Proper local complete intersection morphisms preserve perfect complexes

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Abstract

Let $f : X \longrightarrow Y$ be a proper and local complete intersection morphism of schemes. We prove that $\mathbb{R}f_*$ preserves perfect complexes, without any projectivity or noetherian assumptions. This provides a different proof of a theorem by Neeman and Lipman (see [Li-Ne]) based on techniques from derived algebraic geometry to proceed a reduction to the noetherian case.

Introduction

In [SGA6, Exp. XIV] Grothendieck and its collaborators present a huge list of open problems around the most general version of intersection theory on schemes and of the Grothendieck-Riemann-Roch formula. An important part of these problems concern the question of proving the GRR formula for proper morphisms of schemes without assuming the existence of a global factorization and/or without any noetherian assumption. One of the most basic questions is the existence of push-forward operations in algebraic K-theory of schemes in the most general setting. Directly related to this specific question is the following conjecture.

Conjecture 0.1 *(See [SGA6, ExpIII-2.1]) Let $f : X \longrightarrow Y$ be a proper and pseudo-coherent morphism of schemes. Then, the derived direct image*

$$\mathbb{R}f_* : D_{qcoh}(X) \longrightarrow D_{qcoh}(Y)$$

preserves pseudo-coherent complexes.

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This conjecture has been proven in [Ki], and more recently A. Neeman and J. Lipman deduced from it the following theorem.

Theorem 0.2 *([Li-Ne, Ex. 2.2 (a)]) Let $f : X \longrightarrow Y$ be a proper and perfect morphism of schemes. Then, the derived direct image*

$$\mathbb{R}f_* : D_{qcoh}(X) \longrightarrow D_{qcoh}(Y)$$

preserves perfect complexes.

The purpose of this work is to propose a new proof of the above theorem by assuming further that f is a local complete intersection morphism.

Theorem 0.3 *Let $f : X \longrightarrow Y$ be a proper and local complete intersection morphism of schemes. Then, the derived direct image*

$$\mathbb{R}f_* : D_{qcoh}(X) \longrightarrow D_{qcoh}(Y)$$

preserves perfect complexes.

The proof we propose in this work is based on a reduction to the noetherian case (no surprise) but in the somehow unexpected context of derived algebraic geometry. The general strategy proceeds as follows. The statement being local on Y we can reduce it to the case where Y is an affine scheme $Spec\, A$. We then write A as a filtered colimit of noetherian rings A_i and we try to descend the whole situation over one of the ring A_i. The unexpected fact is that even thought the scheme X descend as a scheme X_i proper and of finite type over some A_i, it does not seem to be true that X_i can be chosen to be itself a local complete intersection. Our main observation is that X_i can be chosen to be a proper and local complete intersection *derived scheme*. This fact can then be used to prove the theorem 0.3 by using some standard facts about derived categories of derived schemes, particularly the base change property (see 1.4), which is a statement specifically true in the derived setting and wrong in the underived setting without extra flatness assumptions. This method of proof also shows that the theorem 0.3 remains true for X and Y being themselves derived schemes, even thought we do not make this statement explicit in this work.

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1 Derived schemes

We have collected in this preliminary section some facts concerning derived schemes that we will use in our proof of the theorem 0.3. They belong to the general properties of derived schemes and derived stacks and are certainly well known to experts. Some of these statements are probably established in the topological setting in [Lu, §2]. We note however some differences between the theory of derived schemes we will be using and the theory of spectral schemes used in [Lu] (notably cotangent complexes are not quite the same). We have therefore included proofs for the three main statements we will be using: base change, continuity and noetherian approximation.

We recall from [To-Ve2, To1, To-Va] the existence of an ∞-category **dSt**, of derived stacks, and its full subcategory $\mathbf{dSch} \subset \mathbf{dSt}$ of derived schemes. The ∞-category **dSch** contains as a full sub-category **Sch**, the category of schemes. The inclusion functor $\mathbf{Sch} \hookrightarrow \mathbf{dSch}$ possesses a right adjoint $h^0 : \mathbf{dSch} \longrightarrow \mathbf{Sch}$, the truncation functor. We note that even thought both ∞-categories **Sch** and **dSch** admit finite limits, **Sch** is not stable by finite limits in **dSch**.

