ON QUADRATIC TRANSPORTATION COST INEQUALITIES.

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Abstract. In this paper we study quadratic transportation cost inequalities. To this end we introduce new families of inequalities (for quadratic transportation cost and for relative entropy) that are shown to be equivalent to the Poincaré inequality. This allows us to give some examples of measures satisfying T_2 but not the logarithmic Sobolev inequality.

Key words : Transportation inequalities, spectral gap, Gaussian concentration.

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1. **Introduction, framework and main results.**

Transportation inequalities recently deserved a lot of interest, especially in connection with the concentration of measure phenomenon (see [15], [16]). Links with others renowned functional inequalities, in particular logarithmic-Sobolev inequalities, were also particularly studied (see [5], [18], [4], [16] ...), as no direct or tractable criteria were available for this kind of inequalities.

Given a metric space (E, d) equipped with its Borel σ field, the $\mathbb{L}^p$ Wasserstein distance between two probability measures μ and ν on E is defined as

$$W_p(\mu,\nu) \,=\, \Big(\inf_\pi \int_{E\times E} d^p(x,y)\,\pi(dx,dy)\Big)^{1/p}, \tag{1.1}$$

where π describes the set of all coupling of (μ, ν) , i.e. the set of all probability measures on the product space with marginal distributions μ and ν.

A probability measure μ is said to satisfy the $T_p(C)$ transportation cost inequality if for all probability measure ν,

$$W_p(\mu,\nu) \,\leq\, \sqrt{2C\,H(\nu,\mu)}\,, \tag{1.2}$$

where $H(\nu, \mu)$ stands for the Kullback-Leibler information (or relative entropy), i.e.

$$H(\nu,\mu) \,=\, \int \log\big(\frac{d\nu}{d\mu}\big)\,d\nu \quad \text{if } \nu\ll\mu \quad ; \quad +\infty \text{ otherwise.}$$

As shown by K. Marton ([17]), T_1 implies a Gaussian type concentration for μ.

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Let us briefly recall the general argument, we shall use later.
For any Borel set A with measure $\mu(A) \geq 1/2$ introduce $A_r^c = \{x\,,\, d(x,A) \geq r\}$ and $d\mu_A = \frac{\mathbf{1}_A}{\mu(A)}\, d\mu$. Set B for A_r^c and assume that $W_1(\nu,\mu) \leq \varphi(H(\nu,\mu))$ for all ν. Then

$$
\begin{aligned}
r \leq W_1(\mu_B,\mu_A) &\leq W_1(\mu_B,\mu) + W_1(\mu,\mu_A) \\
&\leq \varphi(H(\mu_A,\mu)) + \varphi(H(\mu_B,\mu)) \\
&= \varphi\left(\log\frac{1}{\mu(A)}\right) + \varphi\left(\log\frac{1}{\mu(A_r^c)}\right)\,.
\end{aligned}
\tag{1.3}
$$

When $\varphi(u) = \sqrt{2Cu}$ we immediately obtain

$$\mu(A_r^c) \leq \exp\left(-1/2C\left(r - \sqrt{2C\,\log(\frac{1}{\mu(A)})}\right)^2\right)\,.$$

Hence criteria for T_1 to hold are very useful. Such a criterion was first obtained by Bobkov and Götze ([5] Theorem 3.1) and recently discussed by Djellout, Guillin and Wu ([13] Theorem 2.3) where the following is proved

Theorem 1.4. [13] *μ satisfies T_1 if and only if there exist $\varepsilon > 0$ and $x_0 \in E$ such that*

$$(EI_\varepsilon(2)) \qquad \int_E e^{\varepsilon\, d^2(x,x_0)}\,\mu(dx) < +\infty\,.$$

Unfortunately T_1 is not well adapted to dimension free bounds, while T_2 is, as shown by Talagrand ([21]). The first example of measure satisfying T_2 is the standard Gaussian measure ([21]), for which $C = 1$. When E is a complete smooth Riemannian manifold of finite dimension, with d the geodesic distance and dx the volume measure, Otto and Villani ([18]) have studied the T_2 property for absolutely continuous probability measures (Boltzmann measures)

$$(\mathbf{B}.\mathbf{M}) \qquad \mu(dx) = e^{-V(x)}\,dx\,,$$

for $V \in C^2(E)$ in connection with the logarithmic-Sobolev inequality. Their method was recently improved by Wang ([26]) in order to skip the curvature assumption made in [18].

In the sequel we shall assume that μ is a Boltzmann measure with $V \in C^3$, and that the diffusion process built on E with generator $L = 1/2\,div(\nabla) - 1/2\,\nabla V.\nabla$ is non explosive. We denote $(P_t)_{t\geq 0}$ the associated semigroup.
This is assumption (A) in [26]. Conditions for non explosion are known. Here are two different among others when $E = \mathbb{R}^d$:

- there exists some ψ such that $\psi(x) \to +\infty$ as $|x| \to +\infty$ and $\Delta\psi - \nabla V.\nabla\psi$ is bounded from above,
- $\int |\nabla V|^2\, d\mu < +\infty$.

For the first two see e.g. [20] p.26 (replacing V therein by ψ), for the second one see e.g. [10].

Then

Theorem 1.5. [18], [4], [26], *(also see* [12]*)* *If μ satisfies the logarithmic-Sobolev inequality (L.S.I)*

$$\int g^2 \log(g^2)\, d\mu - \Big(\int g^2\, d\mu\Big) \log\Big(\int g^2\, d\mu\Big) \leq 2C \int |\nabla g|^2\, d\mu\,,$$

for all smooth g, then μ satisfies $T_2(C)$.

A partial converse of Theorem 1.5 is also shown in [18] (Corollary 3.1), namely

Theorem 1.6. [18], [4] *Let $E = \mathbb{R}^n$. If μ satisfies $T_2(C)$ and the curvature assumption*

$$Hess(V) \geq K\, Id_n$$

for some $K \in \mathbb{R}$, then μ satisfies a logarithmic-Sobolev inequality (with some new constant $\bar{C}$), provided

$$1 + K\,C > 0\,.$$

The latter restriction is very important and has to be compared with Wang's results ([24] and [25]) telling that a logarithmic-Sobolev inequality holds provided the curvature assumption above and the integrability condition $EI_\varepsilon(2)$ in Theorem 1.4 hold with

$$\varepsilon + K > 0\,.$$

In other words, according to Theorem 1.4 and Theorem 1.6, under the curvature assumption, log-Sobolev, $T_1(C_1)$, $T_2(C_2)$ are all equivalent for appropriate constants C_1 and C_2.

Whether this equivalence holds without restrictions on the constants or not was left open by these authors. One aim of this paper is to show that this equivalence does not hold. Before stating a more precise result, let us complete the picture.

On one hand, as shown by Otto and Villani (see [4] subsection 4.1 for another approach)

Theorem 1.7. *If μ satisfies $T_2(C)$ then μ satisfies the Poincaré (or spectral gap) inequality (S.G.I) i.e. for all smooth f ,*

$$Var_\mu(f) \leq C \int |\nabla f|^2\, d\mu\,.$$

This result gives us a first hint on what should be the difference between T_1 and T_2 as T_1 is well known to hold when $(S.G.I.)$ fails (see [13], Remark 2.4).

On the other hand, the difference between T_2 and T_1 is only concerned with small entropies due to the following elementary

Lemma 1.8. *Assume that μ satisfies $EI_\varepsilon(p)$ for some $\varepsilon > 0$. Then there exists a constant $C(\varepsilon)$ such that for all ν satisfying $H(\nu, \mu) \geq 1$, $W_p^p(\nu, \mu) \leq C(\varepsilon)\, H(\nu, \mu)$.*
Here $EI_\varepsilon(p)$ is defined as in 1.4 with d^p instead of d^2.

Hence the transportation inequalities T_2 and T_1 are "equivalent" for large entropy. Since Marton's method is essentially concerned with large entropy, T_2 cannot furnish a better concentration result than T_1. The main interest of T_2 for the concentration of measure phenomenon is thus that T_2 can be tensorized.

At this point we shall mention that the proof of Lemma 1.8 is using the trivial independent coupling. We learned from F. Bolley and C. Villani [7] that, using a less trivial coupling in [22], this statement can be greatly improved, in particular

Proposition 1.9. Bolley and Villani

$$EI_\varepsilon(p) \qquad \Rightarrow \qquad W_p^p(\nu,\mu) \,\leq\, C(\varepsilon)\left(H(\nu,\mu) \,+\, H^{\frac{1}{2}}(\nu,\mu)\right) .$$

Bolley and Villani are then able to get back Theorem 1.4 i.e. $EI_\varepsilon(2)$ is equivalent to the transportation inequality T_1, but with some better constant than in [13].

Let us come to the contents of the present paper where we shall mainly focus on W_2.

