

QUASI-STATIONARY DISTRIBUTIONS AND DIFFUSION MODELS IN POPULATION DYNAMICS

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In this paper we study quasi-stationarity for a large class of Kolmogorov diffusions. The main novelty here is that we allow the drift to go to $-\infty$ at the origin, and the diffusion to have an entrance boundary at $+\infty$. These diffusions arise as images, by a deterministic map, of generalized Feller diffusions, which themselves are obtained as limits of rescaled birth–death processes. Generalized Feller diffusions take nonnegative values and are absorbed at zero in finite time with probability 1. An important example is the logistic Feller diffusion.

We give sufficient conditions on the drift near 0 and near $+\infty$ for the existence of quasi-stationary distributions, as well as rate of convergence in the Yaglom limit and existence of the Q -process. We also show that, under these conditions, there is exactly one quasi-stationary distribution, and that this distribution attracts all initial distributions under the conditional evolution, if and only if $+\infty$ is an entrance boundary. In particular, this gives a sufficient condition for the uniqueness of quasi-stationary distributions. In the proofs spectral theory plays an important role on L^2 of the reference measure for the killed process.

1. Introduction. The main motivation of this work is the existence, uniqueness and domain of attraction of quasi-stationary distributions for some diffusion models arising from population dynamics. After a change of variable, the problem is stated in the framework of Kolmogorov diffusion processes with a drift behaving like $-1/2x$ near the origin. Hence, we shall study quasi-stationarity for the larger class of one-dimensional Kolmogorov diffusions (drifted Brownian motions), with drift possibly exploding at the origin.

Consider a one-dimensional drifted Brownian motion on $(0, \infty)$

$$(1.1) \quad dX_t = dB_t - q(X_t) dt, \quad X_0 = x > 0,$$

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where q is defined and C^1 on $(0, \infty)$ and $(B_t; t \geq 0)$ is a standard one-dimensional Brownian motion. In particular, q is allowed to explode at the origin. A pathwise unique solution of (1.1) exists up to the explosion time τ . We denote T_y the first time the process hits $y \in (0, \infty)$ (see [11], Chapter VI, Section 3) before the explosion

$$T_y = \inf\{0 \leq t < \tau : X_t = y\}.$$

We denote by $T_\infty = \lim_{n \rightarrow \infty} T_n$ and $T_0 = \lim_{n \rightarrow \infty} T_{1/n}$. Since q is regular in $(0, \infty)$, then $\tau = T_0 \wedge T_\infty$.

The law of the process starting from X_0 with distribution ν will be denoted by \mathbb{P}_ν . A *quasi-stationary distribution* (in short, *q.s.d.*) for X is a probability measure ν supported on $(0, \infty)$ satisfying for all $t \geq 0$

$$(1.2) \quad \mathbb{P}_\nu(X_t \in A \mid T_0 > t) = \nu(A), \quad \forall \text{Borel set } A \subseteq (0, \infty).$$

By definition, a *q.s.d.* is a fixed point of the conditional evolution. The *Yaglom limit* π is defined as the limit in distribution

$$\pi(\bullet) = \lim_{t \rightarrow \infty} \mathbb{P}_x(X_t \in \bullet \mid T_0 > t),$$

provided this limit exists and is independent of the initial condition x . The Yaglom limit is a *q.s.d.* (see Lemma 7.2).

We will also study the existence of the so-called Q -process which is obtained as the law of the process X conditioned to be never extinct, and it is defined as follows. For any $s \geq 0$ and any Borel set $B \subseteq C([0, s])$ consider

$$\mathbb{Q}_x(X \in B) = \lim_{t \rightarrow \infty} \mathbb{P}_x(X \in B \mid T_0 > t).$$

When it exists, this limit procedure defines the law of a diffusion that never reaches 0.

The reason for studying such diffusion processes with a possibly exploding drift at the origin comes from our interest in the following *generalized Feller diffusion processes*:

$$(1.3) \quad dZ_t = \sqrt{\gamma Z_t} dB_t + h(Z_t) dt, \quad Z_0 = z > 0,$$

where h is a nice function satisfying $h(0) = 0$.

Notice that $z = 0$ is an absorbing state for Z . This means that if $Z_0 = 0$, then $Z_t = 0$ for all t is the unique solution of (1.3) (see [11]).

If we define $X_t = 2\sqrt{Z_t/\gamma}$, then

$$(1.4) \quad dX_t = dB_t - \frac{1}{X_t} \left(\frac{1}{2} - \frac{2}{\gamma} h\left(\frac{\gamma X_t^2}{4}\right) \right) dt, \quad X_0 = x = 2\sqrt{z/\gamma} > 0,$$

so that X is a drifted Brownian motion as in (1.1) where $q(x)$ behaves like $1/2x$ near the origin. The process Z is obtained after rescaling some sequences of birth-death processes arising from population dynamics.

Of particular interest is the case $h(z) = rz - cz^2$ (logistic case), for which we obtain

$$q(x) = \frac{1}{2x} - \frac{rx}{2} + \frac{c\gamma x^3}{8}.$$

A complete description of these models is performed in the final section of the paper, where their biological meaning is also discussed. Of course quasi-stationary distributions for Z and X are related by an immediate change of variables, so that the results on X can be immediately translated to results on Z .

The study of quasi-stationarity is a long standing problem (see [24] for a regularly updated extensive bibliography and [7, 10, 28] for the Markov chain case). For Kolmogorov diffusions, the theory started with Mandl’s paper [22] in 1961, and was then developed by many authors (see, in particular, [3, 23, 30]). All these works assume Mandl’s conditions, which are not satisfied in the situation described above, since, in particular, the drift is not bounded near 0. It is worth noticing that the behavior of q at infinity also may violate Mandl’s conditions, since in the logistic case, for instance,

$$\int_1^\infty e^{-Q(z)} \int_1^z e^{Q(y)} dy dz < \infty,$$

where $Q(y) := 2 \int_1^y q(x) dx$.

This unusual situation prevents us from using earlier results on *q.s.d.*’s of solutions of Kolmogorov equations. Hence, we are led to develop new techniques to cope with this situation. In Section 2 we start with the study of a general Kolmogorov diffusion process on the half line and introduce the hypothesis (H1) that ensures the process to reach 0 in finite time with probability 1. Then we introduce the measure μ , not necessarily finite, defined as

$$\mu(dy) := e^{-Q(y)} dy,$$

which is the speed measure of X . We describe the Girsanov transform and show how to use it in order to obtain $\mathbb{L}^2(\mu)$ estimates for the heat kernel (Theorem 2.3). The key is the following: starting from any $x > 0$, the law of the process at time t is absolutely continuous with respect to μ with a density belonging to $\mathbb{L}^2(\mu)$ (and explicit bounds). In the present paper we work in $\mathbb{L}^p(\mu)$ spaces rather than $\mathbb{L}^p(dx)$, since it greatly simplifies the presentation of the spectral theory.

This spectral theory is done in Section 3, where we introduce the hypothesis (H2):

$$\lim_{x \rightarrow \infty} q^2(x) - q'(x) = \infty, \quad C := - \inf_{x \in (0, \infty)} q^2(x) - q'(x) < \infty.$$

This hypothesis ensures the discreteness of the spectrum (Theorem 3.2). The ground state η_1 (eigenfunction associated to the bottom of the spectrum) can be

chosen nonnegative, even positive as we will see, and furnishes the natural candidate $\eta_1 d\mu$ for a *q.s.d.* The only thing to check is that $\eta_1 \in \mathbb{L}^1(\mu)$, which is not immediate since μ is possibly unbounded.

Section 4 gives some sharper properties of the eigenfunctions defined in the previous section, using, in particular, properties of the Dirichlet heat kernel. More specifically, we introduce two independent hypotheses, (H3) and (H4), either of which ensures that the eigenfunctions belong to $\mathbb{L}^1(\mu)$ (Propositions 4.3 and 4.4). Hypothesis (H3) is

$$\int_0^1 \frac{1}{q^2(y) - q'(y) + C + 2} \mu(dy) < \infty,$$

and (H4) is

$$\int_1^\infty e^{-Q(y)} dy < \infty, \quad \int_0^1 ye^{-Q(y)/2} dy < \infty.$$

Section 5 contains the proofs of the existence of the Yaglom limit (Theorem 5.2) as well as the exponential decay to equilibrium (Proposition 5.5), under hypotheses (H1) and (H2), together with either (H3) or (H4). Section 6 contains the results on the Q -process (Corollaries 6.1 and 6.2).

In Section 7 we introduce condition (H5) which is equivalent to the existence of an entrance law at $+\infty$, that is, the repelling force at infinity imposes to the process starting from infinity to reach any finite interval in finite time. The process is then said to “come down from infinity.” We show that the process comes down from infinity if and only if there exists a unique *q.s.d.* which attracts any initial law under the conditional evolution (Theorem 7.3). In particular, this theorem gives sufficient conditions for uniqueness of *q.s.d.*’s. In the context of birth and death chains, the equivalence between uniqueness of a *q.s.d.* and “come down from infinity” has been proved in [5], Theorem 3.2.

The final section contains the description of the underlying biological models, as well as the application of the whole theory developed in the previous sections to these models (Theorem 8.2).

In the following statement (which is basically Theorem 8.2), we record the main results of this paper in terms of the generalized Feller diffusion solution of (1.3). We say that h satisfies the condition (HH) if

$$(i) \quad \lim_{x \rightarrow \infty} \frac{h(x)}{\sqrt{x}} = -\infty, \quad (ii) \quad \lim_{x \rightarrow \infty} \frac{xh'(x)}{h(x)^2} = 0.$$

THEOREM 1.1. *If h satisfies (HH), then for all initial laws with bounded support, the law of Z_t conditioned on $\{Z_t \neq 0\}$ converges exponentially fast to a probability measure ν , called the Yaglom limit.*

The process Z conditioned to be never extinct is well defined and is called the Q -process. The Q -process converges in distribution to its unique invariant probability measure. This probability measure is absolutely continuous w.r.t. ν with a nondecreasing Radon–Nikodym derivative.

If, in addition, the following integrability condition is satisfied:

$$\int_1^\infty \frac{dx}{-h(x)} < \infty,$$

then Z comes down from infinity and the convergence of the conditional one-dimensional distributions holds for all initial laws, so that the Yaglom limit ν is the unique quasi-stationary distribution.

2. One-dimensional diffusion processes on the positive half line. Associated to q , we consider the functions

$$(2.1) \quad \Lambda(x) = \int_1^x e^{Q(y)} dy \quad \text{and} \quad \kappa(x) = \int_1^x e^{Q(y)} \left(\int_1^y e^{-Q(z)} dz \right) dy,$$

where we recall that $Q(y) = \int_1^y 2q(u) du$. Notice that Λ is the scale function for X .

For most of the results in this paper we shall assume sure absorption at zero, that is,

HYPOTESIS (H1).

$$(2.2) \quad \text{For all } x > 0, \quad \mathbb{P}_x(\tau = T_0 < T_\infty) = 1.$$

It is well known (see, e.g., [11], Chapter VI, Theorem 3.2) that (2.2) holds if and only if

$$(2.3) \quad \Lambda(\infty) = \infty \quad \text{and} \quad \kappa(0^+) < \infty.$$

We notice that (H1) can be written as $\mathbb{P}_x(\lim_{t \rightarrow \infty} X_{t \wedge \tau} = 0) = 1$.

EXAMPLE 2.1. The main cases that we are interested in are the following ones:

- (1) When X is defined by (1.4) associated to the generalized Feller diffusion Z . It is direct to show that $Q(x)$ behaves like $\log(x)$ near 0, hence, $\kappa(0^+) < \infty$. The logistic Feller diffusion corresponds to $h(z) = rz - cz^2$ for some constants c and r . It is easily seen that (2.3) is satisfied in this case provided $c > 0$ or $c = 0$ and $r < 0$.
- (2) When the drift is bounded near 0, in which case $\kappa(0^+) < \infty$.

We shall now discuss some properties of the law of X up to T_0 . The first result is a Girsanov type result.

PROPOSITION 2.2. Assume (H1). For any Borel bounded function $F : C([0, t], (0, \infty)) \rightarrow \mathbb{R}$ it holds

$$\begin{aligned} & \mathbb{E}_x[F(X)\mathbb{1}_{t < T_0}] \\ &= \mathbb{E}^{\mathbb{W}_x} \left[F(\omega)\mathbb{1}_{t < T_0(\omega)} \exp \left(\frac{1}{2}Q(x) - \frac{1}{2}Q(\omega_t) - \frac{1}{2} \int_0^t (q^2 - q')(\omega_s) ds \right) \right], \end{aligned}$$

where \mathbb{E}^{W_x} denotes the expectation w.r.t. the Wiener measure starting from x , and \mathbb{E}_x denotes the expectation with respect to the law of X starting also from x .

PROOF. It is enough to show the result for F nonnegative and bounded. Let $x > 0$ and consider $\varepsilon \in (0, 1)$ such that $\varepsilon \leq x \leq 1/\varepsilon$. Also we define $\tau_\varepsilon = T_\varepsilon \wedge T_{1/\varepsilon}$. Choose some ψ_ε which is a nonnegative C^∞ function with compact support, included in $] \varepsilon/2, 2/\varepsilon[$ such that $\psi_\varepsilon(u) = 1$ if $\varepsilon \leq u \leq 1/\varepsilon$. The law of the diffusion (1.1) coincides up to τ_ε with the law of a similar diffusion process X^ε obtained by replacing q with the cutoff $q_\varepsilon = q\psi_\varepsilon$. For the latter we may apply the Novikov criterion ensuring that the law of X^ε is given via the Girsanov formula. Hence,

$$\begin{aligned} & \mathbb{E}_x[F(X)\mathbb{1}_{t < \tau_\varepsilon}] \\ &= \mathbb{E}^{W_x} \left[F(\omega)\mathbb{1}_{t < \tau_\varepsilon(\omega)} \exp \left(\int_0^t -q_\varepsilon(\omega_s) d\omega_s - \frac{1}{2} \int_0^t (q_\varepsilon)^2(\omega_s) ds \right) \right] \\ &= \mathbb{E}^{W_x} \left[F(\omega)\mathbb{1}_{t < \tau_\varepsilon(\omega)} \exp \left(\int_0^t -q(\omega_s) d\omega_s - \frac{1}{2} \int_0^t q^2(\omega_s) ds \right) \right] \\ &= \mathbb{E}^{W_x} \left[F(\omega)\mathbb{1}_{t < \tau_\varepsilon(\omega)} \exp \left(\frac{1}{2} Q(x) - \frac{1}{2} Q(\omega_t) - \frac{1}{2} \int_0^t (q^2 - q')(\omega_s) ds \right) \right]. \end{aligned}$$

The last equality is obtained integrating by parts the stochastic integral. But $\mathbb{1}_{t < \tau_\varepsilon}$ is nondecreasing in ε and converges almost surely to $\mathbb{1}_{t < T_0}$ both for \mathbb{P}_x [thanks to (H1)] and \mathbb{W}_x . It remains to use the Lebesgue monotone convergence theorem to finish the proof. \square

The next theorem is inspired by the calculation in Theorem 3.2.7 of [27]. It will be useful to introduce the following measure defined on $(0, \infty)$:

$$(2.4) \quad \mu(dy) := e^{-Q(y)} dy.$$

Note that μ is not necessarily finite.

THEOREM 2.3. Assume (H1). For all $x > 0$ and all $t > 0$ there exists some density $r(t, x, \cdot)$ that satisfies

$$\mathbb{E}_x[f(X_t)\mathbb{1}_{t < T_0}] = \int_0^\infty f(y)r(t, x, y)\mu(dy)$$

for all bounded Borel f .