Affine derived schemes are of the form $Spec\, A$ for some commutative simplicial ring $A \in sComm$. The $Spec$ ∞-functor defines an equivalence of ∞-categories

$$Spec : \mathbf{sComm}^{op} \simeq \mathbf{dAff},$$

where $\mathbf{dAff} \subset \mathbf{dSch}$ is the full sub ∞-category of derived affine schemes, and $\mathbf{sComm}^{op}$ is the ∞-categories of simplicial commutative rings. Restricted to affine objects the inclusion ∞-functor from schemes to derived schemes, and its right adjoint h^0, is equivalent to the adjunction of ∞-categories

$$\pi_0 : \mathbf{sComm} \longleftrightarrow \mathbf{Comm} : i,$$

where i sends a ring to the constant simplicial ring and π_0 is the connected component ∞-functor.

For all derived scheme X, the scheme $h^0(X)$, considered as an object in **dSch**, has an adjuntion morphism $j : h^0(X) \longrightarrow X$ which is a closed immersion. When $X = Spec\, A$ is affine, this morphism is equivalent to $Spec\, \pi_0(A) \longrightarrow Spec\, A$ corresponding to the natural projection $A \longrightarrow \pi_0(A)$. The higher homotopy groups $\pi_i(A)$ are $\pi_0(A)$-modules, and thus define quasi-coherent sheaves on $Spec\, \pi_0(A)$. These sheaves will be denoted by $h^{-i}(X) := \pi_i(A)$. This construction glue in the non-affine case: for any derived scheme X the scheme $h^0(X)$ carries natural quasi-coherent sheaves $h^i(X) := \pi_i(\mathcal{O}_X)$.

Any derived scheme X possesses a cotangent complex $\mathbb{L}_X \in L_{qcoh}(X)$ (see [To-Ve2, §1.4], [To1, §4.2]), which is a quasi-coherent complex on X (see our next subsection for quasi-coherent complexes and the definition of $L_{qcoh}(X)$). When $X = Spec\, A$ is affine, $\mathbb{L}_X$ is the cotangent complex of A introduced by Quillen (it is a simplicial A-module than can be turned into a dg-module over the normalization of A to consider it as an object in $L_{qcoh}(X)$). For a morphism of derived schemes $f : X \longrightarrow Y$ we define the relative cotangent complex $\mathbb{L}_f$, or $\mathbb{L}_{X/Y}$, as the cone, in the dg-category $L_{qcoh}(X)$, of the natural morphism

$$f^*(\mathbb{L}_Y) \longrightarrow \mathbb{L}_X.$$

Definition 1.1 *1. Let X be a derived scheme. We say that X is* quasi-compact *(resp.* quasi-separated*, resp.* separated*) if the scheme $h^0(X)$ is so.*

2. Let $f : X \longrightarrow$ be a morphism of derived schemes. We say that f is proper *if $h^0(f) : h^0(X) \longrightarrow h^0(Y)$ is a proper morphism of schemes.*

3. Let $f : X \longrightarrow Y$ be a morphism of derived schemes. We say that f is locally of finite presentation *if $h^0(f) : h^0(X) \longrightarrow h^0(Y)$ is a morphism locally of finite presentation of schemes and if the relative cotangent complex $\mathbb{L}_f$ is perfect on X.*

We note that the definition above of morphism locally of finite presentation possesses several equivalent versions (see [To-Ve2, Prop. 2.2.2.4]). In particular, a morphism of derived affine schemes $f : Spec\, A \longrightarrow$

$Spec\,B$ is locally of finite presentation if and only if B is equivalent to a retract of a finite cell commutative simplicial B-algebra (see [To-Ve2, Prop. 1.2.3.5], or [To-Va, Prop. 2.2]).

We will also be using the following notion.

Definition 1.2 *Let $X = Colim\, X_\alpha$ be a colimit in the ∞-category* **dSch** *of derived schemes. We will say that the colimit is* strong *if it is also a colimit in the bigger ∞-category* **dSt** *of derived stacks.*