In section 2 we shall show that (S.G.I) implies some quadratic transportation inequality for measures ν with a *bounded* density. Actually we prove an interpolation result between (S.G.I) and (L.S.I) through a family of inequalities $I(\alpha)$ introduced by Latala and Oleszkiewicz (see [14]) for $0 \leq \alpha \leq 1$,

$$I(\alpha) \qquad \sup_{p\in[1,2)} \frac{\int f^2\,d\mu \,-\, \left(\int |f|^p\,d\mu\right)^{\frac{2}{p}}}{(2\,-\,p)^\alpha} \,\leq\, C(\alpha)\int |\nabla f|^2\,d\mu\,. \tag{1.10}$$

Note that $I(0)$ is the Poincaré inequality and $I(1)$ reduces to the logarithmic-Sobolev inequality. Our first result is the following

Theorem 1.11.
Let μ be as above. If $I(\alpha)$ holds then for all ν such that $\parallel \frac{d\nu}{d\mu} \parallel_\infty \leq\, K$ the following modified transportation inequality holds

$$W_2(\nu,\mu) \,\leq\, D(\alpha,K)\,\sqrt{C(\alpha)\,H(\nu,\mu)}\,,$$

where

$$D(\alpha,K) \,=\, 8\,\exp\left(\frac{1-\alpha}{2}\,(1\,-\,\log(1-\alpha))\right)\,(\log\,K)^{\frac{1-\alpha}{2}}\ ,\ \textit{for } K\geq e^{1-\alpha}\,,$$

and

$$D(\alpha,K) \,=\, 8\,\sqrt{K}\ \ \textit{for } K\leq e^{1-\alpha}\,.$$

Remark that the previous Theorem and Marton's trick allow to recover the concentration property shown in [14]. Indeed, recall (1.3) and remark that the interesting K is given by $K \,=\, 1/\mu(A_r^c)$. We immediately see that if $I(\alpha)$ holds, $\mu(A_r^c)$ behaves like $\exp\left(-C\,r^{\frac{2}{2-\alpha}}\right)$. Also note that for $\alpha = 1$ we recover Otto-Villani result, since K can be arbitrarily chosen.

We refer to [26], [23], [3], [11] and [2] for more refined results in connection with $I(\alpha)$.

If the meaning of a transportation inequality reduced to bounded densities is not clear, the previous Theorem nevertheless allows us to obtain as a first consequence the following (weak) transportation inequality proved in section 3

Corollary 1.12.
Let μ be as above. If $EI_\varepsilon(2)$ and $I(\alpha)$ are satisfied, there exists some constant C such that

$$W_2^2(\nu,\mu) \,\leq\, C\,\left(1\,+\,(1-\alpha)^{-\,\frac{1-\alpha}{2-\alpha}}\,\left(\log^+(1/H(\nu,\mu))\right)^{\frac{1-\alpha}{2-\alpha}}\right)\,H(\nu,\mu)\,.$$

Remark that this corollary presents an interpolation between Poincaré and log-Sobolev inequality, which is expected to tensorize, but it is not dimension free, as a $(\log(n))^{\frac{1-\alpha}{2-\alpha}}$ appears in the tensorization procedure, which is however better than the factor n obtained with the sole T_1. We will see that we have to impose conditions slightly stronger than Poincaré's inequality (at least in the real line case) to get rid of the extra "$\sqrt{log}$" term leading thus to the true T_2 inequality.

Actually one can get some equivalences between various inequalities for *bounded* functions. This will be done in Section 2. The result is the following

Theorem 1.13.
Let μ be as above. Then (up to the constants) the following statements are equivalent

(1) $I(0)$ *(i.e. the Poincaré inequality) holds,*
(2) *the (modified) transportation inequality in Theorem 1.11 holds for $\alpha = 0$,*
(3) *the following (restricted) log-Sobolev inequality holds: for all nonnegative h such that $\int h\, d\mu = 1$,*

$$\int h\, \log h\, d\mu \,\leq\, C(1+\log(\parallel h \parallel_\infty)) \int \frac{|\nabla h|^2}{h}\, d\mu\,,$$

(4) *there exist some C and some $K > 1$ such that the previous log-Sobolev inequality holds for all h as above and bounded by K.*

Point (3) is some kind of modified log-Sobolev inequality, i.e. available for some subset of probability densities. Such modified inequalities were first introduced by Bobkov and Ledoux [6] who have considered the set of h such that $|\nabla \log(h)|$ is bounded by a small enough constant. These inequalities are particularly well suited for concentration estimates (as we said we recover some weaker form of concentration results). However the set considered by Bobkov and Ledoux is not P_t stable. The set of densities bounded by some constant K is P_t stable, so that (3) tells that relative entropy is exponentially decaying i.e for some C

$$H(P_t h\, \mu\,,\, \mu) \,\leq\, e^{-\frac{Ct}{1+\log(\|h\|_\infty)}}\, H(h\mu\,,\, \mu)\,. \tag{1.14}$$

If $I(\alpha)$ holds we think that the decay is controlled by $\log^{1-\alpha}(\parallel h \parallel_\infty)$.

In order to deduce Corollary 1.12 from Theorem 1.11 we are using a simple truncation argument (truncating with a constant). If one wants to improve this result, one has to truncate with some function $C\, \exp(\eta\, d^2)$. This is explained in Section 4. This section is devoted to various comments and technical results allowing to build measures satisfying T_2. Some of them may have their own interest. We then obtain the following corollaries

Corollary 1.15. *Assume that $EI_\varepsilon(2)$ holds. Assume in addition that there exist some $q > 1$ and some $M > 0$ such that for all nonnegative h with $\int h\, d\mu = 1$ and $\int h^q\, d\mu \leq M$,*

$$\int h\, \log h\, d\mu \,\leq\, C(M) \int \frac{|\nabla h|^2}{h}\, d\mu\,,$$

then T_2 holds.

Corollary 1.16. *In the previous corollary we can replace the condition $\int h^q\, d\mu \leq M$ by $h(x) \,\leq\, K\, e^{\eta\, d^2(x,x_0)}$ for some small enough η, (hence the constant $C(M)$ by $C(K,\eta)$) provided $Hess(V) \geq RId$ for some $R \in \mathbb{R}$ and $E = \mathbb{R}^N$.*

We shall give a proof of Corollary 1.16 in section 4.3. In both Corollaries one may reduce a little bit the set of allowed densities assuming in addition that $h \geq a$ for some $a > 0$.

To get rid of the curvature assumption one has to call upon the methods in [4], namely Herbst's argument and the beautiful characterization of T_2 obtained by Bobkov-Götze [5]

Theorem 1.17. *Assume that $EI_\varepsilon(2)$ holds. If the restricted logarithmic Sobolev inequality*

$$\int f^2 \log f^2 \, d\mu - \left(\int f^2 \, d\mu\right) \log \left(\int f^2 \, d\mu\right) \leq C \int |\nabla f|^2 \, d\mu \, ,$$

holds for all

$$f^2 \leq \left(\int f^2 \, d\mu\right) e^{\eta(d^2(x,x_0)+\int d^2(y,x_0)\, \mu(dy))} \tag{1.18}$$

for some $\eta < \varepsilon/2$, then T_2 holds.

We have chosen to write the hypotheses in a slightly different form but this result is of course the same as Corollary 1.16 without the curvature assumption. Note that it generalizes slightly the principal result in [4]: a full logarithmic Sobolev inequality is too strong to get T_2.

The proof of this Theorem will be given at the end of section 4.4. It is an almost immediate adaptation of the section 3.3 in [4]. Since it is the most general result of the section, the reader should ask about the interest of the remainder of section 4. As we said some of the results therein have their own interest, but the comparison between both approaches (Otto and Villani coupling on one hand, Infimum convolution on the other hand) is more interesting. Indeed both approaches are qualitatively very different : Otto-Villani's coupling yield local results (if one wants to get some estimate for the Wasserstein distance for a single h, one only needs to look at $P_t h$) while the infimum convolution approach is global (since variational) in nature. In particular, for the latter approach we did not succeed in reducing the problem to small entropies and/or bounded below densities.

In the final Section 5, we give some Hardy's like conditions implying a T_2 inequality for measure on the real line. It finally enables us to build explicitly a potential V such that μ satisfies T_2 but does not satisfy a logarithmic Sobolev inequality, however with unbounded below curvature. These examples show that T_2 is strictly weaker than (L.S.I), which was in fact the primary goal of the authors, the second one being still an open question: an explicit characterization of the T_2 inequality.

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2. **Modified transportation and functional inequalities.**

In this section we shall discuss several functional inequalities involving bounded functions. We start with the proof of Theorem 1.11

Proof. of Theorem 1.11.
Let ν be a probability measure such that $h = \frac{d\nu}{d\mu}$ satisfies $0 < \beta \leq h(x) \leq K$. We assume first that $h \in \mathbb{D}$ i.e. is the sum of a constant and a C^∞ function with compact support.
Let P_t denotes the μ symmetric semigroup with generator $L = 1/2\, div(\nabla) - 1/2\, \nabla V.\nabla$, and define $\nu_t = (P_t h)\mu$.
Our method relies on Otto-Villani's coupling [18], refined by Wang [26], whose idea is the following: to provide a coupling between ν_t and ν_{t+s} as $\pi_s(dx,dy) = \nu_t(dx)\delta_{\varphi_s(x)}(dy)$ where φ_s is the well defined unique (under our assumptions) solution of the p.d.e.

$$\frac{\mathrm{d}}{\mathrm{d}s}\varphi_s = -\xi_{t+s} \circ \varphi_s, \quad \varphi_0 = Id, S \geq 0$$

with $\xi_{t+s}(x) = \nabla \log P_{t+s}h(x)$.
Then, according to Otto and Villani [18], Lemma 2 (or more exactly its proof), or Wang [26] section 3,

$$A = \frac{d^+}{dt}\left(-W_2(\nu_t,\mu)\right) \leq \limsup_{s \to 0^+} \frac{1}{s} W_2(\nu_t, \nu_{t+s}) \leq 2\left(\int |\nabla \sqrt{P_t h}|^2\, d\mu\right)^{\frac{1}{2}}. \tag{2.1}$$

Using $I(\alpha)$ we obtain for all $1 \leq p < 2$,

$$A \leq \frac{2\sqrt{C(\alpha)\,(2-p)^\alpha}\ \int |\nabla\sqrt{P_t h}|^2\, d\mu}{\sqrt{1 - \left(\int (P_t h)^{\frac{p}{2}}\, d\mu\right)^{\frac{2}{p}}}}. \tag{2.2}$$