If, in addition, there exists some $C > 0$ such that $q^2(y) - q'(y) \geq -C$ for all $y > 0$, then for all $t > 0$ and all $x > 0$,

$$\int_0^\infty r^2(t, x, y)\mu(dy) \leq (1/2\pi t)^{1/2} e^{Ct} e^{Q(x)}.$$

PROOF. Define

$$G(\omega) = \mathbb{1}_{t < T_0(\omega)} \exp\left(\frac{1}{2}Q(\omega_0) - \frac{1}{2}Q(\omega_t) - \frac{1}{2} \int_0^t (q^2 - q')(\omega_s) ds\right).$$

Denote by

$$e^{-v(t,x,y)} = (2\pi t)^{-1/2} \exp\left(-\frac{(x-y)^2}{2t}\right)$$

the density at time t of the Brownian motion starting from x . According to Proposition 2.2, we have

$$\begin{aligned} \mathbb{E}_x[f(X_t)\mathbb{1}_{t < T_0}] &= \mathbb{E}^{\mathbb{W}_x}[f(\omega_t)\mathbb{E}^{\mathbb{W}_x}[G|\omega_t]] \\ &= \int f(y)\mathbb{E}^{\mathbb{W}_x}[G|\omega_t = y]e^{-v(t,x,y)} dy \\ &= \int_0^\infty f(y)\mathbb{E}^{\mathbb{W}_x}[G|\omega_t = y]e^{-v(t,x,y)+Q(y)} \mu(dy), \end{aligned}$$

because $\mathbb{E}^{\mathbb{W}_x}[G|\omega_t = y] = 0$ if $y \leq 0$. In other words, the law of X_t restricted to nonextinction has a density with respect to μ given by

$$r(t, x, y) = \mathbb{E}^{\mathbb{W}_x}[G|\omega_t = y]e^{-v(t,x,y)+Q(y)}.$$

Hence,

$$\begin{aligned} &\int_0^\infty r^2(t, x, y)\mu(dy) \\ &= \int (\mathbb{E}^{\mathbb{W}_x}[G|\omega_t = y]e^{-v(t,x,y)+Q(y)})^2 e^{-Q(y)+v(t,x,y)} e^{-v(t,x,y)} dy \\ &= \mathbb{E}^{\mathbb{W}_x}[e^{-v(t,x,\omega_t)+Q(\omega_t)} (\mathbb{E}^{\mathbb{W}_x}[G|\omega_t])^2] \\ &\leq \mathbb{E}^{\mathbb{W}_x}[e^{-v(t,x,\omega_t)+Q(\omega_t)} \mathbb{E}^{\mathbb{W}_x}[G^2|\omega_t]] \\ &\leq e^{Q(x)} \mathbb{E}^{\mathbb{W}_x}[\mathbb{1}_{t < T_0(\omega)} e^{-v(t,x,\omega_t)} e^{-\int_0^t (q^2 - q')(\omega_s) ds}], \end{aligned}$$

where we have used Cauchy–Schwarz’s inequality. Since $e^{-v(t,x,\cdot)} \leq (1/2\pi t)^{1/2}$, the proof is complete. \square

REMARK 2.4. It is interesting to briefly discuss the conditions we have introduced:

- (1) Since q is assumed to be regular, the condition $q^2 - q'$ bounded from below has to be checked only near infinity or near 0.
- (2) Consider the behavior near infinity. Let us show that if $\liminf_{y \rightarrow \infty} (q^2(y) - q'(y)) = -\infty$, then $\limsup_{y \rightarrow \infty} (q^2(y) - q'(y)) > -\infty$, that is, the drift q is strongly oscillating. Indeed, assume that $q^2(y) - q'(y) \rightarrow -\infty$ as $y \rightarrow \infty$. It follows that $q'(y) \rightarrow \infty$, hence, $q(y) \rightarrow \infty$. For y large enough we may

thus write $q(y) = e^{u(y)}$ for some u going to infinity at infinity. So $e^{2u(y)}(1 - u'(y)e^{-u(y)}) \rightarrow -\infty$, implying that $u'e^{-u} \geq 1$ near infinity. Thus, if $g = e^{-u}$, we have $g' \leq -1$, that is, $g(y) \rightarrow -\infty$ as $y \rightarrow \infty$, which is impossible since g is nonnegative.

(3) If X is given by (1.4), we have

$$q(y) = \frac{1}{y} \left(\frac{1}{2} - \frac{2}{\gamma} h \left(\frac{\gamma y^2}{4} \right) \right).$$

Hence, since h is of class C^1 and $h(0) = 0$, $q^2(y) - q'(y)$ behaves near 0 like $\frac{3}{4y^2}$ so that $q^2 - q'$ is bounded from below near 0 (see the Appendix for further conditions fulfilled by h to get the same result near ∞).

3. \mathbb{L}^2 and spectral theory of the diffusion process. Theorem 2.3 shows that for a large family of initial laws, the distribution of X_t before extinction has a density belonging to $\mathbb{L}^2(\mu)$. The measure μ is natural since the kernel of the killed process is symmetric in $\mathbb{L}^2(\mu)$, which allows us to use spectral theory.

Let $C_0^\infty((0, \infty))$ be the vector space of infinitely differentiable functions on $(0, \infty)$ with compact support. We denote

$$\langle f, g \rangle_\mu = \int_0^\infty f(u)g(u)\mu(du).$$

Consider the symmetric form

$$(3.1) \quad \mathcal{E}(f, g) = \langle f', g' \rangle_\mu, \quad D(\mathcal{E}) = C_0^\infty((0, \infty)).$$

This form is Markovian and closable. The proof of the latter assertion is similar to the one of Theorem 2.1.4 in [8] just replacing the real line by the positive half line. Its smallest closed extension, again denoted by \mathcal{E} , is thus a Dirichlet form which is actually regular and local. According to the theory of Dirichlet forms (see [8] or [9]), we thus know the following:

- there exists a nonpositive self adjoint operator L on $\mathbb{L}^2(\mu)$ with domain $D(L) \supseteq C_0^\infty((0, \infty))$ such that for all f and g in $C_0^\infty((0, \infty))$ the following holds (see [8], Theorem 1.3.1):

$$(3.2) \quad \mathcal{E}(f, g) = -2 \int_0^\infty f(u)Lg(u)\mu(du) = -2\langle f, Lg \rangle_\mu.$$

We point out that, for $g \in C_0^\infty((0, \infty))$,

$$Lg = \frac{1}{2}g'' - qg'.$$

- L is the generator of a strongly continuous symmetric semigroup of contractions on $\mathbb{L}^2(\mu)$ denoted by $(P_t)_{t \geq 0}$. This semigroup is (sub)-Markovian, that is, $0 \leq P_t f \leq 1 \mu$ a.e. if $0 \leq f \leq 1$ (see [8], Theorem 1.4.1).

- There exists a unique μ -symmetric Hunt process with continuous sample paths (i.e., a diffusion process) up to its explosion time τ whose Dirichlet form is \mathcal{E} (see [8], Theorem 6.2.2).

The last assertion implies that, for μ quasi all $x > 0$ (that is, except for a set of zero capacity, see [8] for details), one can find a probability measure \mathbb{Q}_x on $C(\mathbb{R}^+, (0, \infty))$ such that, for all $f \in C_0^\infty((0, \infty))$,

$$f(\omega_{t \wedge \tau}) - f(x) - \int_0^{t \wedge \tau} Lf(\omega_s) ds$$

is a local martingale with quadratic variation $\int_0^{t \wedge \tau} |f'|^2(\omega_s) ds$. Due to our hypothesis $q \in C^1(0, \infty)$, we know that this martingale problem admits a unique solution (see, e.g., [12], page 444). On the other hand, using Itô's formula, we know that, under \mathbb{P}_x , the law of $(X_{t \wedge \tau})$ is also a solution to this martingale problem.

The conclusion is that the semigroup P_t and the semigroup induced by the strong Markov process $(X_{t \wedge \tau})$ coincide on the set of smooth and compactly supported functions. Therefore, for all $f \in \mathbb{L}^2(\mu)$ we have that

$$P_t f(x) = \mathbb{E}[f(X_t) \mathbb{1}_{t < \tau}].$$

Let $(E_\lambda : \lambda \geq 0)$ be the spectral family of $-L$. We can restrict ourselves to the case $\lambda \geq 0$ because $-L$ is nonnegative. Then $\forall t \geq 0, f, g \in \mathbb{L}^2(\mu)$,

$$(3.3) \quad \int P_t f g d\mu = \int_0^\infty e^{-\lambda t} d\langle E_\lambda f, g \rangle_\mu.$$

We notice that if absorption is sure, that is, (H1) holds, this semigroup coincides with the semigroup of X killed at 0, that is, $P_t f(x) = \mathbb{E}[f(X_t^x) \mathbb{1}_{t < T_0}]$.

Note that for $f \in \mathbb{L}^2(\mu)$ and all closed interval $K \subset (0, \infty)$,

$$\begin{aligned} \int (P_t f)^2 d\mu &= \int (P_t(f \mathbb{1}_K + f \mathbb{1}_{K^c}))^2 d\mu \\ &\leq 2 \int (P_t(f \mathbb{1}_K))^2 d\mu + 2 \int (P_t(f \mathbb{1}_{K^c}))^2 d\mu \\ &\leq 2 \int (P_t(f \mathbb{1}_K))^2 d\mu + 2 \int (f \mathbb{1}_{K^c})^2 d\mu. \end{aligned}$$

We may choose K large enough in order that the second term in the latter sum is bounded by ε . Similarly, we may approximate $f \mathbb{1}_K$ in $\mathbb{L}^2(\mu)$ by $\tilde{f} \mathbb{1}_K$ for some continuous and bounded \tilde{f} , up to ε (uniformly in t). Now, thanks to (H1), we know that $P_t(\tilde{f} \mathbb{1}_K)(x)$ goes to 0 as t goes to infinity for any x . Since

$$\int (P_t(\tilde{f} \mathbb{1}_K))^2 d\mu = \int_K \tilde{f} P_{2t}(\tilde{f} \mathbb{1}_K) d\mu,$$

we may apply the Lebesgue bounded convergence theorem and conclude that $\int (P_t(\tilde{f} \mathbb{1}_K))^2 d\mu \rightarrow 0$ as $t \rightarrow \infty$. Hence, we have shown that

$$(3.4) \quad \forall f \in \mathbb{L}^2(\mu) \quad \int (P_t f)^2 d\mu \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Now we shall introduce the main assumption on q for the spectral aspect of the study.

HYPOTESIS (H2).

$$(3.5) \quad C = - \inf_{y \in (0, \infty)} q^2(y) - q'(y) < \infty \quad \text{and} \quad \lim_{y \rightarrow \infty} q^2(y) - q'(y) = +\infty.$$

PROPOSITION 3.1. *Under (H2), $|q(x)|$ tends to infinity as $x \rightarrow \infty$, and $q^-(x)$ or $q^+(x)$ tend to 0 as $x \downarrow 0$. If, in addition, (H1) holds, then $q(x) \rightarrow \infty$, as $x \rightarrow \infty$.*

PROOF. Since $q^2 - q'$ tends to ∞ as $x \rightarrow \infty$, q does not change sign for large x . If q is bounded near infinity, we arrive to a contradiction because q' tends to $-\infty$ and, therefore, q tends to $-\infty$ as well. So q is unbounded. If $\liminf_{x \rightarrow \infty} |q(x)| = a < \infty$, then we can construct a sequence $x_n \rightarrow \infty$ of local maxima, or local minima of q whose value $|q(x_n)| < a + 1$, but then $q^2(x_n) - q'(x_n)$ stays bounded, which is a contradiction.

Now we prove that $q^-(x)$ or $q^+(x)$ tend to 0 as $x \downarrow 0$. In fact, assume there exist an $\epsilon > 0$ and a sequence (x_n) with $0 < x_n \downarrow 0$ such that $q(x_{2n}) = -\epsilon, q(x_{2n+1}) = \epsilon$. Then we can construct another sequence $z_n \downarrow 0$ such that $|q(z_n)| \leq \epsilon$ and $q'(z_n) \rightarrow \infty$, contradicting (H2).

Finally, assume (H1) holds. If $q(x) \leq -1$ for all $x > x_0$, we arrive to a contradiction. Indeed, for all t

$$\mathbb{P}_{x_0+1}(T_0 > t) \geq \mathbb{P}_{x_0+1}(T_{x_0} > t) \geq \mathbb{P}_{x_0+1}(T_{x_0} = \infty).$$

The assumption $q(x) \leq -1$ implies that $X_t \geq B_t + t$ while $t \leq T_{x_0}$ and, therefore,

$$\mathbb{P}_{x_0+1}(T_0 > t) \geq \mathbb{P}_{x_0+1}(B_t + t \text{ hits } \infty \text{ before } x_0) = \frac{e^{-2(x_0+1)} - e^{2x_0}}{e^{-2\infty} - e^{2x_0}} = 1 - e^{-2},$$

where we have used that $(\exp(-2(B_t + t)))$ is a martingale. This contradicts (H1) and we have $q(x) \rightarrow \infty$ as $x \rightarrow \infty$. \square

We may now state the following result.

THEOREM 3.2. *If (H2) is satisfied, $-L$ has a purely discrete spectrum $0 \leq \lambda_1 < \lambda_2 < \dots$. Furthermore, each λ_i ($i \in \mathbb{N}$) is associated to a unique (up to a multiplicative constant) eigenfunction η_i of class $C^2((0, \infty))$, which also satisfies the o.d.e.*

$$(3.6) \quad \frac{1}{2}\eta_i'' - q\eta_i' = -\lambda_i\eta_i.$$

The sequence $(\eta_i)_{i \geq 1}$ is an orthonormal basis of $\mathbb{L}^2(\mu)$; η_1 can be chosen to be strictly positive in $(0, \infty)$.

For $g \in \mathbb{L}^2(\mu)$,

$$P_t g = \sum_{i \in \mathbb{N}} e^{-\lambda_i t} \langle \eta_i, g \rangle_\mu \eta_i \quad \text{in } \mathbb{L}^2(\mu),$$

then for $f, g \in \mathbb{L}^2(\mu)$,

$$\lim_{t \rightarrow \infty} e^{\lambda_1 t} (g, P_t f)_\mu = \langle \eta_1, f \rangle_\mu \langle \eta_1, g \rangle_\mu.$$

If, in addition, (H1) holds, then $\lambda_1 > 0$.

PROOF. For $f \in \mathbb{L}^2(dx)$, define $\tilde{P}_t(f) = e^{-Q/2} P_t(f e^{Q/2})$, which exists in $\mathbb{L}^2(dx)$ since $f e^{Q/2} \in \mathbb{L}^2(\mu)$. $(\tilde{P}_t)_{t \geq 0}$ is then a strongly continuous semigroup in $\mathbb{L}^2(dx)$, whose generator \tilde{L} coincides on $C_0^\infty((0, \infty))$ with $\frac{1}{2} \frac{d^2}{dx^2} - \frac{1}{2}(q^2 - q')$ since $C_0^2(0, \infty) \subset D(L)$, and $e^{Q/2} \in C^2(0, \infty)$. The spectral theory of such a Schrödinger operator on the line (or the half line) is well known, but here the potential $v = (q^2 - q')/2$ does not necessarily belong to \mathbb{L}_{loc}^∞ near 0 as it is generally assumed. We shall use [2], Chapter 2.

First we follow the proof of Theorem 3.1 in [2]. Since we have assumed that v is bounded from below by $-C/2$, we may consider $H = \tilde{L} - (C/2 + 1)$, that is, replace v by $v + C/2 + 1 = w \geq 1$, hence translate the spectrum. Since for $f \in C_0^\infty(0, \infty)$

$$\begin{aligned} -(Hf, f) &:= - \int_0^\infty Hf(u) f(u) du = \int_0^\infty (|f'(u)|^2/2 + w(u) f^2(u)) du \\ (3.7) \quad &\geq \int_0^\infty f^2(u) du, \end{aligned}$$

H has a bounded inverse operator. Hence, the spectrum of H (and then the one of \tilde{L}) will be discrete as soon as H^{-1} is a compact operator, that is, as soon as $M = \{f \in D(H); -(Hf, f) \leq 1\}$ is relatively compact. This is shown in [2] when w is locally bounded, in particular, bounded near 0. If w goes to infinity at 0, the situation is even better since our set M is included into the corresponding one with $w \approx 1$ near the origin, which is relatively compact thanks to the asymptotic behavior of v . The conclusion of Theorem 3.1 in [2] is thus still true in our situation, that is, the spectrum is discrete.

The discussion in Section 2.3 of [2], pages 59–69, is only concerned with the asymptotic behavior (near infinity) of the solutions of $f'' - 2wf = 0$. Nevertheless, the results there apply to our case. All eigenvalues of \tilde{L} are thus simple (Proposition 3.3 in [2]), and of course the corresponding set of normalized eigenfunctions $(\psi_k)_{k \geq 1}$ is an orthonormal basis of $\mathbb{L}^2(dx)$.

The system $(e^{Q/2} \psi_k)_{k \geq 1}$ is thus an orthonormal basis of $\mathbb{L}^2(\mu)$, each $\eta_k = e^{Q/2} \psi_k$ being an eigenfunction of L . We can choose them to be $C^2((0, \infty))$ and they satisfy (3.6).