1.1 Review of derived categories of derived schemes

According to [To2, §2], for any derived stack $X \in \mathbf{dSt}$, we have a natural ($\mathbb{Z}$-linear) dg-category $L_{qcoh}(X)$ of quasi-coherent complexes over X. This can be made into an ∞-functor

$$L_{qcoh} : \mathbf{dSt}^{op} \longrightarrow dg - \mathcal{C}at,$$

where $dg - \mathcal{C}at$ is the ∞-category of (locally presentable, see [To2]) dg-categories For a morphism of derived stacks $f : X \longrightarrow Y$, we have an adjunction of dg-categories

$$f^* : L_{qcoh}(Y) \longleftrightarrow L_{qcoh}(X) : f_*.$$

When $X = Spec\, A$ is an affine derived scheme, the dg-category $L_{qcoh}(X)$ can be explicitely described, up to a natural equivalence, as follows. The commutative simplicial ring A possesses a normalisation $N(A)$, which is a (commutative) dg-ring. The dg-category $L(A)$ of cofibrant $N(A)$-dg-modules is naturally equivalent to $L_{qcoh}(X)$. This description is moreover functorial in the following way. A morphism of derived affine schemes $f : X = Spec\, A \longrightarrow Y = Spec\, B$ corresponds to a morphism of commutative simplicial rings $B \longrightarrow A$, and thus to a morphism of dg-rings $N(B) \longrightarrow N(A)$. The adjunction

$$f^* : L_{qcoh}(Y) \longleftrightarrow L_{qcoh}(X) : f_*$$

is then equivalent to the following adjunction

$$- \otimes_A B : L(A) \longleftrightarrow L(B) : f,$$

where we have written $\otimes_A$ symbolically for $\otimes_{N(A)} N(B)$, and where f is the forgetful functor.

In the general case, we write a general derived stack X as a colimit in **dSt**, $X = Colim\, Spec\, A_\alpha$, and we have

$$L_{qcoh}(X) \simeq Lim\, L_{qcoh}(Spec\, A_\alpha) \simeq Lim\, L(A_\alpha),$$

where the limit here is taken in the ∞-category of dg-categories. When X is a derived scheme we can take a rather simple colimit description by taking all the $Spec\, A_\alpha$ to belong to a basis of opens for the Zariski topology on X.

Finally, the dg-category $L_{qcoh}(X)$ of any derived scheme X possesses a natural non-degenerate t-structure (by which we mean that the associated triangulated category $[L_{qcoh}(X)]$ has such a t-structure), whose heart is canonically equivalent to the category of quasi-coherent sheaves on the scheme $h^0(X)$. Locally on X, over an affine open $U = Spec\, A \subset X$ this t-structure can be described as follows. The dg-category $L_{qcoh}(U)$ is identified with $L(A)$ the dg-category of cofibrant $N(A)$-dg-modules. An object E in $L(A)$ is declared to belong to $L(A)^{\leq 0}$ if it is such that $H^i(E) = 0$ for all $i \geq 0$. It is easy to see that this defines an aisle of a non-degenerate t-structure on $D(A) = [L(A)]$, the derived category of $N(A)$-dg-modules. The heart is the full sub-category of $D(A)$ consisting of dg-module E with $H^i(E) = 0$ except for $i = 0$. This sub-category is equivalent, via the functor $E \mapsto H^0(E)$, to the category of $\pi_0(A)$-modules.

Definition 1.3 *Let X be a derived stack. An object $E \in L_{qcoh}(X)$ is* perfect *if for all affine derived scheme $Z = Spec\, A$ and all morphism $f : Z \longrightarrow X$, the pull-back $f^*(E)$ is a compact object in $L_{qcoh}(Z)$.*

The full sub-dg-category of $L_{qcoh}(X)$ consisting of perfect complexes will be denoted by $L_{parf}(X)$.

Note that for $Z = Spec\, A$ affine, the compact objects in $L_{qcoh}(Z)$ can also be described in several different ways: as strongly dualizable objects or as retracts of finite cell A-dg-modules (see discussion after [To-Va, Def. 2.3]).

Suppose now that we have a cartesian square of derived stacks

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "q"'] & Y' \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

by adjunction there is, for any object $E \in L_{qcoh}(Y')$, a natural morphism in the dg-category $L_{qcoh}(X)$

$$\rho_E : f^*p_*(E) \longrightarrow q_*g^*(E).$$

Proposition 1.4 *If*

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "q"'] & Y' \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

is a cartesian square of quasi-compact and quasi-separated derived schemes, then for all $E \in L_{qcoh}(Y')$ the morphism $\rho_E : f^*p_*(E) \longrightarrow q_*g^*(E)$ above is an equivalence.