Now using a similar argument as in Lemma 3.1 in [26] or simply the fact that $\mathbb{D}$ is a nice core for the diffusion semigroup, the following computation is rigorous

$$\begin{aligned} \frac{d}{dt}\left(1 - \left(\int (P_t h)^{\frac{p}{2}}\, d\mu\right)^{\frac{1}{p}}\right) &= -\frac{1}{2}\left(\int (P_t h)^{\frac{p}{2}}\, d\mu\right)^{\frac{1}{p}-1} \int (P_t h)^{\frac{p}{2}-1}\, LP_t h\, d\mu \\ &= \frac{1}{4}\left(\int (P_t h)^{\frac{p}{2}}\, d\mu\right)^{\frac{1}{p}-1} \int (\frac{p}{2}-1)\,(P_t h)^{\frac{p}{2}-2}\,|\nabla P_t h|^2\, d\mu \\ &= \frac{1}{2}\left(\int (P_t h)^{\frac{p}{2}}\, d\mu\right)^{\frac{1}{p}-1} \int (p-2)\,(P_t h)^{\frac{p}{2}-1}\,|\nabla\sqrt{P_t h}|^2\, d\mu \\ &\leq 0\,. \end{aligned} \tag{2.3}$$

But since $h \leq K$, $P_t h \leq K$ hence according to (2.2) and (2.3)

$$A \leq \frac{2\sqrt{C(\alpha)\,(2-p)^\alpha}\ \int |\nabla\sqrt{P_t h}|^2\, \frac{K^{1-\frac{p}{2}}}{(P_t h)^{1-\frac{p}{2}}}\, d\mu}{\sqrt{1 - \left(\int (P_t h)^{\frac{p}{2}}\, d\mu\right)^{\frac{1}{p}}}\sqrt{1 + \left(\int (P_t h)^{\frac{p}{2}}\, d\mu\right)^{\frac{1}{p}}}} \tag{2.4}$$

$$\leq \; - \frac{4\sqrt{C(\alpha)\,(2-p)^{\alpha}}}{\sqrt{1-\left(\int (P_t h)^{\frac{p}{2}}\,d\mu\right)^{\frac{1}{p}}}} \; \frac{K^{1-\frac{p}{2}}}{(2-p)\left(\int (P_t h)^{\frac{p}{2}} d\mu\right)^{\frac{1}{p}-1}} \frac{d}{dt}\left(1-\left(\int (P_t h)^{\frac{p}{2}}\,d\mu\right)^{\frac{1}{p}}\right)$$

$$\leq \; 8\sqrt{C(\alpha)}\,(2-p)^{\frac{\alpha}{2}-1}\,K^{1-\frac{p}{2}}\left(-\frac{d}{dt}\sqrt{\left(1-\left(\int (P_t h)^{\frac{p}{2}}\,d\mu\right)^{\frac{1}{p}}\right)}\right).$$

For the latter inequality we have used $\int (P_t h)^{\frac{p}{2}}\,d\mu \leq 1$.

It remains to integrate in t. Since $I(\alpha)$ implies (S.G.I), we know that $P_t h$ goes to 1 in $\mathbb{L}^2(\mu)$ as t goes to infinity. Arguing as in [26] p.10, one can show that $W_2(\nu_t,\mu)$ goes to 0 as t goes to ∞, so that we have obtained

$$\begin{aligned} W_2(\nu,\mu) \;&\leq\; 8\sqrt{C(\alpha)}\,(2-p)^{\frac{\alpha}{2}-1}\,K^{1-\frac{p}{2}}\sqrt{\left(1-\left(\int h^{\frac{p}{2}}\,d\mu\right)^{\frac{1}{p}}\right)} \\ &\leq\; 8\sqrt{C(\alpha)}\,(2-p)^{\frac{\alpha}{2}-1}\,K^{1-\frac{p}{2}}\sqrt{\left(1-\left(\int h^{\frac{p}{2}}\,d\mu\right)^{\frac{2}{p}}\right)}. \end{aligned} \tag{2.5}$$

Now we shall use the two following elementary inequalities for $p \in [1,2)$:

- $1 - u^{\frac{2}{p}} \leq \frac{2}{p}(1-u)$ for $u \in [0,1]$,
- $\xi \log\xi + 1 - \xi \geq 0$ for $\xi > 0$.

The latter yields $\log\xi^k \geq 1 - \xi^{-k}$, hence $\xi \log\xi^k \geq \xi - \xi^{1-k}$ and finally for $k = 1-\frac{p}{2}$, $(1-\frac{p}{2})\,\xi \log\xi \geq \xi - \xi^{\frac{p}{2}}$. We apply this with $h(x) = \xi$, integrate with respect to μ and use the former inequality in order to get

$$1 - \left(\int h^{\frac{p}{2}} d\mu\right)^{\frac{2}{p}} \leq \frac{2}{p}\,(1-\frac{p}{2})\,H(\nu,\mu)\,. \tag{2.6}$$

Plugging (2.6) into (2.5) furnishes (using $p \geq 1$)

$$W_2(\nu,\mu) \leq 8\sqrt{C(\alpha)}\,(2-p)^{\frac{\alpha-1}{2}}\,K^{1-\frac{p}{2}}\sqrt{H(\nu,\mu)}\,. \tag{2.7}$$

It is now enough to optimize in p, just taking care that $p \geq 1$. The optimal value is obtained for $2-p = \frac{1-\alpha}{\log K}$ if $K \geq e^{1-\alpha}$ and for $p=1$ otherwise. A simple calculation yields the exact bound in Theorem 1.11.

It remains to extend the result to densities h that are no more bounded away from 0, by using standard tools. □

This modified transportation inequality does not seem tensorizable. Now we come to the proof of Theorem 1.13.

Proof. of Theorem 1.13.

The first implication is given by the previous theorem. The equivalence between (1) and (2) follows from Otto-Villani's way of proof of $T_2 \implies SGI$. Namely choose some smooth

f with compact support (hence bounded) such that $\int f d\mu = 0$, and set for ϵ small enough $\mu_\epsilon = (1+\epsilon f)\mu$. Recall that

$$H(\mu_\epsilon, \mu)/\epsilon^2 \to \int f^2 d\mu \,.$$

By Taylor formula at order 2, as f is smooth and compactly supported, one may find a constant C such that for all x, y

$$f(x) - f(y) \leq |\nabla f(y)||x-y| + C|x-y|^2.$$

Denote by π_ϵ an "optimal coupling" for the Wasserstein distance between μ and μ_ϵ, then for ϵ small enough

$$\begin{aligned}
\int f d\left(\frac{\mu_\epsilon - \mu}{\epsilon}\right) &= \frac{1}{\epsilon}\int (f(x)-f(y)) d\pi_\epsilon \\
&\leq \frac{1}{\epsilon}\int |\nabla f(y)||x-y| d\pi_\epsilon + \frac{C}{\epsilon}\int |x-y|^2 d\pi_\epsilon \\
&\leq \frac{1}{\epsilon}\sqrt{\int |\nabla f|^2 d\mu}\, W_2(\mu_\epsilon,\mu) \,+\, \frac{C}{\epsilon}\, W_2^2(\mu_\epsilon,\mu) \\
&\leq 8\sqrt{1+\epsilon||f||_\infty}\sqrt{\int |\nabla f|^2 d\mu}\,\sqrt{C(0)H(\mu_\epsilon,\mu)/\epsilon^2} \,+ \\
&\qquad + \frac{64\, C}{\epsilon}\,(1+\epsilon||f||_\infty)\, C(0) H(\mu_\epsilon,\mu)\,.
\end{aligned}$$

Let ϵ tend to 0. In the limit we obtain that for all those f

$$\int f^2 d\mu \,\leq\, 8\sqrt{C(0)}\sqrt{\int |\nabla f|^2 d\mu}\sqrt{\int f^2 d\mu}$$

which gives the SGI (but with a worse constant) for all those f, and then extend by density.

Let us come to the restricted log-Sobolev inequality. That (4) implies (1) is standard. To prove (2) implies (3), one can get a precise result by using the robust version of the logarithmic Sobolev inequality proved in [8] (formula (2.6)) namely

$$\int f^2 \log f^2\, d\mu \,\leq\, \frac{t}{\beta}\int |\nabla f|^2\, d\mu \,+\, \frac{2}{\beta}\log\left(\int f^{1+\beta}\, P_t f\, d\mu\right), \tag{2.8}$$

that holds for any P_t (satisfying the assumptions in the introduction), any nonnegative β and any nonnegative f such that $\int f^2 d\mu = 1$.

Indeed, for $\beta \leq 1$, $\int f\, d\mu$ and $\int f^{1+\beta} d\mu$ are less or equal to 1, hence

$$\log\left(\int f^{1+\beta}\, P_t f\, d\mu\right) \,\leq$$

$$\leq \log\left(1 + \int f^{1+\beta}\, P_t(f - \int f d\mu)\, d\mu\right)$$
$$\leq \int f^{1+\beta}\, P_t(f - \int f d\mu)\, d\mu$$
$$\leq Var_\mu^{\frac{1}{2}}(f^{1+\beta})\, Var_\mu^{\frac{1}{2}}(P_t f)\,.$$

If Poincaré holds with constant C_P we thus obtain

$$\int f^2\, \log f^2\, d\mu \,\leq\, \frac{t}{\beta} \int |\nabla f|^2\, d\mu + \frac{2(1+\beta)}{\beta}\, C_P\, e^{-\frac{t}{C_P}} \left(\int |\nabla f|^2 d\mu\right)^{\frac{1}{2}} \left(\int |\nabla f|^2 f^{2\beta} d\mu\right)^{\frac{1}{2}},$$

and finally

$$\int f^2\, \log f^2\, d\mu \,\leq\, \left(\frac{t}{\beta} + \frac{2(1+\beta)}{\beta}\, C_P\, e^{-\frac{t}{C_P}}\, \parallel f^2 \parallel_\infty^{\frac{\beta}{2}}\right) \int |\nabla f|^2 d\mu\,.$$

An easy optimization in t shows that the best choice of β is $\beta = 1$ and yields, for $h = f^2$,

$$\int h\, \log h\, d\mu \,\leq\, C_P\, (2\log 2 + \frac{1}{2}\, \log \parallel h \parallel_\infty) \int \frac{|\nabla h|^2}{h}\, d\mu\,. \tag{2.9}$$

□

Theorem 1.13 gives another characterization of the Spectral Gap property in terms of transportation inequalities. It has to be compared with Corollary 5.1 in [4], where (S.G.I) is shown to be equivalent to some W_L transportation inequality (see section 3).