For every $t > 0$, and for every $g, f \in \mathbb{L}^2(\mu)$, we have

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \langle \eta_k, g \rangle_{\mu} \langle \eta_k, f \rangle_{\mu} = \langle g, P_t f \rangle_{\mu}.$$

In addition, if g and f are nonnegative, we get

$$0 \leq \lim_{t \rightarrow \infty} e^{\lambda_1 t} \langle g, P_t f \rangle_{\mu} = \langle \eta_1, f \rangle_{\mu} \langle \eta_1, g \rangle_{\mu},$$

since $\lambda_1 < \lambda_2 \leq \dots$ and the sum $\sum_{k=1}^{\infty} |\langle \eta_k, g \rangle_{\mu} \langle \eta_k, f \rangle_{\mu}|$ is finite. It follows that $\langle \eta_1, f \rangle_{\mu}$ and $\langle \eta_1, g \rangle_{\mu}$ have the same sign. Changing η_1 into $-\eta_1$ if necessary, we may assume that $\langle \eta_1, f \rangle_{\mu} \geq 0$ for any nonnegative f , hence, $\eta_1 \geq 0$. Since $P_t \eta_1(x) = e^{-\lambda_1 t} \eta_1(x)$ and η_1 is continuous and not trivial, we deduce that $\eta_1(x) > 0$ for all $x > 0$.

Since L is nonpositive, $\lambda_1 \geq 0$. Now assume that (H1) holds. Using (3.4), we get for $g \in \mathbb{L}^2(\mu)$

$$0 = \lim_{t \rightarrow \infty} \langle P_t g, P_t g \rangle_{\mu} = \lim_{t \rightarrow \infty} e^{-2\lambda_1 t} \langle g, \eta_1 \rangle_{\mu}^2,$$

showing that $\lambda_1 > 0$. \square

Moreover, we are able to obtain a pointwise representation of the density r .

PROPOSITION 3.3. *Under (H1) and (H2) we have*

$$(3.8) \quad r(t, x, y) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \eta_k(x) \eta_k(y),$$

uniformly on compact sets of $(0, \infty) \times (0, \infty) \times (0, \infty)$.

Therefore, on compact sets of $(0, \infty) \times (0, \infty)$ we get

$$(3.9) \quad \lim_{t \rightarrow \infty} e^{\lambda_1 t} r(t, x, y) = \eta_1(x) \eta_1(y).$$

PROOF. Using Theorems 2.3 and 3.2, for every smooth function g compactly supported on $(0, \infty)$, we have

$$\begin{aligned} \sum_{k=1}^n e^{-\lambda_k t} \langle \eta_k, g \rangle_{\mu}^2 &\leq \sum_{k=1}^{\infty} e^{-\lambda_k t} \langle \eta_k, g \rangle_{\mu}^2 \\ &= \iint g(x) g(y) r(t, x, y) e^{-Q(x)-Q(y)} dx dy. \end{aligned}$$

Then using the regularity of η_k and r , we obtain, by letting $g(y) dy$ tend to the Dirac measure in x , that

$$\sum_{k=1}^n e^{-\lambda_k t} \eta_k(x)^2 \leq r(t, x, x).$$

Thus, the series $\sum_{k=1}^{\infty} e^{-\lambda_k t} \eta_k(x)^2$ converges pointwise, which by the Cauchy–Schwarz inequality implies the pointwise absolute convergence of $\zeta(t, x, y) := \sum_{k=1}^{\infty} e^{-\lambda_k t} \eta_k(x) \eta_k(y)$ and the bound for all n

$$\sum_{k=1}^n e^{-\lambda_k t} |\eta_k(x) \eta_k(y)| \leq \sqrt{r(t, x, x)} \sqrt{r(t, y, y)}.$$

Using the Harnack inequality (see, e.g., [17]), we get

$$\sqrt{r(t, x, x)} \sqrt{r(t, y, y)} \leq C_K r(t, x, y)$$

for any x and y in the compact subset K of $(0, \infty)$. Using the dominated convergence theorem, we obtain that for all Borel functions g, f with compact support in $(0, \infty)$

$$\begin{aligned} \int \int g(x) f(y) \zeta(t, x, y) e^{-Q(x)-Q(y)} dx dy \\ = \int \int g(x) f(y) r(t, x, y) e^{-Q(x)-Q(y)} dx dy. \end{aligned}$$

Therefore, $\zeta(t, x, y) = r(t, x, y)$ $dx dy$ -a.s., which proves the almost sure version of (3.8).

Since η_k are smooth eigenfunctions, we get the pointwise equality

$$\begin{aligned} e^{-\lambda_k t} \eta_k(x)^2 &= e^{-\lambda_k t/3} \langle r(t/3, x, \bullet), \eta_k \rangle_{\mu} \langle r(t/3, x, \bullet), \eta_k \rangle_{\mu} \\ &= \int \int r(t/3, x, y) r(t/3, x, z) e^{-\lambda_k t/3} \eta_k(y) \eta_k(z) e^{-Q(z)-Q(y)} dy dz, \end{aligned}$$

which together with the fact $r(t/3, x, \bullet) \in \mathbb{L}^2(\mu)$ and Theorem 2.3 allow us to deduce

$$\begin{aligned} \sum_{k=1}^{\infty} e^{-\lambda_k t} \eta_k(x)^2 \\ = \int \int r(t/3, x, y) r(t/3, x, z) \sum_{k=1}^{\infty} e^{-\lambda_k t/3} \eta_k(y) \eta_k(z) e^{-Q(z)-Q(y)} dy dz \\ = \int \int r(t/3, x, y) r(t/3, x, z) r(t/3, y, z) e^{-Q(z)-Q(y)} dy dz = r(t, x, x). \end{aligned}$$

Dini’s theorem then proves the uniform convergence in compacts of $(0, \infty)$ for the series

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \eta_k(x)^2 = r(t, x, x).$$

By the Cauchy–Schwarz inequality, we have for any n

$$\left| \sum_{k=n}^{\infty} e^{-\lambda_k t} \eta_k(x) \eta_k(y) \right| \leq \left(\sum_{k=n}^{\infty} e^{-\lambda_k t} \eta_k(x)^2 \right)^{1/2} \left(\sum_{k=n}^{\infty} e^{-\lambda_k t} \eta_k(y)^2 \right)^{1/2}.$$

This together with the dominated convergence theorem yields (3.9). \square

In the previous theorem, notice that $\sum_k e^{-\lambda_k t} = \int r(t, x, x) e^{-Q(x)} dx$ is the $\mathbb{L}^1(\mu)$ norm of $x \mapsto r(t, x, x)$. This is finite if and only if P_t is a trace-class operator on $\mathbb{L}^2(\mu)$.

4. Properties of the eigenfunctions. In this section we study some properties of the eigenfunctions η_i , including their integrability with respect to μ .

PROPOSITION 4.1. *Assume that (H1) and (H2) are satisfied. Then $\int_1^\infty \eta_1 \times e^{-Q} dx < \infty$, $F(x) = \eta_1'(x)e^{-Q(x)}$ is a nonnegative decreasing function and the following limits exist:*

$$F(0^+) = \lim_{x \downarrow 0} \eta_1'(x)e^{-Q(x)} \in (0, \infty], \quad F(\infty) = \lim_{x \rightarrow \infty} \eta_1'(x)e^{-Q(x)} \in [0, \infty).$$

Moreover,

$$\int_0^\infty \eta_1(x)e^{-Q(x)} dx = \frac{F(0^+) - F(\infty)}{2\lambda_1}.$$

In particular,

$$\eta_1 \in \mathbb{L}^1(\mu) \quad \text{if and only if} \quad F(0^+) < \infty.$$

The function η_1 is increasing and $\int_1^\infty e^{-Q(y)} dy < \infty$.

REMARK 4.2. Note that $g = \eta_1 e^{-Q}$ satisfies the adjoint equation $\frac{1}{2}g'' + (qg)' = -\lambda_1 g$, and then $F(x) = g'(x) + 2q(x)g(x)$ represents the flux at x . Then $\eta_1 \in \mathbb{L}^1(\mu)$ or, equivalently, $g \in \mathbb{L}^1(dx)$ if and only if the flux at 0 is finite.

PROOF. Since η_1 satisfies $\eta_1''(x) - 2q\eta_1'(x) = -2\lambda_1\eta_1(x)$, we obtain for x_0 and x in $(0, \infty)$

$$(4.1) \quad \eta_1'(x)e^{-Q(x)} = \eta_1'(x_0)e^{-Q(x_0)} - 2\lambda_1 \int_{x_0}^x \eta_1(y)e^{-Q(y)} dy,$$

and $F = \eta_1'e^{-Q}$ is decreasing. Integrating further gives

$$\eta_1(x) = \eta_1(x_0) + \int_{x_0}^x \left(\eta_1'(x_0)e^{-Q(x_0)} - 2\lambda_1 \int_{x_0}^z \eta_1(y)e^{-Q(y)} dy \right) e^{Q(z)} dz.$$

If for some $z_0 > x_0$ it holds that $\eta_1'(x_0)e^{-Q(x_0)} - 2\lambda_1 \int_{x_0}^{z_0} \eta_1(y)e^{-Q(y)} dy < 0$, then this inequality holds for all $z > z_0$ since the quantity

$$\eta_1'(x_0)e^{-Q(x_0)} - 2\lambda_1 \int_{x_0}^z \eta_1(y)e^{-Q(y)} dy$$

is decreasing in z . This implies that for large x the function η_1 is negative, because $e^{Q(z)}$ tends to ∞ as $z \rightarrow \infty$. This is a contradiction and we deduce that for all $x > 0$

$$2\lambda_1 \int_x^\infty \eta_1(y)e^{-Q(y)} dy \leq \eta_1'(x)e^{-Q(x)}.$$

This implies that η_1 is increasing and, being nonnegative, it is bounded near 0. In particular, $\eta_1(0^+)$ exists. Also, we deduce that $F \geq 0$ and that $\int_1^\infty e^{-Q(y)} dy < \infty$. We can take the limit as $x \rightarrow \infty$ in (4.1) to get

$$F(\infty) = \lim_{x \rightarrow \infty} \eta_1'(x)e^{-Q(x)} \in [0, \infty),$$

and $\eta_1'(x_0)e^{-Q(x_0)} = F(\infty) + 2\lambda_1 \int_{x_0}^\infty \eta_1(y)e^{-Q(y)} dy$. From this equality the result follows. \square

In the next results we give some sufficient conditions, in terms of q , for the integrability of the eigenfunctions. A first useful condition is the following one:

HYPOTESIS (H3).

$$\int_0^1 \frac{1}{q^2(y) - q'(y) + C + 2} e^{-Q(y)} dy < \infty,$$

where, as before, $C = -\inf_{x>0}(q^2(x) - q'(x))$.

PROPOSITION 4.3. *Assume that (H1)–(H3) are satisfied. Then η_i belongs to $\mathbb{L}^1(\mu)$ for all i .*

PROOF. Recall that $\psi_i = e^{-Q/2}\eta_i$ is an eigenfunction of the Schrödinger operator H introduced in the proof of Theorem 3.2. Replacing f by ψ_i in (3.7) thus yields

$$(C/2 + 1 + \lambda_i) \int_0^\infty \psi_i^2(y) dy = \int_0^\infty (|\psi_i'|^2(y)/2 + w(y)\psi_i^2(y)) dy.$$

Since the left-hand side is finite, the right-hand side is finite, in particular,

$$\int_0^\infty w(y)\eta_i^2(y)\mu(dy) = \int_0^\infty w(y)\psi_i^2(y) dy < \infty.$$

As a consequence, using the Cauchy–Schwarz inequality, we get, on one hand,

$$\int_0^1 |\eta_i(y)|\mu(dy) \leq \left(\int_0^1 w(y)\eta_i^2(y)\mu(dy) \right)^{1/2} \left(\int_0^1 \frac{1}{w(y)}\mu(dy) \right)^{1/2} < \infty,$$

thanks to (H3). On the other hand,

$$\int_1^\infty |\eta_i(y)|\mu(dy) \leq \left(\int_1^\infty \eta_i^2(y)\mu(dy) \right)^{1/2} \left(\int_1^\infty \mu(dy) \right)^{1/2} < \infty,$$

according to Proposition 4.1. We have thus proved that $\eta_i \in \mathbb{L}^1(\mu)$. \square

We now obtain sharper estimates using properties of the Dirichlet heat kernel. For this reason we introduce the following:

HYPOTESIS (H4).

$$\int_1^\infty e^{-Q(x)} dx < \infty \quad \text{and} \quad \int_0^1 x e^{-Q(x)/2} dx < \infty.$$

PROPOSITION 4.4. *Assume (H2) and (H4) hold. Then all eigenfunctions η_k belong to $\mathbb{L}^1(\mu)$, and there is a constant $K_1 > 0$ such that for any $x \in (0, \infty)$ and any k*

$$|\eta_k(x)| \leq K_1 e^{\lambda_k} e^{Q(x)/2}.$$

Moreover, η_1 is strictly positive on \mathbb{R}^+ , and there is a constant $K_2 > 0$ such that for any $x \in (0, 1]$ and any k

$$|\eta_k(x)| \leq K_2 x e^{2\lambda_k} e^{Q(x)/2}.$$

PROOF. In Section 3 we introduced the semigroup \tilde{P}_t associated with the Schrödinger equation and showed that $\eta_k = e^{Q/2} \psi_k$, where ψ_k is the unique eigenfunction related to the eigenvalue λ_k for \tilde{P}_t . Using estimates on this semigroup, we will get some properties of ψ_k , and we will prove the proposition.

The semigroup \tilde{P}_t is given for $f \in \mathbb{L}^2(\mathbb{R}^+, dx)$ by

$$\tilde{P}_t f(x) = \mathbb{E}^{\mathbb{W}_x} \left[f(\omega(t)) \mathbb{1}_{t < T_0} \exp\left(-\frac{1}{2} \int_0^t (q^2 - q')(\omega_s) ds\right) \right],$$

where $\mathbb{E}^{\mathbb{W}_x}$ denotes the expectation w.r.t. the Wiener measure starting from x . We first establish a basic estimate on its kernel $\tilde{p}_t(x, y)$.

LEMMA 4.5. *Assume condition (H2) holds. There exists a constant $K_3 > 0$ and a continuous increasing function B defined on $[0, \infty)$ satisfying $\lim_{z \rightarrow \infty} B(z) = \infty$, such that for any $x > 0, y > 0$ we have*

$$(4.2) \quad 0 < \tilde{p}_1(x, y) \leq e^{-(x-y)^2/4} e^{-B(\max\{x, y\})}$$

and

$$(4.3) \quad \tilde{p}_1(x, y) \leq K_3 p_1^D(x, y),$$

where p_t^D is the Dirichlet heat kernel in \mathbb{R}^+ given for $x, y \in \mathbb{R}^+$ by

$$p_t^D(x, y) = \frac{1}{\sqrt{2\pi t}} (e^{-(x-y)^2/(2t)} - e^{-(x+y)^2/(2t)}).$$

The proof of this lemma is postponed to the [Appendix](#).

It follows immediately from the previous lemma that the kernel $\tilde{p}_1(x, y)$ defines a bounded operator \tilde{P}_1 from $\mathbb{L}^2(\mathbb{R}^+, dx)$ to $\mathbb{L}^\infty(\mathbb{R}^+, dx)$. As a byproduct, we get that all eigenfunctions ψ_k of \tilde{P}_1 are bounded and, more precisely,

$$|\psi_k| \leq K_1 e^{\lambda_k}.$$

One also deduces from the previous lemma that the kernel defined for $M > 0$ by

$$\tilde{p}_1^M(x, y) = \mathbb{1}_{x < M} \mathbb{1}_{y < M} \tilde{p}_1(x, y)$$

is a Hilbert–Schmidt operator in $\mathbb{L}^2(\mathbb{R}^+, dx)$, in particular, is a compact operator (see, e.g., [4], pages 177, 267). In addition, it follows at once again from Lemma 4.5 that if \tilde{P}_1^M denotes the operator with kernel \tilde{p}_1^M , we have the following estimate, in the norm of operators acting on $\mathbb{L}^2(\mathbb{R}^+, dx)$:

$$\|\tilde{P}_1^M - \tilde{P}_1\|_{\mathbb{L}^2(\mathbb{R}^+, dx)} \leq C' e^{-B(M)},$$

where C' is a positive constant independent of M . Since $\lim_{M \rightarrow \infty} B(M) = \infty$, the operator \tilde{P}_1 is a limit in norm of compact operators in $\mathbb{L}^2(\mathbb{R}^+, dx)$ and hence compact. Since $\tilde{p}_1(x, y) > 0$, the operator \tilde{P}_1 is positivity improving (that is, if $0 \neq f \geq 0$, then $\tilde{P}_1 f > 0$), implying that the eigenvector ψ_1 is positive.