Proof: Localising on the Zariski topology on X we can assume that X and Y are affine derived schemes. Let us write the cartesian square as follows

$$\begin{tikzcd} Z_B := Z \times_{Spec\, A} Spec\, B \arrow[r, "g"] \arrow[d, "q"'] & Y' = Z \arrow[d, "p"] \\ X = Spec\, B \arrow[r, "f"'] & Y = Spec\, A. \end{tikzcd}$$

For $E \in L_{qcoh}(Z)$, the object $f^*p_*(E)$ is $\mathbf{H}(Z,E) \otimes^{\mathbb{L}}_A B$, where $\mathbf{H}(Z,E)$ is the cohomology A-dg-module of cohomology of Z with coefficients in E. The object $q_*g^*(E)$ is $\mathbf{H}(Z_B, g^*(E))$, the cohomology B-dg-module of Z_B with coefficients in $g^*(E)$. The morphism ρ_E is then the natural morphism $g^* : \mathbf{H}(Z,E) \longrightarrow \mathbf{H}(Z_B, g^*(E))$, extended to $\mathbf{H}(Z,E) \otimes^{\mathbb{L}}_A B$ by linearity.

As Z is quasi-compact and quasi-separated it belongs to the smallest full sub ∞-category of **dSch** containing affines and which is stable by finite strong colimits in **dSt** (see definition 1.2). Therefore, to prove our proposition it is enough to prove the following two individual statements:

1. If Z is affine, then for all $E \in L_{qcoh}(Z)$, the morphism ρ_E is an equivalence.

2. The full sub ∞-category of objects $Z \in \mathbf{dSch}/Spec\, A$ for which ρ_E is an equivalence for all $E \in L_{qcoh}(Z)$ is stable by finite strong colimits.

The property (1) follows directly from the shape of fiber products of derived affine schemes (see [To-Ve2, Prop. 1.1.0.8]). For the property (2), let $Z \simeq Colim\, Z_i$ be a finite strong colimit in $\mathbf{dSch}$ for which we know that the proproposition holds for all the derived schemes Z_i. We let $Z_{i,B} = Z_i \times_{Spec\, A} Spec\, B$, and we notice that as colimits are universal in $\mathbf{dSt}$ (because it is an ∞-topos, see [To-Ve1, To-Ve2, To1]), we have an induced strong colimit $Z_B \simeq Colim\, Z_{i,B}$. As the colimit is strong we have moreover

$$\mathbf{H}(Z, E) \simeq Lim\, \mathbf{H}(Z_i, E_{|Z_i}),$$

where Lim stands for the limit in the dg-category $L_{qcoh}(Spec\, A)$. As this limit is finite, we have

$$\mathbf{H}(Z, E) \otimes^{\mathbb{L}}_A B \simeq Lim\, (\mathbf{H}(Z_i, E_{|Z_i}) \otimes^{\mathbb{L}}_A B) \simeq Lim\, \mathbf{H}(Z_{i,B}, g^*(E)_{|Z_i}) \simeq \mathbf{H}(Z_B, g^*(B)).$$

$\square$

The second formal property we will need is continuity, relating the dg-category of quasi-coherent complexes of certain limits of derived schemes to the colimit (inside the ∞-category of dg-categories) of the dg-categories of quasi-coherent complexes on each individual derived schemes.

For this we let A be a commutative ring wich is written as a filtered colimit

$$A = Colim\, A_i.$$

We will suppose that the indexing category I has an initial obect $0 \in I$. Let $X_0 \longrightarrow Spec\, A_0$ be a derived scheme, and let set $X_i := X_0 \times_{Spec\, A_0} Spec\, A_i$ its base change to A_i. We will also denote by $X := X \times_{Spec\, A_0} Spec\, A$ the base change to A itself. In this situation we have a natural morphism of dg-categories

$$Colim\, L_{qcoh}(X_i) \longrightarrow L_{qcoh}(X).$$

Proposition 1.5 *With the same notations as above, if the derived scheme X_0 is quasi-compact and quasi-separated then the morphism*

$$Colim\, L_{parf}(X_i) \longrightarrow L_{parf}(X)$$

is an equivalence of dg-categories.

Proof: It follows the same lines as the proof of the proposition 1.4. We reduce the proposition to the following two individual statements.

1. Proposition 1.5 holds for X_0 affine.

2. The full sub ∞-category of objects $X_0 \in \mathbf{dSch}/Spec\, A_0$ for which the proposition holds is stable by strong finite colimits.