Remark 2.10. One may prove (2) implies (3) in an elementary way using once again a truncation argument and careful calculus but with less precise constants.

Remark 2.11. One can easily get a similar but weaker statement without any effort. Indeed recall that $u \log u - u + 1 \geq 0$. Writing this inequality with $v = 1/u$ and then multiplying by v^2 yields $v \log v \leq v^2 - v$. Applying this with $v = h(x)$ and integrating with respect to μ yields

$$\int h\, \log h\, d\mu \,\leq\, Var_\mu(h)\,,$$

if h is a density of probability.
Hence if Poincaré holds

$$\int h\, \log h\, d\mu \,\leq\, C_P \int |\nabla h|^2 d\mu \,\leq\, C_P\, \parallel h \parallel_\infty \int \frac{|\nabla h|^2}{h}\, d\mu\,.$$

3. **Application to Transportation inequalities.**

In this section we shall see how to use the functional inequalities of the previous section in order to obtain transportation inequalities (in particular Corollary 1.12. We shall also compare this results with other similar results in the literature.

We start this section by the proof of the elementary Lemma 1.8 showing that the obstruction for T_2 to hold is in a neighborhood of μ. Notice that many results in this section are available in a general metric space.

Proof. of Lemma 1.8.
Introduce the Young function

$$\tau(u) \,=\, u\,\log^+(u)\,, \tag{3.1}$$

and its Legendre conjugate function $\tau^*(v) \,=\, v\,\mathbb{I}_{v<1} \,+\, e^{v-1}\,\mathbb{I}_{v\geq 1}$.
Among all possible coupling of (μ,ν), the simplest one is the independent one i.e. if we denote $h \,=\, \frac{d\nu}{d\mu}$,

$$\pi(dx,dy) \,=\, h(x)\,\mu(dx)\,\mu(dy)\,.$$

Accordingly

$$\begin{aligned} W_p^p(\nu,\mu) \;&\leq\; \int d^p(x,y)\,h(x)\,\mu(dx)\,\mu(dy) \\ &\leq\; 2\,N_\tau(h)\,N_{\tau^*}(d^p)\,, \end{aligned}$$

where N_τ and N_{τ^*} are the gauge norms in the corresponding Orlicz spaces, the second inequality being the classical Hölder-Orlicz inequality (see e.g. [19] for all concerned with Orlicz spaces). Recall that the gauge norm for a general Young function ψ is defined as

$$N_\psi(g) \,=\, \inf\{\lambda>0\,, \int \psi(g/\lambda)(x,y)\,\mu(dx)\,\mu(dy) \,\leq\, 1\}\,,$$

such that an easy convexity argument yields

$$N_\psi(g) \,\leq\, \max\{1\,, \int \psi(g) d\mu\otimes d\mu\}\,. \tag{3.2}$$

In addition remark that

$$\int h\,\log^+(h) \,=\, \int h\,\log(h) \,-\, \int_{h<1} h\,\log(h) \,\leq\, \int h\,\log(h) \,+\, 1/e\,.$$

Hence if $H(\nu,\mu) \,\geq\, 1$,

$$1 \,\leq\, \int h\,\log^+(h) \,\leq\, (1+1/e)\,H(\nu,\mu)\,,$$

and according to (3.2) and what precedes

$$W_p^p(\nu,\mu) \,\leq\, 2(1+1/e)\,N_{\tau^*}(d^p)\,H(\nu,\mu)\,.$$

Finally, thanks to $I_\varepsilon(p)$, $N_{\tau^*}(d^p) \,<\, +\infty$ and the result follows. □

One can improve the preceding result by showing that (up to the constant) it holds for $H(\nu,\mu)$ bounded away from 0. But as quoted in Proposition 1.9 one can also get a precise bound for the behavior of the Wasserstein distances when entropy goes to 0.

Theorem 1.13 suggests that working with bounded density is natural with regard to transportation cost inequalities, starting with Poincaré inequality. We are so tempted to use some truncation for ν i.e. if $a>0$ we define

$$\nu_a \,=\, (1/\nu(h\leq a))\,h\,\mathbb{I}_{h\leq a}\,\mu\,, \tag{3.3}$$

and look at what happens. According to Lemma 1.8 and (3.2) we may and will assume that $H(\nu,\mu)$ is small enough. We start with two elementary lemmata.

Lemma 3.4. *Let $\nu \,=\, h\,\mu$ be a probability measure. If $a>e$, then*

(1) $H(\nu,\mu) \geq \big(1 - 1/\log a\big) \int_{h>a} h \log h \, d\mu$,

(2) $\nu(h>a) \leq \big(1 / (\log a - 1)\big) \, H(\nu,\mu)$.

Proof. Again we start with $u \log u + 1 - u \geq 0$ which yields

$$\int_{h\leq a} h \log h \, d\mu + 1 - \int_{h \leq a} h \, d\mu \geq 0 \,,$$

hence

$$H(\nu,\mu) \geq \int_{h>a} h \log h \, d\mu - \nu(h>a) \,.$$

(2) follows immediately since $\log h > \log a$ on $\{h>a\}$. For (1) we have

$$\nu(h>a) \leq \int_{h>a} \frac{\log h}{\log a} \, h \, d\mu = (1/\log a) \int_{h>a} h \log h \, d\mu \,.$$

□

Lemma 3.5. *Let $\nu = h\,\mu$ be a probability measure such that $H(\nu,\mu) \leq 1/2$. If $a > e^{\frac{3}{2}}$ and ν_a is given by (3.3), then*

$$H(\nu_a,\mu) \leq \left(1 + \frac{1}{2(\log a - 3/2)} + \frac{2}{\log a - 1}\right) H(\nu,\mu) \,.$$

Proof.

$$\begin{aligned} H(\nu_a,\mu) &= \int \frac{h \, 1\!\!\mathrm{I}_{h\leq a}}{\nu(h\leq a)} \log\left(\frac{h}{\nu(h\leq a)}\right) d\mu \\ &\leq H(\nu,\mu) + ((1/\nu(h\leq a)) - 1) \int_{h\leq a} h \log h \, d\mu \\ &\quad - \log(\nu(h\leq a)) - \int_{h>a} h \log h \, d\mu \\ &\leq H(\nu,\mu) + \frac{\nu(h>a)}{\nu(h\leq a)} H(\nu,\mu) - \log(1-\nu(h>a)) \,. \end{aligned}$$

But if $0 \leq x \leq 1/2$, $-\log(1-x) \leq 2x$, hence according to (3.4)(2), if $H(\nu,\mu) \leq 1/2$, $-\log(1-\nu(h>a)) \leq (2/(\log a - 1)) \, H(\nu,\mu)$ and

$$\frac{\nu(h>a)}{\nu(h\leq a)} \leq \frac{H(\nu,\mu)}{\log a - 1 - H(\nu,\mu)}$$

and we get the desired result. □

We turn now to the proof of Corollary 1.12.

Proof. of Corollary 1.12. We assume that $I(\alpha)$ and the exponential integrability condition $EI_\varepsilon(2)$ are satisfied. Let consider a positive constant C that may change line to line, but which does not depend neither on a nor α.

Since W_2 is a distance, it holds

$$W_2(\nu,\mu) \leq W_2(\nu_a,\mu) + W_2(\nu_a,\nu) \,. \tag{3.6}$$

But, on one hand, since $I(\alpha)$ holds, according to Theorem 1.11, for a large enough

$$\begin{aligned} W_2^2(\nu_a,\mu) &\leq C\,\log^{1-\alpha}(a/\nu(h\leq a))\,H(\nu_a,\mu) \\ &\leq 2C\,\log^{1-\alpha}(a/\nu(h\leq a))\,H(\nu,\mu)\,, \end{aligned}$$

provided $H(\nu,\mu)\leq 1/2$ thanks to Lemma 3.5.

On the other hand, a classical result in mass transportation theory (see [22] Proposition 7.10) tells that for any x_0

$$W_2^2(\nu',\nu)\;\leq\;2\int d^2(x,x_0)\,|h'\,-\,h|\,d\mu\,. \tag{3.7}$$

Applying (3.7) with $\nu'=\nu_a$ yields, assuming again that $H(\nu,\mu)\leq 1/2$

$$\begin{aligned} W_2^2(\nu_a,\nu) &\leq 2\,\frac{\nu(h>a)}{\nu(h\leq a)}\int_{h\leq a} d^2(x,x_0)\,h\,d\mu\,+\,2\int_{h>a} d^2(x,x_0)\,d\nu \\ &\leq C\,\left(H(\nu,\mu)\,+\,N_\tau(h\,\mathbb{1}_{h>a})\right)\,, \end{aligned}$$

according to Lemma 3.4 (2) and Orlicz-Hölder inequality (see the proof of Lemma 1.8 at the beginning of the section), since $EI_\varepsilon(2)$ is satisfied for the latter.