We now claim that $|\psi_k(x)| \leq K_2 x e^{2\lambda_k}$ for $0 < x \leq 1$. We have from Lemma 4.5 and the explicit expression for $p_1^D(x, y)$ the existence of a constant K_3 such that

$$\begin{aligned} |e^{-\lambda_k} \psi_k(x)| &\leq K_3 \int_0^\infty p_1^D(x, y) |\psi_k(y)| dy \\ &\leq K_3 \|\psi_k\|_\infty \sqrt{\frac{2}{\pi}} e^{-x^2/2} \int_0^\infty e^{-y^2/2} \sinh(xy) dy. \end{aligned}$$

We now estimate the integral in the right-hand side. Using the convexity property of \sinh , we get $\sinh(xy) \leq x \sinh(y) \leq \frac{x}{2} e^y$, for $x \in [0, 1]$, $y \geq 0$, which yields

$$\int_0^\infty e^{-y^2/2} \sinh(xy) dy \leq \frac{x}{2} \int_0^\infty e^{-y^2/2} e^y dy,$$

proving the claim. Together with hypothesis (H4), this estimate implies that η_k belongs to $\mathbb{L}^1((0, 1), d\mu)$.

Since

$$\eta_k(x) = \psi_k(x) e^{Q(x)/2},$$

we have

$$\int_1^\infty \eta_k d\mu = \int_1^\infty \psi_k(x) e^{-Q(x)/2} dx,$$

which implies $\eta_k \in \mathbb{L}^1((1, \infty), d\mu)$ using Cauchy–Schwarz’s inequality. This finishes the proof of Proposition 4.4. \square

REMARK 4.6. Let us discuss some easy facts about the hypotheses introduced:

- (1) If q and q' extend continuously up to 0, hypotheses (H2)–(H4) reduce to their counterpart at infinity.
- (2) Consider $q(x) = \frac{a}{x} + g(x)$ with g a C^1 function up to 0. In order that (2.3) holds at the origin, we need $a > -\frac{1}{2}$. Then $\mu(dx) = \Theta(x)x^{-2a} dx$ with Θ bounded near the origin, while $q^2(x) - q'(x) \approx (a + a^2)/x^2$. Hence, for (H2) to hold, we need $a \geq 0$. Now we have the estimates

$$\int_0^\varepsilon \frac{1}{(q^2(x) - q'(x) + C + 2)} \mu(dx) \approx \int_0^\varepsilon \Theta(x)x^{2(1-a)} dx,$$

and

$$\int_0^\varepsilon x e^{-Q(x)/2} dx \approx \int_0^\varepsilon \Theta(x)x^{1-a} dx.$$

Therefore, (H3) holds for $a < \frac{3}{2}$ and (H4) holds (at 0) for $a < 2$. The conclusion is that $a \in [0, \frac{3}{2})$.

We recall that $a = \frac{1}{2}$ if X comes from a generalized Feller diffusion.

- (3) If $q(x) \geq 0$ for x large, hypothesis (H2) implies the first part of hypothesis (H4). Indeed, take $a > 0$ to be such that for any $x \geq a$ we have $q(x) > 0$ and $q^2(x) - q'(x) > 1$. Consider the function $y = e^{-Q/2}$ which satisfies $y' = -qy$ and $y'' = (q^2 - q')y$. For $b > a$ we get after integration by parts

$$\begin{aligned} 0 &= \int_a^b ((q^2 - q')y^2 - yy'') dx \\ &= \int_a^b ((q^2 - q')y^2 + y'^2) dx - y(b)y'(b) + y(a)y'(a). \end{aligned}$$

Using $y' = -qy$, we obtain

$$\begin{aligned} \int_a^b y^2 dx &\leq \int_a^b ((q^2 - q')y^2 + y'^2) dx \\ &= q(a)y(a)^2 - q(b)y(b)^2 \leq q(a)y(a)^2 < \infty \end{aligned}$$

and the result follows by letting b tend to infinity.

5. Quasi-stationary distribution and Yaglom limit. Existence of the Yaglom limit and of $q.s.d.$ for killed one-dimensional diffusion processes have already been proved by various authors, following the pioneering work by Mandl [22] (see, e.g., [3, 23, 30] and the references therein). One of the main assumptions in these papers is $\kappa(\infty) = \infty$ and

$$\int_1^\infty e^{-Q(y)} \left(\int_1^y e^{Q(z)} dz \right) dy = +\infty,$$

which is not necessarily satisfied in our case. Indeed, under mild conditions, the Laplace method yields that $\int_1^y e^{Q(z)} dz$ behaves like $e^{Q(y)}/2q(y)$ when y tends to infinity, so the above equality will not hold if q grows too fast to infinity at infinity. Actually, we will be particularly interested in these cases [our forthcoming assumption (H5)], since they are exactly those when the diffusion “comes down from infinity,” which ensures uniqueness of the *q.s.d.* The second assumption in the aforementioned papers is that q is C^1 up to the origin which is not true in our case of interest.

It is useful to introduce the following condition:

DEFINITION 5.1. We say that *hypothesis* (H) is satisfied if (H1) and (H2) hold, and, moreover, $\eta_1 \in \mathbb{L}^1(\mu)$ [which is the case, e.g., under (H3) or (H4)].

We now study the existence of *q.s.d.* and the Yaglom limit in our framework. When $\eta_1 \in \mathbb{L}^1(\mu)$, a natural candidate for being a *q.s.d.* is the normalized measure $\eta_1\mu/\langle\eta_1, 1\rangle_\mu$, which turns out to be the conditional limit distribution.

THEOREM 5.2. Assume that *hypothesis* (H) holds. Then

$$dv_1 = \frac{\eta_1 d\mu}{\langle\eta_1, 1\rangle_\mu}$$

is a quasi-stationary distribution, that is for every $t \geq 0$ and any Borel subset A of $(0, \infty)$,

$$\mathbb{P}_{v_1}(X_t \in A \mid T_0 > t) = v_1(A).$$

Also for any $x > 0$ and any Borel subset A of $(0, \infty)$,

$$(5.1) \quad \begin{aligned} \lim_{t \rightarrow \infty} e^{\lambda_1 t} \mathbb{P}_x(T_0 > t) &= \eta_1(x) \langle\eta_1, 1\rangle_\mu, \\ \lim_{t \rightarrow \infty} e^{\lambda_1 t} \mathbb{P}_x(X_t \in A, T_0 > t) &= v_1(A) \eta_1(x) \langle\eta_1, 1\rangle_\mu. \end{aligned}$$

This implies, since $\eta_1 > 0$ on $(0, \infty)$,

$$\lim_{t \rightarrow \infty} \mathbb{P}_x(X_t \in A \mid T_0 > t) = v_1(A),$$

and the probability measure v_1 is the Yaglom limit distribution. Moreover, for any probability measure ρ with compact support in $(0, \infty)$, we have

$$(5.2) \quad \lim_{t \rightarrow \infty} e^{\lambda_1 t} \mathbb{P}_\rho(T_0 > t) = \langle\eta_1, 1\rangle_\mu \int \eta_1(x) \rho(dx);$$

$$(5.3) \quad \lim_{t \rightarrow \infty} e^{\lambda_1 t} \mathbb{P}_\rho(X_t \in A, T_0 > t) = v_1(A) \langle\eta_1, 1\rangle_\mu \int \eta_1(x) \rho(dx);$$

$$(5.4) \quad \lim_{t \rightarrow \infty} \mathbb{P}_\rho(X_t \in A \mid T_0 > t) = v_1(A).$$

PROOF. Thanks to the symmetry of the semigroup, we have for all f in $\mathbb{L}^2(\mu)$,

$$\int P_t f \eta_1 d\mu = \int f P_t \eta_1 d\mu = e^{-\lambda t} \int f \eta_1 d\mu.$$

Since $\eta_1 \in \mathbb{L}^1(\mu)$, this equality extends to all bounded f . In particular, we may use it with $f = \mathbb{1}_{(0,\infty)}$ and with $f = \mathbb{1}_A$. Noticing that

$$\int P_t(\mathbb{1}_{(0,\infty)})\eta_1 d\mu = \mathbb{P}_{v_1}(T_0 > t)\langle \eta_1, 1 \rangle_\mu$$

and $\int P_t \mathbb{1}_A \eta_1 d\mu = \mathbb{P}_{v_1}(X_t \in A, T_0 > t)\langle \eta_1, 1 \rangle_\mu$, we have shown that v_1 is a *q.s.d.*

The rest of the proof is divided into two cases. First assume that μ is a bounded measure. Thanks to Theorem 2.3, we know that, for any $x > 0$, any set $A \subset (0, \infty)$ such that $\mathbb{1}_A \in \mathbb{L}^2(\mu)$ and for any $t > 1$,

$$\begin{aligned} \mathbb{P}_x(X_t \in A, T_0 > t) &= \int \mathbb{P}_y(X_{t-1} \in A, T_0 > t-1) r(1, x, y) \mu(dy) \\ &= \int P_{t-1}(\mathbb{1}_A)(y) r(1, x, y) \mu(dy) \\ &= \int \mathbb{1}_A(y) (P_{t-1} r(1, x, \cdot))(y) \mu(dy). \end{aligned}$$

Since both $\mathbb{1}_A$ and $r(1, x, \cdot)$ are in $\mathbb{L}^2(\mu)$ and since (H2) is satisfied, we obtain, using Theorem 3.2,

$$(5.5) \quad \lim_{t \rightarrow \infty} e^{\lambda(t-1)} \mathbb{P}_x(X_t \in A, T_0 > t) = \langle \mathbb{1}_A, \eta_1 \rangle_\mu \langle r(1, x, \cdot), \eta_1 \rangle_\mu.$$

Since

$$\int r(1, x, y) \eta_1(y) \mu(dy) = (P_1 \eta_1)(x) = e^{-\lambda} \eta_1(x),$$

we get that v_1 is the Yaglom limit.

If μ is not bounded [i.e., $\mathbb{1}_{(0,\infty)} \notin \mathbb{L}^2(\mu)$], we need an additional result to obtain the Yaglom limit.

LEMMA 5.3. *Assuming $\eta_1 \in \mathbb{L}^1(\mu)$, then for all $x > 0$, there exists a locally bounded function $\Theta(x)$ such that, for all $y > 0$ and all $t > 1$,*

$$(5.6) \quad r(t, x, y) \leq \Theta(x) e^{-\lambda t} \eta_1(y).$$

We postpone the proof of the lemma and indicate how it is used to conclude the proof of the theorem.

If (5.6) holds, for $t > 1$, $e^{\lambda t} r(t, x, \cdot) \in \mathbb{L}^1(\mu)$ and is dominated by $\Theta(x) \eta_1$. Since $r(1, x, \cdot) \in \mathbb{L}^2(\mu)$ by Theorem 2.3, using Theorem 3.2 and writing again $r(t, x, \cdot) = P_{t-1} r(1, x, \cdot)$ μ a.s., we deduce that $\lim_{t \rightarrow \infty} e^{\lambda t} r(t, x, \cdot)$ exists in $\mathbb{L}^2(\mu)$ and is equal to

$$e^{\lambda} \langle r(1, x, \cdot), \eta_1 \rangle_\mu \eta_1(\cdot) = \eta_1(x) \eta_1(\cdot).$$

Recall that convergence in \mathbb{L}^2 implies almost sure convergence along subsequences. Therefore, for any sequence $t_n \rightarrow \infty$ there exists a subsequence t'_n such that

$$\lim_{n \rightarrow \infty} e^{\lambda_1 t'_n} r(t'_n, x, y) = \eta_1(x)\eta_1(y) \quad \text{for } \mu\text{-almost all } y > 0.$$

Since

$$\mathbb{P}_x(T_0 > t'_n) = \int_0^\infty r(t'_n, x, y)\mu(dy),$$

the Lebesgue bounded convergence theorem yields

$$\lim_{n \rightarrow \infty} e^{\lambda_1 t'_n} \mathbb{P}_x(T_0 > t'_n) = \eta_1(x) \int_0^\infty \eta_1(y)\mu(dy).$$

That is, (5.5) holds with $A = (0, \infty)$ for the sequence t'_n . Since the limit does not depend on the subsequence, $\lim_{t \rightarrow \infty} e^{\lambda_1 t} \mathbb{P}_x(T_0 > t)$ exists and is equal to the previous limit, hence, (5.5) is still true. The rest of this part follows as before.

For the last part of the theorem, that is, passing from the initial Dirac measures at every fixed $x > 0$ to the compactly supported case, we just use that $\Theta(\bullet)$ is bounded on compact sets included in $(0, \infty)$.

PROOF OF LEMMA 5.3. According to the parabolic Harnack’s inequality (see, for example, [31]), for all $x > 0$, one can find $\Theta_0(x) > 0$, which is locally bounded, such that, for all $t > 1$, $y > 0$ and z with $|z - x| \leq \rho(x) = \frac{1}{2} \wedge \frac{x}{4}$,

$$r(t, x, y) \leq \Theta_0(x)r(t + 1, z, y).$$

It follows that

$$\begin{aligned} r(t, x, y) &= \frac{(\int_{|z-x| \leq \rho(x)} r(t, x, y)\eta_1(z)\mu(dz))}{(\int_{|z-x| \leq \rho(x)} \eta_1(z)\mu(dz))} \\ &\leq \Theta_0(x) \frac{(\int_{|z-x| \leq \rho(x)} r(t + 1, z, y)\eta_1(z)\mu(dz))}{(\int_{|z-x| \leq \rho(x)} \eta_1(z)\mu(dz))} \\ &\leq \Theta_0(x) \frac{(\int r(t + 1, z, y)\eta_1(z)\mu(dz))}{(\int_{|z-x| \leq \rho(x)} \eta_1(z)\mu(dz))} \\ &\leq \Theta_0(x) \frac{e^{-\lambda_1(t+1)}\eta_1(y)}{(\int_{|z-x| \leq \rho(x)} \eta_1(z)\mu(dz))}, \end{aligned}$$

since $P_{t+1}\eta_1 = e^{-\lambda_1(t+1)}\eta_1$. But $\Theta_1(x) = \int_{|z-x| \leq \rho(x)} \eta_1(z)\mu(dz) > 0$, otherwise η_1 , which is a solution of the linear o.d.e. $\frac{1}{2}g'' - qg' + \lambda_1g = 0$ on $(0, \infty)$, would vanish on the whole interval $|z - x| \leq \rho(x)$, hence on $(0, \infty)$ according to the uniqueness theorem for linear o.d.e’s. The proof of the lemma is thus completed with $\Theta = e^{-\lambda_1}\Theta_0/\Theta_1$. \square

The positive real number λ_1 is the natural killing rate of the process. Indeed, the limit (5.1) obtained in Theorem 5.2 shows that, for any $x > 0$ and any $t > 0$,

$$\lim_{s \rightarrow \infty} \frac{\mathbb{P}_x(T_0 > t + s)}{\mathbb{P}_x(T_0 > s)} = e^{-\lambda_1 t}.$$

Let us also remark that

$$\mathbb{P}_{\nu_1}(T_0 > t) = e^{-\lambda_1 t}.$$

In order to control the speed of convergence to the Yaglom limit, we first establish the following lemma.

LEMMA 5.4. *Under conditions (H2) and (H4), the operator P_1 is bounded from $\mathbb{L}^\infty(\mu)$ to $\mathbb{L}^2(\mu)$. Moreover, for any compact subset K of $(0, \infty)$, there is a constant C_K such that, for any function $f \in \mathbb{L}^1(\mu)$ with support in K , we have*

$$\|P_1 f\|_{\mathbb{L}^2(\mu)} \leq C_K \|f\|_{\mathbb{L}^1(\mu)}.$$

PROOF. Letting $g \in \mathbb{L}^\infty(\mu)$, since

$$|P_1 g| \leq P_1 |g| \leq \|g\|_{\mathbb{L}^\infty(\mu)},$$

we get from (H4)

$$\int_1^\infty |P_1 g|^2 d\mu \leq \|g\|_{\mathbb{L}^\infty(\mu)}^2 \int_1^\infty e^{-Q(x)} dx.$$

We now recall that (see Section 3)

$$P_1 g(x) = e^{Q(x)/2} \tilde{P}_1(e^{-Q/2} g)(x).$$

It follows from Lemma 4.5 that uniformly in $x \in (0, 1]$ we have [using hypothesis (H4)]

$$|\tilde{P}_1(e^{-Q/2} g)(x)| \leq \mathcal{O}(1) \|g\|_{\mathbb{L}^\infty(\mu)} \int_0^\infty e^{-Q(y)/2} e^{-y^2/4} y dy \leq \mathcal{O}(1) \|g\|_{\mathbb{L}^\infty(\mu)}.$$

This implies

$$\int_0^1 |P_1 g|^2 d\mu = \int_0^1 |\tilde{P}_1(e^{-Q/2} g)(x)|^2 dx \leq \mathcal{O}(1) \|g\|_{\mathbb{L}^\infty(\mu)}^2,$$

and the first part of the lemma follows. For the second part, we have from the Gaussian bound of Lemma 4.5 that, for any $x > 0$ and for any f integrable and with support in K ,

$$\begin{aligned} |\tilde{P}_1(e^{-Q/2} f)(x)| &\leq \mathcal{O}(1) \int_K e^{-Q(y)/2} e^{-(x-y)^2/2} |f(y)| dy \\ &\leq \mathcal{O}(1) \sup_{z \in K} e^{Q(z)/2} \sup_{z \in K} e^{-(x-z)^2/2} \int_K e^{-Q(y)} |f(y)| dy \\ &\leq \mathcal{O}(1) e^{-x^2/4} \int_K e^{-Q(y)} |f(y)| dy \end{aligned}$$

since K is compact. This implies

$$\int_0^\infty |P_1 f|^2 d\mu = \int_0^\infty |\tilde{P}_1(e^{-Q/2} f)(x)|^2 dx \leq \mathcal{O}(1) \|f\|_{\mathbb{L}^1(\mu)}^2. \quad \square$$

We can now use the spectral decomposition of $r(1, x, \cdot)$ to obtain the following convergence result.