   The property (1) simply states that for a filtered colimit of commutative simplicial rings $B = Colim\, B_i$, we have
   $$Colim\, L_{parf}(Spec\, B_i) \simeq L_{parf}(Spec\, B),$$

which is a particular case of [To-Va, Lem. 2.10]. The second property is proven as follows. Let $X_0 = Colim\, X_{0,\alpha}$ be a finite strong colimit in **dSch** such that the proposition 1.5 holds for all $X_{0,\alpha}$. We let $X_{i,\alpha} := X_i \times_{X_0} X_{0,\alpha}$ and $X_\alpha = X \times_{X_0} X_{0,\alpha}$. As the colimit is filtered, it commutes with finite limits, and thus we have

$$Colim_i\, L_{parf}(X_i) \simeq Colim_i\, Lim_\alpha\, L_{parf}(X_{i,\alpha}) \simeq Lim_\alpha\, Colim_i\, L_{parf}(X_{i,\alpha}) \simeq Lim_\alpha\, L_{parf}(X_\alpha).$$

$\square$

1.2 Noetherian approximation for derived schemes

We let A be a commutative ring which is written as a filtered colimit $A = Colim\, A_i$. The ∞-category of derived schemes over $Spec\, A_i$ (resp. over $Spec\, A$) will be denoted by $\mathbf{dSch}_{A_i}$ (resp. $\mathbf{dSch}_A$). We will study the ∞-functor

$$Colim\, \mathbf{dSch}_{A_i} \longrightarrow \mathbf{dSch}_A.$$

For this we denote by $\mathbf{dSch}_A^{\leq n}$ the full sub ∞-category of $\mathbf{dSch}_A$ consisting of derived schemes $f : X \longrightarrow Spec\, A$, for which f is of locally of finite presentation, quasi-compact, quasi-separated, and such that the cotangent complex $\mathbb{L}_f$ is of amplitude $[-n, 0]$ (see [To-Va, §2.4]). We use the same notation for $\mathbf{dSch}_{A_i}^{\leq n} \subset \mathbf{dSch}_{A_i}$. The base change ∞-functors

$$\mathbf{dSch}_{A_i} \longrightarrow \mathbf{dSch}_A$$

preserves cotangent complexes (see [To-Ve2, Lem. 1.4.1.16 (2)]), and thus restrict to ∞-functors

$$\mathbf{dSch}_{A_i}^{\leq n} \longrightarrow \mathbf{dSch}_A^{\leq n}.$$

Proposition 1.6 *The ∞-functor*

$$Colim\, \mathbf{dSch}_{A_i}^{\leq n} \longrightarrow \mathbf{dSch}_A^{\leq n}$$

is an equivalence of ∞-categories.

Proof: The proof again goes along the same general lines as for the proofs of propositions 1.4 and 1.5. We start by considering $\mathbf{dAff}_A^{\leq n} \subset \mathbf{dSch}_A^{\leq n}$ the full sub ∞-category consisting of affine objects. We define in the same way $\mathbf{dAff}_{A_i}^{\leq n} \subset \mathbf{dSch}_{A_i}^{\leq n}$.

Lemma 1.7 *The ∞-functor*

$$Colim\, \mathbf{dAff}_{A_i}^{\leq n} \longrightarrow \mathbf{dAff}_A^{\leq n}$$

is an equivalence.

Proof of the lemma: We start by proving that the ∞-functor is fully faithful. Let B and C be two simplicial A_i-algebras of finite presentation (and with cotangent complexes relative to A_i of amplitude $[-n, 0]$). As B is a finitely presented A_i-algebra, we have

$$Map_A(B \otimes^{\mathbb{L}}_{A_i} A, C \otimes^{\mathbb{L}}_{A_i} A) \simeq Map_{A_i}(B, Colim_{j \geq i}\, (C \otimes^{\mathbb{L}}_{A_i} A_j)) \simeq Colim_{j \geq i}\, Map_{A_i}(B, C \otimes^{\mathbb{L}}_{A_i} A_j),$$

where we have denoted by Map_A the mapping spaces of the ∞-category of commutative simplicial A-algebras (and similarly for A_i). This proves fully faithfulness.