Plugging all this into (3.6), we get that there exists a constant C such that

$$W_2^2(\nu,\mu)\;\leq\;C\,\left(\log^{1-\alpha}(a)\,H(\nu,\mu)\,+\,N_\tau(h\,\mathbb{1}_{h>a})\right)\,. \tag{3.8}$$

Now we choose $a\,=\,1/H^q(\mu,\nu)$ for some $q>0$ (recall that we may assume that $H(\nu,\mu)$ is small enough). Lemma 3.4(2) furnishes

$$\nu(h>a)\;\leq\;C\frac{H(\nu,\mu)}{q\,\log(1/H(\nu,\mu))}\,,$$

so that it is easily seen that

$$\begin{aligned} N_\tau(h\,\mathbb{1}_{h>a}) &= \inf\{\lambda>0;\,\int\tau(h\,\mathbb{1}_{h>a})d\mu\} \\ &\leq \int h\,\mathbb{1}_{h>a}\log^+ h d\mu \\ &\leq C\,q^{-1}\,H(\nu,\mu)\,. \end{aligned}$$

We have thus obtained

$$W_2^2(\nu,\mu)\;\leq\;C\,\left((q\,\log(1/H(\nu,\mu)))^{1-\alpha}\,+\,1/q\right)\,H(\nu,\mu)\,,$$

so that optimizing in q ($q^{2-\alpha}=1/((1-\alpha)\log^{1-\alpha}(1/H))$ so that a is big for small entropy H) we complete the proof of Corollary 1.12. □

Remark 3.9. It is worthwhile noticing that $N_\tau(h\,\mathbb{1}_{h>a})$ behaves like

$$H(\nu,\mu)\,\frac{\log(1/H(\nu,\mu))}{\log a}$$

for small entropies. This is why some extra logarithm appears in Corollary 1.12.

Also notice that a similar result can be directly obtained using the W_L transportation inequality in [4], when $\alpha=0$ i.e. when Poincaré holds. We briefly indicate how to do below (some of the bounds are clearly non sharp).

Indeed taking an optimal coupling Π (or an almost optimal coupling, and then taking limits) for the L cost introduced in [4] section 5.2 (recall that L is, up to constants, the square of the distance for small distances and the distance for large ones), it is immediate that

$$\begin{aligned} W_2^2(\nu,\mu) &\leq C\,\sqrt{\log(1/H)}\,\int \mathbb{1}_{d^2(x,x_0)\leq q\log(1/H)}\,\mathbb{1}_{d^2(y,x_0)\leq q\log(1/H)}\,L(x,y)\,d\Pi \\ &\quad + \int (\mathbb{1}_{d^2(x,x_0)\geq q\log(1/H)}\,+\,\mathbb{1}_{d^2(y,x_0)\geq q\log(1/H)})\,d^2(x,y)\,d\Pi \\ &\leq C\,\sqrt{\log(1/H)}\,W_L\,+\,2\int \mathbb{1}_{d^2(x,x_0)\geq q\log(1/H)}\,(d^2(x,x_0)+C')\,(d\nu\,+\,d\mu)\,, \end{aligned}$$

where $C' = \int d^2(x,x_0)\,(d\mu + d\nu)$. Hence if Poincaré holds, according to [4] the first term in the right hand side is less than $C\,H\,\sqrt{\log(1/H)}$. For the second term first write

$$\begin{aligned} \int_{e^{d^2}\geq 1/H^q} d^2(x,x_0)\,d\mu &\leq \left(\int d^4\,d\mu\right)^{\frac{1}{2}}\,\left(\mu(d^2\,\geq\,q\,\log(1/H))\right)^{\frac{1}{2}} \\ &\leq C\left(\int d^4\,d\mu\right)^{\frac{1}{2}}\,H^{q\eta\,/\,2} \end{aligned}$$

where η is the constant of the gaussian concentration of μ, namely for which $EI_\eta(2)$ holds. It is now enough to choose q large enough for this term to be less than $C\,H$. It remains to study

$$\int_{e^{d^2}\geq 1/H^q} d^2(x,x_0)\,d\nu\,.$$

First remark that we can also assume that $h > K$, for K large enough, in this integral, since on the complement set $h \leq K$ we may use the previous inequality.
But according to Young's inequality

$$d^2(x,x_0)\,h(x)\,\leq\,(1/\varepsilon)\left(h(x)\,\log(h(x))\,+\,e^{\varepsilon\,d^2(x,x_0)}\right)\,,$$

so that the previous quantity can be controlled by

$$\int_{h>K} h\,\log h\,d\mu\,,$$

and

$$\int_{e^{d^2}\geq 1/H^q} e^{\varepsilon\,d^2(x,x_0)}\,d\mu\,.$$

Choosing ε small enough, and using $EI_\varepsilon(2)$ we may argue as before (replacing d^4 by $e^{2\,\varepsilon\,d^2}$) to bound the latter term by (constant times) H again, while for the former we can use Lemma 3.4 (1).

This remark shows that the optimal coupling (if it exists) for W_L achieves (up to the constants) the bound we obtained in Corollary 1.12. One can thus think that this bound is not optimal.

4. **Towards a criterion for Talagrand inequality.**

For simplicity for now on we assume that $E = \mathbb{R}^n$.
First recall (2.1)

$$\begin{aligned} A = \frac{d^+}{dt}\left(-W_2(\nu_t,\mu)\right) &\leq \limsup_{s\to 0^+} \frac{1}{s} W_2(\nu_t,\nu_{t+s}) \\ &\leq 2\left(\int |\nabla \sqrt{P_t h}|^2 \, d\mu\right)^{\frac{1}{2}} \\ &= \left(-\frac{d}{dt} H(\nu_t,\mu)\right)^{\frac{1}{2}} . \end{aligned}$$

Hence using the restricted log-Sobolev inequality given by Theorem 1.13 (3) for bounded functions we should directly recover the modified transportation inequality in Theorem 1.11 for $\alpha = 0$ (up to the constants). More generally the proof of Theorem 1.11 works for any P_t stable subset of functions for which a (restricted) logarithmic Sobolev inequality holds.

Since truncating by constants is not sufficient (in view of the preceding section), we shall first explain what kind of truncation is useful.

4.1. **A first reduction.**

Lemma 4.1. *If*

$$W_2\left(\frac{\nu+\mu}{2},\mu\right) \leq \sqrt{C\, H\left(\frac{\nu+\mu}{2},\mu\right)}$$

then

$$W_2(\nu,\mu) \leq \frac{\sqrt{C}}{\sqrt{2}-1}\sqrt{H(\nu,\mu)}\,.$$

Proof. Since W_2 is a distance

$$W_2(\nu,\mu) \leq W_2(\frac{\nu+\mu}{2},\mu) + W_2(\frac{\nu+\mu}{2},\nu)\,.$$

But W_2^2 is convex in each argument, hence

$$W_2(\frac{\nu+\mu}{2},\nu) \leq \sqrt{1/2}\, W_2(\nu,\mu)\,,$$

hence

$$W_2(\nu,\mu) \leq \frac{\sqrt{2}}{\sqrt{2}-1} W_2(\frac{\nu+\mu}{2},\mu)\,.$$

In addition since relative entropy is also convex, $H(\frac{\nu+\mu}{2},\mu) \leq \frac{1}{2} H(\nu,\mu)$ and we get the result. □

The meaning of this Lemma is clear: it is enough to show T_2 for densities h such that

$$\text{for all } x \quad , \quad h(x) \geq \frac{1}{2}\,. \tag{4.2}$$

Note that this set is P_t stable.

More useful is the following

Lemma 4.3. *Let $\gamma_\eta(x) = e^{\eta\, d^2(x,x_0)}$ for some nonnegative η, and define*

$$d\nu_{\gamma_\eta}^K \,=\, h_{K\gamma_\eta}\, d\mu \,=\, z^{-1}\, \min(h\,,\, K\gamma_\eta)\, d\mu\,,$$

where z is a normalizing constant, assuming that $K \geq e^2$.
Assume that $EI_\varepsilon(2)$is satisfied.
If for some $\eta < \varepsilon$,

$$W_2(\nu_{\gamma_\eta}^K, \mu) \,\leq\, \sqrt{C\, H(\nu_{\gamma_\eta}^K, \mu)}$$

then provided $H(\nu,\mu) \leq 1/2$ it holds

$$W_2(\nu,\mu) \,\leq\, \sqrt{C(\eta, K, C)\, H(\nu,\mu)}\,.$$

Proof. Note that $1 \geq z \geq \nu(h \leq K)$. So on one hand

$$\begin{aligned} H(\nu_{\gamma_\eta}^K, \mu) \;&=\; z^{-1}\, H(\nu,\mu) \,-\, z^{-1}\, \log(z) \\ &\quad -z^{-1} \int_{h\geq K\gamma_\eta} (h - K\gamma_\eta)\, \log(h/z)\, d\mu \,+ z^{-1} \int_{h\geq K\gamma_\eta} K\gamma_\eta\, \log(K\gamma_\eta/h)\, d\mu \\ &\leq\; z^{-1}\, H(\nu,\mu) \,-\, z^{-1}\, \log(z) \\ &\leq\; C(1 \,+\, \frac{2}{\log K \,-\, 1})\, H(\nu,\mu)\,, \end{aligned} \tag{4.4}$$