PROPOSITION 5.5. *Under conditions (H2) and (H4), for all $x > 0$ and any measurable subset A of $(0, \infty)$, we have*

$$(5.7) \quad \begin{aligned} & \lim_{t \rightarrow \infty} e^{(\lambda_2 - \lambda_1)t} (\mathbb{P}_x(X_t \in A \mid T_0 > t) - \nu_1(A)) \\ &= \frac{\eta_2(x)}{\eta_1(x)} \left(\frac{\langle 1, \eta_1 \rangle_\mu \langle \mathbb{1}_A, \eta_2 \rangle_\mu - \langle 1, \eta_2 \rangle_\mu \langle \mathbb{1}_A, \eta_1 \rangle_\mu}{\langle 1, \eta_1 \rangle_\mu^2} \right). \end{aligned}$$

PROOF. Let h be a nonnegative bounded function, with compact support in $(0, \infty)$. By using the semigroup property, Lemma 5.4 and the spectral decomposition for compact self adjoint semigroups (see Theorem 3.2), we have for any $t > 2$,

$$\begin{aligned} & \int \mathbb{P}_x(X_t \in A, T_0 > t) h(x) dx \\ &= \langle h e^Q, P_t \mathbb{1}_A \rangle_\mu = \langle P_1(h e^Q), P_{(t-2)} P_1 \mathbb{1}_A \rangle_\mu \\ &= \langle P_1(h e^Q), \eta_1 \rangle_\mu \langle \eta_1, P_1 \mathbb{1}_A \rangle_\mu e^{-\lambda_1(t-2)} \\ & \quad + \langle P_1(h e^Q), \eta_2 \rangle_\mu \langle \eta_2, P_1 \mathbb{1}_A \rangle_\mu e^{-\lambda_2(t-2)} + R(h, A, t) \end{aligned}$$

with

$$\begin{aligned} |R(h, A, t)| &\leq \sum_{i \geq 3} e^{-\lambda_i(t-2)} |\langle P_1(h e^Q), \eta_i \rangle_\mu \langle \eta_i, P_1 \mathbb{1}_A \rangle_\mu| \\ &\leq e^{-\lambda_3(t-2)} \|P_1(h e^Q)\|_{\mathbb{L}^2(\mu)} \|P_1 \mathbb{1}_A\|_{\mathbb{L}^2(\mu)}, \end{aligned}$$

due to $\lambda_1 < \lambda_2 < \lambda_3 < \dots$, the Cauchy–Schwarz inequality and Parseval’s identity. Note that since P_1 is symmetric with respect to the scalar product, we have $\langle P_1(h e^Q), \eta_1 \rangle_\mu = e^{-\lambda_1} \langle h e^Q, \eta_1 \rangle_\mu$ and similarly for η_2 . We also have $\langle \eta_1, P_1 \mathbb{1}_A \rangle_\mu = e^{-\lambda_1} \langle \eta_1, \mathbb{1}_A \rangle_\mu$ and similarly for η_2 . It follows immediately from Lemma 5.4 that, for any fixed compact subset K of $(0, \infty)$, any A and any h satisfying the hypothesis of the proposition with support contained in K ,

$$|R(h, A, t)| \leq \mathcal{O}(1) e^{-\lambda_3(t-2)} \|h e^Q\|_{\mathbb{L}^1(\mu)} \leq \mathcal{O}(1) e^{-\lambda_3(t-2)} \|h\|_{\mathbb{L}^1(dx)},$$

since h has compact support in $(0, \infty)$. Therefore, letting h tend to a Dirac mass, we obtain that, for any compact subset K of $(0, \infty)$, there is a constant D_K such

that, for any $x \in K$, for any measurable subset A of $(0, \infty)$, and for any $t > 2$, we have

$$\begin{aligned} & |\mathbb{P}_x(X_t \in A, T_0 > t) - e^{Q(x)}\eta_1(x)\langle \eta_1, \mathbb{1}_A \rangle_\mu e^{-\lambda_1 t} - e^{Q(x)}\eta_2(x)\langle \eta_2, \mathbb{1}_A \rangle_\mu e^{-\lambda_2 t}| \\ & \leq D_K e^{-\lambda_3 t}. \end{aligned}$$

The proposition follows at once from

$$\mathbb{P}_x(X_t \in A \mid T_0 > t) = \frac{\mathbb{P}_x(X_t \in A, T_0 > t)}{\mathbb{P}_x(X_t \in (0, \infty), T_0 > t)}. \quad \square$$

6. The Q -process. As in [3] (Theorem B), we can also describe the law of the process conditioned to be never extinct, usually called the Q -process (also see [19]).

COROLLARY 6.1. *Assume (H) holds. For all $x > 0$ and $s \geq 0$ we have*

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{P}_x(X \in B \mid T_0 > t) &= \mathbb{Q}_x(B) \\ &\text{for all } B \text{ Borel measurable subsets of } C([0, s]), \end{aligned}$$

where \mathbb{Q}_x is the law of a diffusion process on $(0, \infty)$, with transition probability densities (w.r.t. the Lebesgue measure) given by

$$q(s, x, y) = e^{\lambda_1 s} \frac{\eta_1(y)}{\eta_1(x)} r(s, x, y) e^{-Q(y)},$$

that is, \mathbb{Q}_x is locally absolutely continuous w.r.t. \mathbb{P}_x and

$$\mathbb{Q}_x(X \in B) = \mathbb{E}_x \left(\mathbb{1}_B(X) e^{\lambda_1 s} \frac{\eta_1(X_s)}{\eta_1(x)}, T_0 > s \right).$$

PROOF. First check, thanks to Fubini’s theorem and $\kappa(0^+) < \infty$ in hypothesis (H1), that $\Lambda(0^+) > -\infty$. We can thus slightly change the notation (for this proof only) and define Λ as $\Lambda(x) = \int_0^x e^{Q(y)} dy$. From standard diffusion theory, $(\Lambda(X_{t \wedge T_0}); t \geq 0)$ is a local martingale, from which it is easy to derive that, for any $y \geq x \geq 0$, $\mathbb{P}_y(T_x < T_0) = \Lambda(y)/\Lambda(x)$.

Now define $v(t, x) = \frac{\mathbb{P}_x(T_0 > t)}{\mathbb{P}_1(T_0 > t)}$. As in [3], proof of Theorem B, one can prove for any $x \geq 1$, using the strong Markov property at T_x of the diffusion X starting from 1, that $v(t, x) \leq \Lambda(x)/\Lambda(1)$. On the other hand, for $x \leq 1$ we obtain $v(t, x) \leq 1$. Then for any $x \geq 0$ we get $v(t, x) \leq 1 + \Lambda(x)/\Lambda(1)$.

Now, thanks to Theorem 5.2, for all x , $e^{\lambda_1 t} \mathbb{P}_x(T_0 > t) \rightarrow \eta_1(x)\langle 1, \eta_1 \rangle_\mu$ as $t \rightarrow \infty$, and

$$\lim_{t \rightarrow \infty} v(t, x) = \frac{\eta_1(x)}{\eta_1(1)}.$$

Using the Markov property, it is easily seen that, for large t ,

$$\mathbb{P}_x(X \in B \mid T_0 > t) = \mathbb{E}_x[\mathbb{1}_B(X)v(t - s, X_s), T_0 > s] \frac{\mathbb{P}_1(T_0 > t - s)}{\mathbb{P}_x(T_0 > t)}.$$

The random variable in the expectation is (positive and) bounded from above by $1 + \Lambda(X_s)/\Lambda(1)$, which is integrable (see below), so we obtain the desired result using the Lebesgue bounded convergence theorem.

To see that $\mathbb{E}_x(\Lambda(X_s)\mathbb{1}_{s < T_0})$ is finite, it is enough to use Itô's formula with the harmonic function Λ up to time $T_0 \wedge T_M$. Since Λ is nonnegative, it easily yields $\mathbb{E}_x(\Lambda(X_s)\mathbb{1}_{s < T_0 \wedge T_M}) \leq \Lambda(x)$ for all $M > 0$. Letting M go to infinity, the indicator converges almost surely to $\mathbb{1}_{s < T_0}$ [thanks to hypothesis (H1)], so the monotone convergence theorem yields $\mathbb{E}_x(\Lambda(X_s)\mathbb{1}_{s < T_0}) \leq \Lambda(x)$. \square

Recall that ν_1 is the Yaglom limit.

COROLLARY 6.2. *Assume (H) holds. Then for any Borel subset $B \subseteq (0, \infty)$ and any x ,*

$$\lim_{s \rightarrow \infty} \mathbb{Q}_x(X_s \in B) = \int_B \eta_1^2(y)\mu(dy) = \langle \eta_1, 1 \rangle_\mu \int_B \eta_1(y)\nu_1(dy).$$

PROOF. We know from the proof of Theorem 5.2 that $e^{\lambda_1 s}r(s, x, \cdot)$ converges to $\eta_1(x)\eta_1(\cdot)$ in $\mathbb{L}^2(\mu)$ as $s \rightarrow \infty$. Hence, since $\mathbb{1}_B\eta_1 \in \mathbb{L}^2(\mu)$,

$$\eta_1(x)\mathbb{Q}_x(X_s \in B) = \int \mathbb{1}_B(y)\eta_1(y)e^{\lambda_1 s}r(s, x, y)\mu(dy) \rightarrow \eta_1(x) \int_B \eta_1^2(y)\mu(dy)$$

as $s \rightarrow \infty$. We remind the reader that $d\nu_1 = \eta_1 d\mu / \langle \eta_1, 1 \rangle_\mu$. \square

REMARK 6.3. The stationary measure of the Q -process is absolutely continuous w.r.t. ν_1 , with Radon–Nikodym derivative $\langle \eta_1, 1 \rangle_\mu \eta_1$, which thanks to Proposition 4.1 is *nondecreasing*. In particular, the ergodic measure of the Q -process dominates stochastically the Yaglom limit. We refer to [19, 23] for further discussion of the relationship between *q.s.d.* and the ergodic measure of the Q -process.

7. Infinity is an entrance boundary and uniqueness of *q.s.d.* We start with the notion of quasi limiting distribution *q.l.d.*

DEFINITION 7.1. A probability measure π supported on $(0, \infty)$ is a *q.l.d.* if there exists a probability measure ν such that the following limit exists in distribution:

$$\lim_{t \rightarrow \infty} \mathbb{P}_\nu(X_t \in \bullet \mid T_0 > t) = \pi(\bullet).$$

We also say that ν is attracted to π , or is in the domain of attraction of π , for the conditional evolution.

Obviously every *q.s.d.* is a *q.l.d.*, because such measures are fixed points for the conditional evolution. We prove that the reciprocal is also true, so both concepts coincide.

LEMMA 7.2. *Let π be a probability measure supported on $(0, \infty)$. If π is a *q.l.d.*, then π is a *q.s.d.* In particular, there exists $\alpha \geq 0$ such that, for all $s > 0$,*

$$\mathbb{P}_\pi(T_0 > s) = e^{-\alpha s}.$$

PROOF. By hypothesis, there exists a probability measure ν such that $\lim_{t \rightarrow \infty} \mathbb{P}_\nu(X_t \in \bullet \mid T_0 > t) = \pi(\bullet)$, in distribution. That is, for all continuous and bounded functions f we have

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}_\nu(f(X_t), T_0 > t)}{\mathbb{P}_\nu(T_0 > t)} = \int f(x)\pi(dx).$$

If we take $f(x) = \mathbb{P}_x(X_s \in A, T_0 > s)$, since $f(x) = \int_A r(t, x, y)\mu(dy)$, an application of Harnack’s inequality and of the dominated convergence theorem ensures that f is continuous in $(0, \infty)$.

First, take $A = (0, \infty)$, so that $f(x) = \mathbb{P}_x(T_0 > s)$. Then, we obtain for all $s \geq 0$

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}_\nu(T_0 > t + s)}{\mathbb{P}_\nu(T_0 > t)} = \mathbb{P}_\pi(T_0 > s).$$

The left-hand side is easily seen to be exponential in s and then there exists $\alpha \geq 0$ such that

$$\mathbb{P}_\pi(T_0 > s) = e^{-\alpha s}.$$

Second, take $f(x) = \mathbb{P}_x(X_s \in A, T_0 > s)$ to conclude that

$$\begin{aligned} \mathbb{P}_\pi(X_s \in A, T_0 > s) &= \lim_{t \rightarrow \infty} \mathbb{P}_\nu(f(X_t) \mid T_0 > t) \\ &= \lim_{t \rightarrow \infty} \mathbb{P}_\nu(X_{t+s} \in A \mid T_0 > t + s) \frac{\mathbb{P}_\nu(T_0 > t + s)}{\mathbb{P}_\nu(T_0 > t)} \\ &= e^{-\alpha s} \pi(A), \end{aligned}$$

and then π is a *q.s.d.* \square

Recall from Theorem 5.2 that, under hypothesis (H), the measure $d\nu_1 = \eta_1 d\mu / \langle \eta_1, 1 \rangle_\mu$ is the Yaglom limit, which, in addition, is a *q.l.d.* attracting all initial distribution with compact support on $(0, \infty)$. It is natural to ask about the uniqueness of the *q.s.d.* Here again, our assumptions on the behavior of q at infinity will allow us to characterize the domain of attraction of the *q.s.d.* ν_1 associated to η_1 . This turns out to be entirely different from the cases studied in [3], for instance.

We say that the diffusion process X comes down from infinity if there is $y > 0$ and a time $t > 0$ such that

$$\lim_{x \uparrow \infty} \mathbb{P}_x(T_y < t) > 0.$$

This terminology is equivalent to the property that ∞ is an entrance boundary for X (for instance, see [26], page 283).

Let us introduce the following condition:

HYPOTESIS (H5).

$$\int_1^\infty e^{Q(y)} \int_y^\infty e^{-Q(z)} dz dy < \infty.$$

Tonelli’s theorem ensures that (H5) is equivalent to

$$(7.1) \quad \int_1^\infty e^{-Q(y)} \int_1^y e^{Q(z)} dz dy < \infty.$$

If (H5) holds, then for $y \geq 1$, $\int_y^\infty e^{-Q(z)} dz < \infty$. Applying the Cauchy–Schwarz inequality, we get $(x - 1)^2 = (\int_1^x e^{Q/2} e^{-Q/2} dz)^2 \leq \int_1^x e^Q dz \int_1^x e^{-Q} dz$ and, therefore, (H5) implies that $\Lambda(\infty) = \infty$.

Now we state the main result of this section.

THEOREM 7.3. *Assume (H) holds. Then the following are equivalent:*

- (i) X comes down from infinity;
- (ii) (H5) holds;
- (iii) ν_1 attracts all initial distributions ν supported in $(0, \infty)$, that is,

$$\lim_{t \rightarrow \infty} \mathbb{P}_\nu(X_t \in \bullet \mid T_0 > t) = \nu_1(\bullet).$$

In particular, any of these three conditions implies that there is a unique q.s.d.

REMARK 7.4. It is not obvious when condition (H5) holds. In this direction, the following explicit conditions on q , all together, are sufficient for (H5) to hold:

- $q(x) \geq q_0 > 0$ for all $x \geq x_0$,
- $\limsup_{x \rightarrow \infty} q'(x)/2q^2(x) < 1$,
- $\int_{x_0}^\infty \frac{1}{q(x)} dx < \infty$.

Indeed, check first that these conditions imply that $q(x)$ goes to infinity as $x \rightarrow \infty$. Then defining $s(y) := \int_y^\infty e^{-Q(z)} dz$, the first condition above implies that $s(y)e^{Q(y)}$ is bounded in $y \geq x_0$. Integrating by parts on $\int s e^Q dz$ gives

$$\int_{x_0}^x s e^Q dz = \int_{x_0}^x \frac{s}{2q} e^Q 2q dz = \frac{s}{2q} e^Q \Big|_{x_0}^x + \int_{x_0}^x \frac{1}{2q} dz + \int_{x_0}^x s e^Q \frac{q'}{2q^2} dz.$$

Since $se^Q/2q$ vanishes at infinity, the third condition implies that $se^Q(1 - q'/2q^2)$ is integrable and, thanks to the second condition, we conclude that (H5) holds.