Now, let B a commutative simplicial A-algebra of finite presentation and with cotangent complex $\mathbb{L}_{B/A}$ of amplitude in $[-n, 0]$. We know that B is equivalent to a retract of a finite cell commutative A-algebra B'. In particular there is an index i and a finite cell commutative A_i-algebra B'_i such that $B' \simeq B'_i \otimes^{\mathbb{L}}_{A_i} A$.

The object B defines a projector up to homotopy on B', that is a projector p on B' considered as an object in the homotopy category $Ho(A-CAlg)$ of commutative A-algebras. By chosing i big enough we can moreover assume that this projector is induced by a projector p_i on B'_i in $Ho(A_i - sComm)$. By [To3, Sublemma 3] the category $Ho(A_i - sComm)$ is Karoubian closed, so p_i splits as a composition in $Ho(A_i - sComm)$

$$p = vu : \; B'_i \xrightarrow{u} B_i \xrightarrow{v} B'_i$$

for some commutative A_i-algebra B_i and with $uv = id$.

The A_i-algebra B_i is of finite presentation (because it is a retract of a finite cell commutative A_i-algebra) and we have $B_i \otimes^{\mathbb{L}}_{A_i} A \simeq B$. It remains to show that i can be chosen so that the cotangent complex $\mathbb{L}_{B_i/A_i}$ has amplitude contained in $[-n, 0]$. We let $Z_i := Spec\, B_i$, $X_i := Spec\, A_i$, $Z := Spec\, B$ and $X = Spec\, A$. The locus in Z_i in which $\mathbb{L}_{B_i/A_i}$ has amplitude contained in $[-n, 0]$ is an open derived sub-scheme $U_i \subset Z_i$. Moreover, we have $U_i \times_{X_i} X_j \simeq U_j$ for all $j \geq i$, because cotangent complexes are stable by base changes. As $U_i \times_{X_i} X \simeq X$ there is an index j with $U_i \times_{X_i} X_j \simeq X_j$ (we can cover U_i by elementary opens $Spec\, B_i[f_\alpha^{-1}]$, and as 1 is a linear combination of the $f'_\alpha s$ in $\pi_0(B)$ it must be so in some $\pi_0(B_j)$). This finishes the proof of the lemma. $\square$

Lemma 1.8 *Let $X_i, Y_i \in \mathbf{dSch}^{\leq n}_{A_i}$, and denote by $X_j := X_i \times_{Spec\, A_i} Spec\, A_j$ and $Y_j := Y_i \times_{Spec\, A_i} Spec\, A_j$ for $j \geq i$, and $X := X_i \times_{Spec\, A_i} Spec\, A$, $Y := Y_i \times_{Spec\, A_i} Spec\, A$. Then, the natural morphism*

$$Colim_{j \geq i}\, Map_{\mathbf{dSch}_{A_j}}(X_j, Y_j) \longrightarrow Map_{\mathbf{dSch}_A}(X, Y)$$

is an equivalence.

Proof of the lemma: Let us first assume that $X_i = Spec\, B_i$ is affine. We have $X = Spec\, B$ with $B = Colim\, B_i$. We thus have

$$Map_{\mathbf{dSch}_A}(X, Y) \simeq Map_{\mathbf{dSch}_{A_i}}(X, Y_i) \simeq Colim_{j \geq i}\, Map_{\mathbf{dSch}_{A_i}}(X_j, Y_i) \simeq Colim_{j \geq i}\, Map_{\mathbf{dSch}_{A_j}}(X_j, Y_j)$$

because Y is locally of finite presentation over $Spec\, A$. To pass from the case where X_i is affine to the general case we use the same argument as for the proof of propositions 1.4 and 1.5. The full sub ∞-category of $\mathbf{dSch}^{\leq n}_{A_i}$ for which the lemma is true contains affine and is stable by finite strong colimits. Therefore it contains all quasi-compact and quasi-separated derived schemes. $\square$

To finish proposition 1.6 it remains to prove essential surjectivity.

Lemma 1.9 *Let $f_i : X_i \longrightarrow Y_i$ be a morphism in $\mathbf{dSch}^{\leq n}_{A_i}$. Let $f : X \longrightarrow Y$ be the induced morphism by base change along $Spec\, A \longrightarrow Spec\, A_i$. If f is a Zariski open immersion then there is $j \geq i$ for which*

$$f_j : X_j = X_i \times_{Spec\, A_i} Spec\, A_j \longrightarrow Y_j = X_i \times_{Spec\, A_i} Spec\, A_j$$

is so.

