{X}_\pi) \simeq \mathit{Jac}(\pi) = \mathcal{O}_{X,x}/(\partial\pi/\partial x_i).$$

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In general $MF(X, \pi)$ concentrated on singularities $MF(X, f) = 0$ when X is smooth over S.

Bloch's formula and non-commutative schemes

Theorem (T., Vezzosi)

1. *The non-commutative scheme $\mathcal{X}_\pi$ is saturated over some funny non-commutative base ring B.*
2. $\chi(HH(\mathcal{X}_\pi)) = [\Delta_X . \Delta_X]_0$

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1. *The non-commutative scheme $\mathcal{X}_\pi$ is saturated over some funny non-commutative base ring B.*
2. $\chi(HH(\mathcal{X}_\pi)) = [\Delta_X.\Delta_X]_0$
3. *If the monodromy acts unipotently on $H^i_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)$ then*

$$\chi(H^*(\mathcal{X}_\pi, \mathbb{Q}_\ell)) = \chi(X_{\bar{k}}) - \chi(X_{\bar{K}}).$$

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$$\chi(H^*(\mathcal{X}_\pi, \mathbb{Q}_\ell)) = \chi(X_{\bar{k}}) - \chi(X_{\bar{K}}).$$

Corollary

The Bloch's conductor conjecture is true when the monodromy is unipotent.

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for S'/S ramified finite covering of S. How this implies the general case of the Bloch's formula is still under investigation.

A final comment

The non-commutative scheme $\mathcal{X}_\pi$ sees many interesting aspects

1. Topological: $H^*(\mathcal{X}_\pi, \mathbb{Q}_\ell)$.
2. Algebraic: $HH(\mathcal{X}_\pi)$.
3. Arithmetic: $\mathcal{X}_\pi \otimes_B \mathcal{S}'_\pi$.

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The (3) above recovers the whole action of Galois group G_K on $H^*_{et}(X, \nu) =$ vanishing cohomology. $\mathcal{X}_\pi$ is surely useful beyond the Bloch's formula.

Algebraic geometry, categories and trace formula

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