On the other hand, if (H5) holds, $q'(x) \geq 0$ for $x \geq x_0$ and $q(x_0) > 0$, then $q(x)$ goes to infinity as $x \rightarrow \infty$ and $\int_{x_0}^\infty \frac{1}{q(x)} dx < \infty$.

We can retain that under the assumption that $q'(x) \geq 0$ for $x \geq x_0$ and $q(x)$ goes to infinity as $x \rightarrow \infty$, then

$$(H5) \iff \int_1^\infty \frac{1}{q(x)} dx < \infty.$$

Indeed, the only thing left to prove is the sufficiency of (H5). Since $s(y)$ tends to 0 as $y \rightarrow \infty$ (because Q grows at least linearly), then by the mean value theorem we have

$$\frac{\int_y^\infty e^{-Q(z)} dz}{e^{-Q(y)}} = \frac{1}{2q(\xi)},$$

where $\xi \in [y, \infty)$. Using that q is monotone, we obtain the bound

$$\frac{\int_y^\infty e^{-Q(z)} dz}{e^{-Q(y)}} \leq \frac{1}{2q(y)},$$

and the equivalence is shown.

The proof of Theorem 7.3 follows from Propositions 7.5–7.7.

PROPOSITION 7.5. *Assume (H1) holds. If there is a unique q.s.d. that attracts all initial distributions supported in $(0, \infty)$, then X comes down from infinity.*

PROOF. Let π be the unique q.s.d. that attracts all distributions. We know that $\mathbb{P}_\pi(T_0 > t) = e^{-\alpha t}$ for some $\alpha \geq 0$. Since absorption is certain, then $\alpha > 0$. For the rest of the proof let ν be any initial distribution supported on $(0, \infty)$, which by hypothesis is in the domain of attraction of π , that is, for any bounded and continuous function f we have

$$\lim_{t \rightarrow \infty} \int_0^\infty \mathbb{P}_\nu(X_t \in dx \mid T_0 > t) f(x) = \int_0^\infty f(x) \pi(dx).$$

We now prove that, for any $\lambda < \alpha$, $\mathbb{E}_\nu(e^{\lambda T_0}) < \infty$. As in Lemma 7.2, we have for any s

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}_\nu(T_0 > t + s)}{\mathbb{P}_\nu(T_0 > t)} = e^{-\alpha s}.$$

Now pick $\lambda \in (0, \alpha)$ and $\varepsilon > 0$ such that $(1 + \varepsilon)e^{\lambda - \alpha} < 1$. An elementary induction shows that there is t_0 such that, for any $t > t_0$, and any integer n ,

$$\frac{\mathbb{P}_\nu(T_0 > t + n)}{\mathbb{P}_\nu(T_0 > t)} \leq (1 + \varepsilon)^n e^{-\alpha n}.$$

Breaking down the integral $\int_0^\infty \mathbb{P}_\nu(T_0 > s)e^{\lambda s} ds$ over intervals of the form $(n, n + 1]$ and using the previous inequality, it is easily seen that this integral converges. This proves that $\mathbb{E}_\nu(e^{\lambda T_0}) < \infty$ for any initial distribution ν .

Now fix $\lambda = \alpha/2$ and for any $x \geq 0$, let $g(x) = \mathbb{E}_x(e^{\lambda T_0}) < \infty$. We want to show that g is bounded, which trivially entails that X comes down from infinity. Thanks to the previous step, for any nonnegative random variable Y with law ν ,

$$\mathbb{E}(g(Y)) = \mathbb{E}_\nu(e^{\lambda T_0}) < \infty.$$

Since Y can be any random variable, this implies that g is bounded. Indeed, observe that g is increasing and $g(0) = 1$, so that $a := 1/g(\infty)$ is well defined in $(0, 1)$. Then check that

$$\nu(dx) = \frac{g'(x)}{(1 - a)g(x)^2} dx$$

is a probability density on $(0, \infty)$. To conclude, we use the fact that $\int g d\nu < \infty$ to get

$$\int g d\nu = \int_0^\infty \frac{g'(x)}{(1 - a)g(x)} dx = \frac{1}{1 - a} \ln g(x)|_0^\infty = \frac{\ln g(\infty)}{1 - a},$$

and then g is bounded. \square

PROPOSITION 7.6. *The following are equivalent:*

- (i) X comes down from infinity;
- (ii) (H5) holds;
- (iii) for any $a > 0$ there exists $y_a > 0$ such that $\sup_{x > y_a} \mathbb{E}_x[e^{aT_{y_a}}] < \infty$.

PROOF. Since (i) is equivalent to ∞ being an entrance boundary and (ii) is equivalent to (7.1), we must show that “ ∞ is an entrance boundary” and (7.1) are equivalent. This will follow from [14], Theorem 20.12, (iii). For that purpose consider $Y_t = \Lambda(X_t)$. Under each one of the conditions (i) or (ii), we have $\Lambda(\infty) = \infty$. It is direct to prove that Y is in natural scale on the interval $(\Lambda(0), \infty)$, that is, for $\Lambda(0) < a \leq y \leq b < \infty = \Lambda(\infty)$,

$$\mathbb{P}_y(T_a^Y < T_b^Y) = \frac{b - y}{b - a},$$

where T_a^Y is the hitting time of a for the diffusion Y . Then, ∞ is an entrance boundary for Y if and only if

$$\int_0^\infty ym(dy) < \infty,$$

where m is the speed measure of Y , which is given by

$$m(dy) = \frac{2 dy}{(\Lambda'(\Lambda^{-1}(y)))^2}$$

(see [15], formula (5.51)), because Y satisfies the SDE

$$dY_t = \Lambda'(\Lambda^{-1}(Y_t)) dB_t.$$

After a change of variables, we obtain

$$\int_0^\infty ym(dy) = \int_1^\infty e^{-Q(y)} \int_1^x e^{Q(z)} dz dx.$$

Therefore, we have shown the equivalence between (i) and (ii).

We continue the proof with (ii) \Rightarrow (iii). Let $a > 0$, and pick x_a large enough so that

$$\int_{x_a}^\infty e^{Q(x)} \int_x^\infty e^{-Q(z)} dz dx \leq \frac{1}{2a}.$$

Let J be the nonnegative increasing function defined on $[x_a, \infty)$ by

$$J(x) = \int_{x_a}^x e^{Q(y)} \int_y^\infty e^{-Q(z)} dz dy.$$

Then check that $J'' = 2qJ' - 1$, so that $LJ = -1/2$. Set now $y_a = 1 + x_a$, and consider a large $M > x$. Itô's formula gives

$$\begin{aligned} & \mathbb{E}_x(e^{a(t \wedge T_M \wedge T_{y_a})} J(X_{t \wedge T_M \wedge T_{y_a}})) \\ &= J(x) + \mathbb{E}_x\left(\int_0^{t \wedge T_M \wedge T_{y_a}} e^{as} (aJ(X_s) + LJ(X_s)) ds\right). \end{aligned}$$

But $LJ = -1/2$, and $J(X_s) < J(\infty) \leq 1/(2a)$ for any $s \leq T_{y_a}$, so that

$$\mathbb{E}_x[e^{a(t \wedge T_M \wedge T_{y_a})} J(X_{t \wedge T_M \wedge T_{y_a}})] \leq J(x).$$

But J is increasing, hence, for $x \geq y_a$ one gets $1/(2a) > J(x) \geq J(y_a) > 0$. It follows that $\mathbb{E}_x(e^{a(t \wedge T_M \wedge T_{y_a})}) \leq 1/(2aJ(y_a))$ and, finally $\mathbb{E}_x(e^{aT_{y_a}}) \leq 1/(2aJ(y_a))$, by the monotone convergence theorem. So (iii) holds.

Finally, it is clear that (iii) \Rightarrow (i). \square

PROPOSITION 7.7. *Assume (H) holds. If there is x_0 such that $\sup_{x \geq x_0} \mathbb{E}_x(e^{\lambda_1 T_{x_0}}) < \infty$, then v_1 attracts all initial distribution supported in $(0, \infty)$.*

The proof of this result requires the following control near 0 and ∞ .

LEMMA 7.8. *Assume (H) holds, and $\sup_{x \geq x_0} \mathbb{E}_x(e^{\lambda_1 T_{x_0}}) < \infty$. For $h \in \mathbb{L}^1(\mu)$ strictly positive in $(0, \infty)$, we have*

$$(7.2) \quad \lim_{\epsilon \downarrow 0} \limsup_{t \rightarrow \infty} \frac{\int_0^\epsilon h(x) \mathbb{P}_x(T_0 > t) \mu(dx)}{\int h(x) \mathbb{P}_x(T_0 > t) \mu(dx)} = 0,$$

$$(7.3) \quad \lim_{M \uparrow \infty} \limsup_{t \rightarrow \infty} \frac{\int_M^\infty h(x) \mathbb{P}_x(T_0 > t) \mu(dx)}{\int h(x) \mathbb{P}_x(T_0 > t) \mu(dx)} = 0.$$

PROOF. We start with (7.2). Using Harnack’s inequality, we have for $\epsilon < 1$ and large t

$$\frac{\int_0^\epsilon h(x)\mathbb{P}_x(T_0 > t)\mu(dx)}{\int h(x)\mathbb{P}_x(T_0 > t)\mu(dx)} \leq \frac{\mathbb{P}_1(T_0 > t) \int_0^\epsilon h(z)\mu(dz)}{Cr(t-1, 1, 1) \int_1^2 h(x)\mu(dx) \int_1^2 \mu(dy)},$$

then

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{\int_0^\epsilon h(x)\mathbb{P}_x(T_0 > t)\mu(dx)}{\int h(x)\mathbb{P}_x(T_0 > t)\mu(dx)} \\ & \leq \limsup_{t \rightarrow \infty} \frac{\mathbb{P}_1(T_0 > t) \int_0^\epsilon h(z)\mu(dz)}{Cr(t-1, 1, 1) \int_1^2 h(x)\mu(dx) \int_1^2 \mu(dy)} \\ & = \frac{e^{-\lambda_1} \langle \eta_1, 1 \rangle_\mu \int_0^\epsilon h(z)\mu(dz)}{C\eta_1(1) \int_1^2 h(x)\mu(dx) \int_1^2 \mu(dy)}, \end{aligned}$$

and the first assertion of the lemma is proven.

For the second limit, we set $A_0 := \sup_{x \geq x_0} \mathbb{E}_x(e^{\lambda_1 T_{x_0}}) < \infty$. Then for large $M > x_0$, we have

$$\mathbb{P}_x(T_0 > t) = \int_0^t \mathbb{P}_{x_0}(T_0 > u)\mathbb{P}_x(T_{x_0} \in d(t-u)) + \mathbb{P}_x(T_{x_0} > t).$$

Using that $\lim_{u \rightarrow \infty} e^{\lambda_1 u} \mathbb{P}_{x_0}(T_0 > u) = \eta_1(x_0) \langle \eta_1, 1 \rangle_\mu$, we obtain that $B_0 := \sup_{u \geq 0} e^{\lambda_1 u} \mathbb{P}_{x_0}(T_0 > u) < \infty$. Then

$$\begin{aligned} \mathbb{P}_x(T_0 > t) & \leq B_0 \int_0^t e^{-\lambda_1 u} \mathbb{P}_x(T_{x_0} \in d(t-u)) + \mathbb{P}_x(T_{x_0} > t) \\ & \leq B_0 e^{-\lambda_1 t} \mathbb{E}_x(e^{\lambda_1 T_{x_0}}) + e^{-\lambda_1 t} \mathbb{E}_x(e^{\lambda_1 T_{x_0}}) \leq e^{-\lambda_1 t} A_0 (B_0 + 1), \end{aligned}$$

and (7.3) follows immediately. \square

PROOF OF PROPOSITION 7.7. Let ν be any fixed probability distribution whose support is contained in $(0, \infty)$. We must show that the conditional evolution of ν converges to ν_1 . We begin by claiming that ν can be assumed to have a strictly positive density h , with respect to μ . Indeed, let

$$\ell(y) = \int r(1, x, y)\nu(dx).$$

Using Tonelli’s theorem, we have

$$\begin{aligned} \int \int r(1, x, y)\nu(dx)\mu(dy) & = \int \int r(1, x, y)\mu(dy)\nu(dx) \\ & = \int \mathbb{P}_x(T_0 > 1)\nu(dx) \leq 1, \end{aligned}$$

which implies that $\int r(1, x, y)v(dx)$ is finite dy -a.s. Also, ℓ is strictly positive by Harnack’s inequality. Finally, define $h = \ell / \int \ell d\mu$. Notice that, for $d\rho = h d\mu$,

$$\mathbb{P}_v(X_{t+1} \in \bullet \mid T_0 > t + 1) = \mathbb{P}_\rho(X_t \in \bullet \mid T_0 > t),$$

showing the claim.

Consider $M > \epsilon > 0$ and any Borel set A included in $(0, \infty)$. Then

$$\left| \frac{\int \mathbb{P}_x(X_t \in A, T_0 > t)h(x)\mu(dx)}{\int \mathbb{P}_x(T_0 > t)h(x)\mu(dx)} - \frac{\int_\epsilon^M \mathbb{P}_x(X_t \in A, T_0 > t)h(x)\mu(dx)}{\int_\epsilon^M \mathbb{P}_x(T_0 > t)h(x)\mu(dx)} \right|$$

is bounded by the sum of the following two terms:

$$I1 = \left| \frac{\int \mathbb{P}_x(X_t \in A, T_0 > t)h(x)\mu(dx)}{\int \mathbb{P}_x(T_0 > t)h(x)\mu(dx)} - \frac{\int_\epsilon^M \mathbb{P}_x(X_t \in A, T_0 > t)h(x)\mu(dx)}{\int \mathbb{P}_x(T_0 > t)h(x)\mu(dx)} \right|,$$

$$I2 = \left| \frac{\int_\epsilon^M \mathbb{P}_x(X_t \in A, T_0 > t)h(x)\mu(dx)}{\int \mathbb{P}_x(T_0 > t)h(x)\mu(dx)} - \frac{\int_\epsilon^M \mathbb{P}_x(X_t \in A, T_0 > t)h(x)\mu(dx)}{\int_\epsilon^M \mathbb{P}_x(T_0 > t)h(x)\mu(dx)} \right|.$$

We have the bound

$$I1 \vee I2 \leq \frac{\int_0^\epsilon \mathbb{P}_x(T_0 > t)h(x)\mu(dx) + \int_M^\infty \mathbb{P}_x(T_0 > t)h(x)\mu(dx)}{\int \mathbb{P}_x(T_0 > t)h(x)\mu(dx)}.$$

Thus, from Lemma 7.8 we get

$$\lim_{\epsilon \downarrow 0, M \uparrow \infty} \limsup_{t \rightarrow \infty} \left| \frac{\int \mathbb{P}_x(X_t \in A, T_0 > t)h(x)\mu(dx)}{\int \mathbb{P}_x(T_0 > t)h(x)\mu(dx)} - \frac{\int_\epsilon^M \mathbb{P}_x(X_t \in A, T_0 > t)h(x)\mu(dx)}{\int_\epsilon^M \mathbb{P}_x(T_0 > t)h(x)\mu(dx)} \right| = 0.$$

On the other hand, we have, using (5.4),

$$\lim_{t \rightarrow \infty} \frac{\int_\epsilon^M \mathbb{P}_x(X_t \in A, T_0 > t)h(x)\mu(dx)}{\int_\epsilon^M \mathbb{P}_x(T_0 > t)h(x)\mu(dx)} = \frac{\int_A \eta_1(z)\mu(dz)}{\int_{\mathbb{R}^+} \eta_1(z)\mu(dz)} = \nu_1(A),$$

independently of $M > \epsilon > 0$, and the result follows. \square

The following corollary of Proposition 7.6 describes how fast the process comes down from infinity.

COROLLARY 7.9. *Assume (H) and (H5) hold. Then for all $\lambda < \lambda_1$, $\sup_{x>0} \mathbb{E}_x[e^{\lambda T_0}] < \infty$.*

PROOF. We have seen in Section 5 (Theorem 5.2) that, for all $x > 0$, $\lim_{t \rightarrow \infty} e^{\lambda t} \mathbb{P}_x(T_0 > t) = \eta_1(x) \langle \eta_1, 1 \rangle_\mu < \infty$, that is, $\mathbb{E}_x[e^{\lambda T_0}] < \infty$ for all $\lambda < \lambda_1$. Applying Proposition 7.6 with $a = \lambda$ and the strong Markov property, it follows

that $\sup_{x > y_\lambda} \mathbb{E}_x[e^{\lambda T_0}] < \infty$. Furthermore, thanks to the uniqueness of the solution of (1.1), $X_t^x \leq X_t^{y_\lambda}$ a.s. for all $t > 0$ and all $x < y_\lambda$, hence, $\mathbb{E}_x[e^{\lambda T_0}] \leq \mathbb{E}_{y_\lambda}[e^{\lambda T_0}]$ for those x , completing the proof. \square

The previous corollary states that the killing time for the process starting from infinity has exponential moments up to order λ_1 . In [18] an explicit calculation of the law of T_0 is done in the case of the logistic Feller diffusion Z (hence the corresponding X) and also for other related models. In particular, it is shown in Corollary 3.10 therein that the absorption time for the process starting from infinity has a finite expectation. As we remarked in studying examples, a very general family of diffusion processes (including the logistic one) satisfy all assumptions in Corollary 7.9, which is thus an improvement of the quoted result.