Proof of the lemma: As f is a Zariski open immersion it is an étale monomorphism. Therefore, the cotangent complex $\mathbb{L}_f$ is zero. By compatibility of cotangent complexes by base changes, and by the proposition 1.5 we must have $\mathbb{L}_{f_j}$ for some $j \geq i$. Therefore, there is $j \geq i$ such that f_j is étale. Moreover, as f is a monomorphism the diagonal morphism $X \longrightarrow X \times_Y X$ is an equivalence. By the lemma 1.8 we must have a $j \geq i$ such that $X_j \longrightarrow X_j \times_{Y_j} X_j$ is an equivalence, or in other words that f_j is a monomorphism of derived schemes. Therefore, there is a $j \geq i$ such that f_j is an étale monomorphism and thus an open immersion. $\square$

We finally finish the proof of the proposition 1.6. By lemma 1.7 we know that affine derived schemes in $\mathbf{dSch}_A^{\leq n}$ belongs to the essential image. We first extend this to any quasi-compact quasi-affine derived scheme X (i.e. any quasi-compact open of a derived affine scheme). Indeed, any such object is the image of a finite family of Zariski open affines

$$\{Spec\, B_\alpha \subset Spec\, B\}_\alpha.$$

By the lemmas 1.7 and 1.9, this family is induced by a finite family of opens

$$\{Spec\, B_{i,\alpha} \subset Spec\, B_i\}_\alpha,$$

in $\mathbf{dSch}_{A_i}^{\leq n}$ for some i. The image of this family defines a derived scheme $X_i \in \mathbf{dSch}_{A_i}^{\leq n}$ such that $X_i \times_{Spec\, A_i} Spec\, A \simeq X$.

Finally, we proceed by induction on the number of affines in an open covering. We assume that we have proven that all $X \in \mathbf{dSch}_A^{\leq n}$ covered by k affine opens are in the essential image. If $X \in \mathbf{dSch}_A$ is covered by $(k+1)$ affine opens, we can form a push-out square in $\mathbf{dSch}_A^{\leq n}$

$$\begin{tikzcd} W \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & X, \end{tikzcd}$$

where all morphisms are Zariski open immersions, V is affine and U can be covered by k affine opens. By what we have seen for quasi-affines, by induction and by lemmas 1.7 and 1.9, this push-out square descent to a diagram of open immersions in $\mathbf{dSch}_{A_i}^{\leq n}$ for some i

$$\begin{tikzcd} W_i \arrow[r] \arrow[d] & U_i \\ V_i. & \end{tikzcd}$$

Taking the push-out in $\mathbf{dSch}_{A_i}$ defines a derived scheme $X_i \in \mathbf{dSch}_{A_i}^{\leq n}$ such that $X_i \times_{Spec\, A_i} Spec\, A \simeq X$.
$\square$

2 Proof of the main theorem

We are now ready to prove the main theorem of this work.

Theorem 2.1 *Let $f : X \longrightarrow Y$ be a local complete intersection and proper morphism schemes, then $f_* : L_{qcoh}(X) \longrightarrow L_{qcoh}(Y)$ preserves perfect complexes.*

Proof: The statement is local on the Zariski topology of Y, so we can assume that $Y = Spec\,A$ is affine (note that when doing so X and Y become automatically quasi-compact and separated). We write $A = Colim\,A_i$, a filtered colimit such that A_i is noetherian for all i. As f is lci, $\mathbb{L}_f$ is perfect of amplitude $[-1, 0]$, so X lies in $\mathbf{dSch}_A^{\leq 1}$. By proposition 1.6 there is an index i and an object $X_i \in \mathbf{dSch}_{A_i}^{\leq 1}$ with $X_i \times_{Spec\,A_i} Spec\,A \simeq X$. On the level of truncations we have a fiber product in the category of schemes

$$h^0(X_i) \times_{Spec\,A_i} Spec\,A \simeq h^0(X).$$

We can thus use [EGAIV-3, Thm. 8.10.5] to show that if i is taken big enough the scheme $h^0(X_i)$ is proper over $Spec\,A_i$. Therefore, $X_i \longrightarrow Spec\,A_i$ is a proper lci morphism of derived schemes.