as soon as $H(\nu,\mu) \leq 1/2$. Hence

$$H(\nu_{\gamma_\eta}^K, \mu) \,\leq\, C(K)\, H(\nu,\mu)\,.$$

On the other hand, (3.7) with $\nu' = \nu_{\gamma_\eta}^K$ furnishes

$$W_2^2(\nu_{\gamma_\eta}^K, \nu) \,\leq\, 2 \int d^2(x,x_0)\, |h \,-\, h_{K\gamma_\eta}|\, d\mu\,. \tag{4.5}$$

and thus we get

$$\begin{aligned} W_2^2(\nu_{\gamma_\eta}^K, \nu) \;&\leq\; 2\,(z^{-1} - 1) \int_{h\leq K\gamma_\eta} d^2\, h\, d\mu \,+\, 2 \int_{h>K\gamma_\eta} d^2\, |1 - z^{-1} \frac{K\gamma_\eta}{h}|\, h\, d\mu \\ &\leq\; 2\frac{\nu(h>K)}{\nu(h\leq K)} \int d^2\, h\, d\mu \,+\, \frac{4}{\nu(h\leq K)} \int_{h>K\gamma_\eta} \frac{1}{\eta}\, \log(h/K)\, h\, d\mu \\ &\leq\; M(K,\eta)\, H(\nu,\mu)\,, \end{aligned} \tag{4.6}$$

where we used Lemma 3.4 and the smallness of $H(\nu,\mu)$ (in particular $N_\tau(h)$ is bounded, hence $\int d^2\, h\, d\mu$ is bounded by some constant only depending of η).
Putting all this together, we thus have shown for $H(\nu,\mu) \leq 1/2$

$$\begin{aligned} W_2(\nu,\mu) \;&\leq\; W_2(\nu_{\gamma_\eta}^K, \nu) \,+\, W_2(\nu_{\gamma_\eta}^K, \mu) \\ &\leq\; \sqrt{C(\eta, K, C)\, H(\nu,\mu)}\,. \end{aligned}$$

□

Once again this Lemma shows that we may assume that $h \leq K\gamma_\eta$ for some $\eta < \varepsilon$. But unfortunately this set does not seem to be P_t stable in general.
However it is included into the subset of densities h such that $\int h^q\, d\mu \leq M$ for some $q > 1$ and $M < +\infty$. Hence as a consequence we obtain Corollary 1.15.

Unfortunately we do not know whether the restricted log-Sobolev inequality stated in Corollary 1.15 is strictly weaker than the (full) log-Sobolev inequality or not. Notice that the method used in Remark 2.10 cannot be extended to the case $q < +\infty$. Indeed in this case one has to call upon Hölder inequality, thus introduce some power of the Dirichlet form.

4.2. **Decay of entropy.** In section 2 we bound the right hand side in (2.1) by some derivative, using Poincaré inequality, and then we integrated in time. One can also first integrate in time and then use inequalities. The result presented here means that this methodology is promised to failure, at least in the bounded curvature case.
Choose some weight function ξ such that $\int_0^{+\infty} \xi^{-1}(t)\,dt \,=\, 1$. Integrating (2.1) with respect to time yields (recall that $W_2(\nu_t, \mu)$ goes to 0 as t tends to infinity)

$$
\begin{aligned}
W_2(\nu,\mu) &\leq \int_0^{+\infty} \left(-\frac{d}{dt}\,H(\nu_t,\mu)\right)^{\frac{1}{2}}\,dt \\
&= \int_0^{+\infty} \left(-\,\xi^2(t)\,\frac{d}{dt}\,H(\nu_t,\mu)\right)^{\frac{1}{2}}\,\xi^{-1}(t)\,dt \\
&\leq \left(\int_0^{+\infty} -\,\xi(t)\,\frac{d}{dt}\,H(\nu_t,\mu)\,dt\right)^{\frac{1}{2}}\,,
\end{aligned}
\tag{4.7}
$$

where we have used Cauchy-Schwarz inequality for the probability measure $\xi^{-1}(t)\,dt$ to get the latter inequality. Hence provided

$$\xi(t)\,H(\nu_t,\mu) \;\text{ goes to }\; 0 \;\text{ as } t \text{ goes to }\; +\infty\,,$$

we have obtained

$$W_2^2(\nu,\mu) \,\leq\, \xi(0)\,H(\nu,\mu) \,+\, \int_0^{+\infty} \xi'(t)\,H(\nu_t,\mu)\,dt\,, \tag{4.8}$$

where the right hand side is finite provided the relative entropy goes to 0 quickly enough.

Remark 4.9. Note that if we choose $\xi^{-1}(t) = (1/T)\,\mathbb{1}_{t\leq T}$, the derivation in (4.7) furnishes

$$W_2(\nu,\mu) \,\leq T^{\frac{1}{2}}\,\left(H(\nu,\mu) \,-\, H(\nu_T,\mu)\right)^{\frac{1}{2}} \,+\, W_2(\nu_T,\mu)\,.$$

Hence a uniform decay of the Wasserstein distance implies T_2.
But this result also shows that T_2 holds as soon as it holds for the probability densities of the form $h = P_T g$ for some $T > 0$.

The natural question is thus to know whether one can find other uniform decays than the exponential one for relative entropy. Of course the exponential decay of the relative entropy

$$H(\nu_t,\mu) \,\leq\, e^{-Ct}\,H(\nu,\mu)$$

is known to be equivalent to a logarithmic Sobolev inequality, in which case it is enough to take $\xi(t) = e^{\theta t}$ with θ smaller than the inverse of the log-Sobolev constant.
More generally if for some $s > 0$ and some $\lambda > 0$,

$$H(\nu_s,\mu) \,\leq\, e^{-\lambda}\,H(\nu,\mu)$$

for all ν , using the semi group property and the fact that $t \to H(\nu_t, \mu)$ is non increasing, it is easy to see that

$$H(\nu_t, \mu) \le e^{-(\frac{t}{s}-1)\lambda} H(\nu, \mu)$$

for all $t > s$, hence the relative entropy is exponentially decaying, *but* there is some constant e^{λ} in front of the e^{-Ct}. This kind of exponential decay is no more immediately equivalent to a log-Sobolev inequality, and we may ask whether it is strictly weaker or not.

Unfortunately the following Lemma shows that in many cases both are equivalent

Lemma 4.10. *Assume that the potential V satisfies the curvature condition $Hess(V) \ge R\,Id$ for some $R \in \mathbb{R}$. Then if for some $s > 0$ and some $\lambda > 0$,*

$$H(\nu_s, \mu) \le e^{-\lambda} H(\nu, \mu)$$

for all ν, μ satisfies a logarithmic Sobolev inequality.

Proof. Recall the classical commutation properties (see [1] Thm 5.4.7 and Remark 5.4.8)

$$P_t(h \log h) - P_t h \log(P_t h) \le \frac{1-e^{-Rt}}{2R} P_t\left(\frac{|\nabla h|^2}{h}\right), \tag{4.11}$$

(just being careful since the semi-group therein is our P_{2t}) the constants being $t/2$ for $R = 0$. Integrating the right hand inequality with respect to μ we obtain for h a density of probability and $\nu = h\mu$ (see also [4, Eq. (4.4)],

$$H(\nu, \mu) \le H(\nu_t, \mu) + \frac{1-e^{-Rt}}{2R} \int \frac{|\nabla h|^2}{h}\, d\mu\,.$$

Applying this at time s we finally obtain

$$H(\nu, \mu) \le \frac{1-e^{-Rs}}{2R(1-e^{-\lambda})} \int \frac{|\nabla h|^2}{h}\, d\mu\,.$$

□

A similar result is true for the Spectral Gap Inequality, without any restriction according to the well known robust inequality

$$Var_\mu(g) \le t \int |\nabla g|^2\, d\mu + Var_\mu(P_t g)\,.$$

4.3. **A natural P_t almost stable subset.** The preceding subsection has shown that there is no hope to get some uniform decay of relative entropy without log-Sobolev. Hence we really have to find an appropriate P_t stable subset for which a restricted log-Sobolev inequality is available. As we saw in subsection 4.1 the natural one is the subset of densities smaller than constant times some gaussian density, but these sets do not seem to be P_t stable. Fortunately we can combine the ideas of both previous subsections in order to build an appropriate almost P_t stable subset. The result is the following

Lemma 4.12. *Assume that the potential V satisfies the curvature condition $Hess(V) \ge R\,Id$ for some $R \le 0$ and that $EI_\varepsilon(2)$ is satisfied. Then if $h \le K\gamma_\eta$ for $\eta < \varepsilon/2$, $P_t h \le M(K,R)\,\gamma_\beta$ with $\beta = (2\eta\, R)/(\varepsilon(e^{Rt}-1))$.*

In particular for any $\theta > 0$ there exist $T > 0$ and $\eta > 0$ such that for all $t \ge T$, $P_t h \le M\,\gamma_\theta$.