We end this section by gathering some known results on birth–death processes that are close to our findings. Let Y be a birth–death process with birth rate λ_n and death rate μ_n when in state n . Assume that $\lambda_0 = \mu_0 = 0$ and that extinction (absorption at 0) occurs with probability 1. Let

$$S = \sum_{i \geq 1} \pi_i + \sum_{n \geq 1} (\lambda_n \pi_n)^{-1} \sum_{i \geq n+1} \pi_i,$$

where

$$\pi_n = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}.$$

In this context sure absorption at 0, that is, (H1), is equivalent to $A := \sum_{i \geq 1} (\lambda_i \pi_i)^{-1} = \infty$ (see [16], formula (7.9)). On the other hand, we also have $\mathbb{E}_1(T_0) = \sum_{i \geq 1} \pi_i$. We may state the following:

PROPOSITION 7.10. *For a birth–death process Y that satisfies (H1), the following are equivalent:*

- (i) Y comes down from infinity;
- (ii) There is one and only one $q.s.d.$;
- (iii) $\lim_{n \uparrow \infty} \uparrow \mathbb{E}_n(T_0) < \infty$;
- (iv) $S < \infty$.

PROOF. In [5], Theorem 3.2, it states the following key alternative: $S < \infty$ iff there is a unique $q.s.d.$; if $S = \infty$, then there is no $q.s.d.$ or there are infinitely many ones. Then the equivalence between (ii) and (iv) is immediate. Also, it is well known that

$$\mathbb{E}_n(T_0) = \sum_{i \geq 1} \pi_i + \sum_{r=1}^{n-1} (\lambda_r \pi_r)^{-1} \sum_{i \geq r+1} \pi_i;$$

see, for example, [16], formula 7.10. Therefore, (iii) and (iv) are equivalent. Let us now examine how this criterion is related to the nature of the boundary at $+\infty$.

From the table in [1], Section 8.1, we have that $+\infty$ is an entrance boundary iff $A = \infty$, $\mathbb{E}_1(T_0) < \infty$ and $S < \infty$. Finally, $S < \infty$ implies that $\mathbb{E}_1(T_0) < \infty$ and this ensures $\mathbb{P}_1(T_0 < \infty) = 1$, that is, $A = \infty$. This shows the result. \square

8. Biological models.

8.1. *Population dynamics and quasi-stationary distributions.* Our aim is to model the dynamics of an isolated population by a diffusion $Z := (Z_t; t \geq 0)$. Since competition for limited resources impedes natural populations with no immigration to grow indefinitely, they are all doomed to become extinct at some finite time T_0 . However, T_0 can be large compared to a human timescale and it is common that population sizes fluctuate for a large amount of time before extinction actually occurs. The notion of quasi-stationarity captures this behavior [25, 29].

The diffusions we consider arise as scaling limits of general birth–death processes. More precisely, let $(Z^N)_{N \in \mathbb{N}}$ be a sequence of continuous time birth–death processes $Z^N := (Z_t^N; t \geq 0)$, renormalized by the weight N^{-1} , hence taking values in $N^{-1}\mathbb{N}$. Assume that their birth and death rates from state x are equal to $b_N(x)$ and $d_N(x)$, respectively, and $b_N(0) = d_N(0) = 0$, ensuring that the state 0 is absorbing. We also assume that, for each N and for some constant B_N ,

$$b_N(x) \leq (x + 1)B_N, \quad x \geq 0,$$

and that there exist a nonnegative constant γ and a function $h \in C^1([0, \infty))$, $h(0) = 0$, called the *growth function*, such that

$$(8.1) \quad \lim_{N \rightarrow \infty} \frac{1}{N}(b_N(x) - d_N(x)) = h(x); \quad \lim_{N \rightarrow \infty} \frac{1}{2N^2}(b_N(x) + d_N(x)) = \gamma x$$

$\forall x \in (0, \infty).$

Important ecological examples include the following:

- (i) The *pure branching* case, where the individuals give birth and die independently, so that one can take $b_N(x) = (\gamma N + \lambda)Nx$ and $d_N(x) = (\gamma N + \mu)Nx$. Writing $r := \lambda - \mu$ for the *Malthusian growth parameter* of the population, one gets $h(z) = rz$.
- (ii) The *logistic branching* case, where $b_N(x) = (\gamma N + \lambda)Nx$ and $d_N(x) = (\gamma N + \mu)Nx + \frac{c}{N}Nx(Nx - 1)$. The quadratic term in the death rate describes the interaction between individuals. The number of individuals is of order N , the biomass of each individual is of order N^{-1} , and c/N is the interaction coefficient. The growth function is then $h(z) = rz - cz^2$.
- (iii) Dynamics featuring *Allee effect*, that is, a positive density-dependence for certain ranges of density, corresponding to cooperation in natural populations. A classical type of growth function in that setting is $h(z) = rz(\frac{z}{K_0} - 1)(1 - \frac{z}{K})$. Observe that, in this last case, the individual growth rate is no longer a monotone decreasing function of the population size.

Assuming further that $(Z_0^N)_{N \in \mathbb{N}}$ converges as $N \rightarrow \infty$ (we thus model the dynamics of a population whose size is of order N), we may prove, following Lipow [21] or using the techniques of Joffe–Métivier [13], that the sequence $(Z^N)_{N \in \mathbb{N}}$ converges weakly to a continuous limit Z . The parameter γ can be interpreted as a demographic parameter describing the ecological timescale. There is a main qualitative difference depending on whether $\gamma = 0$ or not.

If $\gamma = 0$, then the limit Z is a deterministic solution to the dynamical system $\dot{Z}_t = h(Z_t)$. Since $h(0) = 0$, the state 0 is always an equilibrium, but it can be unstable. For example, in the logistic case $h(z) = rz - cz^2$ and it is easily checked that, when $r > 0$, the previous dynamical system has two equilibria, 0 which is unstable, and r/c (called *carrying capacity*) which is asymptotically stable. In the Allee effect case, 0 and K are both stable equilibria, but K_0 is an unstable equilibrium, which means the population size has a threshold K_0 to growth, below which it cannot take over.

If $\gamma > 0$, the sequence $(Z^N)_{N \in \mathbb{N}}$ converges in law to the process Z , solution to the following stochastic differential equation

$$(8.2) \quad dZ_t = \sqrt{\gamma Z_t} dB_t + h(Z_t) dt.$$

The acceleration of the ecological process has generated the white noise. Note that $h'(0^+)$ is the mean *per capita* growth rate for *small* populations. The fact that it is finite is mathematically convenient, and biologically reasonable. Since $h(0) = 0$, the population undergoes no immigration, so that 0 is an absorbing state. One can easily check that when time goes to infinity, either Z goes to ∞ or is absorbed at 0.

When $h \equiv 0$, we get the classical Feller diffusion, so we call *generalized Feller diffusions* the diffusions driven by (8.2). When h is linear, we get the general continuous-state branching process with continuous paths, sometimes also called Feller diffusion by extension. When h is concave quadratic, we get the logistic Feller diffusion [6, 18].

DEFINITION 8.1. (HH) We say that h satisfies the condition (HH) if

$$(i) \quad \lim_{x \rightarrow \infty} \frac{h(x)}{\sqrt{x}} = -\infty, \quad (ii) \quad \lim_{x \rightarrow \infty} \frac{xh'(x)}{h(x)^2} = 0.$$

In particular, (HH) holds for any subcritical branching diffusion and any logistic Feller diffusion. Concerning assumption (i), the fact that h goes to $-\infty$ indicates strong competition in large populations, resulting in negative growth rates (as in the logistic case). Assumption (ii) is fulfilled for most classical biological models, and it appears as a mere technical condition.

Gathering all results of the present paper and applying them to our biological model yields the following statement, which will be proved at the end of this section.

THEOREM 8.2. *Let Z be the solution of (8.2). We assume $h \in C^1([0, \infty))$, $h(0) = 0$ and that h satisfies assumption (HH). Then, for all initial laws with bounded support, the law of Z_t conditioned on $\{Z_t \neq 0\}$ converges exponentially fast to a probability measure ν , called the Yaglom limit. The law \mathbb{Q}_x of the process Z starting from x and conditioned to be never extinct exists and defines the so-called Q -process. This process converges, as $t \rightarrow \infty$, in distribution, to its unique invariant probability measure. This probability measure is absolutely continuous w.r.t. ν with a nondecreasing Radon–Nikodym derivative.*

In addition, if the following integrability condition is satisfied,

$$\int_1^\infty \frac{dx}{-h(x)} < \infty,$$

then Z comes down from infinity and the convergence of the conditional one-dimensional distributions holds for all initial laws. In particular, the Yaglom limit ν is then the unique quasi-stationary distribution.

PROOF. For Z solution to (8.2), recall that $X_t = 2\sqrt{Z_t/\gamma}$ satisfies the SDE $dX_t = dB_t - q(X_t) dt$ with

$$q(x) = \frac{1}{2x} - \frac{2h(\gamma x^2/4)}{\gamma x}, \quad x > 0.$$

In particular, we have

$$q'(x) = -\frac{1}{2x^2} + \frac{2h(\gamma x^2/4)}{\gamma x^2} - h'(\gamma x^2/4) \quad \text{and}$$

$$q^2(x) - q'(x) = \frac{3}{4x^2} + h(\gamma x^2/4) \left(\frac{4}{\gamma^2 x^2} h(\gamma x^2/4) - \frac{4}{\gamma x^2} \right) + h'(\gamma x^2/4).$$

Under assumption (HH), we have the following behaviors at 0 and ∞ : $q(x) \sim_{x \downarrow 0} 1/2x$, as well as

$$q^2(x) - q'(x) \sim_{x \downarrow 0} \frac{3}{4x^2} \quad \text{and} \quad (q^2 - q')(2\sqrt{x/\gamma}) \sim_{x \rightarrow \infty} \frac{h(x)^2}{x} \left(\frac{1}{\gamma} + \frac{xh'(x)}{h(x)^2} \right).$$

Then, it is direct to check that hypothesis (H2) holds

$$\lim_{x \rightarrow \infty} q^2(x) - q'(x) = \infty, \quad C := - \inf_{x \in (0, \infty)} q^2(x) - q'(x) < \infty.$$

We recall that

$$Q(x) = \int_1^x 2q(y) dy, \quad \Lambda(x) = \int_1^x e^{Q(y)} dy \quad \text{and}$$

$$\kappa(x) = \int_1^x e^{Q(y)} \left(\int_1^y e^{-Q(z)} dz \right) dy.$$

Straightforward calculations show that

$$\lim_{x \rightarrow \infty} \frac{Q(x)}{x} = \infty \quad \text{and} \quad A := \lim_{x \rightarrow 0^+} (Q(x) - \log(x)) \in (-\infty, \infty).$$

In particular, $\Lambda(\infty) = \infty$ and the integrand in the definition of κ is equivalent to $y \log(y)$, which ensures $\kappa(0^+) < \infty$. Thus, X and, consequently, Z , is absorbed at 0 with probability 1, that is, hypothesis (H1) holds.

We now continue with (H3), which is

$$\int_0^1 \frac{1}{q^2(y) - q'(y) + C + 2} e^{-Q(y)} dy < \infty.$$

This hypothesis holds because near 0 the integrand is of the order

$$\frac{1}{3/(4y^2)} e^{-Q(y)} \sim \frac{4e^{-A}}{3} y.$$

For the first part of the theorem it remains only to show that (H4) holds

$$\int_1^\infty e^{-Q(x)} dx < \infty \quad \text{and} \quad \int_0^1 x e^{-Q(x)/2} dx < \infty.$$

The first integral is finite because Q grows at least linearly near ∞ and the second one is finite because the integrand is of order $1/\sqrt{x}$ for x near 0.

Hence, we can apply Theorem 5.2, Proposition 5.5, and Corollaries 6.1 and 6.2 to finish with the proof of the first part of the theorem.

For the last part of the theorem we need to show that X comes down from infinity, which is equivalent to (H5). Thanks to Remark 7.4, there is a simple sufficient condition for this hypothesis to hold, which has three components. The first one,

$$q(x) \geq q_0 > 0 \quad \text{for all } x \geq x_0,$$

follows from (HH)(i). The second one,

$$\limsup_{x \rightarrow \infty} q'(x)/2q^2(x) < 1,$$

is equivalent to

$$\limsup_{x \rightarrow \infty} -\frac{xh'(x)}{h(x)^2} < \frac{2}{\gamma},$$

which clearly follows from (HH)(ii). Finally the third one,

$$\int_{x_0}^\infty \frac{1}{q(x)} dx < \infty,$$

thanks to (HH)(i), is equivalent to

$$\int^\infty \frac{-\gamma x}{2h(\gamma x^2/4)} dx = \int^\infty \frac{1}{-h(z)} dz < \infty.$$

This is exactly the extra assumption made in the theorem and the result is proven. □

8.2. *The growth function and conditioning.* Referring to the previous construction of the generalized Feller diffusion (8.2), we saw why $h(z)$ could be viewed as the expected growth rate of a population of size z and $h(z)/z$ as the mean *per capita* growth rate. Indeed, $h(z)$ informs of the resulting action of density upon the growth of the population, and $h(z)/z$ indicates the resulting action of density upon each individual. In the range of densities z where $h(z)/z$ increases with z , the most important interactions are of the *cooperative* type, one speaks of *positive density-dependence*. On the contrary, when $h(z)/z$ decreases with z , the interactions are of the *competitive* type, and density-dependence is said to be *negative*. In many cases, such as the logistic one, the limitation of resources forces harsh competition in large populations, so that, as $z \rightarrow \infty$, $h(z)/z$ is negative and decreasing. In particular, $h(z)$ goes to $-\infty$. The shape of h at infinity determines the long time behavior of the diffusion Z .

Actually, if h goes to infinity at infinity, such as in the pure branching process case (where h is linear), Theorem 8.2 still holds if (HH)(i) is replaced with the more general condition $\lim_{x \rightarrow \infty} \frac{h(x)}{\sqrt{x}} = \pm\infty$, provided the generalized Feller diffusion is further *conditioned on eventual extinction*. Indeed, the following statement ensures that conditioning on extinction roughly amounts to replacing h with $-h$.

PROPOSITION 8.3. *Assume that Z is given by (8.2), where $h \in C^1([0, \infty))$, $h(0) = 0$, $\lim_{x \rightarrow \infty} \frac{h(x)}{\sqrt{x}} = \infty$. Define $u(x) := \mathbb{P}_x(\lim_{t \rightarrow \infty} Z_t = 0)$ and let Y be the diffusion Z conditioned on eventual extinction. Then Y is the solution of the SDE, $Y_0 = Z_0$*

$$(8.3) \quad dY_t = \sqrt{\gamma Y_t} dB_t + \left(h(Y_t) + \gamma Y_t \frac{u'(Y_t)}{u(Y_t)} \right) dt.$$

If, in addition, h satisfies (HH)(ii), then

$$h(y) + \gamma y \frac{u'(y)}{u(y)} \underset{y \rightarrow \infty}{\sim} -h(y).$$

PROOF. Let $J(x) := \int_0^x \frac{2h(z)}{\gamma z} dz$, which is well defined since $h \in C^1([0, \infty))$ with $h(0) = 0$. We set

$$v(x) := a \int_x^\infty e^{-J(z)} dz,$$

with $a = (\int_0^\infty e^{-J(z)} dz)^{-1}$ (well defined by the growth of h near ∞). Now we prove that $u = v$. It is easily checked that v is decreasing with $v(0) = 1$, $v(\infty) = 0$, and that it satisfies the equation $\frac{\gamma}{2} x v''(x) + h(x)v'(x) = 0$ for all $x \geq 0$.

As a consequence, $(v(Z_t); t \geq 0)$ is a (bounded hence) uniformly integrable martingale, so that

$$v(x) = \mathbb{E}_x(v(Z_t)) \rightarrow v(0)\mathbb{P}_x\left(\lim_{t \rightarrow \infty} Z_t = 0\right) + v(\infty)\mathbb{P}_x\left(\lim_{t \rightarrow \infty} Z_t = \infty\right) = u(x),$$

as $t \rightarrow \infty$, so that indeed $u = v$.