Let $E \in L_{parf}(X)$. By the proposition 1.5 we can chose i big enough so that E descend to $E_i \in L_{parf}(X_i)$. By the proposition 1.4 the theorem will be proven if we can prove that $\mathbf{H}(X_i, E_i)$ is a perfect A_i-dg-module.

Lemma 2.2 *Let $f : X \longrightarrow S = Spec\,A$ be a proper and lci morphism of derived schemes with A a noetherian ring. Then*

$$f_* : L_{qcoh}(X) \longrightarrow L_{qcoh}(S)$$

preserves perfect complexes.

Proof of the lemma: We will first need to recall the following local structure theorem for derived schemes whose cotangent complexes have amplitude in $[-1, 0]$ (also called *quasi-smooth* in the litterature).

Sublemma 2.3 *With the same notation as above the quasi-coherent sheaves $h^i(X)$ are coherent on $h^0(X)$ and only a finite number of them are non-zero.*

Proof of the sublemma: This is a local statement so we can assume that $X = Spec\,B$ with B a retract of a finite cell commutative simplicial A-algebra with $\mathbb{L}_{B/A}$ of amplitude in $[-1, 0]$. As the statement we would like to prove is stable by retracts, we can even assume that B has a finite cell decomposition:

$$B_0 = A \longrightarrow B_{(1)} \longrightarrow B_{(2)} \longrightarrow \dots \longrightarrow B_{(k)} = B.$$

For all i, there is a push-out of commutative simplicial A-algebras

$$\begin{tikzcd} B_{(i)} \arrow[r] & B_{(i+1)} \\ \otimes^{\alpha_{i+1}} A[\partial\Delta^{i+1}] \arrow[r] \arrow[u] & \otimes^{\alpha_{i+1}} A[\Delta^{i+1}], \arrow[u] \end{tikzcd}$$

where $A[K]$ denotes the free commutative simplicial A-algebra generated by a simplicial set K and α_i is the number of i-dimensional cells. Let us consider the morphism $p : B_{(1)} \longrightarrow B$. This morphism induces an isomorphism on π_0 and an epimorphism on π_1. Its (homotopy) fiber is therefore connected. Using a Postnikov decomposition of the morphism p, obstruction theory (see [To-Ve2, Lem. 2.2.1.1]), and the fact that $\mathbb{L}_{B/A}$ as amplitude $[-1,0]$, we see that the morphism p as a section up to homotopy. In other words, the derived scheme $Spec\, B$ is a retract of $Spec\, B_{(1)}$. Moreover, by definition of cell algebras, $Spec\, B_{(1)}$ sits into a cartesian square

$$\begin{tikzcd} Spec\, B_{(1)} \arrow[r] \arrow[d] & \mathbb{A}^{\alpha_0}_A \arrow[d, "u"] \\ \{0\} \arrow[r] & \mathbb{A}^{\alpha_1}_A, \end{tikzcd}$$

where the morphism u is determined by the attaching map

$$\otimes^{\alpha_1} A[\partial\Delta^1] \longrightarrow B_{(0)} = A[X_1, \ldots, X_{\alpha_0}].$$

This shows that $Spec\, B_{(1)}$ satisfies the conclusion of the sublemma, and thus so does $Spec\, B$. This finishes the proof of 2.3 $\square$

The sublemma and the fact that E is perfect implies that the sheaves $H^i(E)$ are easily seen to be coherent and only a finite number of them are non-zero. Therefore, by Grothendieck's finiteness theorem and dévissage we have that $f_*(E)$ is a bounded coherent complex on S. Moreover, for a closed point $s : Spec\, k(s) \longrightarrow S$, the proposition 1.4 implies that $s^*(f_*(E)) \simeq \mathbf{H}(X_s, E_s)$, where X_s is the fiber of f at s and E_s the pull-back of E on X_s. Another application of the sublemma implies that $h^i(X_s)$ is coherent and non-zero only for a finite number of indices i. Therefore, perfect complexes on X_s are also with coherent and bounded cohomology sheaves. As X_s is proper the complex of $k(s)$-vector spaces $\mathbf{H}(X_s, E_s)$ is cohomologically bounded with finite dimensional cohomology. The fibers of the bounded coherent complex $f_*(E)$ at every closed point is thus cohomologically bounded. It is therefore of finite Tor dimension and is thus a perfect complex on S. $\square$

This finishes the proof of theorem 2.1. $\square$

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