Proof. Recall the beautiful Harnack-Wang inequality (see [24], and [25] (2.1)),

$$|P_t h(x)| \leq (P_t(|h|^q))^{\frac{1}{q}}(y) \exp\left(\frac{R\, d^2(x,y)}{2(q-1)(e^{Rt}-1)}\right), \tag{4.13}$$

that holds for any (x,y), any $q>1$ and any continuous and bounded h. Again we shall integrate with respect to $\mu(dy)$, use the elementary $d^2(x,y) \leq 2d^2(x,x_0) + 2d^2(x_0,y)$ and apply Cauchy-Schwarz in order to get

$$|P_t h(x)| \leq M \exp\left(\frac{R\, d^2(x_0,x)}{(q-1)(e^{Rt}-1)}\right), \tag{4.14}$$

with

$$M = \left(\int (P_t(|h|^q))^{\frac{2}{q}}(y)\,\mu(dy)\right)^{\frac{1}{2}} \left(\int \exp\left(\frac{2R\, d^2(x_0,y)}{(q-1)(e^{Rt}-1)}\right)\mu(dy)\right)^{\frac{1}{2}}.$$

(4.14) is interesting provided M is finite, i.e provided

$$q>2 \quad , \quad \frac{2R}{(q-1)(e^{Rt}-1)} \leq \varepsilon \quad \text{and} \quad h \in \mathbb{L}^q, \tag{4.15}$$

since $u^{2/q} \leq 1+u$ for a nonnegative u if $q>2$. Note that in this case

$$\begin{aligned} \int (P_t(|h|^q))^{\frac{2}{q}}(y)\,\mu(dy) &\leq K^2(1+\int |h/K|^q\, d\mu) \\ &\leq K^2(1+\int e^{\eta\, q\, d^2} d\mu), \end{aligned}$$

so that for $\eta<\varepsilon$ the latter does not depend on η but only depends on R and ε.
Hence if $h \leq K\gamma_\eta$ for $\eta<\varepsilon/2$, we may take $q=\varepsilon/\eta$ and obtain that $P_t h \leq M(K,R)\,\gamma_\beta$ with $\beta = (2\eta\, R)/(\varepsilon(e^{Rt}-1))$. □

How we shall use this result is now clear. According to Lemma 4.3 we may assume that $h \leq K\gamma_\eta$ with η as small as we want, in order to ensure that $P_t h \leq M\gamma_\theta$ for θ small enough, and <u>all</u> t large enough. But thanks to Remark 4.9 it is enough to consider such densities and the required P_s stability is now ensured. So we have shown Corollary 1.16 at least with the additional curvature assumption.

It should be very interesting to know whether the statement of Lemma 4.12 is still true without the curvature assumption or not. This would complete the picture of what can be done using Otto and Villani coupling.

4.4. **The infimum convolution approach.** Let us as an introduction of this method present a refinement of Lemma 4.3. Actually one can obtain a more precise result if instead of Villani's coupling used in (3.7) one uses the inf-convolution method in [4].
Indeed recall that

$$W_2^2(\nu,\mu) = \sup\left(\int g\, d\nu - \int f\, d\mu\right)$$

where the supremum is running over all pairs (f,g) of measurable and bounded functions satisfying $g(x) \leq f(y) + d^2(x,y)$ for all (x,y). Adding a constant to both f and g we may assume that $\int f\,d\mu = 0$. Denote by

$$Qf(x) = \inf_{y\in E} \Big(f(y) + d^2(x,y)\Big)\,,$$

the function achieving the optimal choice. Integrating with respect to μ it holds

$$Qf(x) \le \int d^2(x,y)\,\mu(dy) \le 2d^2(x,x_0) + 2\int d^2(y,x_0)\,\mu(dy)\,,$$

i.e $Qf(x) \le 2d^2(x,x_0) + C(x_0)$.
Recall that by (see [4]), the condition $\int e^{\eta Qf}\,d\mu \le 1$ is equivalent to T_2. Remark now that for $2\eta < \varepsilon$ the density $h = e^{\eta Qf}/\int e^{\eta Qf} d\mu$ is such that either $\int e^{\eta Qf}\,d\mu \ge 1$ and $h \le e^{\eta C(x_0)}\gamma_{2\eta}$ or $\int e^{\eta Qf}\,d\mu \le 1$. We may thus focus on the first condition.
If $\beta\, W_2^2(\nu,\mu) \le H(\nu,\mu)$ for all ν such that $d\nu/d\mu \le K\,\gamma_\theta$, then for $2\eta < \theta$ and such that $e^{\eta C(x_0)} \le K$, $\nu = h\mu$ satisfies the previous condition so that

$$\beta \int Qf\,d\nu \le H(\nu,\mu)\,.$$

If in addition $\eta < \beta$ we may replace β by η in the left hand side of the previous inequality (even if this left hand side is nonpositive, since the right hand side is nonnegative), and thus obtain

$$(\int e^{\eta Qf}\,d\mu)\,\log\left(\int e^{\eta Qf}\,d\mu\right) \le 0$$

i.e $\int e^{\eta Qf}\,d\mu \le 1$ for all f, which leads then to a refined version of Lemma 4.3.

We may now go further in this infimum convolution approach and mimic arguments of [4, Sec. 3.3]. This approach based on Herbst argument is apparently not well suited for restricted logarithmic Sobolev inequalities. Indeed it requires the use of non normalized functions for which the hypothesis in Corollary 1.16 reads as

$$f^2 \le (\int f^2 d\mu)\,K\,e^{\tilde\eta\,d^2(x,x_0)}\,. \tag{4.16}$$

However, and surprisingly enough, a very slight improvement of the argument yields the result. Before starting the proof, remark that we have used a slightly different form for the definition of the infconvolution than in [4], but all calculus presented in there work is only modified by constants.

Proof. of Theorem 1.17. In fact the theorem will be established under some more general hypothesis, namely we do not need that the restricted logarithmic Sobolev inequality be verified for all the functions satisfying (1.18) but only a subclass. It is however more convenient to write the theorem with this larger class and easier to derive conditions on the real line for such a restricted logarithmic Sobolev inequality.

Remember that $Qf(x) \le 2d^2(x,x_0) + C(x_0)$ for all f such that $\int f\,d\mu = 0$. For $2\eta \le \varepsilon$ and <u>all</u> λ introduce $f_\lambda^2 = e^{\eta\, Q(\lambda f)}$ and $G(\lambda) = \int f_\lambda^2\,d\mu$. Then either $G(\lambda) \le 1$ or $G(\lambda) > 1$ and in this case f_λ^2 satisfies (4.16) (for some well choosen $\tilde\eta$ depending on η and λ) .
Assume that $G(1) > 1$ and introduce

$$\lambda_0 = \inf\{\lambda \in [0,1]\,,\, G(u) > 1 \text{ for all } u \ge \lambda\}\,.$$

Then $\lambda_0 < 1$, $G(\lambda_0) = 1$ (remark that $G(0) = 1$) and $G(\lambda) > 1$ on $]\lambda_0, 1]$. Hence the restricted log-Sobolev holds for all $\lambda \in]\lambda_0, 1]$. An easy computation using the Hamilton Jacobi semigroup described in [4] (see Section 2.1 formula 2.6 and 3.3 first formula p.380) yields

$$\lambda\, G'(\lambda) \,=\, \int f_\lambda^2\, \log f_\lambda^2\, mu \,-\, \frac{4}{\eta} \int |\nabla f_\lambda|^2\, d\mu\,.$$

We may always assume that $C\,\eta \,\leq\, 4$ (decreasing η if necessary), so that the latter yields

$$\lambda\, G'(\lambda) \,\leq\, G(\lambda)\, \log G(\lambda) \tag{4.17}$$

on $]\lambda_0, 1]$. This differential inequality can be rewritten

$$\frac{d}{d\lambda}\left(\frac{\log G(\lambda)}{\lambda}\right) \,\leq\, 0\,,$$

so that $\log G(\lambda)\, /\, \lambda$ is non increasing, hence

$$\lambda_0\, \log G(1) \,\leq\, \log G(\lambda_0) \,=\, 0\,.$$

If $\lambda_0 > 0$, we get $G(1) \leq 1$ in contradiction with our assumption $G(1) > 1$. If $\lambda_0 = 0$, $\lim_{\lambda\to 0} \frac{\log G(\lambda)}{\lambda} = \frac{G'(0)}{G(0)} = \eta \int f d\mu = 0$ and the same conclusion holds.

Hence $G(1) \leq 1$ for all f as above, which is known to be equivalent to T_2. □

5. The case of the real line.

As for many functional inequalities the one dimensional case is much simpler thanks to Hardy inequalities. We thus consider a Probability measure μ on the real line such that $\int e^{\varepsilon x^2}\, d\mu \,<\, +\infty$ and denote by v the second moment $v = \int x^2 d\mu$. We also denote by M the quantity $M = e^{2v}$. In the sequel η will be a positive number smaller than $1 \wedge \varepsilon/2$ so that $e^{2\eta v} \leq M$. Recall that $\mu(dx) = e^{-V(x)}dx$.

Let h be such that $\int h d\mu = 1$ and $h \,\leq\, M\, e^{\eta x^2}$. Define $\nu = h\mu$. For K large enough to be chosen later, we get

$$\begin{aligned} \int h\, \log h\, d\mu \;&=\; \int_{h\leq K} h\, \log h\, d\mu \,+\, \int_{h>K} h\, \log h\, d\mu \\ &\leq\; \int_{h\leq K} (h\wedge K)\, \log(h\wedge K)\, d\mu \,+\, \int_{h>K} h\, (\log M + \eta x^2)\, d\mu \\ &\leq\; \int (h\wedge K)\, \log(h\wedge K)\, d\mu \,+\, \log M\, \nu(h>K) \,+\, \int \psi^2(\sqrt{h})\, x^2\, d\mu \end{aligned} \tag{5.1}$$

where $\psi(u) = 0$ if $0 \leq u \leq \sqrt{K/2}$, $\psi(u) = (\sqrt{2}/\sqrt{2}-1)(u - \sqrt{K/2})$ if $\sqrt{K/2} \leq u \leq \sqrt{K}$ and $\psi(u) = u$ if $u \geq \sqrt{K}$. If we choose $K > 2M$ then $\psi(\sqrt{h})(0) = 0$.