Using the strong Markov Property of Z , we obtain that, for any Borel set $A \subset (0, \infty)$ and $s \geq 0$,

$$\begin{aligned} \mathbb{P}_x(Y_s \in A) &= \mathbb{P}_x(Z_s \in A | T_0 < \infty) = \mathbb{E}_x \left(\frac{\mathbb{P}_{Z_s}(T_0 < \infty)}{\mathbb{P}_x(T_0 < \infty)}, Z_s \in A \right) \\ &= \mathbb{E}_x \left(\frac{u(Z_s)}{u(x)}, Z_s \in A \right). \end{aligned}$$

Then for any measurable and bounded function f we get

$$\mathbb{E}_x(f(Y_s)) = \mathbb{E}_x \left(f(Z_s) \frac{u(Z_s)}{u(x)} \right).$$

Now if f is C^2 and has compact support contained in $(0, \infty)$, we get from Itô's formula that uf is in the domain of L^Z , the generator of Z , and then f is in the domain of the generator L^Y of Y and, moreover,

$$L^Y(f)(x) = \frac{1}{u(x)} L^Z(uf)(x) = \frac{\gamma}{2} x f''(x) + \left(h(x) + \gamma x \frac{u'(x)}{u(x)} \right).$$

Then, since h is locally Lipschitz, we obtain that the law of Y is the unique solution to the SDE (8.3).

Let us show the last part of the proposition. Notice that J is strictly increasing after some x_0 , so we consider its inverse φ on $[J(x_0), \infty)$. Next observe that, for $x > x_0$,

$$-\frac{u}{u'}(x) = e^{J(x)} \int_x^\infty e^{-J(z)} dz = e^{J(x)} \int_{J(x)}^\infty e^{-b} \varphi'(b) db,$$

with the change $b = J(z)$. As a consequence, we can write for $y > J(x_0)$

$$(8.4) \quad -\frac{u}{u'}(\varphi(y)) = e^y \int_y^\infty e^{-b} \varphi'(b) db = \int_0^\infty e^{-b} \varphi'(y+b) db.$$

Because h tends to ∞ , $J(x) \geq (1 + \varepsilon) \log(x)$ for x sufficiently large, so that $\varphi(y) \leq \exp(y/(1 + \varepsilon))$, and $\varphi(y) \exp(-y)$ vanishes as $y \rightarrow \infty$. Now, since

$$\varphi'(y) = \frac{\gamma \varphi(y)}{2h(\varphi(y))} = o(\varphi(y)),$$

$\varphi'(y) \exp(-y)$ also vanishes. Since h is differentiable, J is twice differentiable, and so is φ , so performing an integration by parts yields

$$(8.5) \quad \varphi'(y) = \int_0^\infty e^{-b} \varphi'(y+b) db - \int_0^\infty e^{-b} \varphi''(y+b) db.$$

Since $\varphi'(J(x)) = 1/J'(x)$, we get $J'(x)\varphi''(J(x)) = (1/J'(x))'$, so by the technical assumption (HH)(ii),

$$\varphi''(J(x)) = \varphi'(J(x)) \left(\frac{1}{J'(x)} \right)' = \frac{\gamma}{2} \varphi'(J(x)) \left(\frac{1}{h(x)} - \frac{xh'(x)}{h(x)^2} \right) = o(\varphi'(J(x))),$$

as $x \rightarrow \infty$. Then, as $y \rightarrow \infty$, we have $\varphi''(y) = o(\varphi'(y))$. This shows, thanks to (8.5), that

$$\int_0^\infty e^{-b} \varphi'(y + b) db \underset{y \rightarrow \infty}{\sim} \varphi'(y),$$

which entails, thanks to (8.4), that

$$-\frac{u}{u'}(\varphi(y)) \underset{y \rightarrow \infty}{\sim} \varphi'(y).$$

This is equivalent to

$$\gamma x \frac{u'}{u}(x) \underset{x \rightarrow \infty}{\sim} -\gamma x J'(x) = -2h(x),$$

which ends the proof. \square

Let us examine the case of the Feller diffusion (pure branching process), where $h(z) = rz$. First, it is known (see, e.g., [20], Chapter 2) that when $r > 0$, the supercritical Feller diffusion Z conditioned on extinction is *exactly* the subcritical Feller diffusion with $h(z) = -rz$. The previous statement can thus be seen as an extension of this duality to more general population diffusion processes.

Second, in the critical case ($r = 0$), our present results do not apply. Actually, the (critical) Feller diffusion has no *q.s.d.* [19]. Third, in the subcritical case ($r < 0$), our results do apply, so there is a Yaglom limit and a Q -process, but in contrast to the case when $1/h$ is integrable at ∞ , it is shown in [19] that subcritical Feller diffusions have infinitely many *q.s.d.*

APPENDIX: PROOF OF LEMMA 4.5

We first prove the second bound. For any nonnegative and continuous function f with support in \mathbb{R}^+ , we have from hypothesis (H2)

$$\begin{aligned} \int \tilde{p}_1(x, u) f(u) du &= \mathbb{E}^{\mathbb{W}_x} \left[f(\omega(1)) \mathbb{1}_{1 < T_0}(\omega) \exp\left(-\frac{1}{2} \int_0^1 (q^2 - q')(\omega_s) ds\right) \right] \\ &\leq e^{C/2} \mathbb{E}^{\mathbb{W}_x} [f(\omega(1)) \mathbb{1}_{1 < T_0}(\omega)]. \end{aligned}$$

The estimate (4.3) follows by letting $f(z) dz$ tend to the Dirac measure at y with $K_3 = e^{C/2}$, that is,

$$\tilde{p}_1(x, y) \leq K_3 p_1^D(x, y).$$

Here $p_1^D(x, y) = \frac{1}{\sqrt{2\pi}}(e^{-(x-y)^2/2} - e^{-(x+y)^2/2})$ (see, for example, [15], page 97).

Let us now prove the upper bound in (4.2). Let B_1 be the function defined by

$$B_1(z) := \inf_{u \geq z} (q^2(u) - q'(u)).$$

We have

$$\begin{aligned} & \int \tilde{p}_1(x, y) f(y) dy \\ &= \mathbb{E}^{\mathbb{W}_x} \left[f(\omega(1)) \mathbb{1}_{1 < T_0} \mathbb{1}_{1 < T_{x/3}} \exp\left(-\frac{1}{2} \int_0^1 (q^2 - q')(\omega_s) ds\right) \right] \\ & \quad + \mathbb{E}^{\mathbb{W}_x} \left[f(\omega(1)) \mathbb{1}_{1 < T_0} \mathbb{1}_{1 \geq T_{x/3}} \exp\left(-\frac{1}{2} \int_0^1 (q^2 - q')(\omega_s) ds\right) \right]. \end{aligned}$$

For the first expectation, we have

$$\begin{aligned} & \mathbb{E}^{\mathbb{W}_x} \left[f(\omega(1)) \mathbb{1}_{1 < T_0} \mathbb{1}_{1 < T_{x/3}} \exp\left(-\frac{1}{2} \int_0^1 (q^2 - q')(\omega_s) ds\right) \right] \\ & \leq e^{-B_1(x/3)/2} \mathbb{E}^{\mathbb{W}_x} [f(\omega(1)) \mathbb{1}_{1 < T_0}]. \end{aligned}$$

For the second expectation, we obtain

$$\begin{aligned} & \mathbb{E}^{\mathbb{W}_x} \left[f(\omega(1)) \mathbb{1}_{1 < T_0} \mathbb{1}_{1 \geq T_{x/3}} \exp\left(-\frac{1}{2} \int_0^1 (q^2 - q')(\omega_s) ds\right) \right] \\ & \leq e^{C/2} \mathbb{E}^{\mathbb{W}_x} [f(\omega(1)) \mathbb{1}_{1 < T_0} \mathbb{1}_{1 \geq T_{x/3}}] \\ & = e^{C/2} (\mathbb{E}^{\mathbb{W}_x} [f(\omega(1)) \mathbb{1}_{1 < T_0}] - \mathbb{E}^{\mathbb{W}_x} [f(\omega(1)) \mathbb{1}_{1 < T_{x/3}}]). \end{aligned}$$

Using a limiting argument as above and the invariance by translation of the law of the Brownian motion, and first assuming that $y/2 < x < 2y$, we obtain

$$\begin{aligned} \tilde{p}_1(x, y) & \leq e^{-B_1(x/3)/2} p_1^D(x, y) + e^{C/2} (p_1^D(x, y) - p_1^D(2x/3, y - x/3)), \\ p_1^D(x, y) - p_1^D(2x/3, y - x/3) & = \frac{1}{\sqrt{2\pi}} (e^{-(y+x/3)^2/2} - e^{-(x+y)^2/2}) \leq \frac{1}{\sqrt{2\pi}} e^{-\max\{x, y\}^2/18}. \end{aligned}$$

Since the function B_1 is nondecreasing, we get for $y/2 < x < 2y$

$$\tilde{p}_1(x, y) \leq \frac{1}{\sqrt{2\pi}} (e^{-B_1(\max\{x, y\}/6)/2} + e^{-\max\{x, y\}^2/18}).$$

If $x/y \notin]1/2, 2[$, we get from the estimate (4.3)

$$\tilde{p}_1(x, y) \leq \frac{K_3}{\sqrt{2\pi}} e^{-(y-x)^2/2} \leq \frac{K_3}{\sqrt{2\pi}} e^{-\max\{x, y\}^2/8}.$$

We now define the function B by

$$B(z) := \log\left(\frac{K_3 \vee 1}{\sqrt{2\pi}}\right) + \min\{B_1(z/6)/4, z^2/36\}.$$

It follows from hypothesis (H2) that $\lim_{z \rightarrow \infty} B(z) = \infty$. Combining the previous estimates, we get for any x and y in \mathbb{R}^+

$$\tilde{p}_1(x, y) \leq e^{-2B(\max\{x, y\})}.$$

The upper estimate (4.2) follows by taking the geometric average of this result and (4.3). We now prove that $\tilde{p}_1(x, y) > 0$. For this purpose, let $a = \min\{x, y\}/2$ and $b = 2 \max\{x, y\}$. We have as above for every nonnegative continuous function f with support in \mathbb{R}^+

$$\int \tilde{p}_1(x, y) f(y) dy \geq \mathbb{E}^{\mathbb{W}_x} \left[f(\omega(1)) \mathbb{1}_{1 < T_{[a, b]}} \exp\left(-\frac{1}{2} \int_0^1 (q^2 - q')(\omega_s) ds\right) \right],$$

where we denote $T_{[a, b]}$ the exit time from the interval $[a, b]$. Letting

$$R_{a, b} = \sup_{x \in [a, b]} (q^2(x) - q'(x)),$$

this quantity is finite since $q \in C^1((0, \infty))$. We obtain immediately

$$\int \tilde{p}_1(x, y) f(y) dy \geq e^{-R_{a, b}/2} \int p_1^{[a, b]}(x, y) f(y) dy,$$

where we denote $p_t^{[a, b]}$ the heat kernel with Dirichlet conditions in $[a, b]$. The result follows from a limiting argument as above, since $p_1^{[a, b]}(x, y) > 0$.

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REFERENCES

[1] ANDERSON, W. J. (1991). *Continuous-Time Markov Chains*. Springer, New York. [MR1118840](#)

[2] BEREZIN, F. A. and SHUBIN, M. A. (1991). *The Schrödinger Equation. Mathematics and Its Applications (Soviet Series)* **66**. Kluwer Academic, Dordrecht. [MR1186643](#)

[3] COLLET, P., MARTÍNEZ, S. and SAN MARTÍN, J. (1995). Asymptotic laws for one-dimensional diffusions conditioned to nonabsorption. *Ann. Probab.* **23** 1300–1314. [MR1349173](#)

[4] CONWAY, J. B. (1990). *A Course in Functional Analysis*, 2nd ed. *Graduate Texts in Mathematics* **96**. Springer, New York. [MR1070713](#)

[5] VAN DOORN, E. A. (1991). Quasi-stationary distributions and convergence to quasi-stationarity of birth–death processes. *Adv. in Appl. Probab.* **23** 683–700. [MR1133722](#)

[6] ETHERIDGE, A. M. (2004). Survival and extinction in a locally regulated population. *Ann. Appl. Probab.* **14** 188–214. [MR2023020](#)

[7] FERRARI, P. A., KESTEN, H., MARTINEZ, S. and PICCO, P. (1995). Existence of quasi-stationary distributions. A renewal dynamical approach. *Ann. Probab.* **23** 501–521. [MR1334159](#)

[8] FUKUSHIMA, M. (1980). *Dirichlet Forms and Markov Processes*. Kodansha, North-Holland, Amsterdam.

- [9] FUKUSHIMA, M., OSHIMA, Y. and TAKEDA, M. (1994). *Dirichlet Forms and Symmetric Markov Processes. Studies in Mathematics* **19**. de Gruyter, Berlin. MR1303354
- [10] GOSSELIN, F. (2001). Asymptotic behavior of absorbing Markov chains conditional on non-absorption for applications in conservation biology. *Ann. Appl. Probab.* **11** 261–284. MR1825466
- [11] IKEDA, N. and WATANABE, S. (1989). *Stochastic Differential Equations and Diffusion Processes*, 2nd ed. *North-Holland Mathematical Library* **24**. North-Holland, Amsterdam. MR1011252
- [12] JACOD, J. (1979). *Calcul Stochastique et Problèmes de Martingales. Lecture Notes in Math.* **714**. Springer, Berlin. MR542115
- [13] JOFFE, A. and MÉTIVIER, M. (1986). Weak convergence of sequences of semimartingales with applications to multitype branching processes. *Adv. in Appl. Probab.* **18** 20–65. MR827331
- [14] KALLENBERG, O. (1997). *Foundations of Modern Probability*. Springer, New York. MR1464694
- [15] KARATZAS, I. and SHREVE, S. E. (1988). *Brownian Motion and Stochastic Calculus. Graduate Texts in Mathematics* **113**. Springer, New York. MR917065
- [16] KARLIN, S. and TAYLOR, H. M. (1975). *A First Course in Stochastic Processes*, 2nd ed. Academic Press, New York. MR0356197
- [17] KRYLOV, N. and SAFONOV, M. (1981). A certain property of solutions of parabolic equations with measurable coefficients. *Math. USSR-Izv.* **16** 151–164.
- [18] LAMBERT, A. (2005). The branching process with logistic growth. *Ann. Appl. Probab.* **15** 1506–1535. MR2134113
- [19] LAMBERT, A. (2007). Quasi-stationary distributions and the continuous-state branching process conditioned to be never extinct. *Electron. J. Probab.* **12** 420–446 (electronic). MR2299923
- [20] LAMBERT, A. (2008). Population dynamics and random genealogies. *Stoch. Models* **24** 45–163. MR2466449
- [21] LIPOW, C. (1977). Limiting diffusions for population-size dependent branching processes. *J. Appl. Probab.* **14** 14–24. MR0428493
- [22] MANDL, P. (1961). Spectral theory of semigroups connected with diffusion processes and its application. *Czech. Math. J.* **11** 558–569. MR0137143
- [23] MARTÍNEZ, S. and SAN MARTÍN, J. (2004). Classification of killed one-dimensional diffusions. *Ann. Probab.* **32** 530–552. MR2040791
- [24] POLLETT, P. K. (2008). Quasi-stationary distributions: A bibliography. Available at <http://www.maths.uq.edu.au/~pkp/papers/qsds/qsds.html>, regularly updated.
- [25] RENAULT, O., FERRIÈRE, R. and PORTER, J. The quasi-stationary route to extinction. Preprint.
- [26] REVUZ, D. and YOR, M. (1990). *Continuous Martingales and Brownian Motion*. Springer, New York.
- [27] ROYER, G. (1999). *Une Initiation aux Inégalités de Sobolev Logarithmiques*. S.M.F., Paris.
- [28] SENETA, E. and VERE-JONES, D. (1966). On quasi-stationary distributions in discrete-time Markov chains with a denumerable infinity of states. *J. Appl. Probab.* **3** 403–434. MR0207047
- [29] STEINSALTZ, D. and EVANS, S. N. (2004). Markov mortality models: Implications of quasi-stationarity and varying initial distributions. *Theor. Pop. Biol.* **65** 319–337.
- [30] STEINSALTZ, D. and EVANS, S. N. (2007). Quasistationary distributions for one-dimensional diffusions with killing. *Trans. Amer. Math. Soc.* **359** 1285–1324 (electronic). MR2262851

- [31] TRUDINGER, N. S. (1968). Pointwise estimates and quasilinear parabolic equations. *Comm. Pure Appl. Math.* **21** 205–226. MR0226168

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