We shall now bound the three terms in the right hand side of (5.1). To this end we first remark that

$$H(\nu,\mu) \,\leq\, \int M\, e^{\eta x^2}\, (\log M + \eta x^2)\, d\mu \,=\, C(\eta,\mu) \,<\, +\infty\,.$$

Hence, according to Lemma 3.4 (2), as soon as $K > e$,

$$\nu(h > K) \leq \frac{1}{\log K - 1}\, H(\nu,\mu) \leq \frac{1}{\log K - 1}\, C(\eta,\mu)\,.$$

It follows that $\log M\, \nu(h > K) \,\leq\, 1/2\, H(\nu,\mu)$ as soon as $2\log M \leq \log K - 1$. Furthermore

$$1 \geq z_K = \int (h\wedge K) d\mu \geq 1 \,-\, \frac{C(\eta, M)}{\log K - 1}$$

so that for $\log K - 1 \geq 2C(\eta, M)$, $z_K \geq 1/2$. Thus

$$\int (h\wedge K)\, \log(h\wedge K)\, d\mu \quad\leq\quad \int (h\wedge K)\, \log\left(\frac{h\wedge K}{z_K}\right)\, d\mu$$

and $h \wedge K \,\leq\, 2Kz_K$. Hence if Poincaré holds, we may apply (2.9) and get

$$\int (h\wedge K)\, \log(h\wedge K)\, d\mu \,\leq\, C_P(2\log 2 + (1/2)\log(2K)) \int \frac{(h')^2}{h}\, d\mu\,. \tag{5.2}$$

Plugging these two estimates into (5.1) we arrive at

$$(1/2)\, H(\nu,\mu) \,\leq\, C(C_P, K, \mu) \int \frac{(h')^2}{h}\, d\mu \,+ \eta \int \psi^2(\sqrt{h})\, x^2\, d\mu\,. \tag{5.3}$$

In order to obtain the desired restricted logarithmic Sobolev inequality, it remains to bound the second term in the right hand side of (5.3). To this end we shall use Hardy's inequality on the positive and on the negative half line. We only write things on the positive half line. Since $\psi(\sqrt{h})(0) = 0$, Hardy's inequality (see e.g. [1] Theorem 6.2.1) gives

$$\int_0^\infty \psi^2(\sqrt{h})\, x^2\, d\mu \,\leq\, A^+ \int_0^\infty (\psi')^2(\sqrt{h})\, \frac{(h')^2}{h}\, d\mu\,, \tag{5.4}$$

where

$$A^+ \,=\, \sup_{x\geq 0} \left(\int_x^\infty t^2\, e^{-V(t)}\, dt \int_0^x e^{V(t)}\, dt\right)\,.$$

Since ψ' is bounded we have obtained

Proposition 5.5. *Let $E = \mathbb{R}$, $d\mu = e^{-V}dx$. Assume that $EI_\varepsilon(2)$ is satisfied. Then the restricted logarithmic Sobolev inequality in Theorem 1.17 holds as soon as*

$$A^+ \,=\, \sup_{x\geq 0} \left(\int_x^\infty t^2\, e^{-V(t)}\, dt \int_0^x e^{V(t)}\, dt\right)$$

and

$$A^- \,=\, \sup_{x\leq 0} \left(\int_{-\infty}^x t^2\, e^{-V(t)}\, dt \int_x^0 e^{V(t)}\, dt\right)$$

are finite. Hence in this case μ satisfies T_2.

Note first that the boundedness of A^+ and A^- are sufficient for the Poincare inequality to hold (see e.g. [1] chapter 6, also see [9] for d-dimensional general results).

It remains to find sufficient conditions for all these hypotheses to hold. Here we shall follow section 6.4 in [1] to describe some understandable sufficient conditions. To this end we shall assume that

$$\liminf_{\infty}\, V'(x) \,>\, 0 \;\text{ and }\; V''(x)/(V'(x))^2 \,\to\, 0 \text{ when } x \text{ goes to } \infty\,. \tag{5.6}$$

Note that $V - 2\log|x|$ also fulfills (5.6). In this case it is known that μ satisfies Poincaré inequality (see Theorem 6.4.3 (1) in [1]). Furthermore A^+ is finite as soon as

$$\limsup_{\infty} \, x^2/(V'(x))^2 \, < \, +\infty \,, \tag{5.7}$$

thanks to the estimates in Corollaire 6.4.2 in [1].
Note in addition that a logarithmic Sobolev inequality holds if in only if we have (in addition to (5.6))

$$\limsup_{\infty} \, V(x)/(V'(x))^2 \, < \, +\infty \,. \tag{5.8}$$

In particular if $V(x) \leq \alpha x^2$ at infinity, (5.7) implies (5.8). According to Wang's argument, under the curvature assumption, if $V(x) \geq \alpha x^p$ at infinity for some $p > 2$ then μ satisfies a log-Sobolev inequality too. Hence it is not easy with our rough estimate in Proposition 5.5 to build an example of measure with bounded below curvature, satisfying T_2 but not the log-Sobolev inequality. If we relax the curvature assumption then the construction is simpler.

Example 5.9. We only describe the behavior of V near $+\infty$. Thus choose

$$V(x) \, = \, x^3 \, + \, 3x^2 \, \sin^2 x \, + \, x^\beta \,.$$

then

$$V'(x) \, = \, 3x^2(1 + \sin 2x) \, + \, 6x \, \sin^2 x \, + \beta \, x^{\beta - 1} \,,$$

and

$$V''(x) \, = \, 6x^2 \, \cos 2x \, + \, 6x(1 + 2\sin 2x) \, + \, 6\sin^2 x \, + \, \beta(\beta - 1) \, x^{\beta - 2} \,,$$

so that (5.6) and (5.7) are satisfied as soon as $\beta > 2$, but (5.8) is not satisfied if $\beta < 5/2$.

This furnishes an example of a measure satisfying T_2 but not the log-Sobolev inequality.

References

[1] C. Ané, S. Blachère, D. Chafai, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, and G. Scheffer. *Sur les inégalités de Sobolev logarithmiques.*, volume 10 of *Panoramas et Synthèses*. S.M.F., Paris, 2000.

[2] F. Barthe, P. Cattiaux, and C. Roberto. Interpolated inequalities between exponential and gaussian. Orlicz hypercontractivity and application to isoperimetry. To appear in Revista Matematica Iberoamericana, 2005.

[3] F. Barthe and C. Roberto. Sobolev inequalities for probability measures on the real line. *Studia Math.*, 159:481–497, 2003.

[4] S. G. Bobkov, I. Gentil, and M. Ledoux. Hypercontractivity of Hamilton-Jacobi equations. *J. Math. Pures Appl.*, 80:669–696, 2001.

[5] S. G. Bobkov and F. Götze. Exponential integrability and transportation cost related to logarithmic Sobolev inequalities. *J. Func. Anal.*, 163:1–28, 1999.

[6] S. G. Bobkov and M. Ledoux. Poincaré inequalities and Talagrand's concentration phenomenon for the exponential distribution. *Probab. Theory Relat. Fields*, 107:383–400, 1997.

[7] F. Bolley and C. Villani. Weighted Csiszar-Kullback-Pinsker inequalities and applications to transportation inequalities. *Ann. Fac. Sc. de Toulouse* 14(3):331–352, 2005.

[8] P. Cattiaux. A pathwise approach of some classical inequalities. *Potential Analysis*, 20:361–394, 2004.

[9] P. Cattiaux. Hypercontractivity for perturbed diffusion semigroups. to appear in Ann. Fac. Sc. de Toulouse, 2005.

[10] P. Cattiaux and C. Léonard. Minimization of the Kullback information for general Markov processes. Séminaire de Probas XXX. *Lect. Notes Math.*, 1626:283–311, 1996.

[11] D. Chafai. On Φ-entropies and Φ-Sobolev inequalities. *Journal of Mathematics of Kyoto University*, 44(2):325–363, 2004.
[12] D. Cordero-Erausquin. Some applications of mass transport to gaussian-type inequalities. *Arch. Rational Mech. Anal.*, 161:257–269, 2002.
[13] H. Djellout, A. Guillin, and L. Wu. Transportation cost information inequalities for random dynamical systems and diffusions. *Ann. Prob.*, 32(3B):2702–2732, 2004.
[14] R. Latala and K. Oleszkiewicz. Between Sobolev and Poincaré. in geometric aspects of Functional Analysis. *Lect. Notes Math.*, 1745:147–168, 2000.
[15] M. Ledoux. Concentration of measure and logarithmic Sobolev inequalities. Séminaire de Probas XXXIII. *Lect. Notes Math.*, 1709:120–216, 1999.
[16] M. Ledoux. Concentration, transportation and functional inequalities. In Instructional Conference on Combinatorial aspects of Mathematical Analysis. Edinburgh., 2002.
[17] K. Marton. Bounding $\bar{d}$-distance by informational divergence: a method to prove measure concentration. *Ann. Prob.*, 24:857–866, 1996.
[18] F. Otto and C. Villani. Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Func. Anal.*, 173:361–400, 2000.
[19] M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces.* Marcel Dekker Inc., 1991.
[20] G. Royer. *Une initiation aux inégalités de Sobolev logarithmiques.* S.M.F., Paris, 1999.
[21] M. Talagrand. Transportation cost for Gaussian and other product measures. *Geom. Funct. Anal.*, 6:587–600, 1996.
[22] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics.* A.M.S., Providence, 2003.
[23] F. Y. Wang. A generalization of Poincaré and log-Sobolev inequalities,*Potential Analysis*, 22: 1–15, 2005.
[24] F. Y. Wang. Logarithmic Sobolev inequalities on noncompact Riemannian manifolds. *Probab. Theory Relat. Fields*, 109:417–424, 1997.
[25] F. Y. Wang. Logarithmic Sobolev inequalities : conditions and counterexamples. *J. Operator Theory*, 46:183–197, 2001.
[26] F. Y. Wang. Probability distance inequalities on Riemannian manifolds and path spaces. *J.Func.Anal.*, 206:167–190, 2004.

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